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Ramaz Botchorishvili, Benoît Perthame, Alexis Vasseur

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Schemas d'Equilibre pour des Lois de Conservation Scalaires avec des Termes Sources Raides

Ramaz Botchorishvili,* Benoit Perthame,† Alexis Vasseur‡

Thème 4 — Simulation et optimisation
de systèmes complexes
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Résumé : Nous considérons un cas modèle de sources raides dans les systèmes de lois de conservation, le cas d'une loi scalaire avec un terme d'ordre zéro de faible régularité. Le traitement direct de telles sources avec un schémas de volume fini donne des résultats insatisfaisants du point de vue de la condition de CFL réduite et de la faible précision sur les équilibres. Le terme source doit être pris en compte dans le décentrement et discrétisé aux nœuds du maillage. Pour ce faire, nous introduisons un *schéma équilibre* avec les propriétés suivantes: (i) le principe du maximum est réalisé, (ii) toutes les inégalités d'entropie discrètes sont vérifiées (iii) les états d'équilibre du système sont maintenus. Une des difficultés dans l'étude de la convergence est l'absence de bornes VB pour ce problème. Nous introduisons donc une interprétation cinétique qui prend en compte le terme source. Grâce à la formulation cinétique nous donnons une preuve de convergence qui n'utilise que la propriété (ii) ci-dessus pour déduire la compacité forte des solutions approchées. Nous montrons les qualités numériques de notre schéma grâce à des tests qui montrent la rapidité de convergence comparée au schéma standard. De plus, nous montrons que ce schéma permet de traiter des sources singulières ou avec coefficient oscillant.

* VIAM, Tbilissi State University, 2 University Street, 380043 Tbilissi, Georgia et INRIA, M3N, domaine de Voluceau, BP 105, F78153 Le Chesnay. E-mail: Ramaz.Botchorishvili@inria.fr, rd-boch@viam.hepy.edu.ge

† INRIA, M3N, domaine de Voluceau, BP 105, F78153 LeChesnay et ENS, DMA, 45, rue d'Ulm, F75230 Paris cédex 05. E-mail:Benoit.Perthame@ens.fr

‡ Labo. J.A. Dieudonné, UMR 6621, Université Nice-Sophia Antipolis, Parc Valrose, F-06108 Nice Cedex 02, E-mail: vasseur@math.unice.fr

Unité de recherche INRIA Rocquencourt
Domaine de Voluceau, Rocquencourt, BP 105, 78153 Le Chesnay Cedex (France)
Téléphone : +33 1 39 63 55 11 — Télécopie : +33 1 39 63 53 30

Mots-clés : lois de conservation hyperboliques, formulation cinétique, termes sources raides, schémas décentrés, convergence

Equilibrium Schemes for Scalar Conservation Laws with Stiff Sources

Abstract: We consider a simple model case of stiff source terms in hyperbolic conservation laws, namely, the case of scalar conservation laws with a zeroth order source with low regularity. It is well known that a direct treatment of the source term by finite volume schemes, gives unsatisfactory results for both the reduced CFL condition and refined meshes resulting in the lack of accuracy on equilibrium states. The source term should be taken into account in the upwinding and discretized at the nodes of the grid. In order to solve numerically the problem, we introduce a so-called *equilibrium schemes* with the properties that (i) the maximum principle holds true; (ii) discrete entropy inequalities are satisfied; (iii) equilibrium solutions of the problem are maintained. One of the difficulties in studying the convergence is that there are no BV estimates for this problem. We therefore introduce a kinetic interpretation of upwinding taking into account the source terms. Based on the kinetic formulation we give a new convergence proof that only uses property (ii) in order to ensure desired compactness framework for a family of approximate solutions and that relies on minimal assumptions. The computational efficiency of our *equilibrium schemes* is demonstrated by numerical tests that show that, in comparison with an usual upwind scheme, the corresponding *equilibrium* version can converge 100 times faster. Furthermore, numerical computations show that equilibrium schemes enable to treat efficiently the sources with singularities and oscillating coefficients.

Key-words: hyperbolic conservation laws, kinetic formulation, stiff source terms, upwind schemes, convergence

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1 Introduction

Two kinds of singular sources arise classically in the theory of hyperbolic equations. A class of such problems are relaxation terms (see [4], [5], [19]) which force some algebraic relations between unknowns in such a way to ensure the system with a smaller number of unknowns in the relaxation limit. Another class consists in low regularity, possibly concentrations, in the source. The motivation of this paper comes from such a phenomena in the Saint Venant model for shallow water where bottom (bathimetry) can be irregular in practical situations. Note that even in case of sufficiently smooth bottom these terms are very dominating and important in the process of creation of equilibriums, i.e. steady state solutions.

As a simplified model problem, we consider the scalar conservation law

$$\frac{\partial u}{\partial t} + \frac{\partial A(u)}{\partial x} + B(x,u) = 0, \quad t \geq 0,\, x \in \mathbb{R}, \tag{1.1}$$

$$u(x,t=0) = u_0(x), \quad u_0(x) \in L^1 \cap L^\infty(\mathbb{R}), \tag{1.2}$$

with a smooth flux function A(.), where the unknown function $u(t,x)$ belongs to $\mathbb{R}$. Also, the equation (1.1) is endowed with the full family of entropy inequalities

$$\frac{\partial S(u)}{\partial t} + \frac{\partial \eta(u)}{\partial x} + S'(u)B(x,u) \leq 0, \tag{1.3}$$

for all convex entropy functions $S(\cdot)$ and corresponding entropy fluxes $\eta(\cdot)$ defined in accordance with the relation

$$\eta\prime(u) = S'(u)a(u), \tag{1.4}$$

see Kruzkov [13], Lax [16] for more details.

We start our consideration with the special form of the source term

$$B(x,u) = z'(x)b(u), \;\; b(u) \in C^1(\mathbb{R}).$$

We set and assume

$$a(u) = A'(u) \in C^1(\mathbb{R}), \quad \mathrm{D(u)} = \int_0^{\mathrm{u}} \frac{\mathrm{a(s)}}{\mathrm{b(s)}}\mathrm{ds}$$

$$0 < \frac{a(u)}{b(u)} < \infty, \quad D(\pm\infty) = \pm\infty, \tag{1.5}$$

$$z'(x),z(x) \in L^\infty(\mathbb{R}). \tag{1.6}$$

The assumption (1.5) is restrictive because it implies that $D(.)$ is increasing, and therefore it implies that the equilibriums are continuous. This assumption will be relaxed in the section 5 and in the Appendix. However it simplifies greatly the presentation and we first state our results in this framework.

A numerical difficulty which arises in such situations is to preserve, at a discrete level, the "equilibrium", i.e. steady states given by

$$\frac{\partial A(u)}{\partial x} + z'(x)b(u) = 0,$$

which can be explicitized in the form of the algebraic relation

$$D(u) + z(x) = \text{const}. \tag{1.7}$$

Note that condition (1.5) ensures the existence of a unique Lipschitz continuous solution to the equation (1.7) because the function D is increasing. We now consider the discrete version of (1.7)

$$D(u_j) + z_j = \text{const}, \tag{1.8}$$

where z_j represents the average of $z(x)$ over cell C_j, $C_j =]x_{j-1/2},x_{j+1/2}[$, corresponding to a uniform grid with nodal points x_j and space discretization step $\Delta x = x_{j+1/2} - x_{j-1/2}$, and we denote by u_j^n the average at time $t_n = n\Delta t$ of $u(x,t_n)$ on the cell C_j, as usual in the so called finite volume approach. We consider the *equilibrium schemes* as a solver with the properties:

$$\text{equilibrium initial data corresponding to (1.8) are maintained;} \tag{1.9}$$

$$\text{all the discrete entropy inequalities are valid;} \tag{1.10}$$

$$\text{approximate solutions are, locally in time, } L^\infty \text{ bounded.} \tag{1.11}$$

In this paper, see sections 3 and 5 for details, we introduce a class of finite volume schemes which have properties (1.9)–(1.11). In the simplest case, e.g. under the condition (1.5), the scheme has the usual simple form of an upwind finite volume scheme

$$u_j^{n+1} - u_j^n + \frac{\Delta t}{\Delta x}\Big(A(u_{j+1,-}^n,u_j^n) - A(u_j^n,u_{j-1,+}^n)\Big) = 0. \tag{1.12}$$

Here $A(.,.)$ is the usual Engquist-Osher numerical flux function, see [7],

$$A(u,v) = \int_0^u a_-(\xi)\, d\xi + \int_0^v a_+(\xi)\, d\xi, \tag{1.13}$$

with $a_\pm(\xi)$ the positive and negative parts of $a(\xi)$ defined by $a(\xi) = a_+(\xi) + a_-(\xi)$, $|a(\xi)| = a_+(\xi) - a_-(\xi)$. And the so called discrete equilibrium states $u^n_{j\pm1,\mp}$ are defined according to the relations (see (1.8) as well)

$$\begin{array}{l} D(u_{j+1,-}) + z_j = D(u_{j+1}) + z_{j+1}, \\ D(u_{j-1,+}) + z_j = D(u_{j-1}) + z_{j-1}. \end{array} \tag{1.14}$$

Notice that for a general source term $B(x,u)$ with a unique equilibrium we still can define the equilibrium scheme in the simple form (1.12). Namely we introduce the following initial value problem

$$\frac{\partial A(v)}{\partial x} + B(x,v) = 0, \tag{1.15}$$

$$v(x_j) = u_j. \tag{1.16}$$

If (1.15), (1.16) is well posed on the intervals $(x_j,x_{j+1}]$ and $(x_j,x_{j-1}]$, i.e. the unique solution to (1.15), (1.16) exists, then we can define discrete equilibrium states for the numerical equation (1.12) as

$$u_{j,+} = v(x_{j+1}), \quad u_{j,-} = v(x_{j-1})$$

Clearly, when no simple explicit formulaes are available, for numerical purposes one can apply suitable ODE solvers to (1.15), (1.16) for the definition of discrete equilibrium states.

The presented scheme is a particularly simple and efficient variant of the usual Engquist-Osher scheme which plays a particular role here because of its kinetic interpretation ([3], [12]) which allows a very simple interpretation of the relations (1.14). In the special subcase which is presented in the introduction of a single equilibrium it is not completely original and combined with a Riemann solver it can be interpreted as well balanced scheme following denominations and ideas of Greenberg *et al*, [11]. Then a proof of convergence relying on much stronger assumptions can be found in Gosse [10].More generally it falls in a class which has been advocated recently by several authors (see Greenberg *et al*, for source terms which only depends on

x, Gosse and Leroux [9] for sources depending on u only, LeVeque [17], Vazquez-Cendon [26] on upwind discretization of source terms for the Saint-Venant system, Bermudez et.al [1] for upwind treatement of sources for 2D shallow water equations on unstructured meshes). The source $z'(x)b(u)$ is discretized at the nodes of the grid, while the conservative quantities are cell centered. Notice that other strategies to preserve equilibriums also exist, see for instance Russo [22] for a central scheme. Notice also that, as well-known and as we will see later the natural discretisation of the source, as $z'_j b(u^n_j)$ is very inefficient, although it was proved that the splitting algorithm converges with a rate Δt (see [15], [2]). Concerning the case of double equilibriums, presented in Section 5, it seems that the question was not addressed before.

The main result of this paper is the following convergence theorem.

Theorem 1.1. (Convergence of the scheme). We assume (1.5), (1.6) and the CFL condition stated in (3.4), (3.8) below, and we define the approximate solution $u_{\Delta x}(t,x) = u^n_j$ for $t \in [t_n,t_{n+1})$ and $x \in C_j$. Then, the scheme (1.12)–(1.14) satisfies the properties (1.9)–(1.11) as $\Delta x \to 0$, $u_{\Delta x}(t,x)$ converges in $L^p([0,T] \times \mathbb{R})$, for all $1 \leq p < \infty$, and all $T > 0$, towards the unique entropy solution to (1.1), (1.2).

In order to avoid BV estimates, which are not available for the exact or approximate solutions due to the low regularity of the source term,and which are known to be limited to one space dimension, we design a new method of investigation of numerical schemes. It is based on a new tool for the identification of solutions to kinetic equations corresponding to entropy solutions of the problem under consideration, see section 2. It has the advantage that the same strategy extends to multidimensions without any difficulties. The analysis of the properties of the schemes is performed in section 3 and the strong convergence towards entropy solution is proved in section 4. A variant of the scheme for more general cases of functions $D(u)$ is designed in section 5. High computational efficiency of our schemes is demonstrated by numerical tests in the last section.

2 Kinetic Formulation. Uniqueness of Kinetic Solutions.

In this section we introduce a general tool for studying the convergence of numerical schemes for nonlinear scalar conservation laws. One of the most crucial steps in studying the convergence of numerical schemes for nonlinear problems is the derivation

of apriori estimates ensuring compactness of the family of approximate solutions. For hyperbolic conservation laws the suitable compactness framework traditionally is ensured by BV estimates, see [14],[23]. When no BV bounds are available, the convergence study of the schemes has to be based on weaker compactness arguments. Such arguments were introduced by DiPerna [6] with the notion of measure valued solutions. At the numerical level Szepessy [24] has shown all the interest of the method which has been widely used especially in the two dimensional case on unstructured grids.

A more powerful and easy to use approach is the so called kinetic formulation of the problem (Lions, Perthame, Tadmor [18]). Especially, this approach allows a very simple uniqueness proof (Perthame [20]) which simplifies the convergence analysis of numerical schemes. The equation (1.1), and the family of entropy inequalities (1.3) can be written equivalently as a single kinetic equation with a "density" function $\chi\Big(\xi;u(t,x)\Big)$ (but we drop the dependency upon t and x below)

Definition 2.1. The function $u(t,x)$ is called a kinetic solution if

$$\frac{\partial \chi(\xi;u)}{\partial t} + a(\xi)\cdot\frac{\partial \chi(\xi;u)}{\partial x} - b(\xi)z'(x)\frac{\partial \chi(\xi;u)}{\partial \xi} = \frac{\partial m(t,x,\xi)}{\partial \xi} \tag{2.1}$$

for some nonnegative bounded measure $m(t,x,\xi)$ which satisfies

$$m(t,x,\xi) = 0 \quad \text{for} \quad |\xi| > \|u(t,\cdot)\|_{L^\infty}, \tag{2.2}$$

and

$$\chi(\xi;u) = \begin{cases} +1, & 0 < \xi \le u, \\ -1, & u \le \xi < 0, \\ 0, & \text{otherwise.} \end{cases} \tag{2.3}$$

Evidently, equation (2.1) is linear and more attractive for investigation than the nonlinear equations (1.1) or (1.3). It is clear as well that to prove the convergence of schemes for linear equations like (2.1) there is no need in hard estimates like BV, see section 4 for details. But nonlinearities are hidden in the structure of the function $\chi(\xi;u)$ and in the measure valued source term $m(t,x,\xi)$. The relation between a function $u(t,x)$ and the related equilibrium function $\chi\Big(\xi;u(t,x)\Big)$ is given by the formula

$$u(t,x) = \int_{\mathbb{R}} \chi\Big(\xi;u(t,x)\Big)\, d\xi, \tag{2.4}$$

With the same formula and (2.3), we can recover weak distributional solutions to (1.1) just by integrating (2.1) in ξ. Since $m(t,x,\xi)$ is nonnegative in addition, the

family of entropy inequalities (1.3) for all convex entropy functions can be recovered just multiplying (2.1) by $S'(\xi)$ and integrating in ξ. On the other hand, the relation between atomic measure valued solutions by DiPerna [6], or Young measures, $\nu_{t,x}(\xi)$, a probability measure in ξ, and the kinetic equilibrium function can be explicitized as well (see also [21]),

$$\nu_{t,x}(\xi) = \delta(\xi) - \frac{\partial \chi(\xi;u)}{\partial \xi}. \tag{2.5}$$

So in frames of kinetic formulation one can recover all existing properties of entropy solutions to (1.1). Especially concerning weak limits of entropy solutions, we can naturally arrive at the notion of *kinetic solutions* to scalar conservation laws.

Definition 2.2. Let the function $f(t,x,\xi)$ belong to $L^\infty\Big(0,T;L^\infty(\mathbb{R}^2_{x,\xi})\cap \mathrm{L}^1(\mathbb{R}^2_{x,\xi})\Big)$ for all $T\geq 0$. It is called a generalized kinetic solution to equation (1.1), if

$$\frac{\partial f(t,x,\xi)}{\partial t} + a(\xi)\cdot\frac{\partial f(t,x,\xi)}{\partial x} - b(\xi)z'(x)\frac{\partial f(t,x,\xi)}{\partial \xi} = \frac{\partial m(t,x,\xi)}{\partial \xi}, \tag{2.6}$$

in the sense of distribution for some nonnegative measure $m(t,x,\xi)$ bounded on $[0,T]\times \mathbb{R}_x\times\mathbb{R}_\xi$ for all $T>0$ which satisfies

$$0\leq \mathrm{sign}(\xi)\ f(t,x,\xi) = |f(t,x,\xi)|\leq 1, \tag{2.7}$$

$$\frac{\partial f}{\partial \xi} = \delta(\xi) - \nu(t,x,\xi), \tag{2.8}$$

with $\nu(t,x,\xi)$ some nonnegative measure such that $\int_{\mathbb{R}}\nu(t,x,\xi)\,d\xi = 1$ for all t, x.

Note that it is relatively easy to arrive at *generalized kinetic solutions* departing from numerical schemes, e.g. see section 4. But we are interested only in the entropy solutions of the problem (1.1), (1.2). For this purpose we prove that *generalized kinetic solutions* are unique and have the form $f=\chi(\xi;u)$. To do so, we adapt the approach by Perthame [20] for studying the uniqueness for scalar conservation laws. It is simpler than the usual uniqueness proof of Kruzkov [13] or the one by Diperna [6] used for similar identification problem of entropy measure valued solutions, since it does not require doubling of variables. Furthermore, we arrive at simple explicit requirements that can be naturally adapted for investigation of numerical schemes.

Theorem 2.2. (Uniqueness of generalized kinetic solutions). Let $f(t,x,\xi)$ be a *generalized kinetic solution* to (1.1), (1.2), such that for a.e. $t > 0$,

$$\int_{R^2} f(t,x,\xi)\varphi(x)S'(\xi)\,dx\,d\xi + \int_0^t \int_{R^2} a(\xi)\varphi'(x)S'(\xi)f(\tau,x,\xi)\,dx\,d\xi\,d\tau$$
$$- \int_0^t \int_{R^2} (b(\xi)S'(\xi)))_\xi \varphi'(x) f(\tau,x,\xi)\,dx\,d\xi\,d\tau \le \int_{R^2} \chi(\xi;u^0(x))\varphi(x)S'(\xi)\,dx\,d\xi. \tag{2.9}$$

$$\int_R u(t,x)\varphi(x)\,dx \longrightarrow \int_R u^0(x)\varphi(x)\,dx \quad \text{as} \quad t \to 0, \tag{2.10}$$

for any convex entropy functions $S(\xi)$ and all nonnegative test functions $\varphi \in D(R)$. Then $f(t,x,\xi) = \chi(\xi;u)$ where $u(t,x)$ is the entropy solution of (1.1), (1.2) and $f(t,x,\xi) \longrightarrow \chi(\xi;u_0)$ in $L^1(\mathbb{R}^2)$ as $t \to 0$.

Proof of Theorem 2.2. The proof is based on a comparison of $|f|$ and $|f|^2$. To justify the equations on these quantities, we first regularize by setting $f_\varepsilon(t,x,\xi) = f * \omega_\varepsilon(t,x)$, $m_\varepsilon(t,x,\xi) = m * \omega_\varepsilon(t,x)$, where $\omega_\varepsilon(t,x) = \frac{1}{\varepsilon_1}\omega_1(\frac{t}{\varepsilon_1})\frac{1}{\varepsilon_2}\omega_2(\frac{x}{\varepsilon_2})$ with ε_1, $\varepsilon_2 > 0$ and ω_1, ω_2 two nonnegative regularizing kernels with $\text{supp}\Big(\omega_1(.)\Big) \subset [-\infty,0]$ ensuring the possibility to regularize for any $t > 0$. We have

$$\frac{\partial f_\varepsilon}{\partial t} + a(\xi)\cdot\frac{\partial f_\varepsilon}{\partial x} - \xi z'(x)\frac{\partial f_\varepsilon}{\partial \xi} = \frac{\partial m_\varepsilon}{\partial \xi} + \Psi_\varepsilon, \tag{2.11}$$

$$\Psi_\varepsilon = \xi\frac{\partial}{\partial \xi}[(z'(x)f) * \omega_\varepsilon - z'(x)f * \omega_\varepsilon].$$

Then following [20], we divide the proof into several steps.
(i) we prove that

$$\frac{\partial}{\partial t}\int |f|d\xi + \frac{\partial}{\partial x}\int a(\xi)|f|d\xi + z'(x)\int |f|d\xi = -2m(t,x,0). \tag{2.12}$$

This is simply obtained in multiplying (2.11) by $\text{sign}(\xi)$ and letting ε_1, ε_2 vanish, after noting that $\text{sign}(\xi)\, f_\varepsilon = |f_\varepsilon|$ by the definition 2.2.
(ii) we prove that

$$\frac{\partial}{\partial t}\int \frac{f^2}{2}\,d\xi + \frac{\partial}{\partial x}\int a(\xi)\frac{f^2}{2}\,d\xi + z'(x)\int \frac{f^2}{2}\,d\xi \ge -2m(t,x,0). \tag{2.13}$$

To do this, we multiply (2.11) by f_ε and notice that because f is bounded

$$\int \frac{f_\varepsilon^2}{2} d\xi \longrightarrow \int \frac{f^2}{2}\, d\xi, \quad \text{as} \quad \varepsilon \to 0.$$

It is therefore enough to study the righthandside appearing from (2.11). It contains two terms. the first one is treated using the property (2.8)

$$\int \frac{\partial}{\partial \xi} m_\varepsilon \, f_\varepsilon \, d\xi = - \int m_\varepsilon(t,x,\xi) \, \frac{\partial}{\partial \xi} f_\varepsilon \, d\xi$$

$$= -m_\varepsilon(t,x,0) + \int m_\varepsilon \, \nu * \omega_\varepsilon \, d\xi \geq -m_\varepsilon(t,x,0).$$

The second term is estimated as:

$$|\int f_\varepsilon \, \Psi_\varepsilon \, d\xi| = |\int (f_\varepsilon + \xi \partial_\xi f_\varepsilon) \, ((z'f) * \omega_\varepsilon - z' f_\varepsilon) \, d\xi| =$$

$$|\int \Big(f_\varepsilon - \nu(t,x,\xi) * \omega_\varepsilon\Big) \Big(\int (z'(y) - z'(x)) \, f(t,y,\xi) \omega_{1\varepsilon} \, \omega_{2\varepsilon}(x-y) \, dy\Big) \, d\xi| \leq$$

$$\int |f_\varepsilon - \nu * \omega_\varepsilon| \, d\xi \int |z'(x) - z'(y)| \, \omega_{2\varepsilon}(x-y) \, dy.$$

Evidently, this term vanishes a.e. as $\varepsilon \to 0$ since z' belongs to L^∞ according to (1.6) and thus we arrive at (2.13).
(iii) Strong continuity at $t = 0$. From Brenier's lemma [3], we deduce that $u(t,x) = \int_{\mathbb{R}} f(t,x,\xi) \, d\xi$ satisfies, for all convex functions S with $S(0) = 0$,

$$S\Big(u(t,x)\Big) \leq \int_{\mathbb{R}} S'(\xi) \, f(t,x,\xi) \, d\xi.$$

As $t \to 0$, we deduce from (2.9) and the above inequality that

$$\mathrm{w} - \lim S\Big(u(t,x)\Big) \leq \int_{\mathbb{R}} S'(\xi) \, \chi\Big(u^0(x)\Big).$$

This inequality for the weak convergence of convex function implies that

$$u(t,x) \to u^0(x), \quad \text{strongly in all } L^p, \, 1 \leq p < \infty,$$

and also

$$f(t,x,\xi) \to \chi\Big(u^0(x)\Big), \quad \text{strongly in all } L^p, \, 1 \leq p < \infty.$$

(iv) Subtracting (2.12) and (2.13), and estimating the terms with z' with account of (1.6) we have

$$\frac{\partial}{\partial t}\int (|f| - f^2)\, d\xi + \frac{\partial}{\partial x}\int a(\xi)(|f| - f^2)\, d\xi \leq \|z'\|_\infty \int (|f| - f^2)\, d\xi. \qquad (2.14)$$

Next, we now use the strong continuity of $f(t)$ at $t = 0$ (point (iii) of the proof), and we deduce that

$$|f(t=0)| - f^2(t=0) = |\chi(\xi; u^0)| - |\chi(\xi; u^0)|^2 = 0. \qquad (2.15)$$

Finally, since $|f| - f^2 \geq 0$ by the assumption (2.7), we deduce from (2.14), (2.15) that

$$|f(t)| - f^2(t) = 0, \quad \text{for all } t \geq 0.$$

Therefore f takes the values -1, 0 or $+1$. From this and the property $\int \nu\, d\xi = 1$ in (2.8), we deduce that $f = \chi(\xi; u)$ for some $u(t,x)$. Therefore u is the entropy solution (see [18]) and thus the proof is completed.

Remark 2.1 The time continuity result at $t = 0$ is the kinetic version of a similar statement discovered by Eymard *et al* [8]. It is essential to get a simple proof of convergence of numerical schemes and avoids the possible initial layers that can occur for oscillating initial datas. Also for nondegenerate fluxes, the time continuity is proved by A.Vasseur [25] with weaker assumptions. Namely one does not need the weak continuity at $t = 0$. **Remark 2.2** It is clear that the proof of the theorem is independent of the number of spatial variables and thus it holds true in multidimension as well.

Remark 2.3 It is also easy to deduce uniqueness of entropy solutions of the problem (1.1), (1.2) using the same approach as above or in [20].

Remark 2.4 It is also easy to see that under the assumptions of the Theorem 2.2, the measure valued source term in kinetic equation (2.6) satisfies

$$\int_0^t \int_{R^2} m(\tau,x,\xi)\, d\tau\, dx\, d\xi \longrightarrow 0 \quad \text{as} \quad t \to 0.$$

Remark 2.5 Kinetic solutions are more general compared to entropy solutions because thay make sense even for $u_0 \in L^1(R)$ which does not allow to define $A(u)$ when A is superlinear.

3 Main properties of the scheme

In this section, we focus on the derivation of the quantitative properties (bounds and CFL condition, entropy inequality) and we give a first connection to the kinetic formulation. We also indicate a formal reason for consistency. For the sake of simplicity we now consider the simpler case $b(\xi) = \xi$.
First of all, we can prove the *equilibrium* property (1.9) of the scheme (1.12)-(1.14) i.e. that *it preserves the equilibrium*. Indeed, at the equilibrium state then

$$u^n_{j+1,-} = u^n_j, \qquad u^n_{j-1,+} = u^n_j, \tag{3.1}$$

thus resulting in $u^{n+1}_j = u^n_j$ from (1.12). If we are not at the equilibrium state then the choice of $u^n_{j,\pm}$ creates itself the states to which the averaged values u_{j+1},u_{j-1} could be connected via equilibrium's relation from left and right respectively. That is why we call $u_{j+1,-}$,$u_{j-1,+}$ the discrete equilibriums associated with u_{j+1} on the left and the right.

We now explain the *consistency*. Compared to the classical formula (for the homogeneous problem) with fluxes $A(u^n_{j+1},u^n_j)$ at the interface $x_{j+1/2}$, we have in mind that the $u^n_{j+1,-}$ is computed so that $A(u^n_{j+1,-},u^n_j) - A(u^n_j,u^n_{j+1})$ discretizes the term $z'(x)\,u$, in a split way, at the interface $x_{j+1/2}$ for waves incoming in the cell C_j. We can precise this point and prove the consistency of the discretization

$$\Delta x\widetilde{(z'(x)u)}^n_j = [A(u^n_{j+1,-},u^n_j) - A(u^n_{j+1},u^n_j)] + [A(u^n_j,u^n_{j-1}) - A(u^n_j,u^n_{j-1,+})]. \tag{3.2}$$

It can be seen with the help of the definition (1.14) and of the following Taylor expansions

$$u_{j+1,-} - u_{j+1} = \frac{z_{j+1} - z_j}{D'(\Theta_{j+1,-}u_{j+1,-} + (1-\Theta_{j+1,-})u_{j+1})}, \qquad 0 \le \Theta_{j+1,-} \le 1;$$

$$u_{j-1} - u_{j-1,+} = \frac{z_j - z_{j-1}}{D'(\Theta_{j-1,+}u_{j-1,-} + (1-\Theta_{j-1,+})u_{j-1})}, \qquad 0 \le \Theta_{j-1,+} \le 1;$$

which implies that

$$u^n_{j+1,-} \to u^n_{j+1}, \quad u^n_{j-1,+} \to u^n_{j-1}, \quad \text{as } \Delta x \to 0,$$

from which we deduce that for some intermediate state ξ

$$A(u^n_{j+1,-},u^n_j) - A(u^n_{j+1},u^n_j) \sim a_-(\xi)(u_{j+1,-} - u_{j+1})$$

$$\sim \frac{\xi}{a(\xi)} a_-(\xi)(z_{j+1} - z_j),$$

thus showing the consistency relation (3.2) for incoming waves in the first bracket.

With the above notations, (1.12) is therefore written in terms of the usual Engquist-Osher numerical flux function $A_{j+1/2} = A(u_j^n, u_{j+1}^n)$,

$$u_j^{n+1} - u_j^n + \frac{\Delta t}{\Delta x}(A_{j+1/2}^n - A_{j-1/2}^n) + \frac{\Delta t}{\Delta x}\widetilde{(z'(x)u)}_j^n = 0.$$

But this differs deeply from the standard discretization

$$u_j^{n+1} - u_j^n + \frac{\Delta t}{\Delta x}(A_{j+1/2}^n - A_{j-1/2}^n) + \frac{\Delta t}{\Delta x}(z'(x)u)_j^n = 0, \tag{3.3}$$

that does not preserve the equilibrium although it converges, but very slowly, to steady states, see section 6.

Finally, it remains to comment the *stability properties* (1.10), (1.11). Evidently, under usual CFL condition for homogeneous equation

$$C^n \Delta t \leq \Delta x, \tag{3.4}$$

$$C^n = \sup_j(|a(u_j^n)|, |a(u_{j+1,-}^n)|, |a(u_{j-1,+}^n)|),$$

the numerical scheme (1.12) gives the usual estimate

$$\min(u_{j-1,+}^n; u_j^n; u_{j+1,-}^n) \leq u_j^{n+1} \leq \max(u_{j-1,+}^n; u_j^n; u_{j+1,-}^n). \tag{3.5}$$

Notice however that in general the inequality $|u_{j,\pm}^n| \leq |u_j^n|$ is not true and, naturally, we cannot have the same maximum principle as in the homogeneous case.
Because we use the Engquist-Osher numerical flux function, we can reformulate the scheme under a kinetic form that is more suitable for investigation. We set

$$u_j^{n+1} = \int f_j^{n+1}\, d\xi, \tag{3.6}$$

where the "density function" f_j^{n+1} is defined by the identity

$$\begin{aligned} f_j^{n+1}(\xi) - \chi_j^n(\xi) + \frac{\Delta t}{\Delta x}\Big(& a_-(\xi)\chi_{j+1,-}^n(\xi) + a_+(\xi)\chi_j^n(\xi) \\ & - a_-(\xi)\chi_j^n(\xi) - a_+(\xi)\chi_{j-1,+}^n(\xi)\Big) = 0, \end{aligned} \tag{3.7}$$

with the notations $\chi_j^n(\xi) := \chi(\xi; u_j^n)$, $\chi_{j-1,+}^n(\xi) := \chi(\xi; u_{j-1,+}^n)$, $\chi_{j+1,-}^n(\xi) := \chi(\xi; u_{j+1,-}^n)$. Indeed, integrating (3.7) in ξ and using (3.6) gives exactly (1.12), (1.13).

Lemma 3.1 Under the CFL condition (3.4) with

$$C^n = \max_{|u|\le K_\infty} |a(u)|, \tag{3.8}$$

$$K_\infty = \exp(2T\|z'(x)\|_{L^\infty})\ \|u^0\|_{L^\infty}, \tag{3.9}$$

the scheme (1.12)–(1.14) satisfies the maximum principle

$$|u_j^n| \le K_\infty, \quad \forall n\Delta t \le T, \forall j \in \mathbb{Z}, \tag{3.10}$$

and the in-cell entropy inequality for all convex entropy functions S

$$S(u_j^{n+1}) - S(u_j^n) + \frac{\Delta t}{\Delta x}\Big(\eta(u_{j+1,-}^n, u_j^n) - \eta(u_j^n, u_{j-1,+}^n)\Big) \le 0, \tag{3.11}$$

that can be written more precisely as

$$\begin{aligned}\chi_j^{n+1}(\xi) - \chi_j^n(\xi) + \frac{\Delta t}{\Delta x}\Big(&a_-(\xi)\chi_{j+1,-}^n(\xi) + a_+(\xi)\chi_j^n(\xi)\\ &-a_-(\xi)\chi_j^n(\xi) - a_+(\xi)\chi_{j-1,+}^n(\xi)\Big) = \frac{\partial m_j^{n+1}(\xi)}{\partial \xi},\end{aligned} \tag{3.12}$$

for some bounded nonnegative measures $m_j^{n+1}(\xi)$.

Proof of Lemma 3.1. We first prove the L^∞ bound (3.10). One can estimate

$$\begin{aligned}\Big|\int a_-(\xi)\,(\chi_{j+1,-}^n(\xi) - \chi_{j+1}^n(\xi))\,d\xi\Big| &= \Big|\int \xi\, D'(\xi)\,(\chi_{j+1,-}^n(\xi) - \chi_{j+1}^n(\xi))\,d\xi\Big|\\ &\le \max(|u_{j+1}^n|, |u_{j+1,-}^n|)\ |D(u_{j+1,-}^n) - D(u_{j+1}^n)|\\ &\le \Delta x\,\|z'(x)\|_{L^\infty}\ \sup_j |u_j^n|.\end{aligned} \tag{3.13}$$

By analogy, the following estimate holds true:

$$\Big|\frac{\Delta t}{\Delta x}\int a_+(\xi)\,(\chi_{j-1,+}^n(\xi) - \chi_{j-1}^n(\xi))\,d\xi\Big| \le \Delta t\,\|z'(x)\|_{L^\infty}\ \sup_j |u_j^n|. \tag{3.14}$$

Next, we rewrite (3.7) as

$$f_j^{n+1}(\xi) = \chi_j^n(\xi)\left(1 - \tfrac{\Delta t}{\Delta x}|a(\xi)|\right) - \tfrac{\Delta t}{\Delta x}a_-(\xi)\chi_{j+1}^n(\xi) + \tfrac{\Delta t}{\Delta x}a_+(\xi)\chi_{j-1}^n(\xi) \\ -\frac{\Delta t}{\Delta x}a_-(\xi)(\chi_{j+1,-}^n(\xi) - \chi_{j+1}^n(\xi)) + \frac{\Delta t}{\Delta x}a_+(\xi)(\chi_{j-1,+}^n(\xi) - \chi_{j-1}^n(\xi)) \tag{3.15}$$

and we use (3.13), (3.14). This yields

$$\sup_j |u_j^{n+1}| \le (1 + 2\Delta t\|z'(x)\|_{L^\infty}) \sup_j |u_j^n|,$$

that results in (3.10).

Next, we turn to the entropy inequalities. Notice that by definition, see (2.3), $0 \le \mathrm{sign}(\xi)\chi(\xi;v) \le 1$ for any v. Since $f_j^{n+1}(\xi)$ in (3.7) is a convex combination of such χ, we also have

$$0 \le \mathrm{sign}(\xi) f_j^{n+1}(\xi) = |f_j^{n+1}| \le 1. \tag{3.16}$$

Therefore, for some nonnegative compactly supported measure $m_j^{n+1}(\xi)$, we have

$$\chi_j^{n+1}(\xi) - f_j^{n+1}(\xi) = \frac{\partial m_j^{n+1}(\xi)}{\partial \xi}, \quad m_j^{n+1}(\xi) \ge 0, \tag{3.17}$$

as a consequence of (3.16) and Brenier's lemma [3]. Notice that the boundedness of the support of the measure $m_j^{n+1}(\xi)$ can be obtained from (3.17) with account of the uniform L^∞ estimates (3.10). Also its boundedness is a consequence of L^2 bounds on u_j^n (see [18]) after multiplying by ξ and integrating (3.17) over R_ξ. Putting (3.17) in (3.7), multiplying by $S'(\xi)$ and integrating in ξ results in the entropy property (3.12) that concludes the proof.

Remark 3.1. The CFL condition we used is rather restrictive. A better choice of C^n in (3.4) is to use the maximum speed of propagation at time t_n rather then the uniform upper bound in the interval [0,T]. Notice that this approach results in more suitable timestepping for different time levels which is usually used in numerical computations.

Remark 3.2. One can derive better uniform L^∞ as well. Namely, we set:

$$K_0 = \frac{\|z'(x)\|_{L\infty}}{\min_u D'(u) \max_i |u_i^0|},$$

$$K'_\infty = \exp((1 + K_0\Delta x)T\|z'(x)\|_{L^\infty})\,\|u^0\|_{L^\infty}.$$

Then the lemma 3.1 can be proved exactly in the same way with K_∞ replaced by K'_∞ . Notice that, in the limit $\Delta t \to 0$, it results in the natural and more precise L^∞-estimate $\|u\|_{L^\infty} \leq \exp(T\|z'(x)\|_{L^\infty})\,\|u^0\|_{L^\infty}$.

4 Proof of the Convergence Theorem

We now conclude the convergence proof. Based on the kinetic formulation, it is rather simple. Namely, we first adapt the uniqueness theorem of kinetic solutions and formulate it in another form better adapted to the numerical scheme and called 'Main Convergence Theorem'. For this purpose we formulate the requirements (consistency of the scheme, L^∞ and entropy stability, continuous dependency upon initial data) ensuring strong convergence of a family of approximate solutions. In a second step we show that the scheme satisfies the assumption of the Main Convergence Theorem.

Theorem 4.1. (Main Convergence Theorem). Let the family of approximate solutions $u_{\Delta x}(t,x) \in L^\infty(0,T;L^1(\mathbb{R}))$ satisfy, for some constant K_m, K_1, K_∞, some distribution $\Psi_{\Delta x}(t,x,\xi)$, some measure $m_{\Delta x}(t,x,\xi)$ and some function $\Psi_{0,\Delta x}(t)$,

$$\frac{\partial\chi(\xi;u_{\Delta x})}{\partial t} + a(\xi)\frac{\partial\chi(\xi;u_{\Delta x})}{\partial x} - \xi z'(x)\frac{\partial\chi(\xi;u_{\Delta x})}{\partial\xi} = \frac{\partial m_{\Delta x}(t,x,\xi)}{\partial\xi} + \Psi_{\Delta x}, \tag{4.1}$$

$$\Psi_{\Delta x}(t,x,\xi) \to 0 \text{ in } D' \text{ as } \Delta x \to 0; \tag{4.2}$$

$$m_{\Delta x}(t,x,\xi) \geq 0, \quad \|m_{\Delta x}(t,x,\xi)\|_{M^1} \leq K_m, \tag{4.3}$$

$$\|u_{\Delta x}\|_{L^1} \leq K_1, \quad \|u_{\Delta x}\|_{L^\infty} \leq K_\infty, \tag{4.4}$$

$$\begin{aligned}\int_{R^2} \chi(\xi;u_{\Delta x})\varphi(x)S'(\xi)\,dx\,d\xi + \int_0^t\int_{R^2} a(\xi)\varphi'(x)S'(\xi)\chi(\xi;u_{\Delta x})\,dx\,d\xi\,d\tau \\ \leq \int_{R^2} \chi(\xi;u^0(x))\varphi(x)S'(\xi)\,dx\,d\xi + \Psi_{0\Delta x}(t).\end{aligned} \tag{4.5}$$

$$\int u_{\Delta x}\varphi(x)\,dx\,d\xi = \int u^0(x)\varphi(x)\,dx\,d\xi + \Psi_{1\Delta x}(t), \tag{4.6}$$

for all nonnegative test functions $\varphi \in D(\mathbb{R})$ and smooth convex entropy functions S, where $\Psi_{i\Delta x}(t)$, are bounded functions such that for $i = 0,1$,

$$\Psi_{i,\Delta x}(t) \to \Psi_i(t) \text{ in } L^\infty - w*, \quad \Psi_i(t) \text{ is continuous and } \Psi_i(0) = 0. \tag{4.7}$$

Then, as $\Delta x \to 0$, $u_{\Delta x}$ converges strongly in $L^p([0,T] \times \mathbb{R})$, $1 \leq p < \infty$, to the unique entropy solution to (1.1), (1.2).

Proof of Theorem 4.1. The proof consists in proving that, passing to the limit in the above system, we obtain a function $f(t,x,\xi)$ which satisfies the assumptions of the uniqueness theorem (Theorem 2.2.) for kinetic solutions. It is decomposed into four steps: extracting a subsequence, continuity at t=0, uniqueness for the limiting problem and conclusion of the proof.

(i) *Extracting a subsequence.* Due to the uniform estimates (4.3), (4.4) we can extract subsequences and we obtain

$$\chi(\xi; u_{\Delta x}) \to f(t,x,\xi) \text{ in } L^\infty\text{-w}^*,$$

$$m_{\Delta x}(t,x,\xi) \to m(t,x,\xi) \text{ in } M^1 \text{ weak.}$$

Hence we can pass to the limit as $\Delta x \to 0$ in the linear equation (4.1) and we obtain the equation (2.6). Notice that this strategy for passing to the limit is also used to Perthame and Tzavaras [21], with applications to compensated compactness arguments.

(ii) *Continuity at $t = 0$.* Passing to the limit in (4.5)-(4.7) as $\Delta x \to 0$ results in the conditions (2.9)-(2.10) for initial time continuity of Theorem 2.2.

(iii). *Uniqueness, identification of limiting function.* By definition of the χ function one has

$$0 \leq \text{sign}(\xi)\chi(\xi; u_{\Delta x}) = |\chi(\xi; u_{\Delta x})| \leq 1; \tag{4.8}$$

$$\frac{\partial \chi(\xi; u_{\Delta x})}{\partial \xi} = \delta(\xi) - \nu_{\Delta x}, \tag{4.9}$$

and thus $\nu_{\Delta x} = \delta(\xi - u_{\Delta x})$ is a nonnegative measure satisfying $\int \nu_{\Delta x} d\xi = 1$. Passing to the limit in these relations, we arrive at the requirements (2.7), (2.8) of the Definition 2.1 for the *kinetic solution* f. It is easy to see that all requirements of the uniqueness Theorem 2.2 of kinetic solutions are satisfied, and thus we have

$$\chi(\xi; u_{\Delta x}) \to \chi(\xi; u) \text{ in } L^\infty\text{-w}^* \text{ as } \Delta x \to 0,$$

where u is the unique entropy solution to (1.1), (1.2).

(iv) *Conclusion.* From the nonlinearity of χ, and more precisely from the identity $\int S'(\xi)\chi(\xi;u)d\xi = S(u) - S(0)$, we deduce that

$$\chi(\xi;u_{\Delta x}) \longrightarrow \chi(\xi;u) \quad \text{a.e.},$$

$$u_{\Delta x} \longrightarrow u \quad \text{a.e.},$$

and this concludes the proof of the main convergence theorem.

Remark 4.1. One can relax the uniform L^1 boundedness of approximate solutions in (4.4), by means of consequent constriction of approximate solutions, on domains of finite measure and then artificial continuation by zero outside. In this case the conclusion of the theorem is strong converge of $u_{\Delta x}$ in $L^p_{loc}([0,T] \times \mathbb{R})$, $1 \leq p < \infty$, to the unique entropy solution to (1.1), (1.2) as $\Delta x \to 0$.

Remark 4.2. One of the essential features of the main convergence theorem is that it does not require bounded variations on approximate solutions. Also, it does not depend on the space dimension, and the method by means of which the family of approximate solutions is constructed. Note that uniform BV-estimate is known to be an obstacle in proving convergence in case of insufficiently smooth source terms, as here, but also in multidimension on irregular meshes. Notice also that all the requirements of the main convergence theorem are necessary.

We are now ready to prove the convergence theorem stated in the introduction.

Proof of the Convergence Theorem. The proof is based on the verification of the requirements of the main convergence theorem departing from the formulation (3.12) of the scheme. It is decomposed into four steps.

(i) *Derivation of (4.1), (4.2).* We set: $\chi_{\Delta x} := \chi(\xi;u_{\Delta x})$, $\varphi_j^n(\xi) = \varphi(t_n,x_j,\xi)$, with $\varphi(t,x,\xi)$ a test function. Notice that

$$\begin{aligned} \sum_n (\chi_j^{n+1} - \chi_j^n)\varphi_j^n = -\sum_n (\varphi_j^{n+1} - \varphi_j^n)\chi_j^{n+1} = \\ = \int \chi_{\Delta x}\varphi_t dt + \Psi_1(\varphi,\Delta x,u_{\Delta x}), \end{aligned} \tag{4.10}$$

where $\Psi_1(\varphi,\Delta x,u_{\Delta x}) \to 0$ as $\Delta x \to 0$;

$$\begin{aligned} \sum_j a_-(\xi)(\chi_{j+1}^n - \chi_j^n)\varphi_j^n = -\sum_j a_-(\xi)(\varphi_j^n - \varphi_{j-1}^n)\chi_j^n = \\ = \int a_-(\xi)\chi_{\Delta x}\varphi_x dx + \Psi_2(\varphi,\Delta x,u_{\Delta x}), \end{aligned} \tag{4.11}$$

$$\sum_j a_+(\xi)(\chi_j^n - \chi_{j-1}^n)\varphi_j^n = -\sum_j a_+(\xi)(\varphi_{j+1}^n - \varphi_j^n)\chi_j^n = \\ = \int a_+(\xi)\chi_{\Delta x}\varphi_x dx + \Psi_3(\varphi,\Delta x,u_{\Delta x}), \tag{4.12}$$

where $\Psi_2(\varphi,\Delta x,u_{\Delta x}) \to 0$, $\Psi_3(\varphi,\Delta x,u_{\Delta x}) \to 0$ as $\Delta x \to 0$;
with account of the lefthand side of (3.13) one has:

$$\int a_-(\xi)\frac{\chi_{j+1,-}^n - \chi_{j+1}^n}{\Delta x}\varphi_j^n d\xi = -\frac{z_{j+1}-z_j}{\Delta x}\int sgn(a_-(\xi))\xi\varphi_j^n\frac{\partial\chi(\xi,u_{\Delta x,-})}{\partial\xi}d\xi \\ = z'(x)\int \varphi\xi\mathrm{sgn}(a_-(\xi))\frac{\partial\chi_{\Delta x}}{\partial\xi}d\xi + \Psi_4(\varphi,\Delta x,u_{\Delta x},z'), \tag{4.13}$$

where $u_{\Delta x,-} := \Theta_{j+1,-}^n u_{j+1,-}^n + (1-\Theta_{j+1,-}^n)u_{j+1}^n$, $(t,x) \in [t_n,t_{n+1})\times C_j$ $\Psi_4(\varphi,\Delta x,u_{\Delta x},z') \to 0$ as Δx vanishes, since $u_{j+1,-} \to u_{j+1}$. Because of the same reasons

$$\int a_+(\xi)\frac{\chi_{j-1,+}^n - \chi_{j-1}^n}{\Delta x}\varphi_j^n d\xi = z'(x)\int \varphi\xi\mathrm{sgn}(a_+(\xi))\frac{\partial\chi_{\Delta x}}{\partial\xi}d\xi + \Psi_5(\cdot,\cdot,\cdot), \tag{4.14}$$

where $\Psi_5(\varphi,\Delta x,u_{\Delta x},z') \to 0$ as $\Delta x \to 0$; Multiplying of (3.15) on φ_j^n with account of (3.17), (4.9)–(4.13) (or that is the same, summing of (4.9), (4.13)) one arrives at the suitable distributional form of the equation that results in the validity of (4.1), (4.2).

(ii) *Derivation of (4.4).* Uniform L^∞ estimates are already obtained in lemma 3.1. The L^1-estimate is derived with account of (3.16) after multiplying (3.15) by $\mathrm{sign}(\xi)\Delta x$, integrating in ξ and summing in j. Namely one arrives at the estimate

$$\sum_j \Delta x|u_j^{n+1}| \le (1 + 2\Delta t\|z'(x)\|_{L^\infty})\sum_j \Delta x|u_j^n|$$

that results in $K_1 = \exp(1 + 2t\|z'(x)\|_{L^\infty})\|u^0\|_{L^1}$ in (4.4).

(iii) *Derivation of (4.3).* The sign of $m_{\Delta x}$ is already provided by lemma 3.1. To prove the bound on m in (4.3) one can use the uniform L^1 and L^∞ boundedness of the approximate solutions $u_{\Delta x}$. We multiply (3.15) by $\xi\Delta x$. Then after integration over $\mathbb{R}_\xi$ and summing in j one arrives simply at the following rough (but sufficient for our purposes) estimate $K_m = K_1 K_\infty$.
(iv) *Derivation of (4.5)- (4.7).* After multiplying of (3.15) by $\varphi_j\Delta x$, $\varphi := \varphi(x_j,\xi)$, $\varphi \in D(\mathbb{R}^2)$, then integrating in ξ over R_ξ and summing in j and summing in n until any

k,$k\Delta t \leq T$ we obtain the following expression:

$$\Delta x \sum_j \int_{R_\xi} \chi^{k+1}\varphi_j S'(\xi)d\xi = \Delta x \sum_j \int_{R_\xi} \chi^0 \varphi_j S'(\xi)d\xi + \psi_{0\Delta x}(t_{k+1},\varphi,S),$$

where

$$\begin{aligned}\psi_{0\Delta x}(t_{k+1},\varphi,S) = -\Delta t \sum_{i=0}^{k}\sum_j \int \varphi_j S'(\xi)[a_-(\xi)(\chi^i_{j+1}S'(\xi) - \chi^i_j(\xi))+ \\ a_+(\xi)(\chi^i_j(\xi) - \chi^i_{j-1}(\xi)) + a_-(\xi)(\chi^i_{j+1,-}(\xi) - \chi^i_{j+1}(\xi)) \\ +a_+(\xi)(\chi^i_{j-1}(\xi) - \chi^i_{j-1,+}(\xi)) - \Delta x \frac{\partial m^{i+1}_j(\xi)}{\partial \xi}]d\xi \\ = \Psi_{0\Delta x}(t_{k+1},\varphi) + \Delta t \Delta x \sum_{i=0}^{k}\sum_j \int \varphi_j \frac{dS'(\xi)}{d\xi} m^{i+1}_j(\xi)d\xi \leq \Psi_{0\Delta x}(t_{k+1},\varphi),\end{aligned}$$

since $S''(\xi) \geq 0$, $\varphi_j \geq 0$, $m^i_j(\xi) \geq 0$, and

$$\begin{aligned}\Psi_{0\Delta x}(t_{k+1},\varphi) \quad = \\ \Delta t \sum_{i=0}^{k}\sum_j \int [a_-(\xi)(\varphi_{j+1} - \varphi_j) + a_+(\xi)(\varphi_j - \varphi_{j-1})]S'(\xi)\chi^i_j(\xi)d\xi \\ -\Delta t \sum_{i=0}^{k}\sum_j \int \varphi_j S'(\xi)[a_-(\xi)(\chi^i_{j+1,-}(\xi) - \chi^i_{j+1}(\xi)) \\ +a_+(\xi)(\chi^i_{j-1}(\xi) - \chi^i_{j-1,+}(\xi))]d\xi.\end{aligned}$$

Then for this expression of $\Psi_{0\Delta x}$ one can estimate the right hand side using (3.15), uniform boundedness of approximate solutions $u_{\Delta x}$ and the measures $m_{\Delta x}$ that yields for $t_k \leq t \leq t_{k+1}$

$$\begin{aligned}|\Psi_{0\Delta x}| \leq t_{k+1} \max_{|u|\leq K_\infty} |a(u)| \sum_j \int |\varphi_{j+1} - \varphi_j|d\xi + \\ t_{k+1}2K_\infty \|z'(x)\|_{L^\infty} \sum_j \max_\xi |\varphi_j(\xi)| + t_{k+1}K_m \sum_j \int |\frac{\Delta x d\varphi_j(\xi)}{d\xi}|d\xi.\end{aligned}$$

Then for this expression of $\Psi_{0\Delta x}$ one can estimate the right hand side using (3.15), uniform boundedness of approximate solutions $u_{\Delta x}$ and the measures $m_{\Delta x}$ that yields

for $t_k \leq t \leq t_{k+1}$

$$|\Psi_{0\Delta x}| \leq t_{k+1} \max_{|u|\leq K_\infty} |a(u)| \sum_j \int |\varphi_{j+1} - \varphi_j| d\xi +$$
$$t_{k+1} 2K_\infty \|z'(x)\|_{L^\infty} \sum_j \max_\xi |\varphi_j(\xi)| + t_{k+1} K_m \sum_j \int |\frac{\Delta x d\varphi_j(\xi)}{d\xi}| d\xi.$$

Clearly $\Psi_{0\Delta x}(t_{k+1},\varphi)$ vanishes together with t_{k+1} for any $\varphi \in D(\mathbb{R}^2)$ that results results in the validity of (4.5), (4.7) with account of the formulaes given aboveEvidently, with $S'(\xi) = 1$ and by use of the same technique as above one can easily recover the weak continuty requirements (4.6), (4.7) of approximate solutions $u_{\Delta x}$ at $t = 0$.

Finally, applying the main convergence theorem results in the strong convergence of the equilibrium scheme that concludes the proof.

5 Extension to more general fluxes A

In this section, we extend the scheme to a more realistic situation where the function D is not monotone, motivated by the Saint-Venant system (SVS in short). This creates an additional difficulty because some discontinuous equilibriums are unstable (non entropic), and they should not be preserved by the scheme. Again we restrict our attantion to the case $b(\xi) = \xi$.

We assume that the initial data u^0 is nonnegative (and thus $u(t,x) \geq 0$ also, because it plays the role of the height of water in the SVS), that that $a(u_0) = 0$ for some $u_0 > 0$. As before we define

$$A(u) = \int_0^u a(\xi)\, d\xi, \quad D(u) = \int_{u_0}^u \frac{a(\xi)}{\xi}\, d\xi, \tag{5.1}$$

and, our assumptions are summarized by

$$\begin{array}{l} a(\xi) \text{ is increasing on }]0, +\infty[\text{ and } u_0 ge 0, \\ D \text{ is nonnegative, decreasing on }]0,u_0[, \text{ increasing on }]u_0, +\infty[, \\ D(u) \to +\infty, \quad \text{as } u \to 0 \text{ or } +\infty. \end{array} \tag{5.2}$$

Finally, for the sake of simplicity we only consider an increasing bottom z in this section. The general formulas are given in the Appendix. Setting $\Delta z_{j+1/2} = z_{j+1} -$

$z_j \geq 0$, the scheme has this form:

$$u_j^{n+1} - u_j^n + \frac{\Delta t}{\Delta x}[\quad A(u_{j+1,-}^n, u_j^n) - A(u_j^n, u_{j-1,+}^n) \tag{5.3}$$

$$+A(u_0) - A(\tilde{u}_{j,-}^n)] = 0, \tag{5.4}$$

where $A(\cdot,\cdot)$ still denotes the Engquist-Osher scheme defined by (1.13).
We set

$$\underline{u}_j^n = \inf(u_0, u_j^n) \tag{5.5}$$

$$\overline{u}_j^n = \sup(u_0, u_j^n), \tag{5.6}$$

then $u_{j-1,+}^n$, $u_{j+1,-}^n$, $\tilde{u}_{j,-}^n$, $\tilde{u}_{j,+}^n$ are defined by:

$$\begin{cases} D(u_{j-1,+}^n) = \sup(0, D(\overline{u}_{j-1}^n) - \Delta z_{j-1/2}) \\ u_{j-1,+}^n \geq u_0, \end{cases} \tag{5.7}$$

$$\begin{cases} D(u_{j+1,-}^n) = D(\underline{u}_{j+1}^n) + \Delta z_{j+1/2} \\ u_{j+1,-}^n \leq u_0, \end{cases} \tag{5.8}$$

$$\begin{cases} D(\tilde{u}_{j,-}^n) = \inf(\Delta z_{j+1/2}, D(\overline{u}_j^n)) \\ \tilde{u}_{j,-}^n \leq u_0. \end{cases} \tag{5.9}$$

First, let us explain the derivation of the flux functions described above. Equations (1.14) cannot be solved since they have either zero or two solutions in general (like in the corresponding equilibrium equations for SVS). However, we can define a natural scheme at the kinetic level. It is based on the kinetic formulation of the scalar conservation law in (2.1). For a regular solution $m = 0$ and the equation (2.1) can be solved with the help of the method of characteristics. Then, depending on the bottom's jump particules loose or win energy when crossing the cell interfaces (see fig.1). More precisely, the characteristics are given by

$$\frac{dx}{dt} = a(\xi), \qquad \frac{d\xi}{dt} = -z'(x)\,\xi. \tag{5.10}$$

Notice that this system admits the energy $D(\xi) + z(x)$ in other words

$$\frac{d}{dt}\Big(D(\xi(t)) + z(x(t))\Big) = 0, \tag{5.11}$$

and therefore the function $\chi(\xi; u)$ is constant along the trajectories $D(\xi(t))+z(x(t)) =$Cst. This method allows to construct $u_{j,-}^n$ and $u_{j,+}^n$ by an "energy barrier" (the particles

pass through the interface with a loss or gain of velocity), and provides the additional term $\tilde{u}^n_{j,-}$ from a reflexion effect (some particles do not pass through the interface and are reflected into the cell).
Notice also that, with the kinetic functions $f^n_{j,+}(\xi) = \chi(\xi,u^n_{j,+})$, $f^n_{j,-}(\xi) = \chi(\xi,u^n_{j,-})$, and $\tilde{f}^n_{j,-}(\xi) = \mathbf{1}_{\{\tilde{u}^n_{j,-} \le \xi \le u_0\}}$, the scheme corresponds at the kinetic level to the discrete equation

$$\begin{aligned} f^{n+1}_j(\xi) - \chi^n_j(\xi) \quad + \quad & \frac{\Delta t}{\Delta x}\Big(a(\xi)_-[f^n_{j+1,-}(\xi) + \tilde{f}^n_{j,-}(\xi)] + a(\xi)_+\chi^n_j(\xi) \\ & -a(\xi)_-\chi^n_j(\xi) - a(\xi)_+ f^n_{j-1,+}(\xi)\Big) = 0, \end{aligned} \tag{5.12}$$

which replaces equation (3.7) in this non monotonic case.
We show in this section the following convergence theorem

Theorem 5.1. (Convergence of the general scheme). With the notations and assumptions of Theorem 1.1. except that (1.5) is replaced by (5.2), the above scheme satisfies the properties (1.10),(1.11), maintains the equilibrium initial data

$$\begin{aligned} & D(u_j) + z_j = \text{const} \\ & \text{sign}(u_j - u_0) = \text{const}. \end{aligned}$$

As $\Delta x \to 0$, $u_{\Delta x}(t,x)$ converges in $L^p([0,T] \times \mathbb{R})$, for all $1 \le p < \infty$, and all $T > 0$, towards the unique entropy solution to (1.1), (1.2).

Proof of Theorem 5.1. The statement concerning equilibrium preservation is readily proved by a direct computation of fluxes and we skip the proof. We only point out the difference with the proof for the previous simpler case. First notice that D is decreasing for $\xi \le u_0$ so that $\tilde{u}^n_{j,-} \ge u^n_{j+1,-}$ and the support of $f^n_{j+1,-}$ and $\tilde{f}^n_{j,-}$ are disconnected. Therefore $f^n_{j-1,+}$, $f^n_{j+1,-} + \tilde{f}^n_{j,-}$ are valued in [0,1] and Lemma 3.1 is still valid (with the obvious modifications of the flux functions in the entropy inequalities). In order to show Theorem 4.1 using Theorem 2.2 we only have to show that the expression

$$T^n_j(\xi) = \frac{1}{\Delta x}(a(\xi)_-(f^n_{j+1,-} - \chi^n_{j+1}) + a(\xi)_+(\chi^n_{j-1} - f^n_{j-1,+}) + a(\xi)_-\tilde{f}^n_{j,-}) \tag{5.13}$$

converges in the sense of distribution to $z'\xi\frac{\partial f}{\partial \xi}$ which corresponds to (4.13) (4.14). As before, we denote $\chi_{\Delta x}$ the function defined by:

$$\chi_{\Delta x}(t,x,\xi) = \chi^n_j(\xi) \tag{5.14}$$

for $(t,x) \in]n\Delta t,(n+1)\Delta t[\times]j\Delta x,(j+1)\Delta x[$. We introduce a test function $\phi(t,x,\xi)$ and denote

$$\phi_j^n(\xi) = \frac{1}{\Delta t \Delta x} \int_{j\Delta x}^{(j+1)\Delta x} \int_{n\Delta t}^{(n+1)\Delta t} \phi(t,x,\xi)\, dt\, dx. \tag{5.15}$$

So when we multiply (5.13) by ϕ and integrate and summ up, we find

$$\begin{aligned}
& \Delta t\Delta x \sum_{j,n} \int T_j^n(\xi)\phi_j^n(\xi)\, d\xi \\
= \;& \Delta t\Delta x \sum_{j,n} \frac{1}{\Delta x} \Big[\int a(\xi)_-[f_{j,-}^n(\xi) - \chi_j^n(\xi)]\phi_{j-1}^n(\xi)\, d\xi \\
& + \int a(\xi)_+[\chi_j^n(\xi) - f_{j,+}^n(\xi)]\phi_{j+1}^n(\xi)\, d\xi \\
& + \int a(\xi)_- \tilde{f}_{j,-}^n(\xi)\phi_j^n(\xi)\, d\xi \Big] \\
= \;& \Delta t\Delta x \sum_{j,n} \int R_j^n(\xi)\phi_j^n(\xi)\, d\xi + \psi_0,
\end{aligned} \tag{5.16}$$

where $|\psi_0| \le \Delta x C(\partial_x \phi) \left(\sup_{j,n} (\int |T_j^n|\, d\xi) \right)$, and

$$\int R_j^n(\xi)\phi_j^n(\xi)\, d\xi = \frac{1}{\Delta x} \int_0^{u_0} a(\xi)[f_{j,-}^n(\xi) - \chi_j^n(\xi)]\phi_j^n(\xi)\, d\xi \tag{5.17}$$

$$+ \frac{1}{\Delta x} \int_{u_0}^{+\infty} a(\xi)[\chi_j^n(\xi) - f_{j,+}^n(\xi)]\phi_j^n(\xi)\, d\xi \tag{5.18}$$

$$+ \frac{1}{\Delta x} \int_0^{u_0} a(\xi)\tilde{f}_{j,-}^n(\xi)\phi_j^n(\xi)\, d\xi. \tag{5.19}$$

To analyze these terms, we denote by $\mathcal{D}_1 : \mathbb{R}^+ \to]0,u_0[$ (respectively $\mathcal{D}_2 : \mathbb{R}^+ \to]u_0, +\infty[$) the inverse of D on $]0,u_0[$ (respectively on $]u_0, +\infty[$). The right hand side term of (5.17) writes

$$\begin{aligned}
& \frac{1}{\Delta x} \int_0^{u_0} a(\xi)[f_{j,-}^n(\xi) - \chi_j^n(\xi)]\phi_j^n(\xi)\, d\xi \\
= \;& \frac{1}{\Delta x} \int_{\underline{u}_j^n}^{u_{j,-}^n} D'(\xi)\xi\phi_j^n(\xi)\, d\xi \\
= \;& \frac{1}{\Delta x} \int_{D(\underline{u}_j^n)}^{D(u_{j,-}^n)} \mathcal{D}_1(\zeta)\phi_j^n(\mathcal{D}_1(\zeta))\, d\zeta
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\Delta x}\int_{D(\underline{u}_j^n)}^{D(\underline{u}_j^n)+\Delta z_{j-1/2}} \mathcal{D}_1(\zeta)\phi_j^n(\mathcal{D}_1(\zeta))\,d\zeta \\
&= \frac{\Delta z_{j-1/2}}{\Delta x}\underline{u}_{j,-}^{n*}\phi_j^n(\underline{u}_{j,-}^{n*})
\end{aligned}
$$

where $D(\underline{u}_{j,-}^{n*})$ is given by the mean value Theorem. Especially,

$$u_{j,-}^n \le \underline{u}_{j,-}^{n*} \le \underline{u}_j^n. \tag{5.20}$$

But since D is equicontinuous on $]0, \sup u[$, from (5.20) we deduce

$$|\underline{u}_{j,-}^{n*} - \underline{u}_j^n| \le \varepsilon(\Delta z_{j-1/2}) \tag{5.21}$$

where ε is continuous, independent on n,j,Δx and $\varepsilon(0) = 0$. So the right hand side of (5.17) is equal to

$$\frac{\Delta z_{j+1/2}}{\Delta x}\underline{u}_j^n\phi_j^n(\underline{u}_j^n) + \psi_{1,j}^n + \psi_{2,j}^n \tag{5.22}$$

with

$$
\begin{aligned}
&|\psi_{1,j}^n| \le \|z'\|_{L^\infty} C(\phi)\varepsilon(\|z'\|_{L^\infty}\Delta x) \\
&\psi_{2,j}^n \le C(\phi)|z'_{\Delta x}(j\Delta x) - z'_{\Delta x}(j\Delta x + \Delta x)|,
\end{aligned}
$$

where we denote $z_{\Delta x}$ the function which is equal to $z_j + (x - j\Delta x)(z_{j+1} - z_j)/\Delta x$ on $]j\Delta x,(j+1)\Delta x[$.
We consider now two cases: $D(\overline{u}_j^n) \ge \Delta z_{j+1/2}$ and $D(\overline{u}_j^n) \le \Delta z_{j+1/2}$.

(i) *The case* $D(\overline{u}_j^n) \le \Delta z_{j+1/2}$. We have $|D(\underline{u}_j^n) - D(u_j^n)| \le \Delta z_{j+1/2}$ so from (5.22), the right hand side term of (5.17) is equal to

$$\frac{\Delta z_{j+1/2}}{\Delta x}u_j^n\phi_j^n(u_j^n) + \psi_{1,j}^n + \psi_{2,j}^n + \psi_{3,j}^n \tag{5.23}$$

with $|\psi_{3,j}^n| \le \|z'\|_{L^\infty} C(\phi)\varepsilon(\|z'\|_{L^\infty}\Delta x)$. Thanks to (5.7), $u_{j,+}^n = u_0$, so the first term of (5.18) is vanishing. Thanks to (5.9), $D(\tilde{u}_{j,-}^n) = D(\overline{u}_j^n)$, so the terms (5.18) and (5.19) together give

$$
\begin{aligned}
\psi_{4,j}^n &= \frac{1}{\Delta x}\int_{u_0}^{\overline{u}_j^n} D'(\xi)\xi\phi_j^n(\xi)\,d\xi + \frac{1}{\Delta x}\int_{\tilde{u}_{j,-}^n}^{u_0} D'(\xi)\xi\phi_j^n(\xi)\,d\xi \\
&= \frac{1}{\Delta x}\int_0^{D(\overline{u}_j^n)} [\mathcal{D}_2(\zeta)\phi_j^n(\mathcal{D}_2(\zeta)) - \mathcal{D}_1(\zeta)\phi_j^n(\mathcal{D}_1(\zeta))]\,d\zeta
\end{aligned}
$$

$$\leq \frac{\Delta z_{j+1/2}}{\Delta x} \sup_{0\leq\zeta\leq\Delta z_{j+1/2}} |\mathcal{D}_2(\zeta)\phi_j^n(\mathcal{D}_2(\zeta)-\mathcal{D}_1(\zeta))\phi_j^n(\mathcal{D}_2(\zeta))|$$
$$\leq \|z'\|_{L^\infty} C(\phi)\varepsilon(\|z'\|_{L^\infty}\Delta x)$$

since $D(\overline{u}_j^n) \leq \Delta z_{j+1/2}$ and $\mathcal{D}_1$, $\mathcal{D}_2$ are continuous at 0 with the same value.

Finally we find

$$\int R_j^n(\xi)\phi_j^n(\xi)\,d\xi = \frac{\Delta z_{j+1/2}}{\Delta x} u_j^n \phi_j^n(u_j^n) + \psi_j^n \tag{5.24}$$
$$= -z'_{\Delta x}(x)\int \partial_\xi(\xi\phi_j^n(\xi))\chi_{\Delta x}(\xi,u_j^n)\,d\xi + \psi_j^n \tag{5.25}$$

(ii) *The case* $D(\overline{u}_j^n) \geq \Delta z_{j+1/2}$. In this case $\underline{u}_j^n = u_0$, so because of (5.22), the right hand side of (5.17) is equal to

$$\frac{\Delta z_{j+1/2}}{\Delta x} u_0 \phi_j^n(u_0) + \psi_{1,j}^n + \psi_{2,j}^n. \tag{5.26}$$

Thanks to (5.9), $D(\tilde{u}_{j,-}^n) = \Delta z_{j+1/2}$ so (5.19) is equal to

$$\frac{1}{\Delta x}\int_{\tilde{u}_{j,-}^n}^{u_0} D'(\xi)\xi\phi_j^n(\xi)\,d\xi = \frac{1}{\Delta x}\int_{\Delta z_{j+1/2}}^{0} \mathcal{D}_1(\zeta)\phi_j^n(\mathcal{D}_1(\zeta))\,d\zeta$$
$$= -\frac{\Delta z_{j+1/2}}{\Delta x}\tilde{u}_{j,-}^{n*}\phi_j^n(\tilde{u}_{j,-}^{n*})$$
$$= -\frac{\Delta z_{j+1/2}}{\Delta x} u_0\phi_j^n(u_0) + \psi_{5,j}^n,$$

where $D(\tilde{u}_{j,-}^{n*})$ is obtained by the mean value Theorem, so $|D(\tilde{u}_{j,-}^{n*}) - D(u_0)| \leq \Delta z_{j+1/2}$ and consequently $|\psi_{5,j}^n| \leq \|z'\|_\infty C(\phi)\varepsilon(\|z'\|_{L^\infty}\Delta x)$. Thanks to (5.7), $D(u_{j,+}^n) = D(\overline{u}_j^n) - \Delta z_{j+1/2}$ so the term (5.18) gives

$$\frac{1}{\Delta x}\int_{u_{j,+}^n}^{\overline{u}_j^n} D'(\xi)\xi\phi_j^n(\xi)\,d\xi = \frac{1}{\Delta x}\int_{D(\overline{u}_j^n)-\Delta z_{j+1/2}}^{D(\overline{u}_j^n)} \mathcal{D}_2(\zeta)\phi_j^n(\mathcal{D}_2(\zeta))\,d\zeta.$$

Using the main value Theorem and $u_j^n = \overline{u}_j^n$, we obtain for (5.18)

$$\frac{\Delta z_{j+1/2}}{\Delta x}\overline{u}_{j,+}^{n*}\phi_j^n(\overline{u}_{j,+}^{n*}) = \frac{\Delta z_{j+1/2}}{\Delta x} u_j^n\phi_j^n(u_j^n) + \psi_{6,j}^n.$$

We have $|D(\overline{u}^{n*}_{j,+}) - D(u^n_j)| \leq \Delta z_{j-1}$, so $|\psi^n_{6,j}| \leq \|z'\|_\infty C(\phi)\varepsilon(\|z'\|_{L^\infty}\Delta x)$. So, we still recover (5.24) (5.25).
Finally, whatever the sign of $D(\overline{u}^n_j) - \Delta z_{j+1/2}$ is, putting (5.25) in (5.16) and noticing that the sum is taken on $-R/\Delta x \leq j \leq R/\Delta x$ and $0 \leq n \leq T/\Delta t$ if Supp$\phi \subset [0,T] \times [-R,R]$ we find:

$$\begin{aligned}\int T(t,x,\xi)\phi(t,x,\xi)\,dt\,dx\,d\xi \quad &= \quad -\int z'(x)\partial_\xi(\xi\phi(t,x,\xi))\chi_{\Delta x}(\xi,u^n_j)\,d\xi\,dt\,dx \\ &\quad +\tilde{\psi}_1 + \tilde{\psi}_2 + \tilde{\psi}_3\end{aligned}$$

with

$$\begin{aligned}|\tilde{\psi}_1| &\leq C(\phi,z)\epsilon(\|z'\|_{L^\infty}\Delta x) \\ |\tilde{\psi}_2| &\leq C(\phi)\int_{-R}^{R} |z_{\Delta x}(x) - z(x)|\,dx \\ |\tilde{\psi}_3| &\leq C(\phi)\int_{-R}^{R} |z'(x) - z'(x+\Delta x)|\,dx.\end{aligned}$$

Therefore $\tilde{\psi}_1, \tilde{\psi}_2, \tilde{\psi}_3$ converge to 0 when Δx tends to 0, which gives the desired result.

6 Numerical Tests

In order to study the computational capability of the approach developed above, we have considered several test problems.
(i) We use the Burgers-Hopf equation with source term describing bathimetry in the SVS model,

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\frac{u^2}{2} + z'(x)u = 0, \tag{6.1}$$

$$u(x,t=0) = 0 \ \text{ for } x > 0, \qquad u(x=0,t) = 2 \ \text{ for } t > 0. \tag{6.2}$$

The steady state solution of this problem is given by the simple relation

$$u + z = 2, \tag{6.3}$$

and the discrete equilibrium states are defined by the simple formulaes

$$u^-_{j+1} = u_{j+1} - \Delta z_{j+1/2}, \qquad u^+_{j-1} = u_{j-1} + \Delta z_{j-1/2}.$$

For a continuous bottom the function $z(x)$ is chosen as

$$z(x) = \begin{cases} \cos(\pi x), & 4.5 \leq x \leq 5.5, \\ 0, & \text{otherwise.} \end{cases} \tag{6.4}$$

For a discontinuous "bottom", we choose

$$z(x) = \begin{cases} \cos(\pi x), & 5 < x < 6, \\ 0, & \text{otherwise.} \end{cases} \tag{6.5}$$

(ii) A classical singularity arising e.g.in fluid mechanics in case of spherical symmetry leads to the model equation

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\frac{u^2}{2} = \frac{1}{x}u^2. \tag{6.6}$$

We have tested it with the initial and boundary conditions

$$u(x,t=0) = 0, \quad -5 < x < 5, \tag{6.7}$$

$$u(x=-5,t) = 10, \quad u(x=5,t) = -10, \quad t > 0, \tag{6.8}$$

we have also used, with the outflow boundary conditions and

$$u(x,t=0) = x\,\mathbf{1}_{|x|\leq 0.05}. \tag{6.9}$$

This choice of initial and boundary conditions allows to compute the exact steady solutions of the problems, (6.6)–(6.9). They are given by $u = -2x$ and $u = x$ respectively. The discrete equilibrium states $u_{j,+}$,$u_{j,-}$ are defined by solutions of the problem

$$\frac{du}{dx} = \frac{u}{x}, \qquad u(x_j) = u_j$$

in points x_{j-1} and x_{j+1} respectively (and therefore, we are always in the simple situation described in the introduction).

Numerical computations of the problem (6.1), (6.4) are performed by the standard Engquist-Osher scheme with centered finite difference approximation for $z'(x)$, see(3.3), and we compare these results with those obtained by our equilibrium version. Steady state solutions calculated by the Engquist-Osher scheme with 101 and 1001 nodal points are given on *fig.1* and *fig.2* respectively. Notice that in both cases the errors are presented in large domains($x \geq 4.5$). On fig.1 the errors are significant. On fig.2 they are 10 times smaller but still distinguishable. Steady state solutions

calculated by the corresponding equilibrium version of the scheme with 101 nodes in space is given on *fig.3*. The advantage of the equilibrium scheme is evident. It should be emphasized that the standard scheme converges very slowly and mesh refinement does not help much to improve the accuracy of computations, see table 1.

Table 1. Comparison of errors

$Number$	$Engquist-Osher$	$scheme$	$Equilibrium$	$scheme$
$of\ nodes$	$L^\infty - error$	$L^1 - error$	$L^\infty - error$	$L^1 - error$
101	0.1651	0.4880	$6.434 \cdot 10^{-5}$	$1.263 \cdot 10^{-5}$
201	$8.452 \cdot 10^{-2}$	0.2625	–	–
401	$4.272 \cdot 10^{-2}$	0.1360	–	–
801	$2.146 \cdot 10^{-2}$	$6.847 \cdot 10^{-2}$	–	–
1001	$1.718 \cdot 10^{-3}$	$5.532 \cdot 10^{-2}$	–	–
2501	$6.864 \cdot 10^{-3}$	$2.217 \cdot 10^{-2}$	–	–
5001	$3.520 \cdot 10^{-3}$	$1.115 \cdot 10^{-2}$	–	–
10001	$3.694 \cdot 10^{-3}$	$5.561 \cdot 10^{-3}$	–	–

Thus even for this simple test problem numerical results obtained by the Equilibrium schemes are far better then those obtained by centered source terms in the Engquist-Osher scheme with 100 times more nodes in space. So we can suppose that the equilibrium version of Engquist Osher scheme converges at least 100 times faster in comparison with the standard one. Emphasizing from the experience of computations that the errors are usually amplified with the complexity and irregularity of bottoms, we can conjecture that the efficiency of equilibrium schemes will be more significant in higher dimensions and with more complicated bottoms. To demonstrate capabilities of the method to treat bottoms with singularities, computations are performed with discontinuous function z defined according to (6.5). In order to visualize convergence history, numerical results are given in subsequent time moments, see *fig. 4-10*. Deviations from steady state solution given by (6.3) are $6.50883 \cdot 10^{-5}$ and $1.28746 \cdot 10^{-5}$ in L^1 and L^∞ respectively. Notice that the errors are of the same order as in the case of continuous bottom. Also notice that with centered source terms, the scheme with 101 nodes in space is unstable in this case of a discontinuous bottom. The situation improves with 1001 nodes in space and one can perform computations by standard Engquist-Osher version but the errors are too high, see fig. 11.
Thus one can conclude that computational efficiency of our equilibrium scheme is independent of the complexity of the bottom functions. This conclusion is again confirmed by the numerical results visualising the convergence history of the equili-

brium scheme for test problems (6.6)–(6.9) containing another type of singularities in the sources, see fig.13- 19. Notice that the standard version is unstable, see fig.11-12for the corresponding divergence history.

(iii) Next numerical test shows the efficiency of the developed equilibrium approach even in case of simple linear flux function $A(u) = u$. We show that the developed approach can be successfully applied for the numerical treatment of another type of sources, namely to sources containing oscillatory coefficients. As a test problem we consider the linear equation

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = z(x)u, \tag{6.10}$$

$$u(0,x) = 0, \tag{6.11}$$

$$u(t,0) = 2, \tag{6.12}$$

$$z(x) = \cos(\frac{x}{\varepsilon}), \tag{6.13}$$

where $\varepsilon > 0$ is a parameter. For numerical solution of (6.10)-(6.13) the following scheme is applied:

$$\frac{u^{n+1} - u_j^n}{\Delta t} + \frac{u_j^n - u_{j-1,+}^n}{\Delta x} = 0 \tag{6.14}$$

where $u_{j-1,+}^n$ is defined from the equilibrium relation as solution of the problem

$$\frac{du}{dt} = z(x)u, \qquad u(x_{j-1}) = u_{j-1}^n$$

calculated at $x = x_j$ calculations are performed with 101 nodal points in space. Numerical results are given for two different times for different values of ε,$\varepsilon = 0.2$,$\varepsilon = 0.1$,$\varepsilon = 0.05$, see fig. 20,21,23. From these figures good convergence towards exact solution is evident. One can conclude as well that the equilibrium scheme gives maximum possible accuracy even with extremally rough grids. Compare the resolution provided by the grid for $z(x)$ and the corresponding exact and numerical solution. Notice that the standard first order scheme is inaccurate and application of timestepping procedure results in a wrong steady state solution, see fig. 22.

7 Appendix.The scheme for non-monotonic bottom z

We give here the complete scheme of section 5 for a non-monotonic bottom z. Again, we assume that the initial data u^0 is nonnegative, $a(\xi)$ is increasing on $]0, +\infty[$ with

$a(u_0) = 0$ for some $u_0 > 0$ and we still assume (5.2). Then the scheme has the form

$$u_j^{n+1} - u_j^n + \frac{\Delta t}{\Delta x} \quad [A(u_{j+1,-}^n, u_j^n) - A(u_j^n, u_{j-1,+}^n) \tag{7.1}$$

$$+A(\tilde{u}_{j,+}^n) - A(\tilde{u}_{j,-}^n)] = 0. \tag{7.2}$$

Here $A(\cdot,\cdot)$ still denotes the Engquist-Osher scheme

$$A(u_1,u_2) = \int_0^{u_1} a(\xi)_- \, d\xi + \int_0^{u_2} a(\xi)_+ \, d\xi, \tag{7.3}$$

with $a(\xi)_+ = a(\xi)\mathbf{1}_{\{\xi \geq u_0\}}$, $a(\xi)_- = a(\xi)\mathbf{1}_{\{0 \leq \xi \leq u_0\}}$. And, setting

$$\underline{u}_j^n = \inf(u_0, u_j^n) \tag{7.4}$$

$$\overline{u}_j^n = \sup(u_0, u_j^n), \tag{7.5}$$

then the values $u_{j-1,+}^n$, $u_{j+1,-}^n$, $\tilde{u}_{j,-}^n$, $\tilde{u}_{j,+}^n$ are defined by

$$\begin{cases} D(u_{j-1,+}^n) = \sup(0, D(\overline{u}_{j-1}^n) - \Delta z_{j-1/2}) \\ u_{j-1,+}^n \geq u_0, \end{cases} \tag{7.6}$$

$$\begin{cases} D(u_{j+1,-}^n) = \sup(0, D(\underline{u}_{j+1}^n) + \Delta z_{j+1/2}) \\ u_{j+1,-}^n \leq u_0, \end{cases} \tag{7.7}$$

$$\begin{cases} D(\tilde{u}_{j,-}^n) = \sup(0, \inf(\Delta z_{j+1/2}, D(\overline{u}_j^n))) \\ \tilde{u}_{j,-}^n \leq u_0, \end{cases} \tag{7.8}$$

$$\begin{cases} D(\tilde{u}_{j,+}^n) = \sup(0, \inf(-\Delta z_{j-1/2}, D(\underline{u}_j^n))) \\ \tilde{u}_{j,+}^n \geq u_0. \end{cases} \tag{7.9}$$

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Références

[1] Bermudez A., Dervieux A., Desideri J-A., Vazquez M.E., Upwind schemes for two-dimensional shallow water equations with variable depth using unstructured meshes, Comput. Methods Appl. Mech. Engrg., **155**(1998) 49-72.

[2] Bouchut F., Perthame B. Kruzkov's estimates for scalar conservation laws revisited, Trans. A.M.S. **350**(7) (1998) 2847–2870.

[3] Brenier Y., Résolution d'équations d'évolution quasilinéaires en dimensions N d'espace à l'aide d'équations linéaires en dimensions N+1, J. Diff. Eq. **50**(3) (1982) 375–390.

[4] Chen G.-Q., Levermore C.D., Liu T.P., Hyperbolic conservation laws with stiff relaxation terms and entropy, Comm. Pure Appl. math. **48**(7) (1995) 787–830.

[5] Coquel F., Perthame B., Relaxation of energy and approximate Riemann solvers for general pressure laws in fluid dynamics, SIAM J. Num. Anal. **35**(6) (1998) 2223–2249.

[6] DiPerna R.J., Measure valued solutions to conservation laws, Arch. Rat. Mech. Anal. **88** (1985) 223–270.

[7] Engquist B., Osher S., Stable and entropy satisfying approximations for transonic flow calculations, Math. comp. **34** (1980) 45–75.

[8] Eymard R., Gallouët T., Herbin R., Existence and uniqueness of the entropy solution to a nonlinear hyperbolic equation, Chin. Ann. Math. Ser. B **16** (1) (1995) 1–14.

[9] Gosse L., Leroux A.-Y., A well-balanced scheme designed for inhomogeneous scalar conservation laws, C. R. Acad. Sc., Paris, Sér. I **323** (1996) 543–546.

[10] Gosse L., Localization effects and measure source terms in numerical schemes for balance laws, Preprint.

[11] Greenberg J. M., LeRoux A.-Y., Baraille R., Noussair A., Analysis and approximation of conservation laws with source terms, SIAM J. Numer. Anal. **34** (5)(1997) 1980–2007.

[12] Giga Y., Miyakawa T., A kinetic construction of global solutions of first-order quasilinear equations, Duke Math. J. **50** (1983) 505–515.

[13] Kruzkov S.N., Generalized solutions of the Cauchy problem in the large for nonlinear equations of first order, Dokl. Akad. Nauk. SSSR **187**(1) (1970) 29–32; English trans, Soviet Math. Dokl. **10** (1969) .

[14] Kuznetsov N.N., Finite difference schemes for multidimensional first order quasilinear equation in classes of discontinuous functions, in: "Probl. Math. Phys. Vych. Mat.". Moscow: Nauka (1977) 181-194.

[15] J.O. Langseth, A. Teveito and R. Winther, On the convergence of operator splitting applied to conservation laws with source terms, SIAM J. Num. Anal. **33** (1996) 843–863.

[16] Lax P., Shock waves and entropy, in: "Contributions to Nonlinear Functional Analysis." E.H. Zarantonello, ed. New York: Academic Press (1971) 603–634.

[17] Leveque R., *Numerical Methods for Conservation Laws, Lectures in Mathematics*, ETH Zurich, Birkhauser (1992).

[18] Lions P.L., Perthame B., Tadmor E., A kinetic formulation of multidimensional scalar conservation laws and related equations, J. Amer. Math. Soc. **7** (1994) 169–191.

[19] Natalini R., Convergence to equilibrium for the relaxation approximations of conservation laws, Comm. Pure Appl. Math. **49** (1996) 1–30.

[20] Perthame B., Uniqueness and error estimates in first order quasilinear conservation laws via the kinetic entropy defect measure, J. Math. P. et Appl. **77** (1998) 1055–1064.

[21] Perthame B., Tzavaras A., Kinetic formulation for systems of two conservation laws and elastodynamics. To appear.

[22] Russo G., personal communication.

[23] Sanders R., On the convergence of monotone finite difference schemes with variable spatial differencing, Math.Comp., V.40 (161), (1983) 91-106.

[24] Szepessy A., Convergence of a streamline diffusion finite element method for conservation law with boundary conditions, RAIRO Model. Math. et Anal. Num. **25** (1991) 749–783.

[25] Vasseur A., Time regularity for the system of isentropic gas dynamics with $\gamma = 3$, Comm. in P.D.E. **24** (1999) 1987–1997.

[26] Vazquez-Cendon M.E., Improved treatement of source terms in upwind schemes for shallow water equations in channels with irregular geometry, J.Comput.Phys., **148(2)** (1999) 497-526.

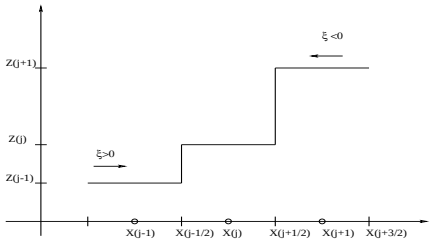


FIG. 1 – *piecewise smooth bottom*

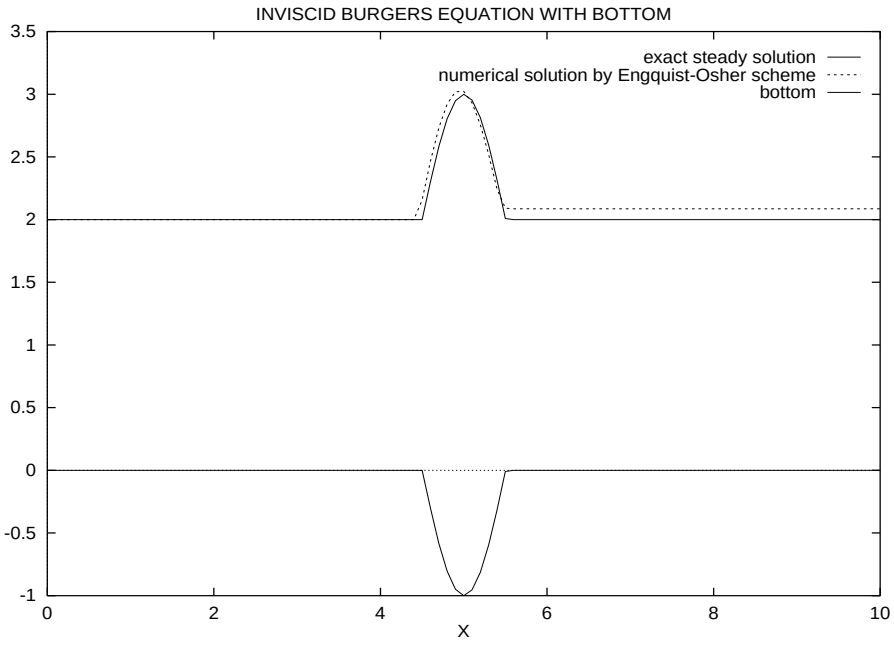


FIG. 2 – *Standard Engquist-Osher scheme, 101 nodes in space*

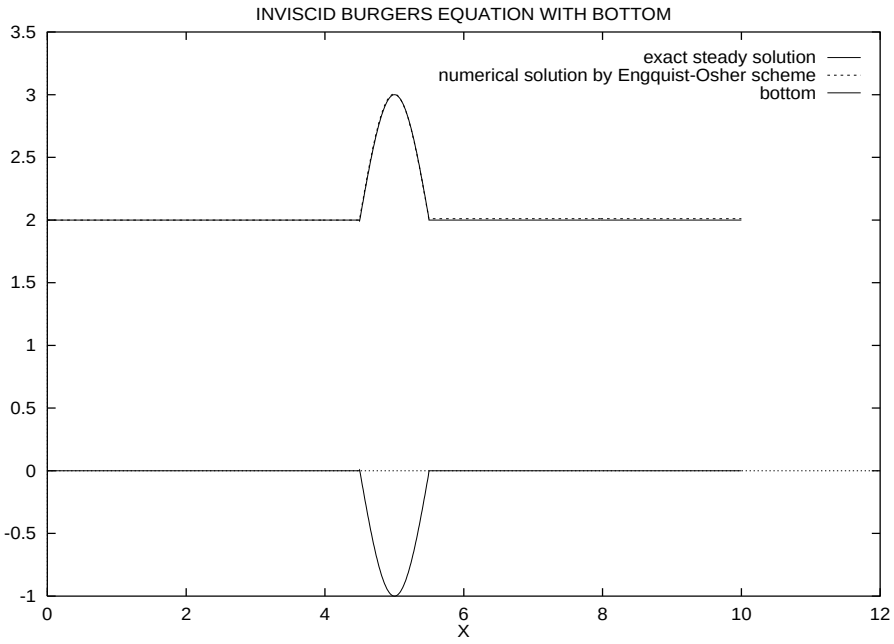


FIG. 3 – *Standard Engquist-Osher scheme, 1001 nodes in space*

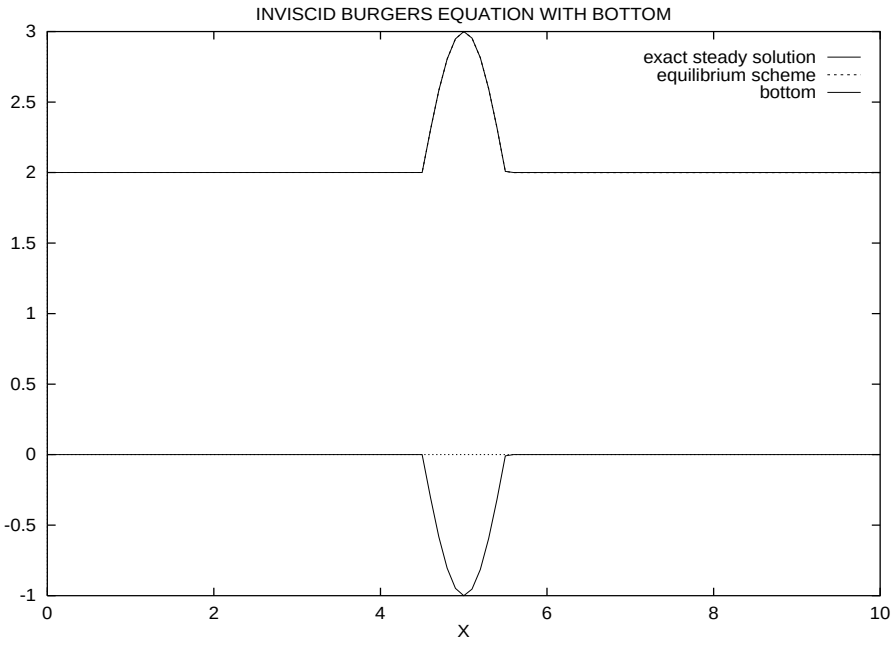


FIG. 4 – *Equilibrium version of Engquist-Osher scheme, 101 nodes in space*

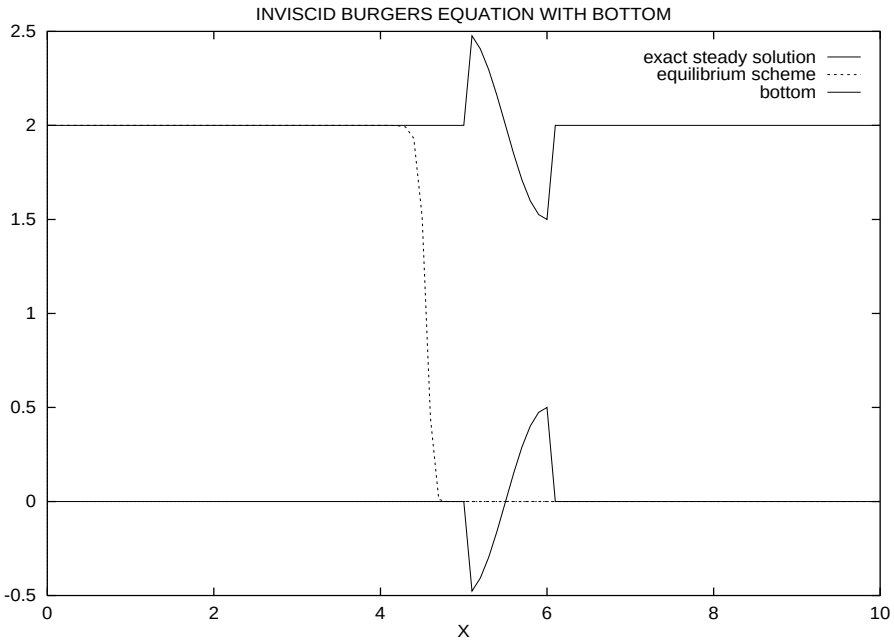


FIG. 5 – *Equilibrium version of Engquist-Osher scheme, 101 nodes in space*

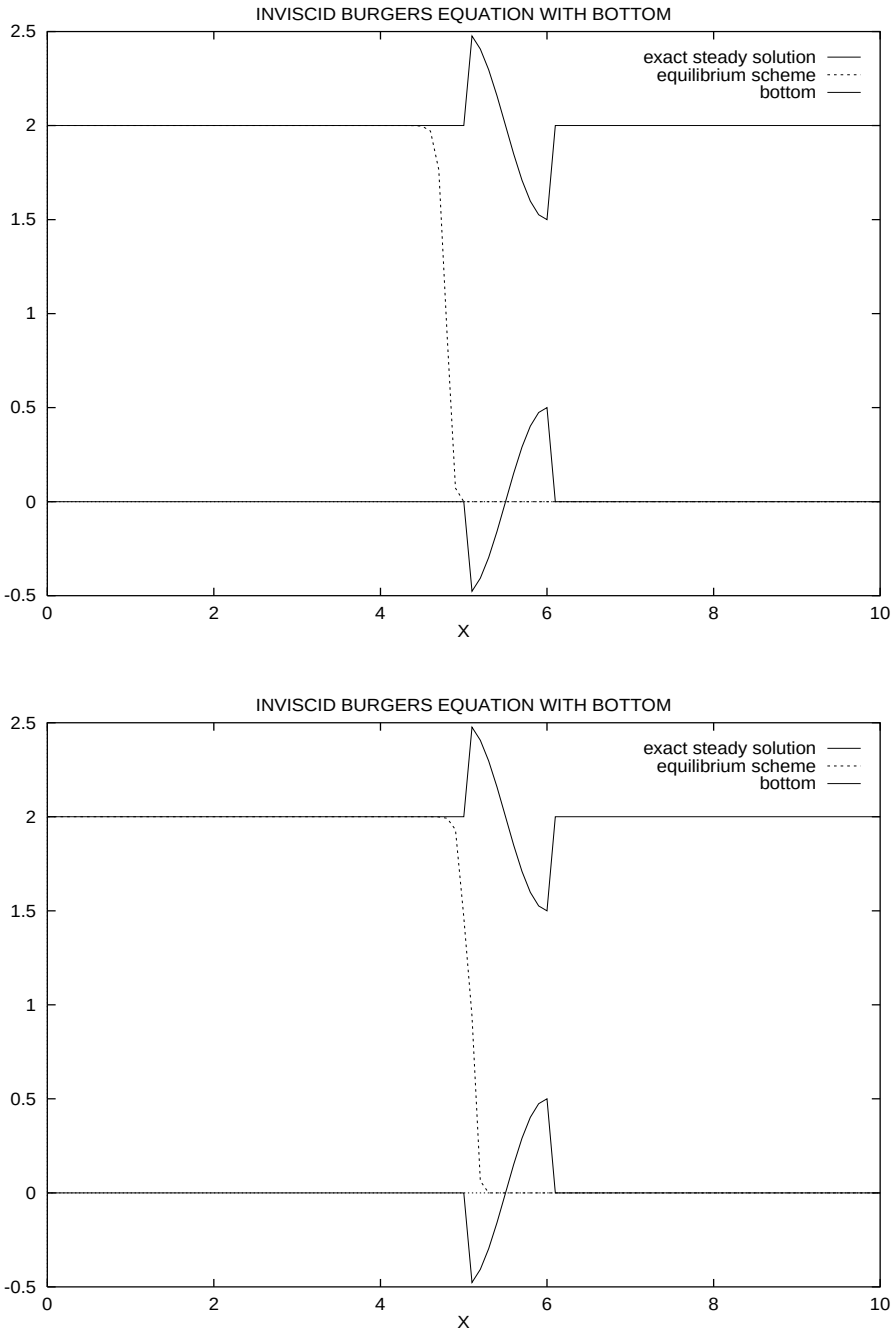


FIG. 6 – *Equilibrium version of Engquist-Osher scheme, 101 nodes in space*

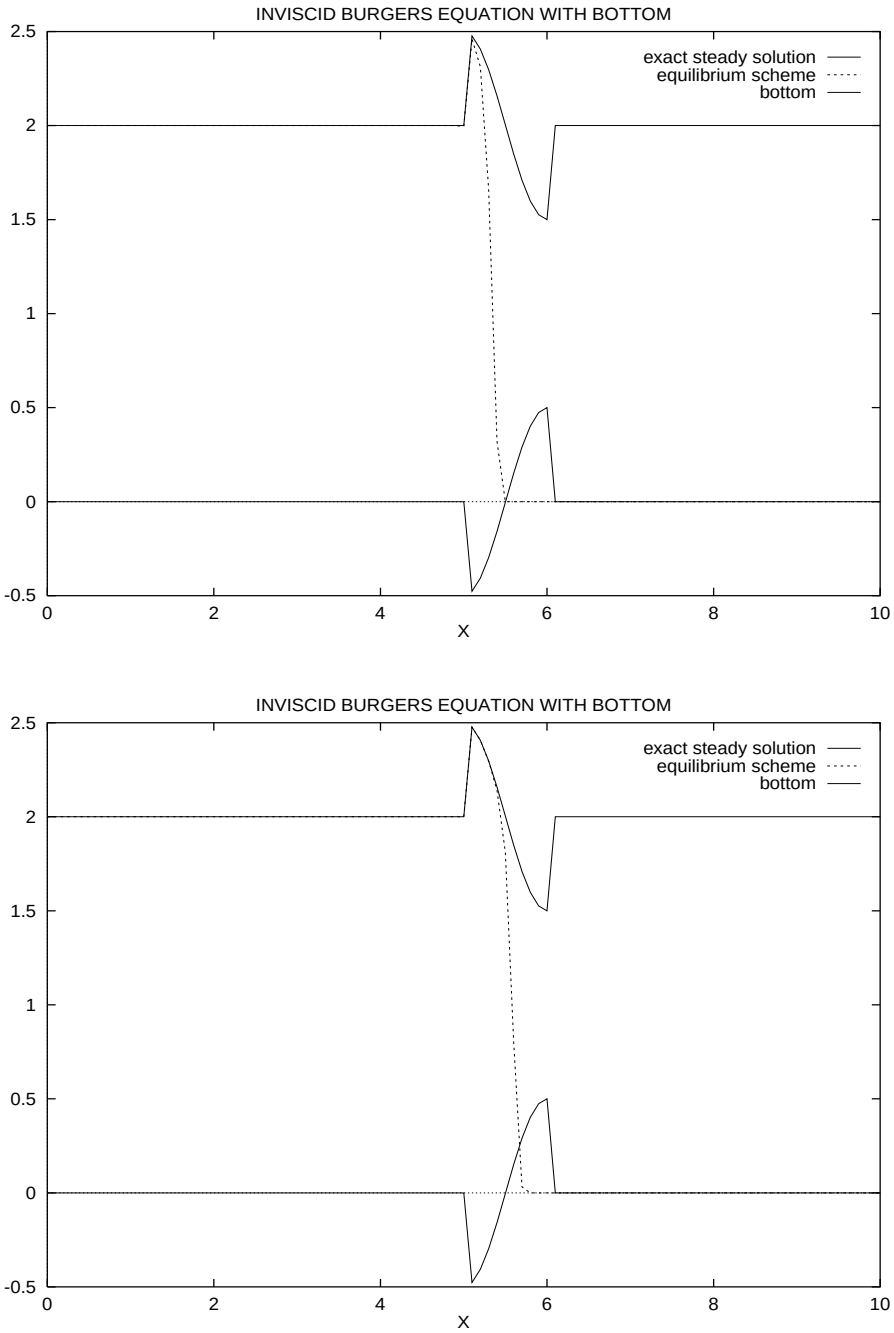


FIG. 7 – *Equilibrium version of Engquist-Osher scheme, 101 nodes in space*

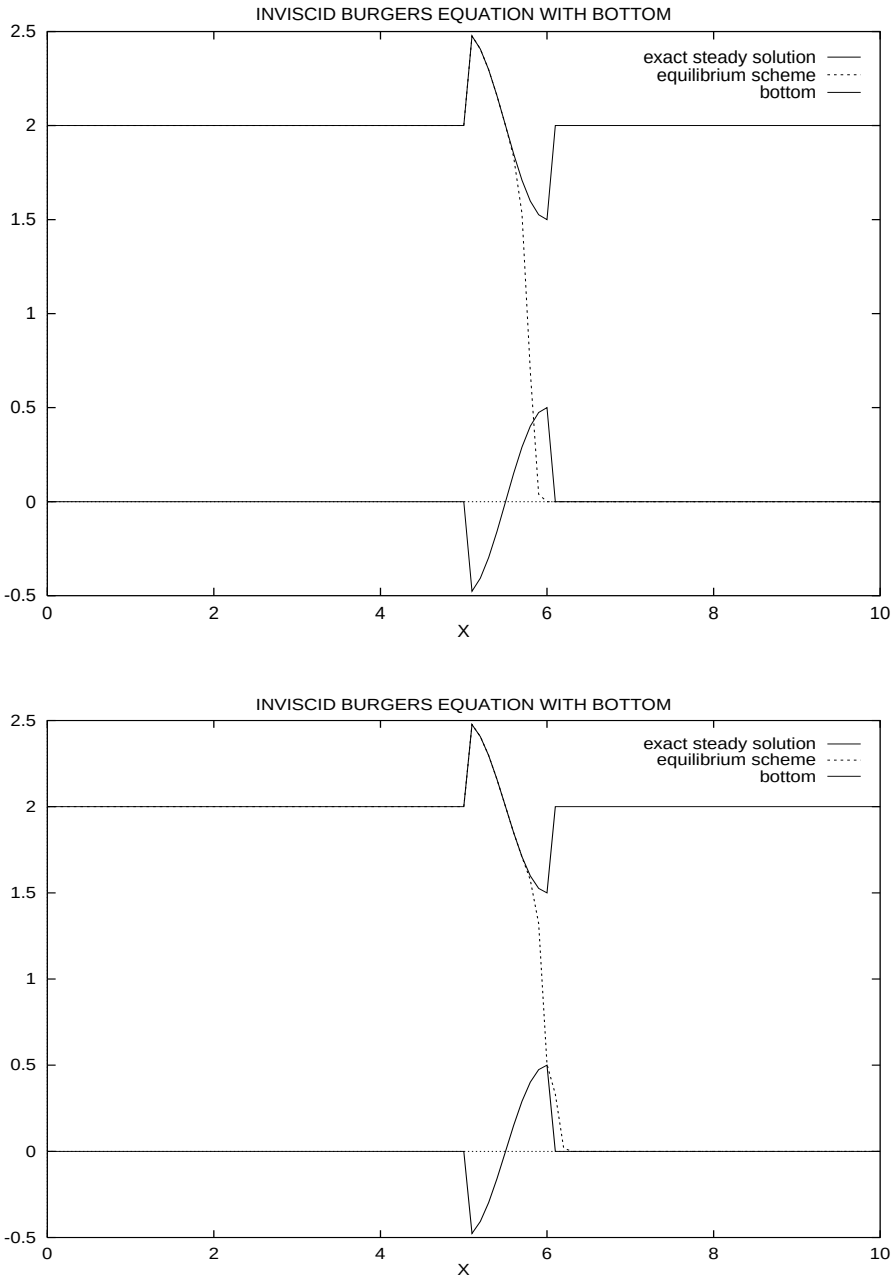


FIG. 8 – *Equilibrium version of Engquist-Osher scheme, 101 nodes in space*

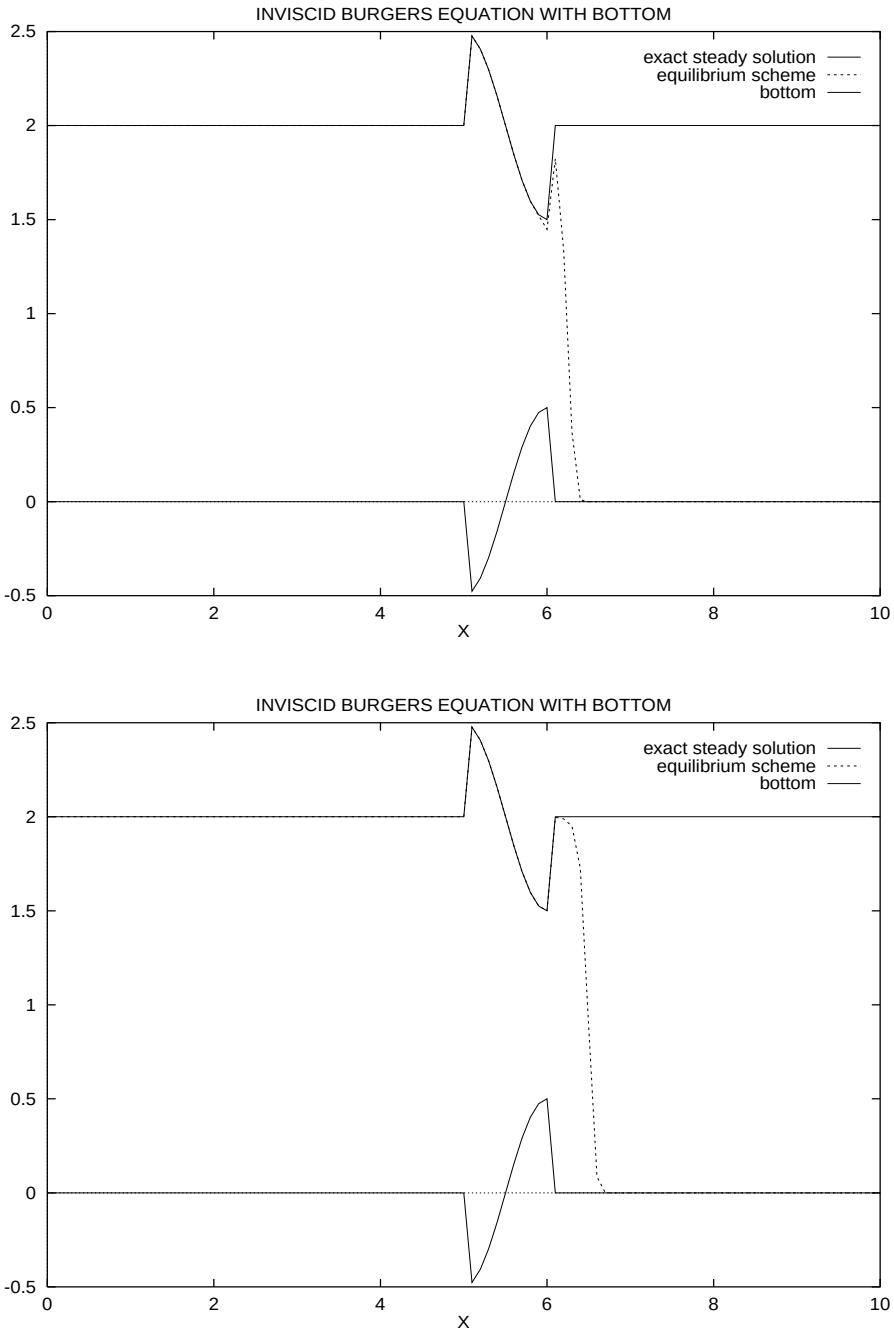


FIG. 9 – *Equilibrium version of Engquist-Osher scheme, 101 nodes in space*

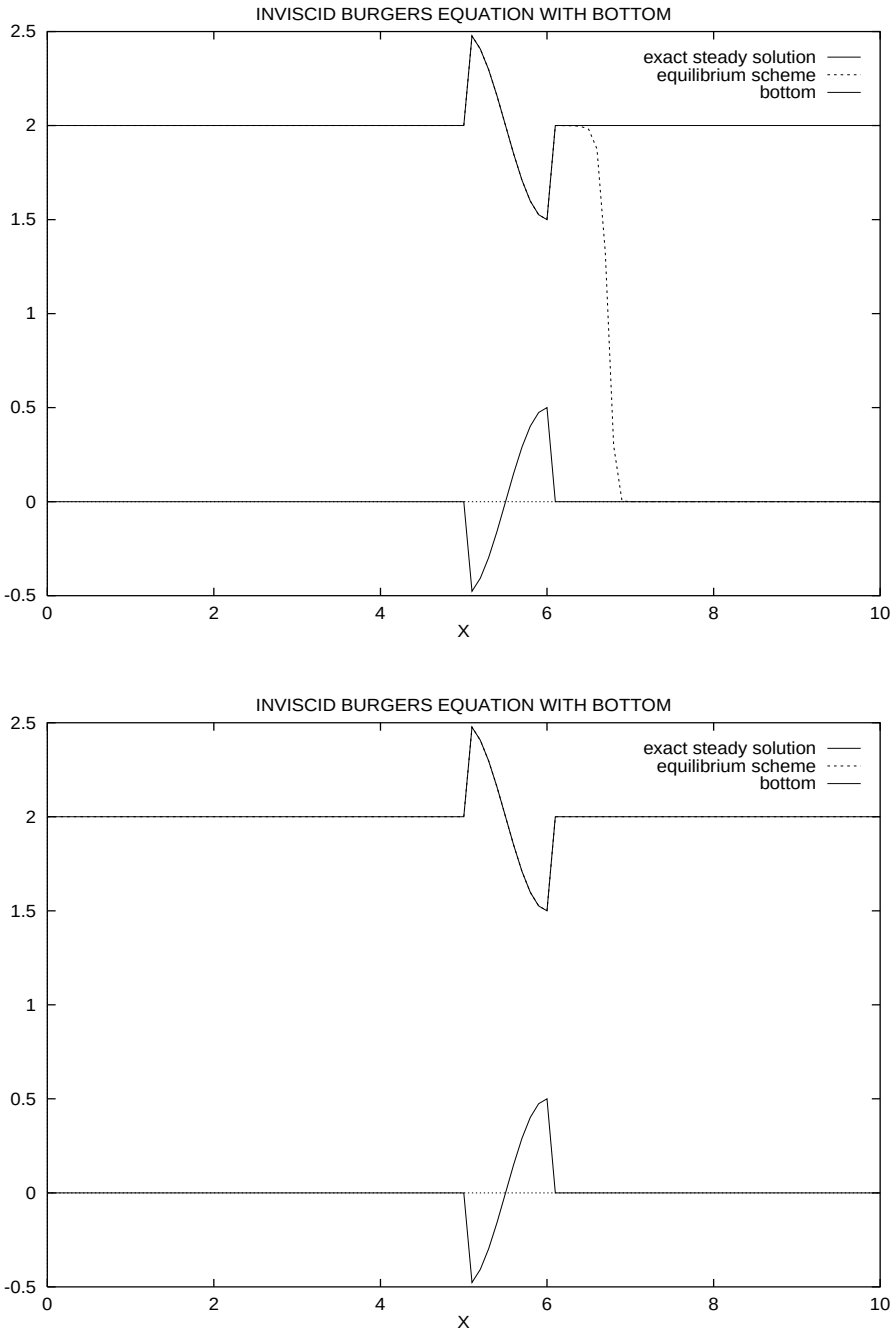


FIG. 10 – *Equilibrium version of Engquist-Osher scheme, 101 nodes in space*

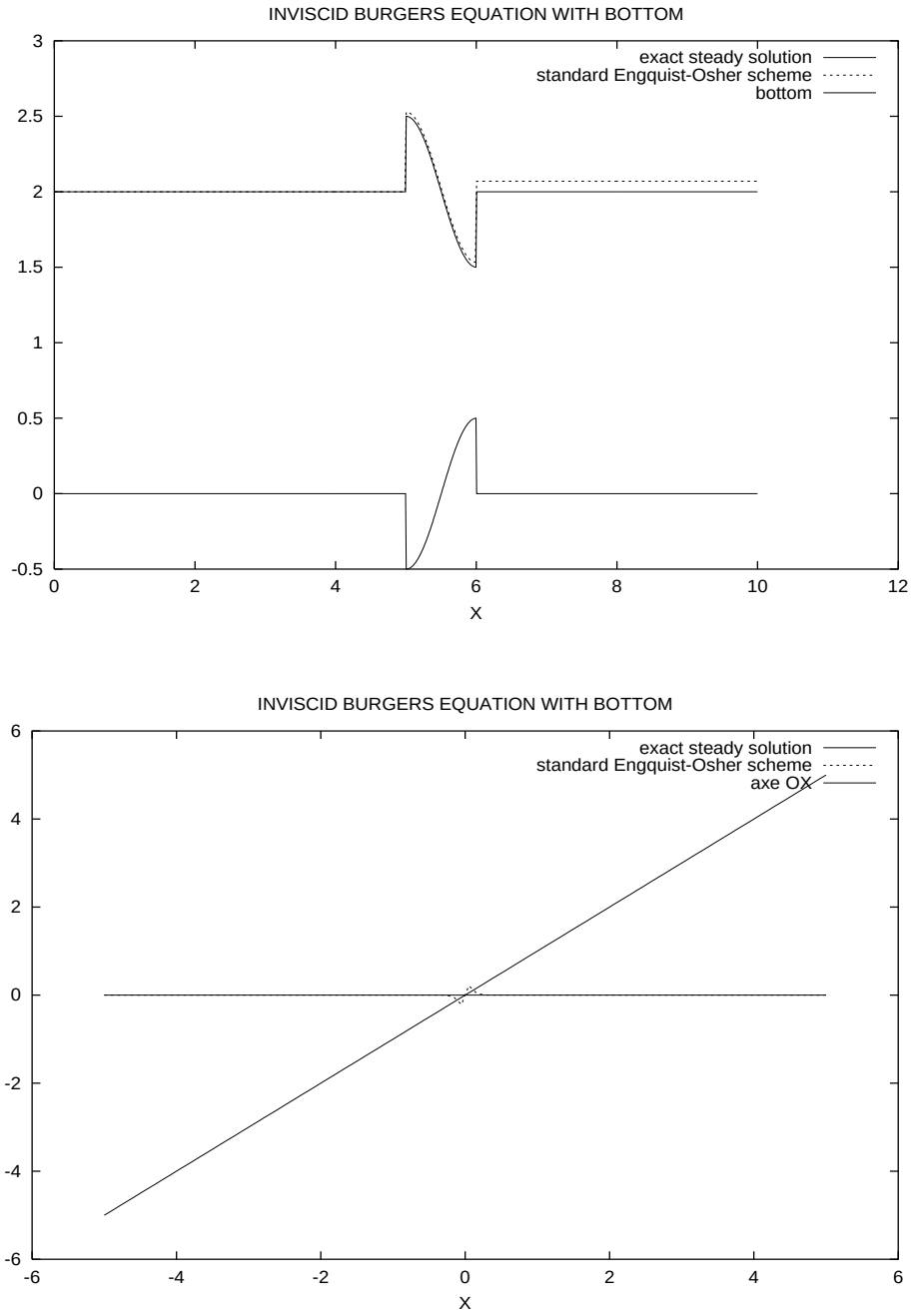


FIG. 11 – *Upper:discontinuous bottom, 1000 nodes in space; Lower: divergence history, 100 nodes in space.*

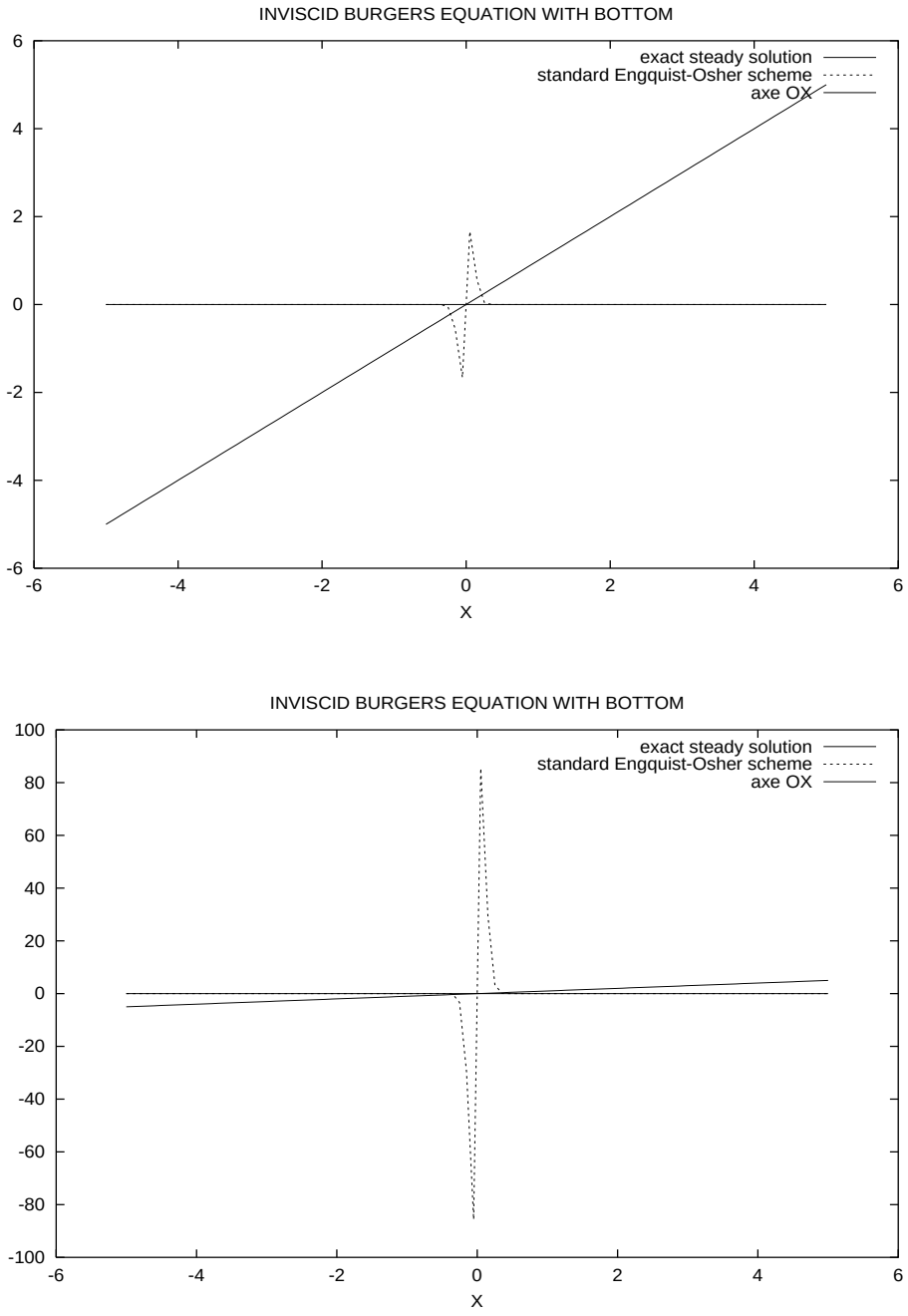


FIG. 12 – *Divergence history, 100 nodes in space.*

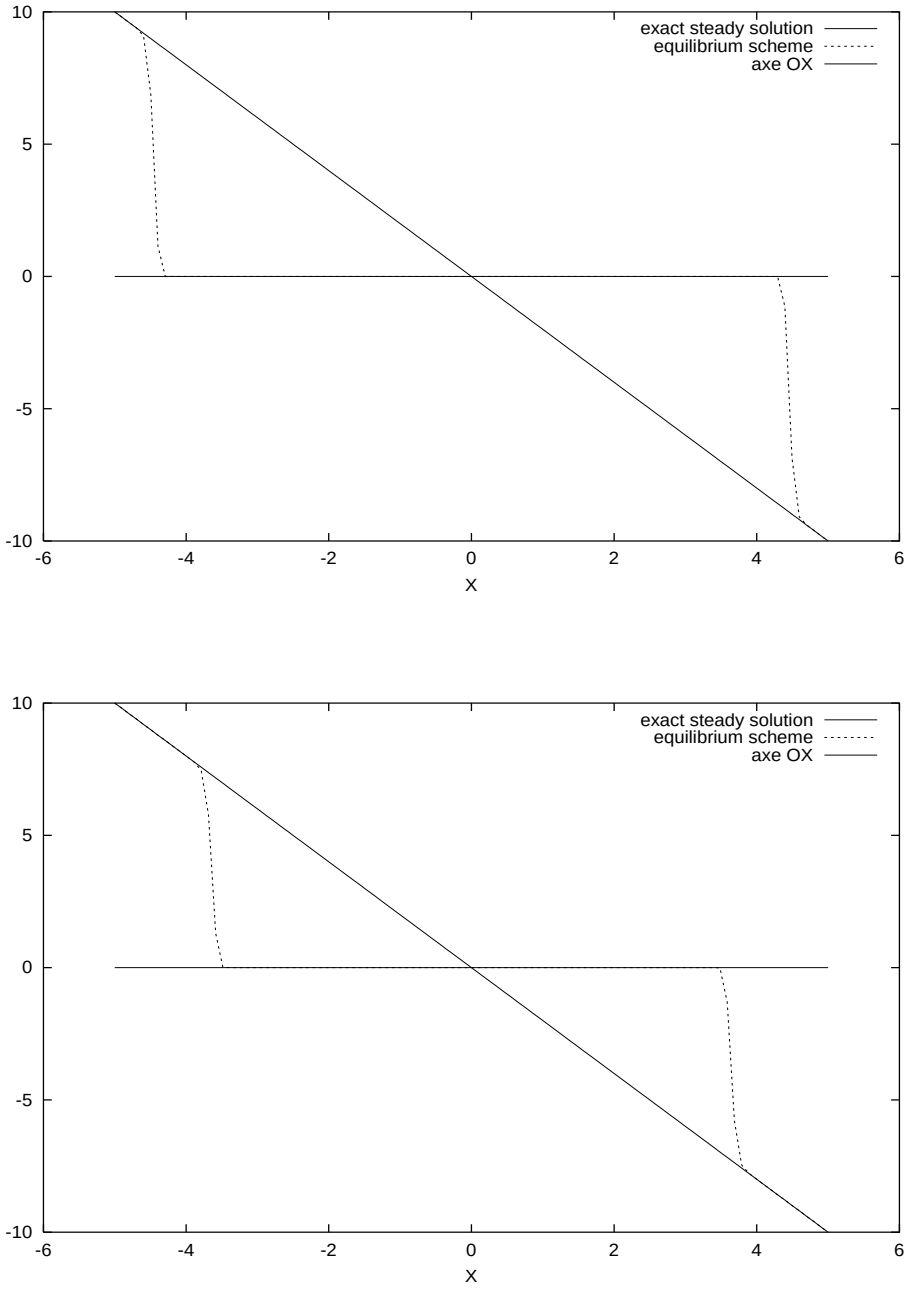


FIG. 13 – *Equilibrium version of Engquist-Osher scheme, 100 nodes in space.*

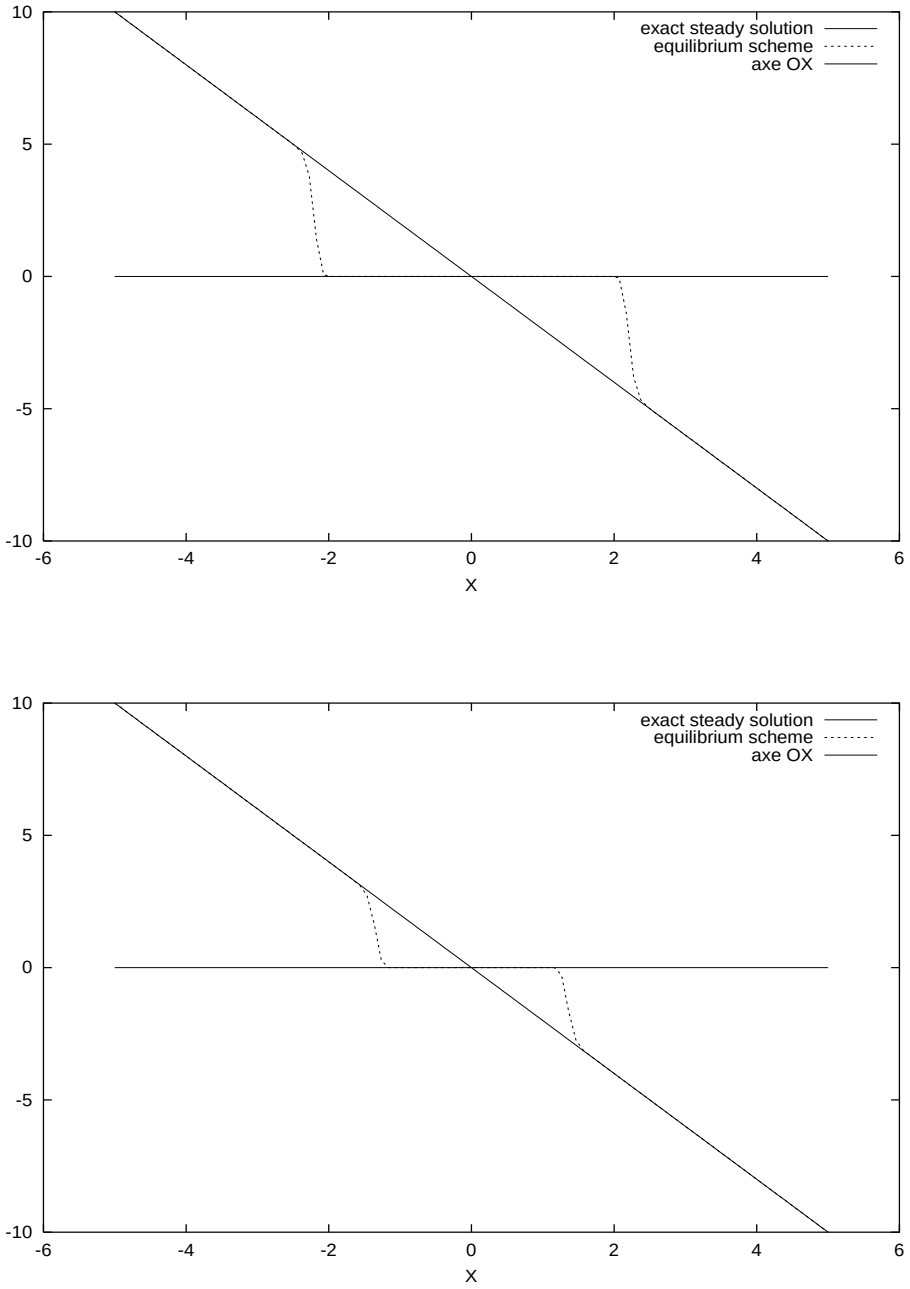


FIG. 14 – *Equilibrium version of Engquist-Osher scheme, 100 nodes in space.*

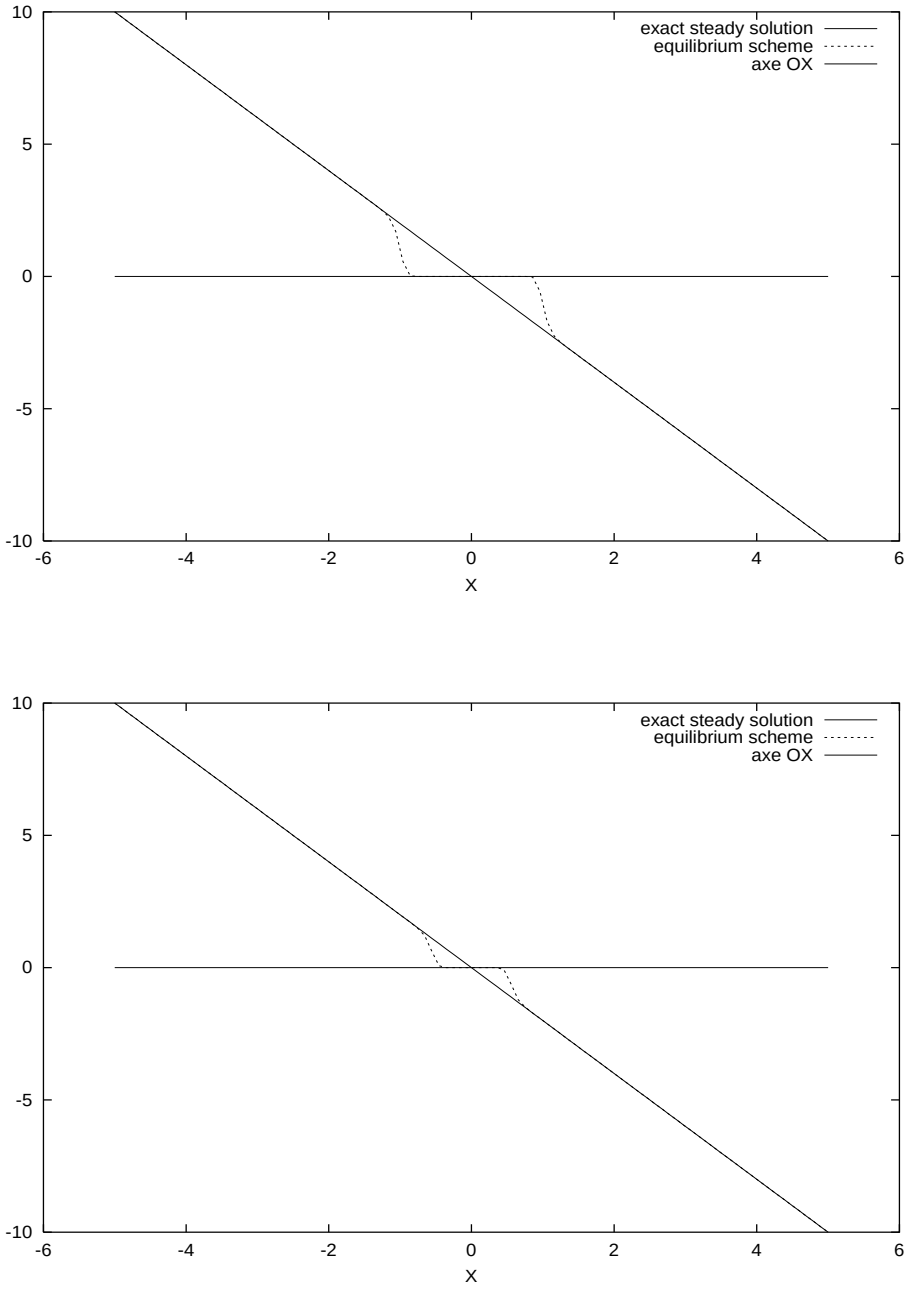


FIG. 15 – *Equilibrium version of Engquist-Osher scheme, 100 nodes in space.*

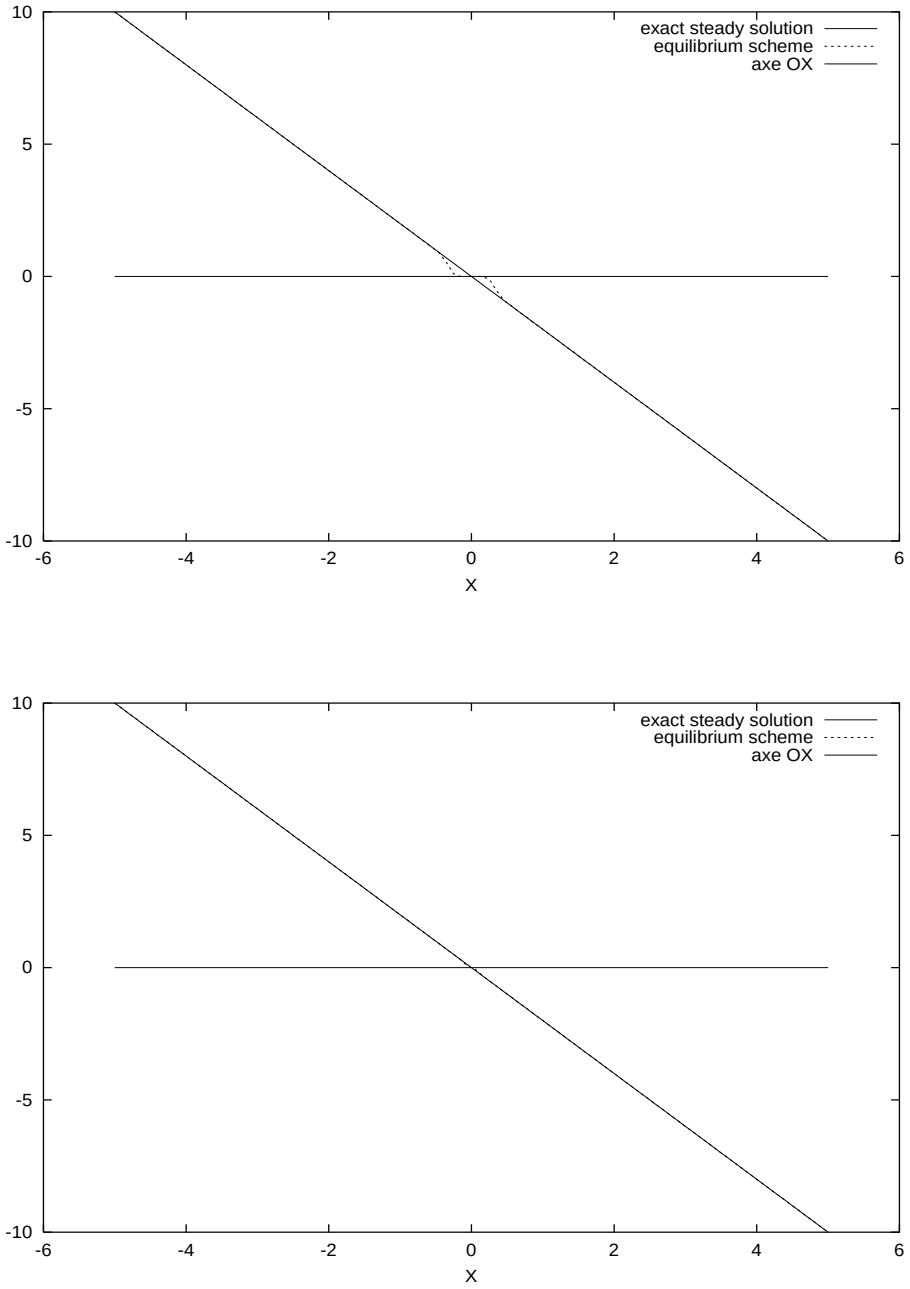


FIG. 16 – *Equilibrium version of Engquist-Osher scheme, 100 nodes in space.*

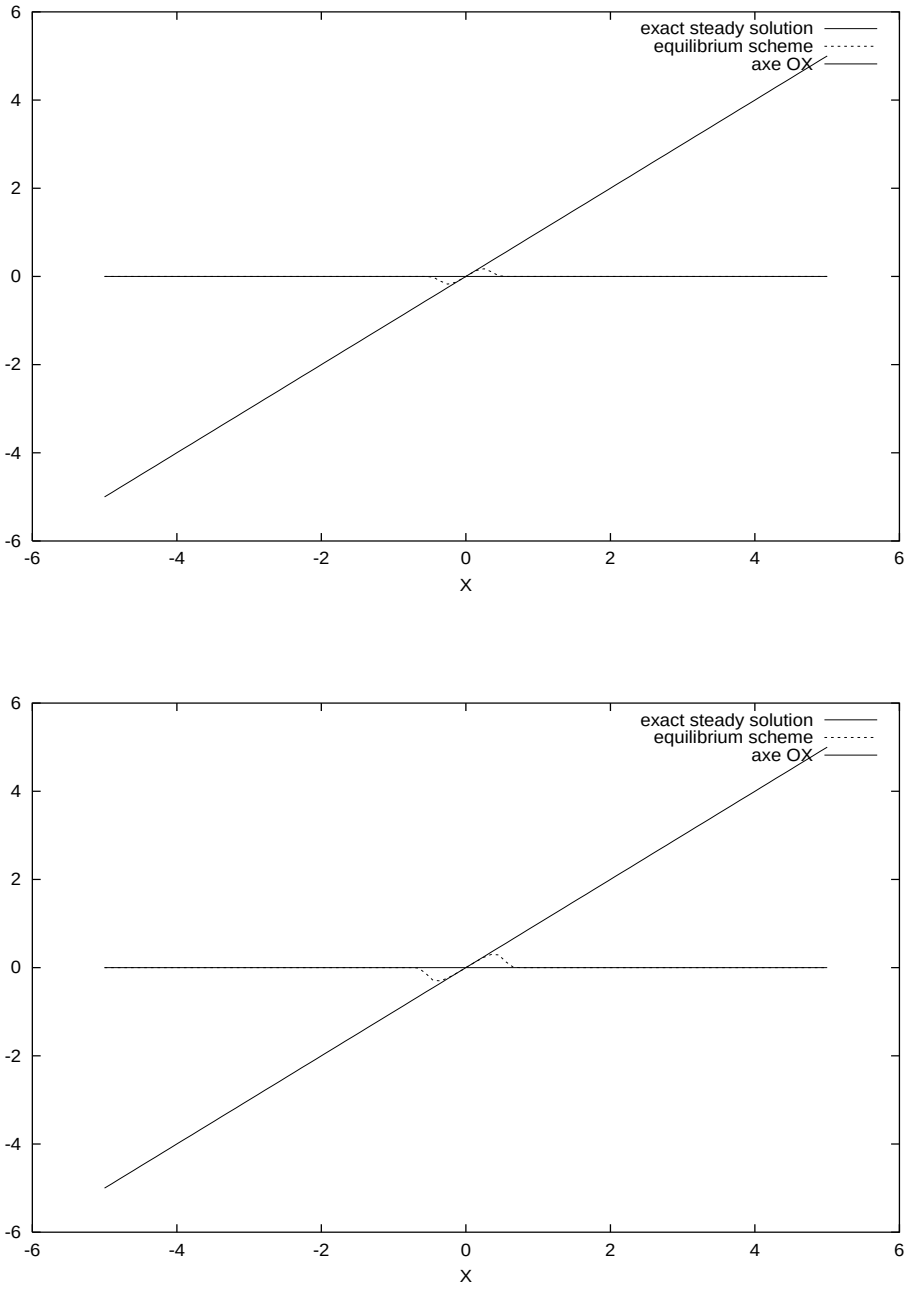


FIG. 17 – *Equilibrium version of Engquist-Osher scheme, 100 nodes in space*

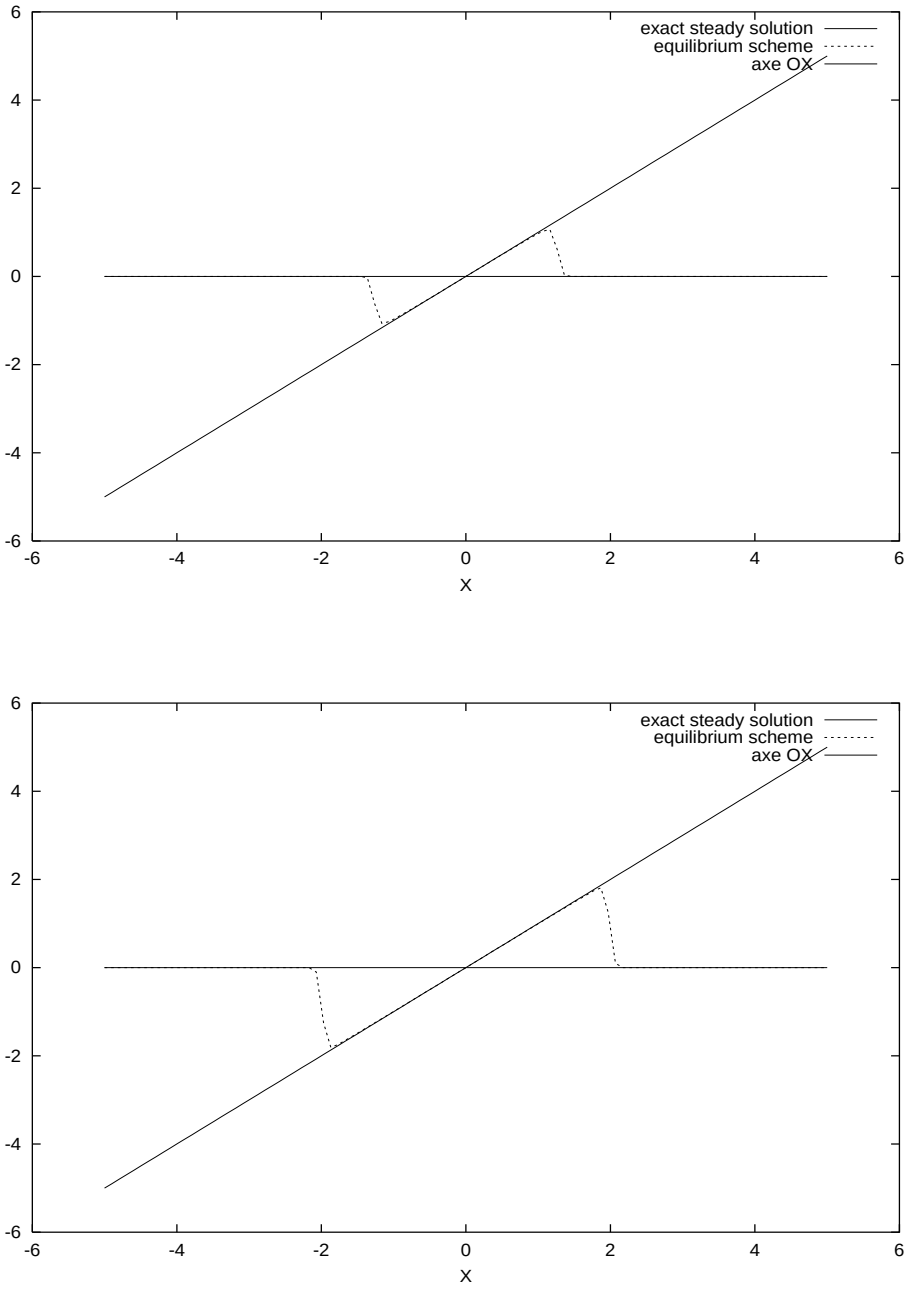


FIG. 18 – *Equilibrium version of Engquist-Osher scheme, 100 nodes in space*

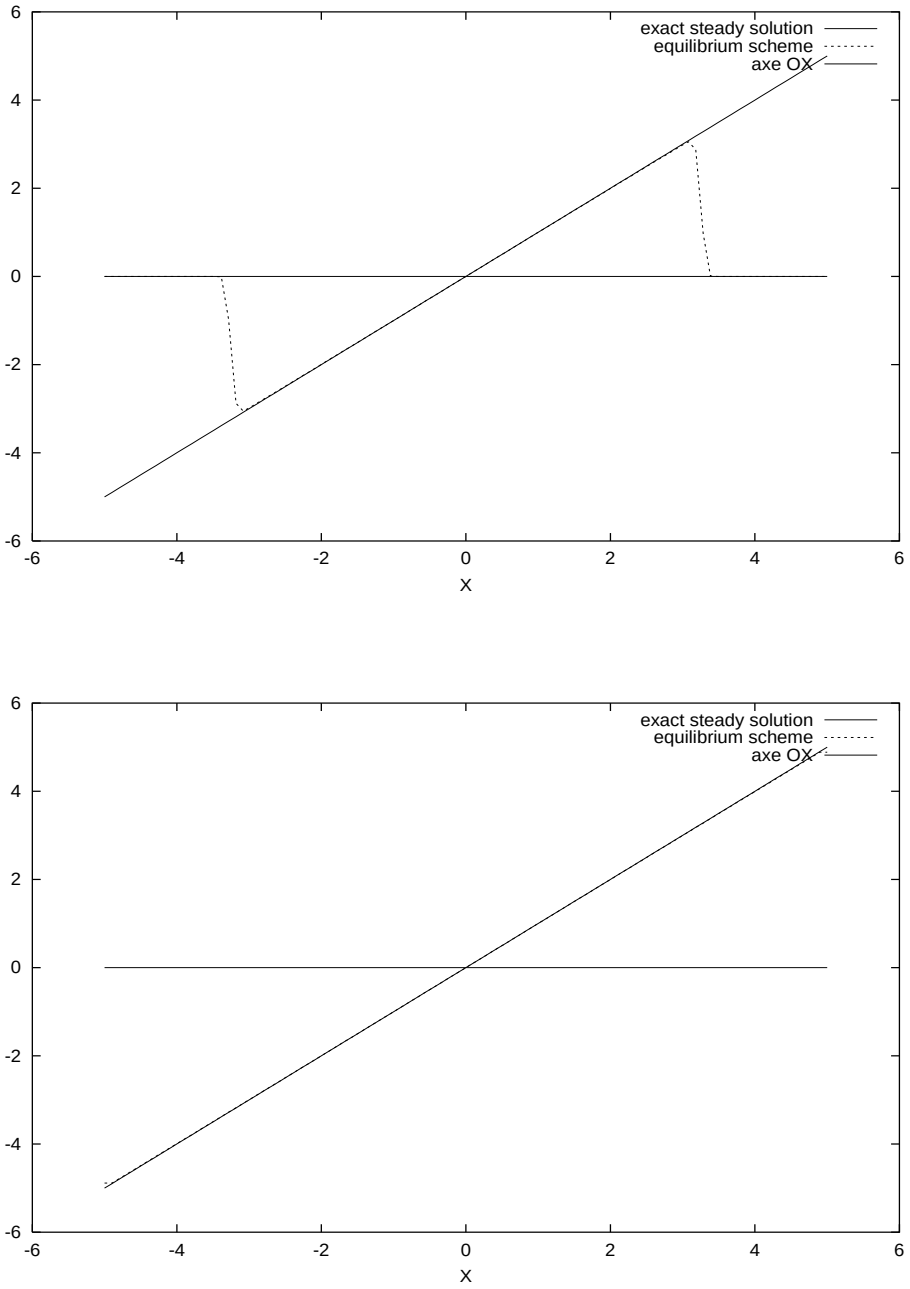


FIG. 19 – *Equilibrium version of Engquist-Osher scheme, 100 nodes in space.*

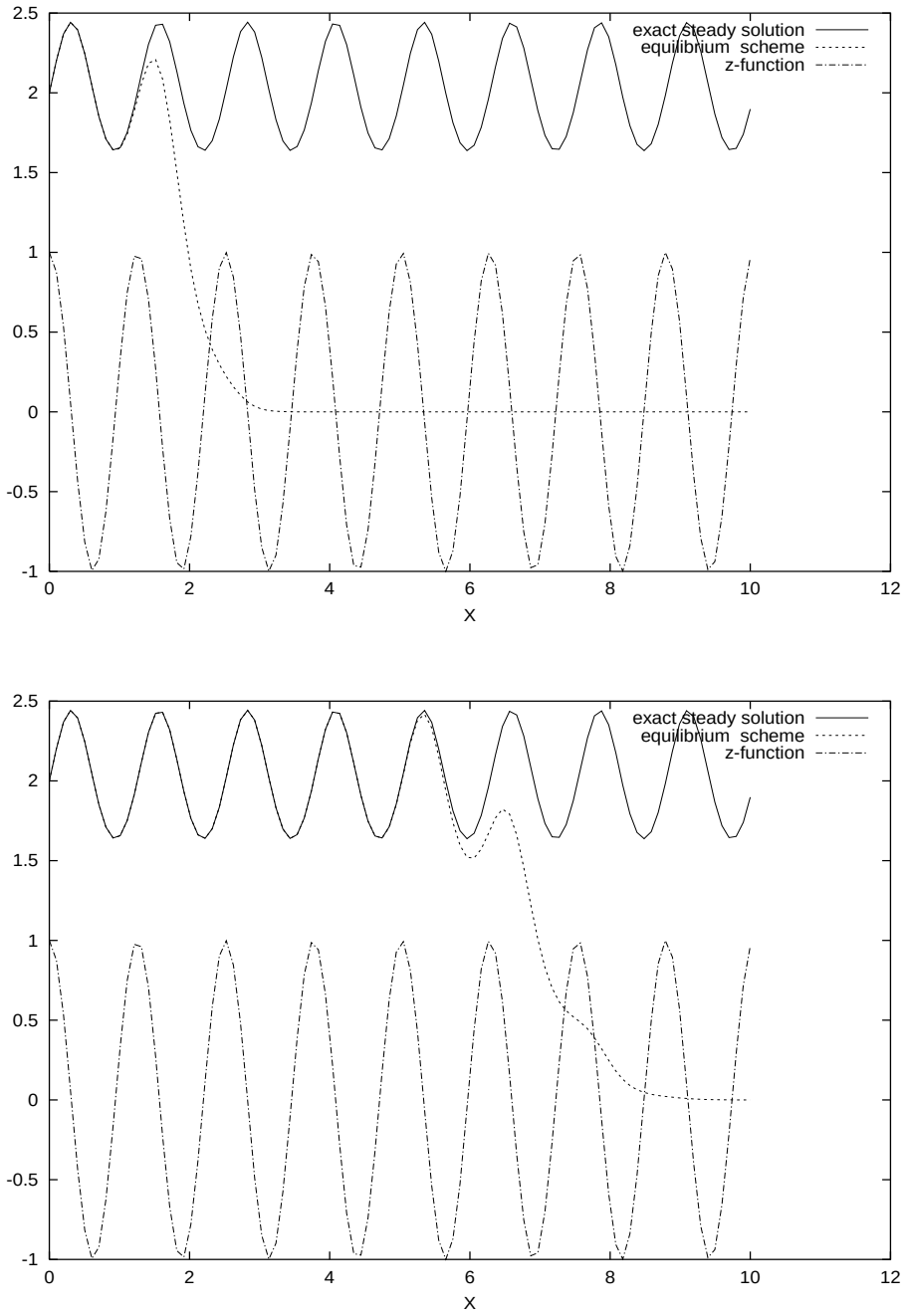


FIG. 20 – *Linear advection equation, 101 nodes,*$\varepsilon = 0.2$

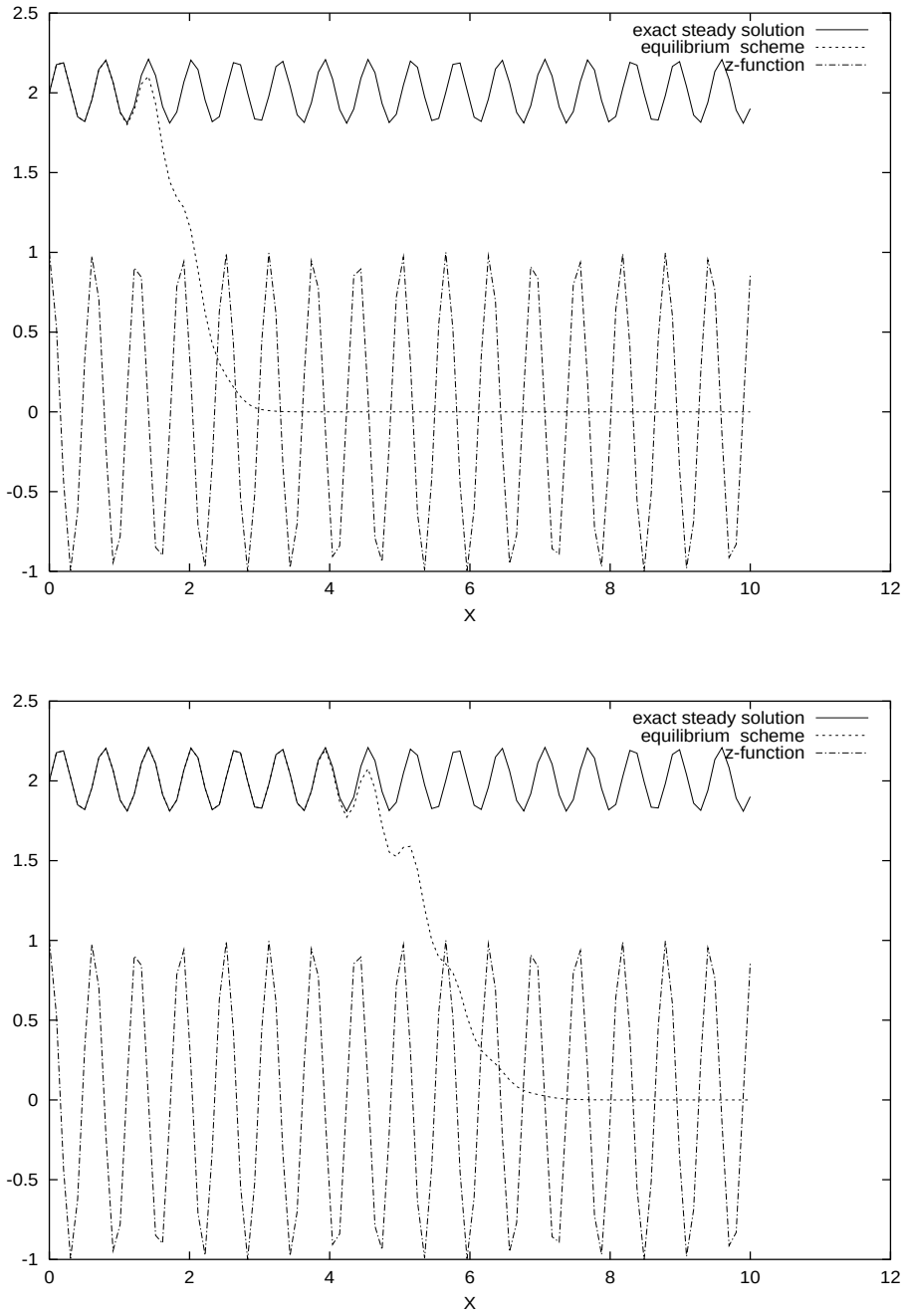


FIG. 21 – *Linear advection equation, 101 nodes,* $\varepsilon = 0.1$

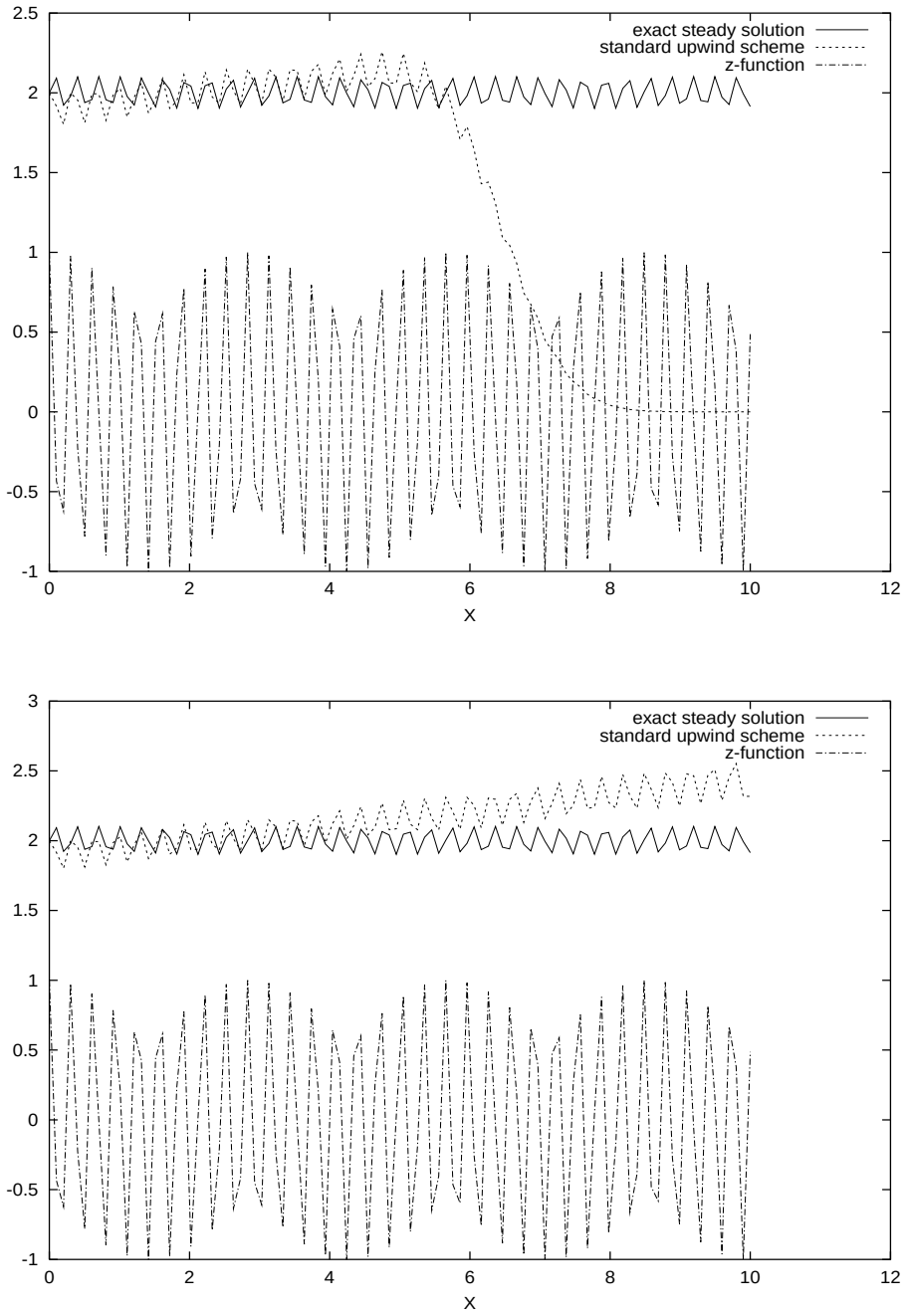
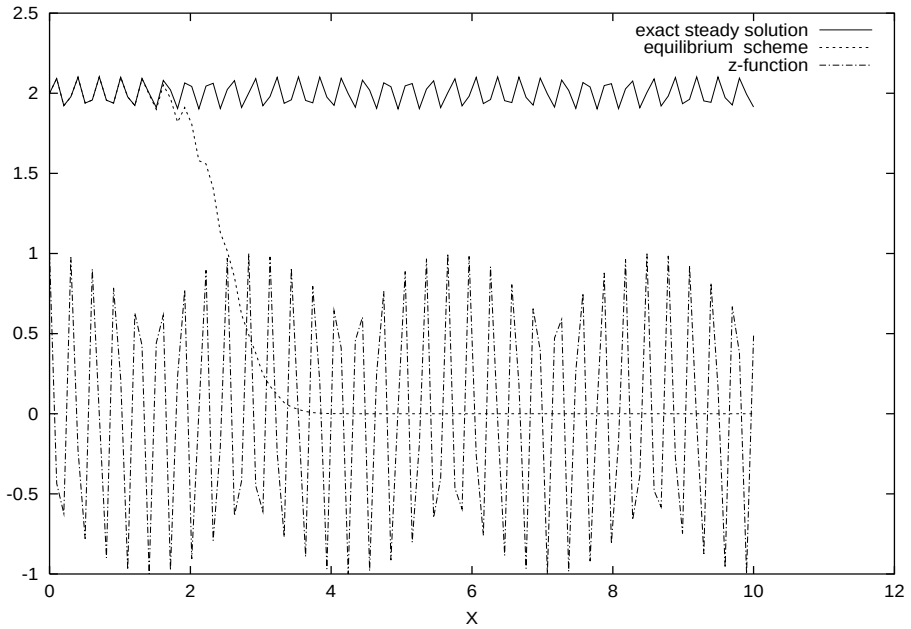


FIG. 22 – *Linear advection equation, 101 nodes,*$\varepsilon = 0.05$.



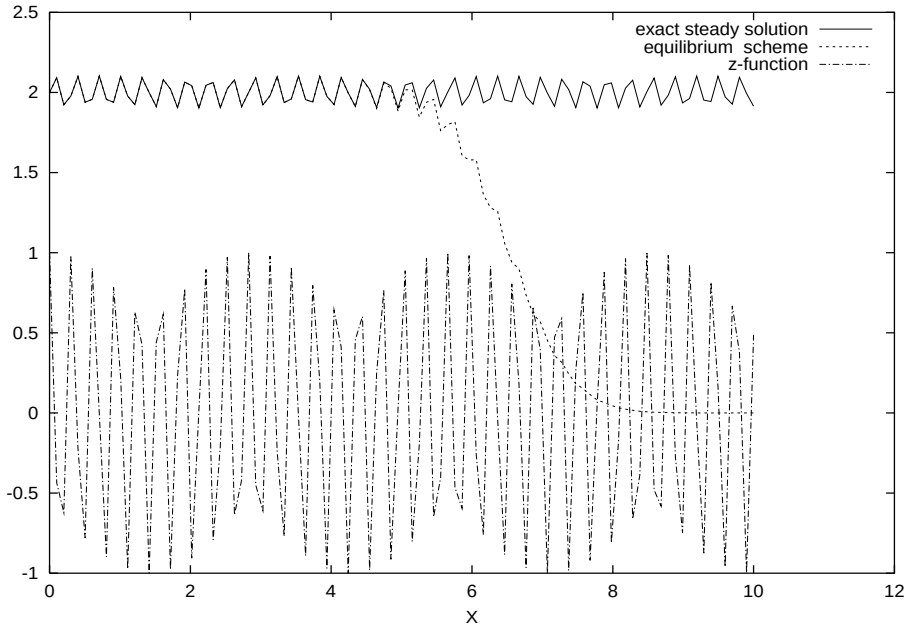


FIG. 23 – *Linear advection equation, 101 nodes,* $\varepsilon = 0.05$.

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