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Abstract: The Lie Algebra Rank Condition (LARC) plays a central role in nonlinear systems control theory. The present paper establishes that the satisfaction of this condition by a set of smooth control vector fields is equivalent to the existence of smooth transverse periodic functions. The proof here enclosed is constructive and provides an explicit method for the synthesis of such functions.

Key-words: nonlinear system, homogeneous system, Lie Algebra

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Une caractérisation de la Condition de Rang d'une Algèbre de Lie à base de fonctions périodiques transverses

Résumé : La Condition de Rang d'une Algèbre de Lie —Lie Algebra Rank Condition (LARC), en anglais— joue un rôle central dans la théorie de la commande des systèmes non-linéaires. Il est montré dans ce rapport que la satisfaction de cette condition par un jeu de champs de vecteurs réguliers équivaut à l'existence de fonctions périodiques transverses. La preuve est constructive et fournit une méthode explicite pour la synthèse de telles fonctions.

Mots-clés : système non linéaire, système homogène, algèbre de Lie

1 Introduction

The main result of the present study is a theorem which basically states that smooth vector fields $X_1, \dots, X_m$ on a finite-dimensional manifold M satisfy the classical Lie Algebra Rank Condition at a point $p \in M$ ($LARC(p)$) if and only if there exist an integer $\bar{n}(> m)$ and, for any neighborhood U_p of p, a smooth function $f : \alpha \mapsto f(\alpha)$ from $\mathbb{R}^{\bar{n}-m}$ to U_p which, for every α, is (maximally) "transversal" to the subspace spanned at $f(\alpha)$ by these vector fields.

The authors believe that the proposed theorem could become instrumental, and a unifying tool, for the development of new solutions to various problems involving nonlinear control systems. Direct application of the theorem concerns, in the first place, "practical" feedback stabilization of either driftless control systems —such as nonholonomic systems—, in relation to time-varying feedback methods, or systems subjected to a non-vanishing drift vector field, in relation to "hybrid" open-loop/feedback control solutions based on the use of "highly oscillatory" terms and averaging techniques. Other applications are also envisioned in the context of nonholonomic motion planning, again in relation to oscillatory open-loop control techniques which have been proposed to approximate arbitrary trajectories in the state space, and —a more tentative guess— in the domain of state estimation and nonlinear observer design. Results in some of these directions have already been obtained and will be reported in forthcoming publications.

The following notation is used throughout the paper.

- For manifolds M and N, M_p denotes the tangent space of M at p, and for $F \in \mathcal{C}^\infty(M; N)$, T_pF denotes the tangent mapping of F at p.
- $\mathbb{T}^k$, with $k \in \mathbb{N}$, denotes the k-dimensional torus.
- $B_n(0, \delta)$ denotes the closed ball in $\mathbb{R}^n$ centered at zero, and of radius δ.
- For $h \in \mathcal{C}^\infty(\mathbb{R}^n; \mathbb{R}^m)$, and $g \in \mathcal{C}^\infty(\mathbb{R}^n; \mathbb{R})$ with $g(x) \neq 0$ for $x \neq 0$, we write $h = o(g)$ when $|h(x)|/|g(x)| \longrightarrow 0$ as $x \longrightarrow 0$.
- $\mathbf{d}$ denotes the exterior derivative.

2 Main result

Theorem 1 *Let $X_1, \dots, X_m$ denote smooth vector fields on a smooth n-dimensional manifold M, such that the accessibility distribution $\Delta(p) \triangleq Span\ \{X(p) : X \in Lie(X_1, \dots, X_m)\}$ is of constant dimension n_0 in a neighborhood of p_0. Then, the following properties are equivalent:*

1. *$n_0 = n$, i.e. the Lie Algebra Rank Condition at p_0, $LARC(p_0)$, is satisfied for the vector fields $X_1, \dots, X_m$.*
2. *There exist $\bar{n} \in \mathbb{N}$ and, for any neighborhood $\mathcal{U}$ of p_0, a function $F \in \mathcal{C}^\infty(\mathbb{T}^{\bar{n}-m}; \mathcal{U})$ such that:*

$$\forall \theta \in \mathbb{T}^{\bar{n}-m}, \quad M_{F(\theta)} = Span\ \{X_1(F(\theta)), \dots, X_m(F(\theta))\} + T_\theta F(\mathbb{T}^{\bar{n}-m}_\theta)\,. \tag{1}$$

Remark 1 *Relation (1) is reminiscent of the transversality property for functions —see e.g. [1, Sec. 3.5] for a definition.*

We next rephrase this theorem by considering a system of local coordinates $x = (x_1, \dots, x_n)$ on M which maps p_0 to $0 \in \mathbb{R}^n$. Throughout the paper, we also denote by $\alpha = (\alpha_{m+1}, \dots, \alpha_{\bar{n}})$ a system of local coordinates on $\mathbb{T}^{\bar{n}-m}$.

Theorem 2 *Let $g_1, \dots, g_m$, resp. f, denote the local representatives on $\mathbb{R}^n$ of $X_1, \dots, X_m$, resp. of F, in Theorem 1. Then, the following properties are equivalent:*

1. *LARC(0): the system*

$$S: \qquad \dot{x} = \sum_{i=1}^{m} u_i g_i(x)$$

satisfies the Lie Algebra Rank Condition at the origin.

2. *TC(0): there exist $\bar{n} \in \mathbb{N}$ and a family $(f_\epsilon)_{\epsilon>0}$ of functions $f_\epsilon \in \mathcal{C}^\infty(\mathbb{T}^{\bar{n}-m}; B_n(0,\epsilon))$ such that, for any $\epsilon > 0$, the following Transversality Condition holds:*

$$\forall \theta \in \mathbb{T}^{\bar{n}-m}, \quad \mathit{Rank}\begin{pmatrix} g_1(f_\epsilon(\theta)) & \dots & g_m(f_\epsilon(\theta)) & \dfrac{\partial f_\epsilon}{\partial \alpha_{m+1}}(\theta) & \dots & \dfrac{\partial f_\epsilon}{\partial \alpha_{\bar{n}}}(\theta) \end{pmatrix} = n\,. \tag{2}$$

3 Proof of Theorem 2

3.1 TC(0) $\Longrightarrow$ LARC(0)

We assume that $LARC(0)$ is not satisfied, and show that $TC(0)$ cannot be satisfied either. By assumption, the accessibility distribution is of constant dimension n_0 in a neighborhood of the origin. Therefore, if $n_0 < n$, the Frobenius theorem guarantees the existence of local coordinates $y = \phi(x)$ such that y_n is constant along the trajectories of S, i.e. for some neighborhood $\mathcal{U}$ of the origin,

$$\forall i = 1, \dots, m, \ \forall x \in \mathcal{U} \qquad \frac{\partial \phi_n}{\partial x}(x) g_i(x) = 0\ . \tag{3}$$

Now assume that $TC(0)$ is satisfied, and choose any f_ϵ satisfying (2) and such that $B_n(0,\epsilon) \subset \mathcal{U}$. By the compacity of $\mathbb{T}^{\bar{n}-m}$, the smooth function $\theta \longmapsto \phi_n(f_\epsilon(\theta))$ from $\mathbb{T}^{\bar{n}-m}$ to $\mathbb{R}$ reaches its upper bound for some $\bar{\theta}$, i.e.

$$\forall i = m+1, \dots \bar{n}, \qquad \frac{\partial \phi_n}{\partial x}(f_\epsilon(\bar{\theta}))\frac{\partial f_\epsilon}{\partial \alpha_i}(\bar{\theta}) = 0\ . \tag{4}$$

From (4), and (3) evaluated at $x = f_\epsilon(\bar{\theta})$, we obtain

$$\frac{\partial \phi_n}{\partial x}(f_\epsilon(\bar{\theta}))\begin{pmatrix} g_1(f_\epsilon(\bar{\theta})) & \dots & g_m(f_\epsilon(\bar{\theta})) & \dfrac{\partial f_\epsilon}{\partial \alpha_{m+1}}(\bar{\theta}) & \dots & \dfrac{\partial f_\epsilon}{\partial \alpha_{\bar{n}}}(\bar{\theta}) \end{pmatrix} = 0\ ,$$

which is in contradiction with $TC(0)$. ∎

3.2 LARC(0) $\Longrightarrow$ TC(0)

3.2.1 Notation and recalls

Prior to addressing the proof itself, we specify some complementary notation and recall a few basic definitions and results —some of which are well known while others are less classical— that are extensively used in the sequel. These recalls are about homogeneity on one hand, and free Lie algebras on the other hand. For a more complete survey about these issues, we refer the reader to [2, 3], as for the properties associated with homogeneity, and to [4, 7] as for the role of free Lie algebras in control theory.

About homogeneity
Given $\mu > 0$ and a *weight vector* $r = (r_1, \dots, r_n)$ $(r_i > 0\ \forall i)$, a *dilation* Δ^r_μ on $\mathbb{R}^n$ is a map from $\mathbb{R}^n$ to $\mathbb{R}^n$ defined by $\forall z = (z_1, \dots, z_n) \in \mathbb{R}^n,\ \Delta^r_\mu z \stackrel{\Delta}{=} (\mu^{r_1} z_1, \dots, \mu^{r_n} z_n)$.
A function $f \in \mathcal{C}^0(\mathbb{R}^n; \mathbb{R})$ is *homogeneous of degree l with respect to the family of dilations* $(\Delta^r_\mu)_{\mu>0}$, or more concisely *Δ^r-homogeneous of degree l*, if $\forall \mu > 0,\ f(\Delta^r_\mu z) = \mu^l f(z)$.
A *Δ^r-homogeneous norm* is defined as a positive definite function on $\mathbb{R}^n$, Δ^r-homogeneous of degree one.
A smooth vector field X on $\mathbb{R}^n$ is *Δ^r-homogeneous of degree d* if, for all $i = 1, \dots, n$, the function $x \longmapsto X_i(x)$ is Δ^r-homogeneous of degree $d + r_i$.
The system

$$S_{ap}: \qquad \dot{z} = \sum_{i=1}^{m} b_i(z) u_i \tag{5}$$

is a *Δ^r-homogeneous approximation* of S if there exists a change of coordinates $\phi:\ x \longmapsto z$ which transforms S into

$$\dot{z} = \sum_{i=1}^{m} \left(b_i(z) + h_i(z)\right) u_i\,, \tag{6}$$

where b_i is Δ^r-homogeneous of degree -1, and h_i denotes higher-order terms, i.e. for any j, the j-th component $h_{i,j}$ of h_i satisfies $h_{i,j} = o(\rho^{r_j - 1})$, where ρ is any Δ^r-homogeneous norm.
The main motivation for introducing such approximations comes from the following result.

Proposition 1 *[2, 6] For any system S of smooth v.f. which satisfies $LARC(0)$, there exists a Δ^r-homogeneous approximation S_{ap} which also satisfies $LARC(0)$.*

Finally, we say that a set $\{b_1, \dots, b_m\}$ of v.f., or the associated system (5), is *nilpotent of order* $d+1$ if any Lie bracket of these v.f. of length larger than, or equal to, $d+1$ is identically zero. It is simple to verify that any set $\{b_1, \dots, b_m\}$ of smooth v.f. with the b_i's Δ^r-homogeneous of degree -1 , is nilpotent of order $1 + Max\{r_i : i = 1, \dots, n\}$.

About free Lie algebras
Let us consider a finite set of indeterminates $X_1, \dots, X_m$, and denote by $\mathcal{L}(X)$ the free Lie algebra over $\mathbb{R}$ generated by the X_i's. We also denote by $\mathcal{F}(X)$ the set of formal brackets in the X_i's. For any set $\{b_1, \dots, b_m\}$ of smooth v.f., and any $B \in \mathcal{F}(X)$, we denote by $\mathrm{Ev}(B)\{b_1, \dots, b_m\}$ the *evaluation map* , i.e. $\mathrm{Ev}(X_i)\{b_1, \dots, b_m\} = b_i$, and

$$\mathrm{Ev}([B_\lambda, B_\rho])\{b_1, \dots, b_m\} = [\mathrm{Ev}(B_\lambda)\{b_1, \dots, b_m\}, \mathrm{Ev}(B_\rho)\{b_1, \dots, b_m\}]\,.$$

The definition of a (generalized) P. Hall basis of $\mathcal{L}(X)$ is recalled below.

Definition 1 *A P. Hall basis $\mathcal{B}$ of $\mathcal{L}(X)$ is a totally ordered subset of $\mathcal{F}(X)$ such that*

1. *Each X_i belongs to $\mathcal{B}$.*
2. *If $B = [B_\lambda, B_\rho] \in \mathcal{F}$ with $B_\lambda, B_\rho \in \mathcal{F}$, then $B \in \mathcal{B}$ if and only if $B_\lambda, B_\rho \in \mathcal{B}$ with $B_\lambda < B_\rho$, and either (i) B_ρ is one of the X_i's or (ii) $B_\rho = [B_{\lambda\rho}, B_{\rho^2}]$ with $B_{\lambda\rho} \leq B_\lambda$.*
3. *If $B \in \mathcal{B}$ is a bracket of length $\ell(B) \geq 2$, i.e. $B = [B_\lambda, B_\rho]$, with $B_\lambda, B_\rho \in \mathcal{B}$, then $B_\lambda < B$.*

In order to simplify the forthcoming analysis we choose a specific P. Hall basis $\mathcal{B}$ obtained by specifying the total order associated with it. It is well known (see e.g. [5, Ch. 4]) that the chosen ordering is compatible with the definition of a P. Hall basis.

Specific order:

$$\begin{cases} \ell(B) < \ell(B') \Longrightarrow B < B' \\ X_i < X_j \Longleftrightarrow i < j \\ \text{For } \ell(B) = \ell(B') > 1, B < B' \Longleftrightarrow B_\lambda < B'_\lambda, \text{ or } B_\lambda = B'_\lambda \text{ and } B_\rho < B'_\rho\,. \end{cases} \tag{7}$$

We denote by

$$\mathcal{B} = \{B_1, B_2, \ldots, B_q, \ldots\},\ B_1 < B_2 < \ldots < B_q < \ldots \tag{8}$$

the P. Hall basis associated with the total order (7), and also by $\ell(i)$ the length of any bracket B_i of this basis. From (7) and the definition of a P. Hall basis,

$$\forall i = 1, \ldots, m, \quad B_i = X_i\,. \tag{9}$$

Note that, for any $i \geq m+1$, there exist unique integers $\lambda(i)$ and $\rho(i)$ such that

$$B_i = [B_{\lambda(i)}, B_{\rho(i)}]\,. \tag{10}$$

Moreover, $i = k$ $(\geq m+1)$ if and only if $\lambda(i) = \lambda(k)$ and $\rho(i) = \rho(k)$. Also, it directly follows from (7) and the definition of a P. Hall basis that $\lambda(i) < \rho(i) < i$. By extension of the above notation, and whenever this will make sense, we will use the symbols $\lambda^2(i), \lambda\rho(i), \rho^2(i), \ldots$ to index the elements of $\mathcal{B}$. For instance, if $\ell(\rho(i)) > 2$, we can write $B_{\rho(i)} = [B_{\lambda\rho(i)}, B_{\rho^2(i)}]$.

Let $0 < d \in \mathbb{N}$, we denote by $\mathcal{L}_d(X)$ the subspace of $\mathcal{L}(X)$ generated by brackets of length at most equal to d. Then, the subset of $\mathcal{B}$ composed of all brackets B_j such that $\ell(j) \leq d$ is a basis of $\mathcal{L}_d(X)$ denoted as $\mathcal{B}_d$. Let $n(d)$ denote the dimension of $\mathcal{L}_d(X)$, so that

$$\mathcal{B}_d = \{B_1, \ldots, B_{n(d)}\} \quad \text{and } \ell(n(d)) = d\,.$$

One can associate the following *free system* with the basis $\mathcal{B}_d$.

$$\begin{cases} \dot{x}_1 &= u_1 \\ & \vdots \\ \dot{x}_m &= u_m \\ \dot{x}_{m+1} &= x_{\lambda(m+1)}\dot{x}_{\rho(m+1)} \\ & \vdots \\ \dot{x}_{n(d)} &= x_{\lambda(n(d))}\dot{x}_{\rho(n(d))}\,. \end{cases} \tag{11}$$

It is straightforward to verify that (11) defines a control-affine driftless system:

$$S(d): \qquad \dot{x} = \sum_{i=1}^{m} u_i b_i(x) \tag{12}$$

where the components $b_{i,j}$ $(j = 1, \ldots, n(d))$ of the v.f. b_i are defined by

$$b_{i,j}(x) = \begin{cases} \delta_i^j & \text{if } \ell(j) = 1 \\ x_{\lambda(j)} b_{i,\rho(j)} & \text{otherwise} \end{cases} \tag{13}$$

(with δ_i^j standing for the Kronecker delta).

The following properties of free systems will be used in the sequel. While the first two properties are well known (see [4]), we are not aware of a reference for the last two ones. For this reason, proofs of these properties are given in the appendix.

Lemma 1 *For $i = m+1, \ldots, n(d)$, let b_i denote the vector field $Ev(B_i)\{b_1, \ldots, b_m\}$. Then, the following properties hold.*

1. *For any $i = 1, \ldots, n(d)$ and any $x \in \mathbb{R}^{n(d)}$, $b_i(x) = a_i \partial/\partial x_i + \sum_{j>i} b_{i,j}(x) \partial/\partial x_j$ for some non-zero constant a_i, so that $S(d)$ satisfies $LARC(x)$ for any $x \in \mathbb{R}^{n(d)}$.*

2. *The vector fields b_i are Δ-homogeneous of degree $-\ell(i)$ with Δ_μ $(\mu > 0)$ the dilation defined by*

$$\Delta_\mu x = (\mu^{\ell(1)} x_1, \ldots, \mu^{\ell(n(d))} x_{n(d)}), \tag{14}$$

   so that $S(d)$ is nilpotent of order $d+1$.

3. *For any $p \in \mathcal{C}^\infty(\mathbb{R}^{n(d)}; \mathbb{R})$, Δ-homogeneous of degree $d' < d$, and any $j \in \{1, \ldots, m\}$, there exists $q^j \in \mathcal{C}^\infty(\mathbb{R}^{n(d)}; \mathbb{R})$, Δ-homogeneous of degree $d'+1$, such that*

$$\forall i = 1, \ldots, m,\ \forall x \in \mathbb{R}^{n(d)}, \qquad \frac{\partial q^j}{\partial x}(x) b_i(x) = \begin{cases} p(x) & \text{if } i = j \\ 0 & \text{otherwise}. \end{cases} \tag{15}$$

4. *For any $i \in \{1, \ldots, n(d)\}$, and any $p \in \mathcal{C}^\infty(\mathbb{R}^{n(d)}; \mathbb{R})$, Δ-homogeneous of degree $d' - \ell(i)$ —with $d' \leq d$—, there exist h_1 and $h_{2,j}$, in $\mathcal{C}^\infty(\mathbb{R}^{n(d)}; \mathbb{R})$, Δ-homogeneous of degree d' and $d' - \ell(j)$ respectively, such that*

$$p(x)\mathbf{d}x_i = \mathbf{d}h_1 + \sum_{j:1<\ell(j)<d'} h_{2,j}(x) \left(\mathbf{d}x_j - x_{\lambda(j)} \mathbf{d}x_{\rho(j)}\right). \tag{16}$$

Remark 2

1. *Although the proof of Property 4 given in the appendix uses the other properties (Property 3, in particular) of the free system (11), Property 4 is not, properly speaking, a property of this system but rather a property of the total order chosen for the P. Hall basis.*

2. *The functions p, q^j, h_1 and $h_{2,j}$ in Properties 3 and 4 are polynomial in x, because they are smooth and homogeneous.*

3. *Since the smooth functions q^j in Property 3 is homogeneous of degree $d'+1$, it can only depend on the $n(d'+1)$ first components of x.*

After these preliminary recalls, we can now proceed with the proof of Theorem 2. It is composed of three steps which are summarized in the following three propositions.

Proposition 2 *If $TC(0)$ holds for a homogeneous approximation S_{ap} of a system S, then $TC(0)$ holds for S also.*

Proposition 3 *If, for any $d \in \mathbb{N}-\{0\}$, $TC(0)$ holds for the free system $S(d)$ with $\bar{n} = n(d)$, then $TC(0)$ holds for any smooth driftless system S_{hom} which satisfies $LARC(0)$ and whose control vector fields are Δ^r-homogeneous of degree -1 for some dilation Δ^r_μ.*

Proposition 4 *For any $d \in \mathbb{N}-\{0\}$, $TC(0)$ holds for the free system $S(d)$ with $\bar{n} = n(d)$.*

From Proposition 1, if S satisfies $LARC(0)$, it has an homogeneous approximation which also satisfies $LARC(0)$. This property, combined with the three propositions above, clearly implies that $LARC(0) \Longrightarrow TC(0)$. There remains to prove these three propositions.

3.2.2 Proof of Proposition 2

S rewrites, in some coordinates $z = \phi(x)$, as

$$\dot{z} = \sum_{i=1}^{m} u_i \left(\tilde{b}_i(z) + h_i(z)\right) \tag{17}$$

where the $\tilde{b}_i$'s, Δ^r-homogeneous of degree -1 (for some dilation Δ^r), are the v.f. of the homogeneous approximation S_{ap}, and h_i denotes higher-order terms, i.e.

$$h_{i,j} = o(\rho^{r_j - 1}), \tag{18}$$

with ρ denoting any Δ^r-homogeneous norm. We want to show that if $TC(0)$ holds for S_{ap}, then it also holds for S. Since $TC(0)$ is independent of the system of coordinates, it is sufficient to show that $TC(0)$ holds in the coordinates z. Let $\bar{n}$ and $(f_\epsilon)_{\epsilon>0}$ denote the integer and family of functions involved in the definition (see Theorem 2) of $TC(0)$ for the approximation S_{ap}. We show below that S satisfies $TC(0)$ by considering the same integer $\bar{n}$ and the family of functions $(\bar{f}_\epsilon)_{\epsilon>0}$ defined by

$$\bar{f}_\epsilon(\theta) = \Delta^r_{\mu(\epsilon)} f_1(\theta), \tag{19}$$

with $\mu(\epsilon)$ denoting a strictly positive number which is i) smaller than some adequately chosen $\mu_0 > 0$, and ii) such that $\sup_{\theta \in \mathbb{T}^{\bar{n}-m}} |\Delta^r_{\mu(\epsilon)} f_1(\theta)| \leq \epsilon$. Note that $\mu(\epsilon)$ always exists because $f_1(\mathbb{T}^{\bar{n}-m})$ is a compact set so that $\lim_{\mu \to 0} \sup_{\theta \in \mathbb{T}^{\bar{n}-m}} |\Delta^r_\mu f_1(\theta)| = 0$.

With z denoting a vector in $\mathbb{R}^n$, one deduces from (18) that

$$\lim_{\mu \to 0} \frac{h_{i,j}(\Delta^r_\mu z)}{\rho^{r_j-1}(\Delta^r_\mu z)} = \lim_{\mu \to 0} \frac{h_{i,j}(\Delta^r_\mu z)}{\mu^{r_j-1}\rho^{r_j-1}(z)} = 0.$$

Therefore,

$$h_{i,j}(\Delta_\mu^r z) = c_{i,j}(\mu, z)\mu^{r_j - 1}$$

where $|c_{i,j}(\mu, z)|$ tends to zero as μ tends to zero. Moreover, the convergence is uniform with respect to the z variable when $z \in B_n(0,1)$. The above equation can also be written in vectorial form as

$$h_i(\Delta_\mu^r z) = \mu^{-1}\Delta_\mu^r c_i(\mu, z) \tag{20}$$

with $c_i = (c_{i,1}, \dots, c_{i,n})$.

Let us now evaluate the rank of the matrix

$$A(\epsilon,\theta) \triangleq \left((\tilde{b}_1 + h_1)(\bar{f}_\epsilon(\theta)) \quad \dots \quad (\tilde{b}_m + h_m)(\bar{f}_\epsilon(\theta)) \quad \frac{\partial \bar{f}_\epsilon}{\partial \alpha_{m+1}}(\theta) \quad \dots \quad \frac{\partial \bar{f}_\epsilon}{\partial \alpha_{\bar{n}}}(\theta) \right).$$

Using (19), (20), and the fact that each $\tilde{b}_i$ is homogeneous of degree -1,

$$A(\epsilon,\theta) \;=\; \bar{A}(\epsilon,\theta)\; D(\mu(\epsilon))$$

with

$$\begin{aligned}\bar{A}(\epsilon,\theta) \triangleq \;& \left(\Delta^r_{\mu(\epsilon)}\tilde{b}_1(f_1(\theta)) \quad \dots \quad \Delta^r_{\mu(\epsilon)}\tilde{b}_m(f_1(\theta)) \quad \Delta^r_{\mu(\epsilon)}\frac{\partial f_1}{\partial \alpha_{m+1}}(\theta) \quad \dots \quad \Delta^r_{\mu(\epsilon)}\frac{\partial f_1}{\partial \alpha_{\bar{n}}}(\theta) \right) \\ & + \left(\Delta^r_{\mu(\epsilon)} c_1(\mu(\epsilon), f_1(\theta)) \quad \dots \quad \Delta^r_{\mu(\epsilon)} c_m(\mu(\epsilon), f_1(\theta)) \quad 0 \quad \dots \quad 0 \right),\end{aligned}$$

and

$$D(\mu(\epsilon)) \triangleq diag\{1/\mu(\epsilon), \dots, 1/\mu(\epsilon), 1, \dots, 1\}.$$

Since $D(\mu(\epsilon))$ is non singular, it readily follows that

$$\begin{aligned}\text{Rank } A(\epsilon,\theta) = \;\; \text{Rank } & \Big(\tilde{b}_1(f_1(\theta)) + c_1(\mu(\epsilon), f_1(\theta)) \quad \dots \quad \tilde{b}_m(f_1(\theta)) + c_m(\mu(\epsilon), f_1(\theta)) \\ & \qquad \frac{\partial f_1}{\partial \alpha_{m+1}}(\theta) \quad \dots \quad \frac{\partial f_1}{\partial \alpha_{\bar{n}}}(\theta) \Big)\end{aligned} \tag{21}$$

Now, by assumption,

$$\forall \theta \in \mathbb{T}^{\bar{n}-m}, \qquad \text{Rank } \left(\tilde{b}_1(f_1(\theta)) \quad \dots \quad \tilde{b}_m(f_1(\theta)) \quad \frac{\partial f_1}{\partial \alpha_{m+1}}(\theta) \quad \dots \quad \frac{\partial f_1}{\partial \alpha_{\bar{n}}}(\theta) \right) = n. \tag{22}$$

In view of (21) and (22), and using the facts that $f_1(\theta) \in B_n(0,1)$ and that $|c_{i,j}(\mu, z)|$ tends uniformly (w.r.t. $z \in B_n(0,1)$) to zero as μ tends to zero, there exists a strictly positive number μ_0 such that

$$(\mu(\epsilon) \leq \mu_0) \implies (\forall \theta \in \mathbb{T}^{\bar{n}-m}\,,\; Rank\; A(\epsilon,\theta) \;=\; n)\,.$$

This concludes the proof of Proposition 2. ∎

Remark 3 *The previous analysis implies —by setting* $\forall i,\ h_i \equiv 0$ *in (17)—, that for a homogeneous system, if a function* $f \in \mathcal{C}^\infty(\mathbb{T}^{\bar{n}-m}; \mathbb{R}^n)$ *satisfies (2) then, for any* $\mu > 0$, $\Delta_\mu f$ *also satisfies (2). Therefore,* $TC(0)$ *is satisfied for this homogeneous system with the functions* $f_\epsilon \triangleq \Delta_{\mu(\epsilon)} f$*, where* $\mu(\epsilon)$ *is any strictly positive value such that* $\sup_{\theta \in \mathbb{T}^{\bar{n}-m}} |\Delta_{\mu(\epsilon)} f(\theta)| \leq \epsilon$.

3.2.3 Proof of Proposition 3

Consider a smooth driftless system

$$S_{hom}: \qquad \dot{z} = \sum_{i=1}^{m} \tilde{b}_i(z) v_i \ . \tag{23}$$

whose v.f. $\tilde{b}_i$ $(i = 1, \ldots, m)$ are Δ^r-homogeneous of degree -1 for some dilation Δ^r_μ, and satisfy $LARC(0)$. Since S_{hom} is nilpotent of some order $d+1$, it can be associated with the free system $S(d)$ whose vector fields b_i are defined in (13). We show below that any family $(f_\epsilon)_{\epsilon>0}$ which satisfies $TC(0)$ for the free system $S(d)$ induces a family $(\tilde{f}_\epsilon)_{\epsilon>0}$ which satisfies $TC(0)$ for S_{hom}. In fact, from Remark 3 above, we only need to show the existence of a single function $\tilde{f} \in \mathcal{C}^\infty(\mathbb{T}^{n(d)-m}; \mathbb{R}^n)$ which satisfies the transversality condition (2) for S_{hom}.

Let f denote any of the functions f_ϵ associated with $S(d)$. From Property 1 of Lemma 1, the vectors $b_1(x), \ldots, b_{n(d)}(x)$ are linearly independent at any $x \in \mathbb{R}^{n(d)}$. Therefore, in the neighborhood of any $\theta \in \mathbb{T}^{n(d)-m}$, there exist (unique) smooth functions $u^\alpha_{i,j}$ such that

$$\forall j = m+1, \ldots, n(d),\ \forall \theta \in \mathrm{Dom}(\alpha), \qquad \frac{\partial f}{\partial \alpha_j}(\theta) = \sum_{i=1}^{n(d)} u^\alpha_{i,j}(\theta) b_i(f(\theta)), \tag{24}$$

with $\mathrm{Dom}(\alpha)$ denoting the domain of the local coordinates α. Also, using the fact that f satisfies the transversality condition (2) for $S(d)$,

$$\forall \theta \in \mathrm{Dom}(\alpha), \qquad \mathrm{Det}\, U^\alpha(\theta) \neq 0 \quad \text{with} \quad U^\alpha(\theta) \triangleq \left(u^\alpha_{i,j}(\theta)\right)_{i,j=m+1,\ldots,n(d)} . \tag{25}$$

Let us now define the function $\tilde{f}$. To this purpose, let us pick an arbitrary couple $(\theta_0, z_0) \in (\mathbb{T}^{n(d)-m} \times \mathbb{R}^n)$, and consider an element θ of $\mathbb{T}^{n(d)-m}$. Consider also a smooth path $\gamma_\theta : t \in [0,1] \longrightarrow \gamma_\theta(t) \in \mathbb{T}^{n(d)-m}$ which connects θ_0 to θ, i.e. such that $\gamma_\theta(0) = \theta_0$, and $\gamma_\theta(1) = \theta$. Let $z_{\gamma_\theta}(t)$ denote the solution, for $t \in [0,1]$, of

$$\dot{z} = \sum_{i=1}^{n(d)} U^\alpha_i(\gamma_\theta(t))\, \dot{\alpha}(\gamma_\theta(t))\, \tilde{b}_i(z) \qquad z(0) = z_0\,, \tag{26}$$

where $U^\alpha_i = (u^\alpha_{i,m+1}, \ldots, u^\alpha_{i,n(d)})$, and for $i = m+1, \ldots, n(d)$, $\tilde{b}_i \triangleq Ev(B_i)\{\tilde{b}_1, \ldots, \tilde{b}_m\}$. Note that $z_{\gamma_\theta}(t)$ is well defined for $t \in [0,1]$. Indeed, finite-time escape is not possible because the v.f. $\tilde{b}_i$ are homogeneous of negative degree (by assumption). Note also that $z_{\gamma_\theta}(t)$ does not depend on the system of local coordinates chosen to parameterize $\mathbb{T}^{n(d)-m}$. Indeed, if α and β are two systems of local coordinates around a point $\gamma_\theta(t_0)$, one easily verifies from (24) that in the neighborhood of t_0, where $\gamma_\theta(t) \in \mathrm{Dom}(\alpha) \cap \mathrm{Dom}(\beta)$,

$$U^\alpha_i(\gamma_\theta(t))\, \dot{\alpha}(\gamma_\theta(t)) = U^\beta_i(\gamma_\theta(t))\, \dot{\beta}(\gamma_\theta(t))\,.$$

Let us show that $z_{\gamma_\theta}(1)$ is independent of the the path γ_θ chosen to connect θ_0 to θ. To this purpose, consider two paths γ^i_θ $(i = 1, 2)$ which map 0 to θ_0 and 1 to θ. We must show that the solution $z^1_{\gamma_\theta}(1)$ of (26) at $t = 1$ with $\gamma_\theta = \gamma^1_\theta$ is the same as the solution $z^2_{\gamma_\theta}(1)$ of (26) at $t = 1$ with $\gamma_\theta = \gamma^2_\theta$. To show this, we will use the results stated in the following lemma, whose proof is given in the appendix.

Lemma 2 *Consider any P. Hall basis $\mathcal{B} = \{B_1, \ldots, B_i, \ldots\}$ of $\mathcal{L}(X_1, \ldots, X_m)$. Then, there exist mappings $(T, u) \longmapsto c_i(T, u)$ such that, for any set $\{g_1, \ldots, g_m\}$ of v.f. nilpotent of order $d+1$, and any $u \in \mathcal{C}^\infty([0,T]; \mathbb{R}^{n(d)})$, the solution at time T of*

$$\dot{x} = \sum_{i=1}^{n(d)} u_i(t)\, g_i(x)\,, \qquad x(0) = x_0 \tag{27}$$

is

$$x(T) = \prod_{i=1}^{n(d)} \exp\left(c_i(T,u)\, g_i\right) x_0 \tag{28}$$

where $g_i \triangleq Ev(B_i)\{g_1, \ldots, g_m\}$ $(i = m+1, \ldots, n(d))$. Furthermore, if $g_1, \ldots, g_m$ are the control v.f. of the —$n(d)$-dimensional— free system $S(d)$, then for any $x_0 \in \mathbb{R}^{n(d)}$, the mapping $(c_1, \ldots, c_{n(d)}) \longmapsto \prod_{i=1}^{n(d)} \exp\left(c_i\, g_i\right) x_0$, from $\mathbb{R}^{n(d)}$ to $\mathbb{R}^{n(d)}$ is one-to-one.

Applying the first result stated in the lemma to equation (26) yields

$$\forall k = 1,2, \qquad z^k_{\gamma_\theta}(1) = \prod_{i=1}^{n(d)} \exp\left(c_i\left(1, U^\alpha(\gamma^k_\theta)\dot{\alpha}(\gamma^k_\theta)\right)\, \tilde{b}_i\right) z_0\,. \tag{29}$$

Consider now the following equation —compare with (26)—

$$\dot{x} = \sum_{i=1}^{n(d)} U^\alpha_i(\gamma^k_\theta(t))\, \dot{\alpha}(\gamma^k_\theta(t))\, b_i(x) \qquad x(0) = f(\theta_0)\,. \tag{30}$$

Applying the first result stated in the lemma to this equation, using (24) and the fact that $f(\theta) = f(\gamma^k_\theta(1))$ for $k = 1,2$ —since $\gamma^k_\theta(1) = \theta$—,

$$\prod_{i=1}^{n(d)} \exp\left(c_i\left(1, U^\alpha(\gamma^1_\theta)\dot{\alpha}(\gamma^1_\theta)\right)\, b_i\right) f(\theta_0) = \prod_{i=1}^{n(d)} \exp\left(c_i\left(1, U^\alpha(\gamma^2_\theta)\dot{\alpha}(\gamma^2_\theta)\right)\, b_i\right) f(\theta_0)$$

The second result stated in the lemma then implies that

$$\forall i = 1, \ldots, n(d), \qquad c_i\left(1, U^\alpha(\gamma^1_\theta)\dot{\alpha}(\gamma^1_\theta)\right) = c_i\left(1, U^\alpha(\gamma^2_\theta)\dot{\alpha}(\gamma^2_\theta)\right) \tag{31}$$

and it follows, in view of (29), that $z^1_{\gamma_\theta}(1) = z^2_{\gamma_\theta}(1)$. This in turn establishes that the mapping $(\theta, \gamma_\theta) \rightarrow z_{\gamma_\theta}(1)$ is a function of θ solely. This is the function $\tilde{f}$ which we were looking for.

At this point, it only remains to verify that the function $\tilde{f}$ so defined satisfies the transversality condition (2) for S_{hom}. Recalling that $\tilde{f}(\theta)$ is obtained as the solution of (26) at $t = 1$, and that this solution does not depend on the path $\gamma_\theta(.)$ which passes thru θ at time $t = 1$, one deduces that along any smooth curve $\theta(.)$ the mapping $t \longmapsto \tilde{f}(\theta(t))$ is differentiable with

$$\frac{d}{dt}\tilde{f}(\theta(t)) = \sum_{i=1}^{n(d)} U^\alpha_i(\theta(t))\, \dot{\alpha}(\theta(t))\, \tilde{b}_i(\tilde{f}(\theta(t)))\,.$$

This in turn implies that $\tilde{f}$ is smooth and satisfies

$$\forall \theta \in \mathbb{T}^{n(d)-m}, \quad \frac{\partial \tilde{f}}{\partial \alpha_j}(\theta) = \sum_{i=1}^{n(d)} u^{\alpha}_{i,j}(\theta)\, \tilde{b}_i(\tilde{f}(\theta))\,. \tag{32}$$

Finally, by using (25) and the fact that S_{hom} satisfies $LARC(0)$ —and therefore, by homogeneity, $LARC(x)$ for any $x \in \mathbb{R}^n$—, one easily deduces from (32) that $\tilde{f}$ satisfies the transversality condition (2) for S_{hom}. ∎

3.2.4 Proof of Proposition 4

From Remark 3, and Property 2 of Lemma 1, it is sufficient to prove the existence of a single function $f \in \mathcal{C}^\infty(\mathbb{R}^{n(d)-m}; \mathbb{R}^{n(d)})$ for which the transversality condition (2) is satisfied. For free systems, it is possible to rewrite this condition in a more explicit fashion.

Lemma 3 *For a free system $S(d)$, $f \in \mathcal{C}^\infty(\mathbb{T}^{n(d)-m}; \mathbb{R}^{n(d)})$ satisfies Condition (2) if and only if*

$$\forall \theta \in \mathbb{T}^{n(d)-m}, \qquad Det\left(\frac{\partial f_k}{\partial \alpha_j}(\theta) - f_{\lambda(k)}(\theta)\frac{\partial f_{\rho(k)}}{\partial \alpha_j}(\theta)\right)_{k,j=m+1,\dots,n(d)} \neq 0\,. \tag{33}$$

This lemma is proved in the appendix.

We next rewrite condition (33) in the formalism of differential forms. The analysis could be carried out in the same way without this formalism, but at the price of more complicated notations. We thus rewrite (33) as[1]

$$\forall \theta \in \mathbb{T}^{n(d)-m}, \quad \left(\omega_{m+1} \wedge \ldots \wedge \omega_{n(d)}\right)(\theta) \neq 0 \tag{34}$$

with ω_i the differential one-form on $\mathbb{T}^{n(d)-m}$ defined by

$$\omega_i = \mathbf{d} f_i - f_{\lambda(i)} \mathbf{d} f_{\rho(i)}\,. \tag{35}$$

The function f which we are trying to construct is obtained by setting $f \triangleq f^{n(d)}$, with the function $f^{n(d)}$ denoting the last function obtained via a recursive construction which starts with some function f^{m+1}. For each $k = m+1, \dots, n(d)$, the function $f^k \in \mathcal{C}^\infty(\mathbb{T}^{k-m}; \mathbb{R}^{n(d)})$ is required to verify the following property:

$$\forall \theta^k = (\theta_{m+1}, \dots, \theta_k) \in \mathbb{T}^{k-m}, \quad \left(\omega^k_{m+1} \wedge \ldots \wedge \omega^k_k\right)(\theta^k) \neq 0\,, \tag{36}$$

with ω^k_i the differential one-form on $\mathbb{T}^{k-m}$ defined by

$$\omega^k_i = \mathbf{d} f^k_i - f^k_{\lambda(i)} \mathbf{d} f^k_{\rho(i)}\,. \tag{37}$$

A possible choice for f^{m+1} is as follows:

$$f^{m+1}_i(\theta_{m+1}) = \begin{cases} \sin\theta_{m+1} & \text{for } i = \lambda(m+1) \\ \cos\theta_{m+1} & \text{for } i = \rho(m+1) \\ \dfrac{1}{4}\sin 2\theta_{m+1} & \text{for } i = m+1 \\ 0 & \text{otherwise}\,. \end{cases} \tag{38}$$

[1] As this is customary, we write $\left(\omega_{m+1} \wedge \ldots \wedge \omega_{n(d)}\right)(\theta)$ for $\left(\omega_{m+1} \wedge \ldots \wedge \omega_{n(d)}\right)(\partial/\partial\alpha_{m+1}, \dots, \partial/\partial\alpha_{n(d)})(\theta)$

Indeed, it readily follows from this definition that[2]

$$\forall \theta^{m+1} \in \mathbb{T}, \quad \omega^{m+1}_{m+1}(\theta^{m+1}) = \left(\frac{\partial f^{m+1}_{m+1}}{\partial \alpha_{m+1}} - f^{m+1}_{\lambda(m+1)} \frac{\partial f^{m+1}_{\rho(m+1)}}{\partial \alpha_{m+1}} \right)(\theta^{m+1}) = \frac{1}{2}\,.$$

Assume now that, for some $k-1 \in \{m+1, \dots, n(d)-1\}$, a function $f^{k-1} \in \mathcal{C}^\infty(\mathbb{T}^{k-1-m}; \mathbb{R}^{n(d)})$ which verifies the property (36) for $k-1$ has been obtained. We show below how to construct from this function a new function $f^k \in \mathcal{C}^\infty(\mathbb{T}^{k-m}; \mathbb{R}^{n(d)})$ which verifies the property (36). The existence of a suitable function f^k is guaranteed by the following lemma.

Lemma 4 *Let (s, c, f) denote an element of $\mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n(d)}$, and let Δ^k_μ $(\mu > 0)$ denote the dilation defined by*

$$\Delta^k_\mu(s, c, f) = (\mu^{\ell(\lambda(k))} s, \mu^{\ell(\rho(k))} c, \Delta_\mu f)\,, \tag{39}$$

where Δ_μ is defined by (14). Let p^k_i $(i = 1, \dots, n(d))$ be the functions defined by

$$p^k_i(s, c) = s\,\delta^{\lambda(k)}_i + c\,\delta^{\rho(k)}_i + \frac{m^k_k}{2} sc\,\delta^k_i \tag{40}$$

where m^k_k is defined via

$$m^k_i = \begin{cases} 0 & \textit{if } \ell(i) \le \ell(\lambda(k)) \textit{ or } \lambda(i) \neq \lambda(k) \\ 1 + m^k_{\rho(i)} & \textit{otherwise}\,. \end{cases} \tag{41}$$

Then, there exist functions $q^k_{i,j} \in \mathcal{C}^\infty(\mathbb{R}^{n(d)}; \mathbb{R})$ $(i = 1, \dots, n(d),\ j = 1, \dots, j_{i,k})$, which are Δ^k-homogeneous of degree $\ell(i) - j\ell(\lambda(k))$, and such that, for η larger than some positive value η_0, the function

$$f^k \triangleq \bar{f}^k \circ \bar{g}^k_\eta \quad \textit{with} \quad \begin{cases} \bar{g}^k_\eta : \theta^k \longmapsto (\eta^{\ell(\lambda(k))} \sin\theta_k, \eta^{\ell(\rho(k))} \cos\theta_k, f^{k-1}(\theta^{k-1})) \\ \bar{f}^k_i : (s, c, f) \longmapsto f_i + p^k_i(s, c) + \sum_{j=1}^{j_{i,k}} s^j q^k_{i,j}(f) \end{cases} \tag{42}$$

$$j_{i,k} \triangleq \max\{j : \ell(i) - j\ell(\lambda(k)) \ge 0\}\,,$$

verifies the property (36).

Remark: It is simple to verify that each function $\bar{f}^k_i$ in (42) is polynomial in its arguments, and Δ^k-homogeneous of degree $\ell(i)$ w.r.t. the dilation defined by (39). The proof of the lemma much relies on this property.

Proof: It is composed of two parts. In the first one (step 1), functions $q^k_{i,j}$ which induce adequate properties are defined. In the second part (step 2), one shows that the function f^k, defined in the lemma from the functions $q^k_{i,j}$, verifies the property (36).

[2] We implicitly identify $\alpha(\theta) \approx \theta \in \mathbb{R}$.

Step 1. Let

$$\forall i = m+1, \ldots, k, \qquad \bar{\omega}_i^k \triangleq \mathbf{d}\bar{f}_i^k - \bar{f}_{\lambda(i)}^k \mathbf{d}\bar{f}_{\rho(i)}^k \,. \tag{43}$$

Our objective is to define functions $q_{i,j}^k$ such that, for any $i = 1, \ldots, n(d)$, the two following properties are verified.

P1(i): Each function $q_{i,j}^k$ is Δ^k-homogeneous of degree $\ell(i) - j\ell(\lambda(k))$, as specified in Lemma 4.

P2(i): If $i \in \{m+1, \ldots, k\}$,

$$\bar{\omega}_i^k = \left(\mathbf{d}f_i - f_{\lambda(i)}\mathbf{d}f_{\rho(i)} + \bar{\gamma}_i^k\right) + \sum_{j=m+1}^{i-1} t_{i,j}(s,f)\left(\mathbf{d}f_j - f_{\lambda(j)}\mathbf{d}f_{\rho(j)} + \bar{\gamma}_j^k\right) \tag{44}$$

where the $t_{i,j}$'s are smooth functions, and $\bar{\gamma}_i^k$ is a differential one-form on $\mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n(d)}$:

$$\bar{\gamma}_i^k = \bar{\gamma}_{i,1}^k \mathbf{d}s + \bar{\gamma}_{i,2}^k \mathbf{d}c$$

with $\bar{\gamma}_{i,1}$, $\bar{\gamma}_{i,2}$, Δ^k-homogeneous of degree $\ell(i) - \ell(\lambda(k))$ and $\ell(i) - \ell(\rho(k))$ respectively, and

$$\left\{\begin{array}{ll} \bar{\gamma}_{i,1}^k \equiv 0 & \text{if } i < \lambda(k) \\ \bar{\gamma}_{i,1}^k \equiv 1 & \text{if } i = \lambda(k) \\ \bar{\gamma}_{i,1}^k(s,c,0) = 0 & \text{if } \lambda(k) < i < k \\ \bar{\gamma}_{i,1}^k(s,c,0) = \dfrac{m_k^k}{2}\, c & \text{for } i = k \end{array}\right. \qquad \left\{\begin{array}{ll} \bar{\gamma}_{i,2}^k \equiv 0 & \text{if } i < \rho(k) \\ \bar{\gamma}_{i,2}^k \equiv 1 & \text{if } i = \rho(k) \\ \bar{\gamma}_{i,2}^k(s,c,0) = 0 & \text{if } \rho(k) < i < k \\ \bar{\gamma}_{i,2}^k(s,c,0) = -\dfrac{m_k^k}{2}\, s & \text{for } i = k\,. \end{array}\right. \tag{45}$$

Let us start the construction of an adequate set of functions $q_{i,j}^k$. We distinguish three cases.

Case 1: $1 \le i \le Max\{m, \lambda(k)\}$. We let

$$\forall i \in \{1, \ldots, Max\{m, \lambda(k)\}\},\ \forall j \in \{1, \ldots, j_{i,k}\}, \qquad q_{i,j}^k \equiv 0 \tag{46}$$

so that **P1(i)** is clearly verified for these values of i. If $i \le m$, **P2(i)** is irrelevant. If $m+1 \le i \le \lambda(k)$, it readily follows from (40), (42), (43), and (46) that

$$\bar{\omega}_i^k = \mathbf{d}f_i - f_{\lambda(i)}\mathbf{d}f_{\rho(i)}^k + \bar{\gamma}_i^k \tag{47}$$

where $\bar{\gamma}_i^k \equiv 0$ if $i < \lambda(k)$, and $\bar{\gamma}_i^k = \mathbf{d}s$ if $i = \lambda(k)$. Therefore, **P2(i)** is also verified.

Case 2: $Max\{m, \lambda(k)\} < i \le k$. We first let

$$\left\{\begin{array}{ll} q_{i,1}^k \equiv 0 & \text{if } \lambda(i) < \lambda(k) \\ q_{i,1}^k(f) = m_i^k f_{\rho(i)} & \text{if } \lambda(i) = \lambda(k)\,. \end{array}\right. \tag{48}$$

which is consistent with **P1(i)**. To define the other functions $q_{i,j}^k$, we consider a construction which is recursive in the index i. More precisely, let us assume that functions $q_{1,j}^k, \ldots, q_{i-1,j}^k$ have been defined so that **P1(1)**, ..., **P1(i-1)** and **P2(1)**, ..., **P2(i-1)** are verified. We show below how to obtain functions $q_{i,j}^k$ so that **P1(i)** and **P2(i)** are also verified.

We first note that

$$\lambda(i) < \rho(k)\,. \tag{49}$$

Assume on the contrary that $\lambda(i) \geq \rho(k)$. Then, from the definition of a P. Hall basis, $\lambda(i) < \rho(i)$. This implies that

$$\ell(i) = \ell(\lambda(i)) + \ell(\rho(i)) \geq 2\ell(\rho(k)) \geq \ell(k)\,.$$

If $\ell(i) > \ell(k)$, then $i > k$, and this contradicts the assumption. Otherwise, $\ell(i) = \ell(k)$, and we also get $i > k$ because of (7) and the fact that $\lambda(i) \geq \rho(k) > \lambda(k)$.

We introduce the following definitions for the sake of simplifying some aspects of the forthcoming analysis.

Definition 2 *A differential one-form $r = r_s\mathbf{d}s + r_c\mathbf{d}c + \sum_{j=1}^{n(d)} r_j\mathbf{d}f_j$, with r_s, r_c, r_j homogeneous of degree $\ell(i) - \ell(\lambda(k))$, $\ell(i) - \ell(\rho(k))$, and $\ell(i) - \ell(j)$ respectively, is said to be of*

- *Type 1 if $r_j \equiv 0$ for each j, and both r_s and r_c are identically zero at $f = 0$.*
- *Type 2 if $r_c \equiv r_j \equiv 0$ for each j, and $r_s = as^\kappa$ with $a \in \mathbb{R}$ and $1 \leq \kappa \in \mathbb{N}$.*
- *Type 3 if $r_s \equiv r_c \equiv 0$ and, for each j, $r_j(s,c,f)$ is in the form $r_j(s,c,f) = s^{2+\kappa_j} r'_j(f)$ with $\kappa_j \in \mathbb{N}$.*

An upper-left index i for a one-form will indicate its type, e.g. ${}^2r$ indicates that ${}^2r$ is of Type 2.

Next, we develop $\bar{\omega}_i^k$ and examine the terms involved in this development. From (42) and (43), we have

$$\begin{aligned}\bar{\omega}_i^k = {} & \mathbf{d}\left(f_i + p_i^k + \sum_{j=1}^{j_{i,k}} s^j q_{i,j}^k\right) \\ & - \left(f_{\lambda(i)} + p_{\lambda(i)}^k + \sum_{j=1}^{j_{\lambda(i),k}} s^j q_{\lambda(i),j}^k\right)\left(\mathbf{d}f_{\rho(i)} + \mathbf{d}p_{\rho(i)}^k + \sum_{j=1}^{j_{\rho(i),k}} \left(j s^{j-1} q_{\rho(i),j}^k \mathbf{d}s + s^j \mathbf{d}q_{\rho(i),j}^k\right)\right)\end{aligned} \tag{50}$$

and, by rearranging the terms in the right-hand side of this equality

$$\bar{\omega}_i^k = \mathbf{d}f_i - f_{\lambda(i)}\mathbf{d}f_{\rho(i)} + \mathbf{d}\left(\sum_{j=2}^{j_{i,k}} s^j q_{i,j}^k\right) + \alpha_1 + \alpha_2 + \alpha_3 + {}^1r + {}^2r + {}^3r \tag{51}$$

with

$$\begin{cases} \alpha_1 = \mathbf{d}p_i^k - p_{\lambda(i)}^k \mathbf{d}p_{\rho(i)}^k \\ \alpha_2 = s\mathbf{d}q_{i,1}^k - sq_{\lambda(i),1}^k \mathbf{d}f_{\rho(i)} - sf_{\lambda(i)}\mathbf{d}q_{\rho(i),1}^k - p_{\lambda(i)}^k \mathbf{d}f_{\rho(i)} \\ \alpha_3 = -s^2 \mathbf{d}p_{\rho(i)}^k \displaystyle\sum_{j=2}^{j_{\lambda(i),k}} s^j q_{\lambda(i),j}^k\,. \end{cases} \tag{52}$$

In (51), ${}^1r$, ${}^2r$, and ${}^3r$ just correspond to terms which do not need to be explicited further and are of Type 1, 2, and 3, following Definition 2. In order to obtain (51), we have used the following two arguments: (i) each function $q^k_{j,1}$ $(j < i)$ vanishes at the origin —if $j \leq Max\{m, \lambda(k)\}$ this follows from (46); otherwise, if $\lambda(j) \leq \lambda(k)$ this follows from (48), and if $\lambda(j) > \lambda(k)$, this follows from the induction hypothesis **P1(j)** which implies that $q^k_{j,1}$ is Δ^k-homogeneous of positive degree—, (ii) from (49), $\lambda(i) < \rho(k)$ so that (40) implies that $p^k_{\lambda(i)}(s,c)$ is either s or zero. Note also that the homogeneity properties of the components of ${}^1r$, ${}^2r$, and ${}^3r$ follow directly from the homogeneity of $\bar{f}^k_i$ (see the remark after Lemma 4).

Let us now focus our attention on the terms α_i which are specified in (52). We first note that

$$\alpha_3 \equiv 0\,. \tag{53}$$

Indeed, assume on the contrary that α_3 is not the null function. Then, in view of (46), it is necessary that $\lambda(i) > \lambda(k)$ (otherwise $q^k_{\lambda(i),j}$, and thus α_3, would be equal to zero). Since $\lambda(i) < \rho(i)$ —from the definition of a P. Hall basis—, we also have $\rho(i) \geq \rho(k)$ (otherwise $p^k_{\rho(i)}$, and thus α_3 would be equal to zero). This implies that $i > k$, which is in contradiction with the assumption.

We now consider the term α_2 in (52). We have

$$\lambda(i) < \lambda(k) \Longrightarrow \alpha_2 \equiv 0\,. \tag{54}$$

This follows from (40), (46), and (48), after noticing that either $\ell(\rho(i)) = 1$, so that $q^k_{\rho(i),1} \equiv 0$, or $\ell(\rho(i)) > 1$ and $\lambda(\rho(i)) \leq \lambda(i) < \lambda(k)$ (from the definition of a P. Hall basis), so that we still obtain $q^k_{\rho(i),1} \equiv 0$. Then,

$$\lambda(i) = \lambda(k) \;\; \text{with} \quad \left.\begin{array}{l} \ell(\rho(i)) \leq \ell(\lambda(k)) \\ \text{or} \\ \lambda\rho(i) < \lambda(k) \end{array}\right\} \Longrightarrow \alpha_2 \equiv 0\,. \tag{55}$$

Indeed, if the left-hand side of the above implication holds, then (40), (41), and (48) imply that

$$\alpha_2 = s\left(m^k_i \mathbf{d}f_{\rho(i)} - f_{\lambda(i)}\mathbf{d}q^k_{\rho(i),1} - \mathbf{d}f_{\rho(i)}\right) = s\left(m^k_i \mathbf{d}f_{\rho(i)} - \mathbf{d}f_{\rho(i)}\right) \equiv 0 \tag{56}$$

From the definition of a P. Hall basis, $\lambda\rho(i) \leq \lambda(i)$, so that the case where $\lambda(i) = \lambda(k)$ with $\lambda\rho(i) > \lambda(k)$ cannot happen. Therefore, if $\lambda(i) = \lambda(k)$, the last possible case is $\lambda\rho(i) = \lambda(k)$. We have

$$\Big(\lambda(i) = \lambda(k) \text{ and } \lambda\rho(i) = \lambda(k)\Big) \Longrightarrow \alpha_2 = s\, m^k_{\rho(i)}\left(\mathbf{d}f_{\rho(i)} - f_{\lambda\rho(i)}\mathbf{d}f_{\rho^2(i)}\right) \tag{57}$$

Indeed, from (41), (48), and (46),

$$\alpha_2 = s\left(m^k_i \mathbf{d}f_{\rho(i)} - f_{\lambda\rho(i)}\mathbf{d}q^k_{\rho(i),1} - \mathbf{d}f_{\rho(i)}\right) = s\left(m^k_i \mathbf{d}f_{\rho(i)} - f_{\lambda\rho(i)} m^k_{\rho(i)}\mathbf{d}f_{\rho^2(i)} - \mathbf{d}f_{\rho(i)}\right)$$

and (57) follows. Concerning α_2, there only remains to examine the case where $\lambda(i) > \lambda(k)$. In this case $p^k_{\lambda(i)} \equiv 0$ —since, by (49), $\lambda(i) < \rho(k)$—, so that

$$\alpha_2 = s\left(\mathbf{d}q^k_{i,1} - q^k_{\lambda(i),1}\mathbf{d}f_{\rho(i)} - f_{\lambda(i)}\mathbf{d}q^k_{\rho(i),1}\right)\,. \tag{58}$$

Each term within the above parenthesis is a sum of terms $p_{i,j}(f)\mathbf{d}f_j$ where each $p_{i,j}$ is homogeneous of degree $\ell(i) - \ell(\lambda(k)) - \ell(j)$. By applying Property 4 in Lemma 1 to the term $q^k_{\lambda(i),1}\mathbf{d}f_{\rho(i)} + f_{\lambda(i)}\mathbf{d}q^k_{\rho(i),1}$, and by replacing x with f in Lemma 1, we obtain

$$\alpha_2 = s\left(\mathbf{d}q^k_{i,1} - \mathbf{d}h_1 + \sum_{1<\ell(j)<\ell(i)-\ell(\lambda(k))} h_{2,j}(f)\left(\mathbf{d}f_j - f_{\lambda(j)}\mathbf{d}f_{\rho(j)}\right)\right)$$

for some functions h_1 and $h_{2,j}$ Δ^k-homogeneous of degree $\ell(i) - \ell(\lambda(k))$ and $\ell(i) - \ell(\lambda(k)) - \ell(j)$ respectively. Furthermore, by choosing

$$q^k_{i,1} = h_1 \qquad \text{when } \lambda(i) > \lambda(k)\,, \tag{59}$$

—this choice is clearly consistent with **P1(i)**—, we get

$$\alpha_2 = s \sum_{1<\ell(j)<\ell(i)-\ell(\lambda(k))} h_{2,j}(f)\left(\mathbf{d}f_j - f_{\lambda(j)}\mathbf{d}f_{\rho(j)}\right)\,. \tag{60}$$

From what precedes, we finally obtain

$$\alpha_2 = \begin{cases} s\displaystyle\sum_{j=m+1}^{\min\{i,\rho(k)\}-1} h_{2,j}(f)\left(\mathbf{d}f_j - f_{\lambda(j)}\mathbf{d}f_{\rho(j)} + \bar{\gamma}^k_j\right) - s\sum_{j=m+1}^{\min\{i,\rho(k)\}-1} h_{2,j}(f)\bar{\gamma}^k_j & \text{if } i<k \\ s\,m^k_{\rho(k)}\left(\mathbf{d}f_{\rho(k)} - f_{\lambda(k)}\mathbf{d}f_{\rho(k)} + \bar{\gamma}^k_{\rho(k)}\right) - s\,m^k_{\rho(k)}\bar{\gamma}^k_{\rho(k)} & \text{if } i=k\,. \end{cases} \tag{61}$$

The second equation is a consequence of (57), when $\lambda\rho(k) = \lambda(k)$, and of (41) and (55) otherwise. As for the first equation, we argue as follows. If $\lambda(i) < \lambda(k)$, the result follows directly from (54) — with $h_{2,j} \equiv 0$. If $\lambda(i) = \lambda(k)$, so that $\rho(i) < \rho(k)$, the result follows from (55) or (57). Finally, if $\lambda(i) > \lambda(k)$, then, by (7) and the assumption — $i \le k$—, $\ell(i) < \ell(k) \Longrightarrow \ell(i) - \ell(\lambda(k)) < \ell(\rho(k))$, and the result follows from (60).

Let us now consider the term ${}^3r$ in (51). From Definition 2, ${}^3r$ is a sum of one-forms $s^{2+\kappa_j}r'_j\mathbf{d}f_j$, where each r'_j is a polynomial function of f, Δ^k-homogeneous of degree

$$\ell(i) - \ell(j) - (2+\kappa_j)\ell(\lambda(k)) < \min\{\ell(i), \ell(\rho(k))\}\,.$$

By applying Property 4 in Lemma 1 to each one-form $r'_j\mathbf{d}f_j$, we get

$${}^3r = s^2\left(\sum_j s^{\kappa_j}\mathbf{d}h_{1,j} + \sum_{j=m+1}^{\min\{i,\rho(k)\}-1} h'_{2,j}(s,f)\left(\mathbf{d}f_j - f_{\lambda(j)}\mathbf{d}f_{\rho(j)} + \bar{\gamma}^k_j\right) - \sum_{j=m+1}^{\min\{i,\rho(k)\}-1} h'_{2,j}(s,f)\bar{\gamma}^k_j\right) \tag{62}$$

where the functions $h_{1,j}$ are Δ^k-homogeneous of non-negative degree and therefore vanish at the origin. If the degree of homogeneity is zero, so that $h_{1,j}$ is constant, we may as well choose $h_{1,j} \equiv 0$.

We can now define the functions $q^k_{i,j}$. Let us note that $q^k_{i,1}$ has already been defined: by (48) if $\lambda(i) \leq \lambda(k)$, and by (59) otherwise. For the definition of $q^k_{i,j}$ with $j > 1$, we distinguish two cases, according to whether i is smaller than or equal to k.

If $i < k$, by using (53), (61), and (62), relation (51) can be rewritten in the form (44), with

$$\bar{\gamma}^k_i = \mathbf{d}\left(\sum_{j=2}^{j_{i,k}} s^j q^k_{i,j}\right) + \alpha_1 +^1 r +^2 r - s \sum_{j=m+1}^{\min\{i,\rho(k)\}-1} \left(h_{2,j} + sh'_{2,j}\right)(s,f)\bar{\gamma}^k_j + \sum_j s^{2+\kappa_j}\mathbf{d}h_{1,j} \quad (63)$$

and smooth functions $t_{i,j}$ which we do not need to specify further. The functions $h_{2,j}$ and $sh'_{2,j}$, involved in the above expression of $\bar{\gamma}^k_i$, are polynomial in s and f. Also, from (45), $\bar{\gamma}^k_j = \bar{\gamma}^k_{j,1}\mathbf{d}s$ —because $j < \rho(k)$—, where $\bar{\gamma}^k_{j,1}$ depends on s and f only. As a consequence, we have

$$-s \sum_{j=m+1}^{\min\{i,\rho(k)\}-1} \left(h_{2,j} + sh'_{2,j}\right)(s,f)\bar{\gamma}^k_j = sh'(s,f)\mathbf{d}s = a_0 s^{\kappa'}\mathbf{d}s + h''\mathbf{d}s \quad (64)$$

with $a_0 \in \mathbb{R}$, $1 \leq \kappa' \in \mathbb{N}$, h' and h'' functions of s and f only, and h'' identically zero when $f = 0$. From Definition 2, (64) can be rewritten as

$$-s \sum_{j=m+1}^{\min\{i,\rho(k)\}-1} \left(h_{2,j} + sh'_{2,j}\right)(s,f)\bar{\gamma}^k_j =^1 r' +^2 r' \quad (65)$$

From (40), (52), and the fact that $i < k$ implies that either $\lambda(i) \neq \lambda(k)$ or $\rho(i) \neq \rho(k)$, we deduce that $\alpha_1 = \mathbf{d}p^k_i$. Therefore, by using (65) in (63)

$$\bar{\gamma}^k_i = \mathbf{d}\left(\sum_{j=2}^{j_{i,k}} s^j q^k_{i,j}\right) + \mathbf{d}p^k_i +^1 r'' + \mathbf{d}as^{2+\kappa} + \sum_j s^{2+\kappa_j}\mathbf{d}h_{1,j} \quad (66)$$

where we have used the fact that any function of Type 2 is the differential of a polynomial as^q with $q \geq 2$. From there, the functions $q^k_{i,j}$ $(j > 1)$ are uniquely defined by setting

$$\sum_{j=2}^{j_{i,k}} s^j q^k_{i,j} \triangleq -as^{2+\kappa} - \sum_j s^{2+\kappa_j} h_{1,j} \quad (67)$$

It is simple to check that $\mathbf{P1(i)}$ is verified with this choice. This yields, in view of (66)

$$\bar{\gamma}^k_i = \mathbf{d}p^k_i +^1 r'' - \sum_j h_{1,j}\mathbf{d}\left(s^{2+\kappa_j}\right) = \mathbf{d}p^k_i +^1 r'''.$$

where the last equality comes from the fact that $h_{1,j}(0) = 0$, as mentioned after Eq. (62). By using the definition of one-forms of Type 1, it follows that (45) is satisfied and thus that $\mathbf{P2(i)}$ is verified —note that, if ${}^1r''' = r_s\mathbf{d}s + r_c\mathbf{d}c$ and $i \leq \rho(k)$, then r_c is homogeneous of non-positive degree so that it is necessarily a constant, which in fact is equal to zero since r_c vanishes at $f = 0$.

For the last case, $i = k$, we proceed similarly. By using (53), (61), and (62), relation (51) can again be rewritten in the form (44), with this time

$$\bar{\gamma}_i^k = \mathbf{d}\left(\sum_{j=2}^{j_{i,k}} s^j q_{i,j}^k\right) + \alpha_1 - s m_{\rho(k)}^k \bar{\gamma}_{\rho(k)}^k +^1 r +^2 r - s^2 \sum_{j=m+1}^{\min\{i,\rho(k)\}-1} h_{2,j}(s,f)\bar{\gamma}_j^k + \sum_j s^{2+\kappa_j}\mathbf{d}h_{1,j} \tag{68}$$

instead of (63). From from (40), (41), (52), and the induction hypothesis $\mathbf{P2(\rho(k))}$ if $\rho(k) > m$,

$$\begin{aligned} \alpha_1 - s\, m_{\rho(k)}^k \bar{\gamma}_{\rho(k)}^k &= \alpha_1 - s\, m_{\rho(k)}^k \mathbf{d}c - s\, m_{\rho(k)}^k \bar{\gamma}_{\rho(k),1}^k \mathbf{d}s \\ &= \tfrac{m_k^k}{2}(c\mathbf{d}s - s\mathbf{d}c) - s\, m_{\rho(k)}^k \bar{\gamma}_{\rho(k),1}^k \mathbf{d}s \end{aligned} \tag{69}$$

where the last equality comes from (40) and (41). Therefore, (68) rewrites as

$$\begin{aligned} \bar{\gamma}_i^k = \mathbf{d}\left(\sum_{j=2}^{j_{i,k}} s^j q_{i,j}^k\right) + \frac{m_k^k}{2}(c\mathbf{d}s - s\mathbf{d}c) +^1 r +^2 r - s^2 \sum_{j=m+1}^{\min\{i,\rho(k)\}-1} h_{2,j}(s,f)\bar{\gamma}_j^k \\ - s\, m_{\rho(k)}^k \bar{\gamma}_{\rho(k),1}^k \mathbf{d}s + \sum_j s^{2+\kappa_j}\mathbf{d}h_{1,j} \end{aligned} \tag{70}$$

From here, we proceed as for the previous case in order to rewrite the above equation as —compare with (66)—

$$\bar{\gamma}_i^k = \mathbf{d}\left(\sum_{j=2}^{j_{i,k}} s^j q_{i,j}^k\right) + \frac{m_k^k}{2}(c\mathbf{d}s - s\mathbf{d}c) +^1 r'' + \mathbf{d}a s^{2+\kappa} - \sum_j s^{2+\kappa_j}\mathbf{d}h_{1,j}\,. \tag{71}$$

Using the same relation (67) to define the functions $q_{i,j}^k$ yields

$$\bar{\gamma}_i^k = \frac{m_k^k}{2}(c\mathbf{d}s - s\mathbf{d}c) +^1 r'''$$

and it is simple to check that the one-form $\bar{\gamma}_i^k$ satisfies (45), so that $\mathbf{P2(i)}$ is verified. This ends the study of **Case 2**.

Case 3: $k < i \leq n(d)$. We simply let $q_{i,j}^k \equiv 0$ for each $j = 1, \ldots, j_{i,k}$ so that both $\mathbf{P1(i)}$ and $\mathbf{P2(i)}$ are readily verified. This ends the first step.

Step 2. Since $f^k = \bar{f}^k \circ \bar{g}_\eta^k$, we deduce from (37) and (44) that, for $i \in \{m+1, \ldots, k\}$,

$$\begin{aligned} \omega_i^k &= \bar{\omega}_i^k \circ \mathbf{d}\bar{g}_\eta^k \\ &= \left(\omega_i^{k-1} + \gamma_i^k \mathbf{d}\alpha_k\right) + \sum_{j=m+1}^{i-1} t_{i,j}(s,f)\left(\omega_j^{k-1} + \gamma_j^k \mathbf{d}\alpha_k\right) \end{aligned} \tag{72}$$

where

$$\gamma_i^k(\theta^k) = \bar{\gamma}_{i,1}^k(\bar{g}_\eta^k(\theta^k))\frac{\partial \eta^{\ell(\lambda(k))}\sin}{\partial \alpha_k}(\theta_k) + \bar{\gamma}_{i,2}^k(\bar{g}_\eta^k(\theta^k))\frac{\partial \eta^{\ell(\rho(k))}\cos}{\partial \alpha_k}(\theta_k)\,. \tag{73}$$

By skew-symmetry of the wedge product, it follows from (72) that

$$\omega_{m+1}^k \wedge \ldots \wedge \omega_k^k = \left(\omega_{m+1}^{k-1} + \gamma_{m+1}^k \mathbf{d}\alpha_k\right) \wedge \ldots \wedge \left(\omega_k^{k-1} + \gamma_k^k \mathbf{d}\alpha_k\right)$$

Since each ω_i^{k-1} is a one-form on $\mathbb{T}^{k-m-1}$, we deduce from the above equation (using multi-linearity and skew-symmetry of the wedge product) that

$$\omega_{m+1}^k \wedge \ldots \wedge \omega_k^k = \sum_{i=m+1}^{k} \gamma_i^k \left(\omega_{m+1}^{k-1} \wedge \cdots \wedge \omega_{i-1}^{k-1} \wedge \mathbf{d}\alpha_k \wedge \omega_{i+1}^{k-1} \wedge \cdots \wedge \omega_k^{k-1}\right). \tag{74}$$

From (39) and (42),

$$\begin{aligned} \bar{\gamma}_{i,1}^k(\bar{g}_\eta^k(\theta^k)) &= \bar{\gamma}_{i,1}^k(\Delta_\eta^k(\sin\theta_k, \cos\theta_k, \Delta_{1/\eta} f^{k-1}(\theta^{k-1}))) \\ &= \eta^{\ell(i)-\ell(\lambda(k))} \bar{\gamma}_{i,1}^k(\sin\theta_k, \cos\theta_k, \Delta_{1/\eta} f^{k-1}(\theta^{k-1})) \\ &= \eta^{\ell(i)-\ell(\lambda(k))} \bar{\gamma}_{i,1}^k(\sin\theta_k, \cos\theta_k, 0) + \sum_{j<\ell(i)-\ell(\lambda(k))} \eta^j \beta_{i,j}^1(\theta^k) \end{aligned} \tag{75}$$

where $\beta_{i,j}^1$ denotes smooth functions on $\mathbb{T}^{k-m}$. The second equality in the above equation comes from the fact that $\bar{\gamma}_{i,1}^k$ is Δ^k-homogeneous of degree $\ell(i) - \ell(\lambda(k))$, and the third one from the fact that $\bar{\gamma}_{i,1}^k(s, c, f)$ is polynomial in s, c, and f. A similar calculation yields

$$\bar{\gamma}_{i,2}^k(\bar{g}_\eta^k(\theta^k)) = \eta^{\ell(i)-\ell(\rho(k))} \bar{\gamma}_{i,2}^k(\sin\theta_k, \cos\theta_k, 0) + \sum_{j<\ell(i)-\ell(\rho(k))} \eta^j \beta_{i,j}^1(\theta^k) \tag{76}$$

From (45), (73), (75), and (76),

$$\gamma_i^k(\theta^k) = \begin{cases} \eta^{\ell(k)} \dfrac{m_k^k}{2} + \displaystyle\sum_{1<j<\ell(k)} \eta^j \beta_{i,j}(\theta^k) & \text{if } i = k \\ \displaystyle\sum_{1<j<\ell(k)} \eta^j \beta_{i,j}(\theta^k) & \text{otherwise}, \end{cases} \tag{77}$$

for some smooth functions $\beta_{i,j}$ on $\mathbb{T}^{k-m}$. In view of (74) and (77),

$$\left(\omega_{m+1}^k \wedge \ldots \wedge \omega_k^k\right)(\theta^k) = \eta^{\ell(k)} \frac{m_k^k}{2} \left(\omega_{m+1}^{k-1} \wedge \ldots \wedge \omega_{k-1}^{k-1}\right)(\theta^{k-1}) + \sum_{1 \le j < \ell(k)} \eta^j \beta'_{k,j}(\theta^k)$$

for some other smooth functions $\beta'_{k,j}$ on $\mathbb{T}^{k-m}$. By the compacity of $\mathbb{T}^{k-m}$ and the induction hypothesis, (36) follows when η is larger than some $\eta_0 > 0$. ∎

Appendix

Proof of Lemma 1

We begin with the proof of Property 3. Consider the system

$$\begin{cases} \dot{x} &= \displaystyle\sum_{i=1}^{m} u_i b_i(x) \qquad x = (x_1, \ldots, x_{n(d'+1)}) \\ \dot{z} &= p(x) u_j \qquad j \in \{1, \ldots, m\} \end{cases} \tag{78}$$

where the sub-system with the state x is obtained by considering the first $n(d'+1)$ equations of $S(d)$. This sub-system is simply the free system $S(d'+1)$. Since p is homogeneous of degree d' with respect to the dilation (14), it can only depend on the variables x_i of weight $\ell(i) \leq d' < \ell(n(d'+1)) = d'+1$. Therefore, p is also Δ^e-homogeneous of degree d with the dilation Δ^e_μ defined by

$$\Delta^e_\mu(x_1, \dots, x_{n(d'+1)}, z) = \left(\mu^{\ell(1)} x_1, \dots, \mu^{\ell(n(d'+1))} x_{n(d'+1)}, \mu^{\ell(n(d'+1))} z\right), \tag{79}$$

and the control vector fields b^e_i $(i = 1, \dots, m)$ of system (78) are Δ^e-homogeneous of degree -1. This implies, in particular, that every Lie bracket of length at least equal to $d'+2$ is homogeneous of degree at most equal to $-(d'+2)$ and must be identically zero. Therefore, at each point (x, z), the involutive closure of the distribution generated by the vector fields b^e_i is of dimension $n(d'+1)$ at most —which is the dimension of the basis $\mathcal{B}_{d'+1}$ of $\mathcal{L}_{d'+1}(X)$—, and it is precisely $n(d'+1)$ because the free system $S(d'+1)$ is controllable. By application of the Frobenius theorem, we deduce the existence of a function $\phi \in \mathcal{C}^\infty(U; \mathbb{R})$, with U a neighborhood of $(0,0) \in \mathbb{R}^{n(d'+1)} \times \mathbb{R}$, such that

$$\phi(0,0) = 0\,, \quad \frac{\partial \phi}{\partial(x,z)}(0,0) \neq 0 \tag{80}$$

and ϕ is constant along the solutions of (78), i.e.

$$\forall i = 1, \dots, m,\ \forall (x,z) \in U \qquad \frac{\partial \phi}{\partial(x,z)}(x,z) b^e_i(x,z) = 0\,. \tag{81}$$

By means of a Taylor expansion in the neighborhood of the origin, we expand ϕ as

$$\phi = \sum_{i=1}^{d'+1} \phi_i + o^{d'+1}(\rho) \tag{82}$$

where ϕ_i is a polynomial function, Δ^e-homogeneous of degree i, and ρ is any Δ^e-homogeneous norm. First, we note that

$$\phi_1 \equiv \dots \equiv \phi_{d'} = 0\,.$$

Indeed, by homogeneity, these functions cannot depend on z which has weight $d'+1$. Then, the polynomials $(\partial \phi_1/\partial x) b^e_i(x,z)$, which are homogeneous of degree zero and thus constant, must be equal to zero since otherwise $(\partial \phi/\partial(x,z)) b^e_i(0,0) = (\partial \phi_1/\partial x) b^e_i(0,0)$ would not be equal to zero, in contradiction with (81). Therefore the homogeneous function ϕ_1 is constant along the solutions of (78). Due to the controllability of the free system involved in (78), this is possible only if ϕ_1 is a constant function which must be identically zero since ϕ_1 is homogeneous of positive degree. Using similar arguments, one can show that $\phi_2, \dots, \phi_{d'}$ are also identically zero. Then, we claim that $\phi_{d'+1}$ can not be identically zero. Indeed, by homogeneity, the derivative of $o^{d'+1}(\rho)$ in (82) at the origin must be zero. Therefore, from (80),

$$\frac{\partial \phi_{d'+1}}{\partial(x,z)}(0,0) \neq 0\,.$$

Furthermore,

$$\frac{\partial \phi_{d'+1}}{\partial z}(0) \neq 0$$

since otherwise the homogeneous polynomial function $\phi_{d'+1}$ would not depend on z, so that we would again obtain that $\phi_{d'+1}$ is identically zero, in contradiction with (80). Therefore

$$\phi_{d'+1}(x,z) = c\left(z - q^j(x)\right) \qquad (c \neq 0)$$

for some polynomial function q^j homogeneous of degree $d'+1$. The fact that $\phi_{d'+1}$ is constant along the solutions of (78), i.e.

$$\forall i = 1, \ldots, m,\ \forall (x,z) \in U \qquad \frac{\partial \phi_{d'+1}}{\partial (x,z)}(x,z) b_i^e(x) = 0\,. \tag{83}$$

yields (15) for x in a neighborhood of the origin, and this extends to $x \in \mathbb{R}^{n(d)}$ since $\phi_{d'+1}$ and the b_i^e's are homogeneous.

We now proceed with the proof of Property 4. We assume that $i \in \{1, \ldots, m\}$, since otherwise a simple algebraic manipulation yields

$$\begin{aligned}\mathbf{d}x_i = (\mathbf{d}x_i - x_{\lambda(i)}\mathbf{d}x_{\rho(i)}) + \sum_{r=1}^{\bar{r}} x_{\lambda(i)} x_{\lambda\rho(i)} \ldots x_{\lambda\rho^{r-1}(i)}(\mathbf{d}x_{\rho^r(i)} - x_{\lambda\rho^r(i)}\mathbf{d}x_{\rho^{r+1}}) \\ + x_{\lambda(i)} x_{\lambda\rho(i)} \ldots x_{\lambda\rho^{\bar{r}}(i)} \mathbf{d}x_{\rho^{\bar{r}+1}(i)}\end{aligned}$$

where $\bar{r}$ is the smallest integer such that $\rho^{\bar{r}+1} \in \{1, \ldots, m\}$. Property 3 in Lemma 1 guarantees the existence of a polynomial function h_1, Δ-homogeneous of degree $= d' - \ell(i) + 1 = d'$ since $i \in \{1, \ldots, m\}$, and such that

$$p(x)\mathbf{d}x_i = \frac{\partial h_1}{\partial x}(x)\left(\sum_{i=1}^{m} \mathbf{d}x_i b_i(x)\right)\,. \tag{84}$$

We rewrite (84) as

$$\begin{aligned}p(x)\mathbf{d}x_i &= \sum_{j=1}^{n(d)} \frac{\partial h_1}{\partial x_j}(x)\mathbf{d}x_j - \sum_{j=1}^{n(d)} \frac{\partial h_1}{\partial x_j}(x)\left(\mathbf{d}x_j - \sum_{i=1}^{m}\mathbf{d}x_i b_{i,j}(x)\right) \\ &= \mathbf{d}h_1 - \sum_{j=1}^{n(d)} \frac{\partial h_1}{\partial x_j}(x)\left(\mathbf{d}x_j - \sum_{i=1}^{m}\mathbf{d}x_i b_{i,j}(x)\right)\end{aligned} \tag{85}$$

To conclude the proof, it is sufficient to prove that for any j, there exist functions $\alpha_{j,p}$, Δ-homogeneous of degree $\ell(j) - \ell(p)$, such that

$$\omega_j \triangleq \mathbf{d}x_j - \sum_{i=1}^{m} \mathbf{d}x_i b_{i,j}(x) = \sum_{1<\ell(p)\leq \ell(j)} \alpha_{j,p}(x)(\mathbf{d}x_p - x_{\lambda(p)}\mathbf{d}x_{\rho(p)})\,. \tag{86}$$

We prove (86) by induction on $\ell(j)$. If $\ell(j)=1$, it follows from (13) that ω_j is zero. The property is thus trivially verified with $\alpha_{j,p}\equiv 0$. We then assume that this property holds for $\ell(j)\leq k$. Now, let $\ell(j)=k+1$. We have

$$\begin{aligned}
\omega_j &= \big(\mathbf{d}x_j - x_{\lambda(j)}\mathbf{d}x_{\rho(j)}\big) + x_{\lambda(j)}\mathbf{d}x_{\rho(j)} - \sum_{i=1}^{m}\mathbf{d}x_i b_{i,j}(x)\\
&= \big(\mathbf{d}x_j - x_{\lambda(j)}\mathbf{d}x_{\rho(j)}\big) + x_{\lambda(j)}\left(\sum_{i=1}^{m}\mathbf{d}x_i b_{i,\rho(j)}(x) + \sum_{1<\ell(p)\leq\ell(\rho(j))}\alpha_{\rho(j),p}(x)(\mathbf{d}x_p - x_{\lambda(p)}\mathbf{d}x_{\rho(p)})\right)\\
&\qquad\qquad - \sum_{i=1}^{m}\mathbf{d}x_i b_{i,j}(x)\\
&= \big(\mathbf{d}x_j - x_{\lambda(j)}\mathbf{d}x_{\rho(j)}\big) + x_{\lambda(j)}\sum_{1<\ell(p)\leq\ell(\rho(j))}\alpha_{\rho(j),p}(x)(\mathbf{d}x_p - x_{\lambda(p)}\mathbf{d}x_{\rho(p)})
\end{aligned}$$

where the last equality comes from (13). From here, (86) readily follows. ∎

Proof of Lemma 2

Let us first show that the solution at time T of (27) can be expressed as (28). The proof is based on Sussmann's product expansion for the Chen series [7]. In order to avoid reintroducing the algebraic machinery used in this reference, we will borrow the notation of [7].

Let $X=\{X_1,\dots,X_m\}$ and $\bar{X}=\{\bar{X}_1,\dots,\bar{X}_m,\bar{X}_{m+1},\dots\bar{X}_{n(d)}\}$ denote two sets of indeterminates, and consider the formal power series —or Chen series—

$$\bar{S}_u(T)=\sum_{\bar{\sigma}}\left(\int_0^T u_{\bar{\sigma}}\right)\bar{X}_{\bar{\sigma}^\star}$$

where the sum is over all multi-indices $\bar{\sigma}$ with value in $\{1,\dots,n(d)\}$. It is well known that the solution at time T of (27) is given by $\bar{S}_u(T)$ after identifying each $\bar{X}_i$ to g_i, applying the differential operator so obtained to the identity function, and evaluating at $x=x_0$. The formal power series $\bar{S}_u(T)$ is also the solution at time T of

$$\dot{\bar{S}}(t)=\bar{S}(t)\left(\sum_{i=1}^{n(d)}u_i(t)\bar{X}_i\right)\,,\qquad \bar{S}(0)=1\,.$$

It is known that $\bar{S}_u(T)$ is an *exponential Lie series*, i.e.

$$\bar{S}_u(T)=exp(\sum_j \bar{S}_j^{\bar{X}}) \tag{87}$$

where

i) for all j, $\bar{S}_j^{\bar{X}}\in\mathcal{L}(\bar{X})$ the Lie algebra generated by the $\bar{X}_i$'s,

ii) the series $\sum_j\bar{S}_j^{\bar{X}}$ is convergent, i.e. $\forall\bar{\sigma},\quad \#\{j\,:\,\bar{S}_{j,\bar{\sigma}}^{\bar{X}}\neq 0\}<\infty$ where $\#$ denotes the cardinal, and $\bar{S}_{j,\bar{\sigma}}^{\bar{X}}$ is the coefficient of $\bar{X}_{\bar{\sigma}}$ in $\bar{S}_j^{\bar{X}}$, i.e.

$$\bar{S}_j^{\bar{X}}=\sum_{\bar{\sigma}}\bar{S}_{j,\bar{\sigma}}^{\bar{X}}\,\bar{X}_{\bar{\sigma}}\,.$$

Now, consider the series $S_u(T)$, solution at time T of

$$\dot{S}(t) = S(t)\left(\sum_{i=1}^{m} u_i(t)X_i + \sum_{i=m+1}^{n(d)} u_i(t)B_i\right), \qquad S(0) = 1\,,$$

where the B_i's are the formal brackets of the P. Hall basis $\mathcal{B}$ of $\mathcal{L}(X)$, identified to the corresponding element in $\mathcal{L}(X)$. Since $S(t)$ can be obtained from $\bar{S}(t)$ by substituting $X_1, \ldots, X_m$ for $\bar{X}_1, \ldots, \bar{X}_m$ and $B_{m+1}, \ldots, B_{n(d)}$ for $\bar{X}_{m+1}, \ldots, \bar{X}_{n(d)}$, it follows from (87) that

$$S_u(T) = exp(\sum_j S_j^X)$$

where

$$S_j^X = \bar{S}_j^{Proj(\bar{X})} \text{ with } Proj(\bar{X}) = (X_1, \ldots, X_m, B_{m+1}, \ldots, B_{n(d)})\,. \tag{88}$$

We claim that $S_u(T)$ is an exponential Lie series (in the indeterminates $X_1, \ldots, X_m$). To prove this, we must show that i) for all j, $S_j^X \in \mathcal{L}(X)$, and ii) $\sum_j S_j^X$ is convergent. Firstly i) follows from (88) and the fact that $\bar{S}_j^{\bar{X}} \in \mathcal{L}(\bar{X})$. Indeed,

$$S_j^X = \bar{S}_j^{Proj(\bar{X})} \in \mathcal{L}(Proj(\bar{X})) = \mathcal{L}(X)$$

since each $B_i\,(i = m+1, \ldots, n(d))$ belongs to $\mathcal{L}(X)$. Secondly, let us show that ii) is satisfied too. From (88), we have

$$\begin{aligned}
\#\left\{j : S_{j,\sigma}^X \neq 0\right\} &= \#\left\{j : \sum_{\bar{\sigma}:[Proj(\bar{X})]_{\bar{\sigma}} = X_\sigma} \bar{S}_{j,\bar{\sigma}}^{Proj(\bar{X})} \neq 0\right\} \\
&\leq \#\left\{j : \exists \bar{\sigma} : \left([Proj(\bar{X})]_{\bar{\sigma}} = X_\sigma \text{ and } \bar{S}_{j,\bar{\sigma}}^{Proj(\bar{X})} \neq 0\right)\right\} \\
&\leq \sum_{\bar{\sigma}:[Proj(\bar{X})]_{\bar{\sigma}} = X_\sigma} \#\left\{j : \bar{S}_{j,\bar{\sigma}}^{Proj(\bar{X})} \neq 0\right\}
\end{aligned}$$

Each number in the last sum is finite because $\sum_j \bar{S}_j^{\bar{X}}$ is convergent, and

$$\#\{\bar{\sigma} : [Proj(\bar{X})]_{\bar{\sigma}} = X_\sigma\}$$

is also finite because

$$[Proj(\bar{X})]_{\bar{\sigma}} = X_\sigma \Longrightarrow |\bar{\sigma}| \leq |\sigma|$$

where $|.|$ here denotes the length of the multi-index. Therefore, the sum $\sum_j S_j^X$ is convergent.

Then, it follows from [7, Lemma], and from the fact that $\mathcal{B}$ is a P. Hall basis for $\mathcal{L}(X)$, that there exists a sequence $\{c_i(T,u)\}$ such that

$$S_u(T) = \prod_{B_i \in \mathcal{B}} exp(c_i(T,u)B_i)$$

Identifying each indeterminates X_i in the above equality with the v.f. g_i, and using the fact that the family $\{g_1, \dots, g_m\}$ is nilpotent of order $d+1$, gives (28). The first result stated in the lemma is thus proved.

There remains to show that, when the g_i's are the v.f. of the free system $S(d)$, the mapping $(c_1, \dots, c_{n(d)}) \longmapsto \prod_{i=1}^{n(d)} exp(c_i\, g_i)x_0$ is one-to-one for any x_0. Firstly, for any $i = 1, \dots, n(d)$,

$$\exp(c_i g_i)\, x^i = x^i + \Big(0, \dots, 0, a_i c_i, p^i_{i+1}(x^i_1, \dots, x^i_i, c_i), \dots, p^i_{n(d)}(x^i_1, \dots, x^i_{n(d)-1}, c_i)\Big) \tag{89}$$

for some polynomial functions p^i_k. This formula can be obtained as follows. The Chen-Fliess series expansion gives

$$\exp(c_i g_i)\, x^i = x^i + c_i g_i(x^i) + \sum_{k\geq 2} \frac{c_i^k}{k!} L^k_{g_i} id(x^i) \tag{90}$$

where id is the identity function. Each component $(L^k_{g_i} id)_j$ of $L^k_{g_i} id$ is homogeneous of degree $\ell(j) - k\ell(i)$. This implies that the first i components of $L^k_{g_i} id$ are identically zero, and that $(L^k_{g_i} id)_j$ cannot depend on the variables $x_j, \dots, x_{n(d)}$. From this, and the fact that (see Property 1 in Lemma 1)

$$g_i(x) = a_i \partial/\partial x_i + \sum_{j>i} g_{i,j}(x)\partial/\partial x_j\ ,$$

one easily shows that (90) rewrites as (89).

Secondly, using (89), a direct induction on k —for decreasing $k = n(d), \dots, 1$— yields

$$\prod_{i=k}^{n(d)} \exp(c_i g_i)\, x_0 = x_0 + \Big(0, \dots, 0, a_k c_k, a_{k+1} c_{k+1} + q^k_{k+1}(c_k, x_0), \dots, \\ a_{n(d)} c_{n(d)} + q^k_{n(d)}(c_k, \dots, c_{n(d)-1}, x_0)\Big)$$

so that

$$\prod_{i=1}^{n(d)} \exp(c_i g_i)\, x_0 = x_0 + \Big(a_1 c_1, a_2 c_2 + q^1_2(c_1, x_0), \dots, a_{n(d)} c_{n(d)} + q^1_{n(d)}(c_1, \dots, c_{n(d)-1}, x_0)\Big)$$

Since each a_i is different from zero, the map $c \longmapsto \prod_{i=1}^{n(d)} \exp(c_i g_i)\, x_0$ is clearly one-to-one. ∎

Proof of Lemma 3

It is clearly sufficient to find a function $f \in \mathcal{C}^\infty(\mathbb{T}^{n(d)-m}; \mathbb{R}^{n(d)})$ such that

$$\forall \theta \in \mathbb{T}^{n(d)-m}, \qquad \text{Det}\, \big(u^\alpha_{i,j}(\theta)\big)_{i,j=m+1,\dots,n(d)} \neq 0 \tag{91}$$

with the functions $u^\alpha_{i,j}$ uniquely defined by

$$\frac{\partial f}{\partial \alpha_j}(\theta) = \sum_{i=1}^{n(d)} u^\alpha_{i,j}(\theta)\, b_i(f(\theta))\,. \tag{92}$$

For any $i \geq m+1$ and $j = 1, \dots, m$, one has $b_{i,j} \equiv 0$ because, from Property 1 in Lemma 1, $b_{i,j}$ is a homogeneous function of strictly negative degree $1 - \ell(i)$. Using this fact and (13), one readily obtains

$$\forall i = 1, \dots, m, \forall j, \quad u^\alpha_{i,j}(\theta) = \frac{\partial f_i}{\partial \alpha_j}(\theta) \tag{93}$$

and (92) rewrites as

$$\frac{\partial f}{\partial \alpha_j}(\theta) - \sum_{i=1}^{m} \frac{\partial f_i}{\partial \alpha_j}(\theta) b_i(f(\theta)) = \sum_{i=m+1}^{n(d)} u^\alpha_{i,j} b_i(f(\theta))\,. \tag{94}$$

Therefore,

$$\left(\frac{\partial f_k}{\partial \alpha_j} - \sum_{i=1}^{m} \frac{\partial f_i}{\partial \alpha_j} b_{i,k}(f)\right)_{k,j=m+1,\dots,n(d)} = \Big(b_{k,i}(f)\Big)^T_{k,i=m+1,\dots,n(d)} \Big(u_{i,j}\Big)_{i,j=m+1,\dots,n(d)} \tag{95}$$

From Property 1 in Lemma 1, the square matrix $(b_{k,i}(f))$ in the above relation is, for any f, non-singular —since it is lower triangular with non-zero constant terms on the diagonal. Therefore, (91) is equivalent to

$$\forall \theta \in \mathbb{T}^{n(d)-m}, \quad \operatorname{Det}\left(\frac{\partial f_k}{\partial \alpha_j}(\theta) - \sum_{i=1}^{m} \frac{\partial f_i}{\partial \alpha_j}(\theta) b_{i,k}(f(\theta))\right)_{k,j=m+1,\dots,n(d)} \neq 0\,. \tag{96}$$

We claim that there exist an invertible lower-triangular matrix T, with 1's on the diagonal, such that

$$\left(\frac{\partial f_k}{\partial \alpha_j} - \sum_{i=1}^{m} \frac{\partial f_i}{\partial \alpha_j} b_{i,k}(f)\right)_{k,j=m+1,\dots,n(d)} = T(f)\left(\frac{\partial f_k}{\partial \alpha_j} - f_{\lambda(k)} \frac{\partial f_{\rho(k)}}{\partial \alpha_j}\right)_{k,j=m+1,\dots,n(d)} \tag{97}$$

which is clearly sufficient to conclude the proof. This can be proved by induction on $\ell(k)$. For $\ell(k) = 2$ —there is no k such that $\ell(k) = 1$, since $k \geq m+1$—, we have $\ell(\rho(k)) = 1$, so that (13) and (93) yield

$$\frac{\partial f_k}{\partial \alpha_j} - \sum_{i=1}^{m} \frac{\partial f_i}{\partial \alpha_j} b_{i,k}(f) = \frac{\partial f_k}{\partial \alpha_j} - \sum_{i=1}^{m} \frac{\partial f_i}{\partial \alpha_j} f_{\lambda(k)} b_{i,\rho(k)}(f) = \frac{\partial f_k}{\partial \alpha_j} - f_{\lambda(k)} \frac{\partial f_{\rho(k)}}{\partial \alpha_j}\,.$$

This is consistent with (97). Now, assume that the equality in (97) holds for all the lines with index k such that $\ell(k) \leq p$ $(p \geq 2)$, we show below that it also holds for the lines with index k

such that $\ell(k) = p + 1$. We have

$$\begin{aligned}
\frac{\partial f_k}{\partial \alpha_j} - \sum_{i=1}^{m} \frac{\partial f_i}{\partial \alpha_j} b_{i,k}(f) &= \frac{\partial f_k}{\partial \alpha_j} - \sum_{i=1}^{m} \frac{\partial f_i}{\partial \alpha_j} f_{\lambda(k)} b_{i,\rho(k)}(f) \\
&= \frac{\partial f_k}{\partial \alpha_j} - f_{\lambda(k)} \frac{\partial f_{\rho(k)}}{\partial \alpha_j} + f_{\lambda(k)} \left(\frac{\partial f_{\rho(k)}}{\partial \alpha_j} - \sum_{i=1}^{m} \frac{\partial f_i}{\partial \alpha_j} b_{i,\rho(k)}(f) \right) \\
&= \frac{\partial f_k}{\partial \alpha_j} - f_{\lambda(k)} \frac{\partial f_{\rho(k)}}{\partial \alpha_j} + f_{\lambda(k)} \sum_{m+1 \leq q \leq \rho(k)} t_{\rho(k),q} \left(\frac{\partial f_q}{\partial \alpha_j} - f_{\lambda(q)} \frac{\partial f_{\rho(q)}}{\partial \alpha_j} \right)
\end{aligned}$$

where the last equality comes from the induction hypothesis. This concludes the proof. ∎

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