A stochastic Model of TCP/IP with Stationary Ergodic Random Losses

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Abstract: We consider here a flow control mechanism in which the rate at which data is sent increases linearly in time until a loss occurs. At that point the transmission rate decreases by a multiplicative factor. This mechanism is a good approximation to TCP/IP, the congestion control in the Internet. Losses are generated by some exogenous random process, which is assumed to be stationary ergodic; no Markovian assumptions are made. We obtain an explicit expression for the average transmission rate and obtain bounds in the case that there is a limit on the maximum rate.

Key-words: TCP/IP, Performance evaluation, Stationary ergodic point processes, Palm calculus.

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Un modèle stochastique de TCP/IP en présence d'un processus de perte de paquets aléatoire, stationnaire et ergodique

Résumé : On présente dans ce papier un modèle stochastique pour un protocole de contrôle de flux similaire à TCP, le fameux protocole contrôlant la congestion dans l'Internet. Le protocole modélisé augmente linéairement le débit de transmission jusqu'à l'apparition d'une congestion. Il divise alors le débit par deux et recommence l'augmentation. Ceci est similaire à TCP où la congestion est détectée par l'apparition d'une perte d'un paquet. On suppose que les pertes de paquets, donc les signaux de congestion, sont générées par un processus aléatoire supposé être stationnaire, ergodique et indépendant du flux contrôlé. Aucune hypothèse Markovienne n'est faite. On obtient une expression explicite du débit moyen en fonction des paramètres du processus de pertes. Dans le cas où une limite existe sur le débit de transmission, on trouve deux bornes pour le débit moyen. Le modèle est à la fin étendue pour modéliser le mécanisme Timeout du protocole TCP.

Mots-clés : TCP/IP, Évaluation de performance, Processus ponctuel ergodique et stationnaire, Calcul de Palm

1 Introduction

In this paper we analyze the performance of a TCP-like flow control mechanism with random losses. We assume that the rate at which data is sent increases linearly in time until a loss occurs. At that point the transmission rate decreases by a multiplicative factor. The linear growth and multiplicative decrease is a good approximation to TCP/IP [17], the congestion control in the Internet, see e.g. [30] for the explanation of these assumptions.

There are many related references on TCP over lossy channels. [22, 32, 20, 30] provide an analysis for the case of independent losses, whereas bursty models have been studied in [1, 2, 3, 7, 10, 21, 30, 34], which use either two state Markovian models or even more simplistic ones.

In this paper we investigate the case of a general, not necessarily Markovian process of losses. All we assume is that this process is stationary ergodic. With this minimal assumption, we are able to obtain explicit expressions for the average send rate of flow control mechanism.

In TCP/IP, the decrease by a multiplicative factor $\nu = 0.5$ occurs if the loss is detected using a mechanism known as duplicate ACKs (denoted TD), see e.g. [30]. There are cases, however, in which the source is not able to use that mechanism and losses are detected by the expiration of some timer TO. In reaction to such losses the window decreases to a minimum size (of one unit). We shall analyze in Sections 2-5 the case of TD, and extend this analysis to TO in Section 6.

In the next section we shall define the mathematical model for the TCP window evolution. In the ensuing Section 3 we shall compute the first two moments of the window size at loss arrivals for the stationary regime as well as we shall obtain a closed form expression to the average transmission rate at an arbitrary time. Different examples of loss processes will then be studied in Section 4: losses which are modeled by an i.i.d. sequence, by a Markov modulated Poisson process and by a Markov arrival process. Bounds are derived in Section 5 to compute a modified model that takes into account non-linearities that occur in TCP when there is a limit on the maximum size of the window. Section 6 extends the analysis to the case where both TD and TO are present. Section 7 concludes with some comments on the assumptions of the model and their justification by numerical experiments.

2 The model

We present in this section a simple model for TCP in which losses are detected by the TD mechanism. The instants of the random losses are modeled by a general stationary ergodic point process [5]. We assume that this loss process has non-null and finite intensity λ. Let $\{T_n\}_{n=-\infty}^{+\infty}$ be a particular realization of the point process. Then, the evolution of the so called TCP window (i.e. the number of packets that the source can sent before receiving an acknowledgment (ACK)) can be described by the following equation

$$X_{n+1} = \nu X_n + \alpha S_n, \tag{1}$$

where X_n is a window size just prior to the arrival of the loss at T_n, $S_n := T_{n+1} - T_n$, ν is a factor by which the window reduces after a loss (usually, it is equal to 0.5) and α is a rate of the window growth in the absence of random losses. The pair $\{T_n, X_n\}$ can be considered as a marked point process [5]. The TCP window is a direct measure for the instantaneous throughput; the latter is given in fact as the window size divided by the round-trip time.

Note that the equation (1) is a particular case of stochastic linear difference equations. Using the results of [8, 13], we conclude that equation (1) has a stationary solution given by

$$X_n^* = \alpha \sum_{k=0}^{\infty} \nu^k S_{n-1-k}. \tag{2}$$

Moreover, if the window evolution starts from an arbitrary window size X_0, it will converge almost sure to the above stationary regime [8, 13]:

$$\lim_{n\to\infty} |X_n - X_n^*| = 0, \quad P - \text{a.s.},$$

3 The computation of the first two moments of X_n and the average throughput

First we calculate the expectation and the second moment of the window size at the instants of random losses. Namely,

Proposition 1 *Let $\lambda = 1/E[S_n]$ be an intensity of the loss process and let $R(k) = E[S_n S_{n+k}]$ be a covariance function for the process $\{S_n\}_{n=-\infty}^{+\infty}$. Then,*

$$E[X_n^*] = \frac{\alpha}{\lambda(1-\nu)} \tag{3}$$

$$E[(X_n^*)^2] = \frac{\alpha^2}{1-\nu^2}[R(0) + 2\sum_{k=1}^{\infty} \nu^k R(k)] \tag{4}$$

PROOF: To calculate (3) and (4), we use the expression for the stationary solution (2).

$$E[X_n^*] = \alpha \sum_{k=0}^{\infty} \nu^k E[S_{n-1-k}] = \frac{\alpha}{\lambda} \sum_{k=0}^{\infty} \nu^k = \frac{\alpha}{\lambda(1-\nu)}$$

Similarly, we obtain

$$E[(X_n^*)^2] = E[\alpha \sum_{j=0}^{\infty} \nu^j S_{n-1-j} \alpha \sum_{k=0}^{\infty} \nu^k S_{n-1-k}]$$

$$= \alpha^2 E[\sum_{k=0}^{\infty} \sum_{j=0}^{k} \nu^j S_{n-1-j} \nu^{k-j} S_{n-1-k+j}] = \alpha^2 \sum_{k=0}^{\infty} \sum_{j=0}^{k} \nu^k E[S_{n-1-j} S_{n-1-k+j}]$$

$$= \alpha^2 \sum_{k=0}^{\infty} \nu^k \begin{cases} R(0) + 2\sum_{j=1}^{r} R(2j), & \text{if} \quad k = 2r, \\ 2\sum_{j=1}^{r} R(2j-1), & \text{if} \quad k = 2r-1. \end{cases}$$

Then, we regroup the terms of the last series to get

$$\begin{aligned} E[(X_n^*)^2] &= \alpha^2 R(0) \sum_{j=0}^{\infty} \nu^{2j} + 2\alpha^2 \sum_{k=1}^{\infty} R(k) \nu^k \sum_{j=0}^{\infty} \nu^{2j} \\ &= \frac{\alpha^2}{1-\nu^2}[R(0) + 2\sum_{k=1}^{\infty} \nu^k R(k)]. \end{aligned}$$

□

Remark 1 We would like to note that the expectation computed in (3) is taken with respect to the loss instants. This expectation is also referred to as Palm expectation in the context of point processes. Of course, one might be interested in computing the expectation of the window size at an arbitrary time moment. For ergodic processes the latter expectation coincides with the average throughput.

Next, by using the expression (2) and the concept of the Palm probability, we calculate the average throughput of our TCP model

$$\bar{X} = \lim_{T \to \infty} \frac{1}{T} \int_0^T X(t) dt.$$

Proposition 2 *The average throughput is given by*

$$\bar{X} = \frac{\lambda\alpha}{2}[R(0) + \frac{1}{\nu}\sum_{k=1}^{\infty} \nu^k R(k)] \tag{5}$$

PROOF: As was noticed in Remark 1, in our case the average throughput coincides with the expectation of the window size $E[X(t)]$ at an arbitrary time point. To compute $E[X(t)]$ one can use the following inversion formula (see e.g., [5] Ch.1 Sec.4)

$$E[X(t)] = \lambda E^0[\int_0^{T_1} X(t)dt] \tag{6}$$

where $E^0[\cdot]$ is an expectation associated with Palm distribution. In particularly, $P^0\{T_0 = 0\} = 1$. Now using formula (6) and expression (2), we can write

$$\begin{aligned}
E[X(t)] &= \lambda E^0[\int_0^{T_1} (\frac{1}{2}X_0 + \alpha t)dt] = \lambda E^0[\frac{1}{2}X_0 S_0 + \frac{\alpha}{2}S_0^2] \\
&= \frac{\lambda}{2}E^0[\alpha \sum_{k=0}^{\infty} \nu^k S_{-1-k} S_0] + \frac{\lambda\alpha}{2}E^0[S_0^2] \\
&= \frac{\lambda\alpha}{2}\sum_{k=0}^{\infty} \nu^k R(k+1) + \frac{\lambda\alpha}{2}R(0) \\
&= \frac{\lambda\alpha}{2}[R(0) + \frac{1}{\nu}\sum_{k=1}^{\infty} \nu^k R(k)]
\end{aligned}$$

□

Remark 2 *Often the covariance function $C(k) = R(k) - E[S_n]^2$ is used instead of the correlation function $R(k)$. Then, the formulae (4) and (5) become*

$$E[(X_n^*)^2] = \frac{\alpha^2}{1-\nu^2}[C(0) + 2\sum_{k=1}^{\infty} \nu^k C(k)] + \frac{\alpha^2}{\lambda^2(1-\nu)^2},$$

$$\bar{X} = \frac{\lambda\alpha}{2}[C(0) + \frac{1}{\nu}\sum_{k=1}^{\infty} \nu^k C(k)] + \frac{\alpha(2-\nu)}{2\lambda(1-\nu)}.$$

4 Examples of loss process

Now let us consider some important particular cases of the general loss process.

4.1 IID random losses (General Renewal Process)

In this subsection we model the loss process as a general renewal process. Namely, we assume that $S_n, n = ..., -1, 0, 1, ...$ are i.i.d. random variables. Then the formulae (3), (4) and (5) take the following form.

Proposition 3 *Let $\{S_n\}_{n=-\infty}^{+\infty}$ are i.i.d. with $d := E[S_n]$ and $d^{(2)} := E[S_n^2]$. Then,*

$$E[X_n^*] = \frac{\alpha d}{1-\nu},$$

$$E[(X_n^*)^2] = \frac{\alpha^2}{1-\nu^2}[d^{(2)} + \frac{2\nu d^2}{1-\nu}],$$

$$\bar{X} = E[X(t)] = \frac{\alpha}{2d}[d^{(2)} + \frac{d^2}{1-\nu}].$$

4.2 Correlated random losses modeled as a Markovian Arrival Process

In this section we consider the correlated losses which are modeled by Markovian Arrival Process (MAP) [24, 25, 28, 29]. It was shown in [4] that for a given general point process, there is a sequence of MAPs which converges to the point process in distribution. In particular, this implies that in principle the general point process can be approximated by appropriate MAPs. Furthermore, PH-renewal process [29] and Markov Modulated Poisson Process (MMPP) [12] are particular case of the Markovian arrival process.

Let us briefly review the definition and some properties of the Markovian Arrival Process. Let $N(t)$ be a counting process associated with MAP, that is $N(t)$ is the number of arrivals (or losses in our setting) in the interval $(0, t]$. Also let $J(t)$ be an auxiliary state variable. Then MAP can be described in terms of a two-dimensional Markov process $\{N(t), J(t)\}$ on the state space $\{(i,j)|i \geq 0, 1 \leq j \leq m\}$ with the following infinitesimal generator

$$Q = \begin{bmatrix} C & D & 0 & 0 & \cdots \\ 0 & C & D & 0 & \cdots \\ 0 & 0 & C & D & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

where the matrix $C \in \mathbb{R}^{m \times m}$ governs the transition of the process J without arrival (loss) and it has negative diagonal elements and nonnegative off-diagonal elements.

The matrix $D \in \mathbb{R}^{m \times m}$ governs the transitions of J with the simultaneous arrivals and it has nonnegative elements. Thus, the underlying Markov process $J(t)$ has the following infinitesimal generator

$$\bar{Q} = C + D.$$

Further, we assume that $\bar{Q} \neq C$ and C is a stable matrix. This ensures that $J(t)$ does not get absorbed in a class of states in which arrivals stop. When $J(t) = i$, the rate of transitions to state $j \neq i$ is $\bar{Q}_{ij}$. If such a transition occurs then an arrival occurs simultaneously with the transition with probability $D_{ij}/(-C_{ii} - D_{ii})$.

Let π be the stationary distribution of the underlying Markov process, that is

$$\pi \bar{Q} = 0, \quad \pi e = 1, \tag{7}$$

where e is the column vector of ones. If the process $J(t)$ has π as its initial distribution, then the arrival process becomes time stationary with the following fundamental arrival rate [28, 29]

$$\lambda = \pi D e.$$

Let $\{S_n\}_{n=1}^{\infty}$ be a sequence of the inter-arrival times for MAP and let $\{J_n\}_{n=1}^{\infty}$ be a sequence of the states of the underlying Markov process at the arrival epochs. Then $\{J_n, D_n\}_{n=1}^{\infty}$ is a Markov renewal process [16] with the following transition probability matrix [29]

$$F(x) = \left(\int_0^x \exp\{Cu\} du \right) D = (I - \exp\{Cx\})(-C)^{-1} D$$

Note that $T = F(\infty) = -C^{-1}D$ is a transition probability matrix of a discrete time Markov chain embedded at the instants of arrivals. Let μ be its stationary distribution. Now if we take the initial distribution of the underlying Markov chain $J(t)$ as μ, the arrival process becomes interval-stationary or event-stationary. Note that there is an elegant relation between the distributions π and μ

$$\pi = \frac{1}{\mu C^{-1} e} \mu C^{-1}, \quad \mu = \frac{1}{\pi D e} \pi D. \tag{8}$$

The derivation of the above formulae is given in the Appendix.

The event-stationary version of MAP has the following joint distribution function for the inter-arrival times [19]

$$F_{S_0 \cdots S_n}(x_0, ..., x_n) = \mu \prod_{i=0}^{n} \{(I - \exp\{Cx_i\})T\} e \tag{9}$$

Consequently, the joint Laplace-Stieltjes transform is given by

$$f(z_0,...,z_n) = E[\exp\{-\sum_{k=0}^{n} z_k S_k\}] = \mu \prod_{k=0}^{n} \{(z_k I - C)^{-1} D\} e. \qquad (10)$$

Next, using the Laplace-Stieltjes transform (10), we can easily calculate the first two moments and the covariance function of the inter-arrival time process. Namely,

$$E[S_n] = -\frac{d}{ds}(\mu(zI-C)^{-1}De)|_{z=0} = \mu C^{-2}De = -\mu C^{-1}Te = -\mu C^{-1}e, \qquad (11)$$

$$E[S_n^2] = \frac{d^2}{ds^2}(\mu(zI-C)^{-1}De)|_{z=0} = -2\mu C^{-3}De = 2\mu C^{-2}Te = 2\mu C^{-2}e \qquad (12)$$

$$R(k) = E[S_n S_{n+k}] = \frac{\partial^2}{\partial z_0 \partial z_k} f(z_0,...,z_k)|_{z_i=0} = \mu C^{-2}DT^{k-1}C^{-2}De. \qquad (13)$$

To derive the expression for the covariance function $R(k)$ we have used the following formula for the differentiation of an inverse matrix-valued function: $(A^{-1}(z))' = -A^{-1}(z)A'(z)A^{-1}(z)$ [6].

Note that MAP becomes MMPP with infinitesimal generator R and arrival rate matrix Λ, if we take $C = R - \Lambda$ and $D = \Lambda$.

Now, employing (11), (12) and (13), we can calculate the first two moments of the window evolution process $\{X_n\}$.

Proposition 4 *Let the loss process $\{S_n\}$ be represented by MAP. Then,*

$$E[X_n^*] = -\frac{\alpha}{1-\nu}\mu C^{-1}e \qquad (14)$$

$$E[(X_n^*)^2] = \frac{\alpha^2}{1-\nu^2}2\mu(C^{-2} + \nu C^{-2}D[I-\nu T]^{-1}C^{-2}D)e \qquad (15)$$

PROOF: The above formulae are immediately obtained from (3), (4) with the help of (11), (12), (13) and the following derivation

$$\sum_{k=1}^{\infty} \nu^k R(k) = \mu C^{-2} D \sum_{k=1}^{\infty} \nu^k T^{k-1} C^{-2} De$$

$$= \mu C^{-2} D \nu \sum_{k=0}^{\infty} \nu^k T^k C^{-2} De = \nu \mu C^{-2} D [I - \nu T]^{-1} C^{-2} De$$

□

Next, using (5), we calculate the average throughput.

Proposition 5 *Let $\{S_n\}$ be a Markovian arrival process. Then, the average throughput is given by*

$$\bar{X} = -\frac{\alpha}{\mu C^{-1} e} \mu (C^{-2} + \frac{1}{2} C^{-2} D [I - \nu T]^{-1} C^{-2} D) e. \tag{16}$$

4.3 Correlated random losses modeled as a Markovian Modulated Poisson Process

As already mentioned, the MMPP is a special case of an MAP process with infinitesimal generator R and arrival rate matrix Λ, if we take $C = R - \Lambda$ and $D = \Lambda$. The previous subsection thus provides a full solution for the MMPP as a special case. In this subsection we further restrict to an MMPP with two states and obtain a simple expression for the TCP average throughput.

In this section let us consider a loss proccess generated by the 2-state MMPP. Note that a number of fitting algorithms [11, 14, 15, 15, 26, 33] are available for the determination of MMPP(2) parameters from the data. Recall that MMPP(2) is decribed by the infinitisimal generator of the underlying Markov process

$$R = \begin{bmatrix} -\sigma_1 & \sigma_1 \\ \sigma_2 & -\sigma_2 \end{bmatrix}$$

and by the matrix of Poisson arrival rates

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}.$$

Note that MMPP is a particular case of MAP with $C = R - \Lambda$ and $D = \Lambda$. The stationary distributions π and μ and the average rate λ are given by

$$\pi = \frac{1}{\sigma_1 + \sigma_2} \begin{bmatrix} \sigma_2 & \sigma_1 \end{bmatrix}, \quad \mu = \frac{1}{\sigma_1 + \sigma_2} \begin{bmatrix} \lambda_1 \sigma_2 & \lambda_2 \sigma_1 \end{bmatrix},$$

$$\lambda = \pi \Lambda e = \frac{\lambda_1 \sigma_2 + \lambda_2 \sigma_1}{\sigma_1 + \sigma_2}.$$

The covariance function of the interarrival times for the event-stationary MMPP is the following negative exponential function [18, 19]

$$C(n) = E[(S_k - E[S_k])(S_{k+n} - E[S_{k+n}])] = c\kappa^{n-1}, \quad n \geq 1, \tag{17}$$

where

$$c = \frac{\sigma_1 \sigma_2 (\lambda_1 - \lambda_2)^2 \lambda_1 \lambda_2}{(\lambda_1 \sigma_2 + \lambda_2 \sigma_1)^2 (\lambda_1 \sigma_2 + \lambda_2 \sigma_1 + \lambda_1 \lambda_2)^2},$$

$$\kappa = \frac{\lambda_1 \lambda_2}{\lambda_1 \sigma_2 + \lambda_2 \sigma_1 + \lambda_1 \lambda_2}.$$

Using formula (12), we also calculate $R(0)$ for the event-stationary MMPP(2) model.

$$R(0) = E[S_n^2] = 2\mu C^{-2} e$$

$$= 2\frac{\lambda_1\sigma_2(\sigma_1+\sigma_2+\lambda_2)^2 + \lambda_2\sigma_1(\sigma_1+\sigma_2+\lambda_1)^2 + (\lambda_1-\lambda_2)^2\sigma_1\sigma_2}{(\lambda_1\sigma_2+\lambda_2\sigma_1)(\lambda_1\sigma_2+\lambda_2\sigma_1+\lambda_1\lambda_2)^2} \tag{18}$$

Now we can express the values of $E[X_n^*]$, $E[(X_n^*)^2]$ and the throughput $\bar{X}$ in terms of the MMPP(2) parameters.

Proposition 6 *Let the loss process be represented by* MMPP(2) *model. Then,*

$$E[X_n^*] = \frac{\alpha(\sigma_1+\sigma_2)}{(1-\nu)(\lambda_1\sigma_2+\lambda_2\sigma_1)}, \tag{19}$$

$$E[(X_n^*)^2] = \frac{\alpha^2}{1-\nu^2}[r + 2\nu(\frac{c}{1-\nu\kappa} + \frac{1}{\lambda^2(1-\nu)}), \tag{20}$$

$$\bar{X} = \frac{\lambda\alpha}{2}[r + \frac{c}{1-\nu\kappa}] + \frac{\alpha}{2\lambda(1-\nu)}, \tag{21}$$

where $r := R(0)$ *is given by* (18).

Proof: Both formula (4) and formula (5) use the series $\sum_{k=1}^{\infty} \nu^k R(k)$. Let us calculate this series for the MMPP(2) model.

$$\begin{aligned}
\sum_{k=1}^{\infty} \nu^k R(k) &= \sum_{k=1}^{\infty} \nu^k C(k) + (E[S_n])^2 \sum_{k=1}^{\infty} \nu^k \\
&= c\sum_{k=1}^{\infty} \nu^k \kappa^{k-1} + \frac{\nu}{\lambda^2(1-\nu)} \\
&= \frac{c\nu}{1-\nu\kappa} + \frac{\nu}{\lambda^2(1-\nu)}
\end{aligned}$$

Then, substituting the above expression into (4) and (5), we obtain the formulae (20) and (21).

□

5 Bounds for the model with the window size limitation

In the previous sections we did not include in the modeling the fact that the window stops growing when it achieves some maximum size. The limitation on the window size comes naturally from the following reasons [27, 30]. Firstly, the maximum window size is limited by default (which is taken in general as 64 KBytes). Secondly, the receiver has a maximal buffer size which determines the maximal window size. We shall assume in this section that the window size is limited by maximum window size of M.

We take $\nu = 0.5$, which is the case in the currently used TCP versions. Then, the example of the window evolution is presented in Figure 1. The stochastic difference equation (1) is respectively modified to the following form.

$$X_{n+1} = M \wedge (\frac{1}{2} X_n + \alpha S_n) \tag{22}$$

Note that the model becomes nonlinear and it is probably not possible to obtain the explicit expressions for $E(X_n)$ and $\bar{X}$ for the general loss process. We shall thus use the results of the previous sections to obtain bounds for the performance measures in this case.

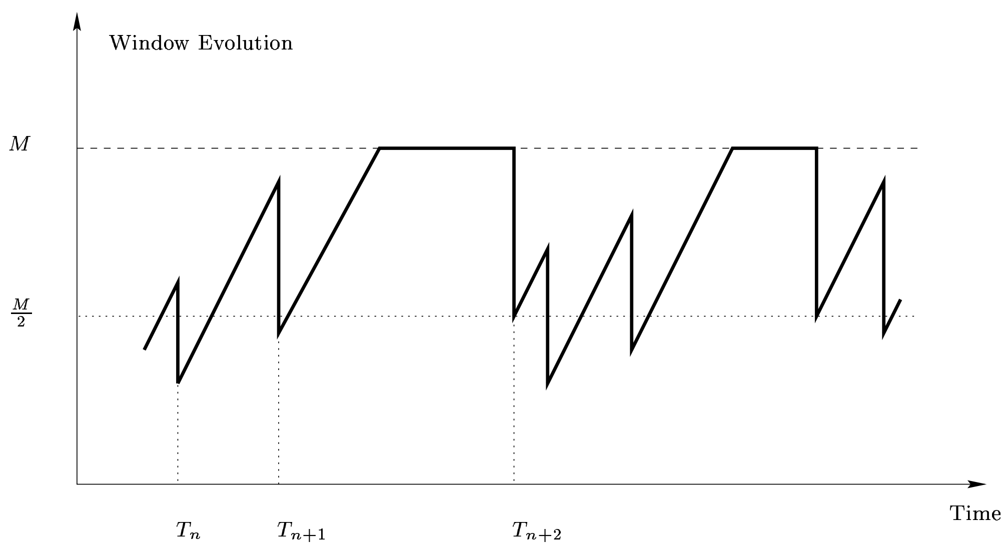

Figure 1: The window evolution in the presence of the window size limitation

Before deriving the bounds we shall establish a stability result for the process X_n using a Loynes-type construction.

Theorem 1 *Assume that $\{S_n\}$ is a stationary process. Then there exists a stationary process $\{X_n^*\}$ defined on the same probability space, and satisfying the recursion (22). Furthermore, for any initial state X_0 we have P-a.s.*

$$\lim_{n\to\infty} \sup_{k\geq n} |X_k - X_k^*| = 0. \tag{23}$$

That is, for any initial state X_0, the process $\{X_k\}_{k\geq n}$ converges in distribution to the stationary process as $n \to \infty$.

PROOF: Define on the same probability space the family of processes $\{X_k^{(n)}, k \in Z\}$, $n = 0, 1, ...$ as follows. $X_k^{(n)} := 0$ for $k \leq -n$, and for $k > -n$ it is given by the recursion (22). For each k, $X_k^{(n)}$ is increasing with respect to n and thus it has a limit (which is obviously finite) which we denote by X_k^*. This limit satisfies (22) since for every n, $X_k^{(n)}$ satisfies it. Finally, the stationarity of the sequence $\{S_n\}$ implies that $\{X_k^*\}$ is stationary as well. The convergence of X_n to X_n^* follows from the fact that

$$\begin{aligned}
|X_{n+1} - X_{n+1}^*| &= |M \wedge (\frac{1}{2}X_n + \alpha S_n) - M \wedge (\frac{1}{2}X_n^* + \alpha S_n)| \\
&\leq \frac{1}{2}|X_n - X_n^*|
\end{aligned}$$

To show that the above inequality holds, one needs to consider four cases. If the both values of $(\frac{1}{2}X_n + \alpha S_n)$ and $(\frac{1}{2}X_n^* + \alpha S_n)$ are less or alternatively greater than M, then the inequality is obvious. Let us consider not so obvious cases. For example, let $(\frac{1}{2}X_n + \alpha S_n) > M$ and $(\frac{1}{2}X_n^* + \alpha S_n) < M$, then

$$\begin{aligned}
|X_{n+1} - X_{n+1}^*| &= M - (\frac{1}{2}X_n^* + \alpha S_n) \\
&\leq (\frac{1}{2}X_n + \alpha S_n) - (\frac{1}{2}X_n^* + \alpha S_n) \\
&\leq \frac{1}{2}|X_n - X_n^*|.
\end{aligned}$$

Thus,

$$|X_n - X_n^*| \leq 2^{-n}|X_0 - X_0^*| \to 0 \quad \text{as} \quad n \to \infty.$$

Since both X_0 and X_0^* are finite (they are bounded between 0 and M) this implies (23).

□

To derive the lower and upper bounds, let us consider the following auxiliary stochastic processes defined on the same probability space as X_n:

$$\check{X}_{n+1} = \frac{1}{2}\check{X}_n + \frac{M}{2} \wedge (\alpha S_n) = \frac{1}{2}\check{X}_n + \alpha(\frac{M}{2\alpha} \wedge S_n), \tag{24}$$

and

$$\hat{X}_{n+1} = \frac{1}{2}\hat{X}_n + M \wedge (\alpha S_n) = \frac{1}{2}\hat{X}_n + \alpha(\frac{M}{\alpha} \wedge S_n). \tag{25}$$

Proposition 7 *Let $\{S_n\}$ be a stationary stochastic point process. Assume that $X_0 = \check{X}_0 = \hat{X}_0$. Then for all $n \geq 0$, $\check{X}_0 \leq X_0 \leq \hat{X}_0$. Moreover,*

$$2\alpha\check{d} \leq E[X_n] \leq 2\alpha\hat{d} \tag{26}$$

where the expectation $E[X_n]$ is taken with respect to the stationary regime, $\check{d} = E[\check{S}_n]$, $\hat{d} = E[\hat{S}_n]$, and where $\check{S}_n := \frac{M}{2\alpha} \wedge S_n$, $\hat{S}_n := \frac{M}{\alpha} \wedge S_n$.

PROOF: We show by induction that $\check{X}_n \leq X_n$. It holds for $n = 0$. Assume it holds for $n = k$. Then, consider two cases $S_k \leq \frac{M}{2\alpha}$ and $S_k > \frac{M}{2\alpha}$. For $S_k \leq \frac{M}{2\alpha}$, one has

$$\begin{aligned} X_{k+1} &= \frac{1}{2}X_k + \alpha S_k \\ &\geq \frac{1}{2}\check{X}_k + \alpha S_k = \check{X}_{k+1}. \end{aligned}$$

And if $S_k > \frac{M}{2\alpha}$, then

$$\begin{aligned} X_{k+1} &= \frac{1}{2}X_k + (M - \frac{1}{2}X_k) \wedge (\alpha S_k) \\ &\geq \frac{1}{2}X_k + (M - \frac{1}{2}X_k) \wedge (\frac{M}{2}) \\ &= \frac{1}{2}X_k + \frac{M}{2} \\ &\geq \frac{1}{2}\check{X}_k + \frac{M}{2} = \check{X}_{k+1} \end{aligned}$$

The first inequality is true, since $X_k \leq M$. Hence, $\check{X}_{k+1} \leq X_{k+1}$ and according to the induction principle, the inequality $\check{X}_n \leq X_n$ holds for all $n \geq 0$. Consequently, $E[\check{X}_n] \leq E[X_n]$ for all $n \geq 0$.

Since by the results of [13] and Theorem 1 both processes $\{\check{X}_n\}$ and $\{X_n\}$ converge to the stationary regime, we can let n go to infinity. This results in the lower bound.

The upper bound is obtained in the similar manner by using the auxiliary process (25).

□

Next we calculate the lower and upper bounds for the throughput.

Proposition 8 *Let* $\check{d}^{(2)} := E[(\check{S}_n)^2]$, $\check{R}(k) := E[\check{S}_{n-k}S_n]$ *and* $\hat{d}^{(2)} := E[(\hat{S}_n)^2]$, $\hat{R}(k) := E[\hat{S}_{n-k}S_n]$. *Then, the lower and upper bounds for the throughput are given by*

$$\bar{X} \geq \alpha\lambda\left(\check{R}(0) - \check{d}^{(2)} + \sum_{k=0}^{\infty}\frac{1}{2^{k+1}}\check{R}(k+1)\right), \tag{27}$$

$$\bar{X} \leq \alpha\lambda\left(\hat{R}(0) - \hat{d}^{(2)} + \sum_{k=0}^{\infty}\frac{1}{2^{k+1}}\hat{R}(k+1)\right). \tag{28}$$

PROOF: To obtain the lower bound for the throughput we again use the auxiliary process (24). Suppose that $\{X_n\}$ is in a stationary regime and define

$$\check{X}(t) = \begin{cases} \frac{1}{2}\check{X}_n^* + \alpha t, & t \in [T_n, \check{T}_n], \\ \frac{1}{2}\check{X}_n^* + \alpha\check{S}_n, & t \in [\check{T}_n, T_{n+1}], \end{cases} \tag{29}$$

where $\check{T}_n = T_n + \check{S}_n$ (See Figure 2). Similarly to (2), one can write the expression for the stationary version of $\{\check{X}_n\}$, that is

$$\check{X}_n^* = \alpha\sum_{k=0}^{\infty}(\frac{1}{2})^k\check{S}_{n-1-k}.$$

Now, using (29) and the above expression for $\check{X}_n^*$, we obtain the lower bound

$$\begin{aligned} \bar{X} &= \lambda E^0[\int_0^{T_1} X(t)dt] \geq \lambda E^0[\int_0^{T_1}\check{X}(t)dt] \\ &= \lambda E^0[\int_0^{\check{S}_0}(\frac{\check{X}_0^*}{2} + \alpha t)dt + \int_{\check{S}_0}^{S_0}(\frac{\check{X}_0^*}{2} + \alpha\check{S}_0)dt] \\ &= \lambda E^0[\frac{1}{2}\check{X}_0^*\check{S}_0 + \frac{\alpha}{2}\check{S}_0^2 + (\frac{1}{2}\check{X}_0^* + \alpha\check{S}_0)(S_0 - \check{S}_0)] \\ &= \lambda E^0[\frac{1}{2}\check{X}_0^*S_0 + \alpha\check{S}_0S_0 - \frac{\alpha}{2}\check{S}_0^2] \end{aligned}$$

$$= \lambda E^0[\frac{1}{2}\alpha\sum_{k=0}^{\infty}(\frac{1}{2})^k \check{S}_{-1-k}S_0 + \alpha\check{S}_0 S_0 - \frac{\alpha}{2}\check{S}_0^2]$$

$$= \alpha\lambda\left(\sum_{k=0}^{\infty}(\frac{1}{2})^{k+1}\check{R}(k+1) + \check{R}(0) - \frac{1}{2}\check{d}^{(2)}\right).$$

Now, by using the auxilliary process (25), one can calculate the upper bound for the throughput in the similar fashion.

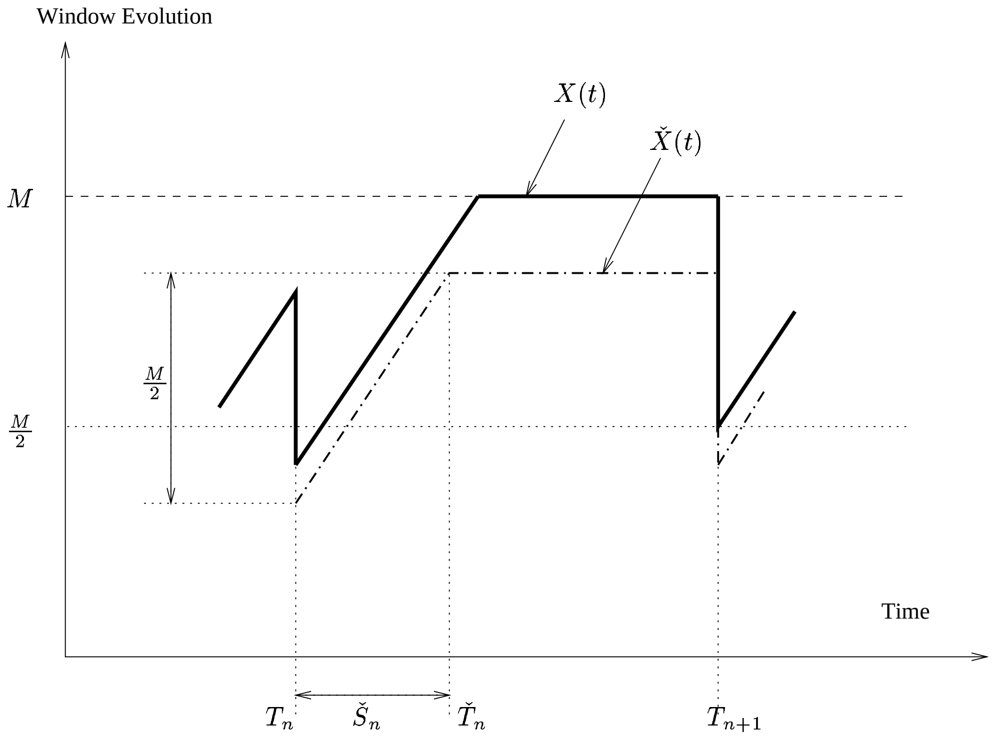

Figure 2: A lower bound for the window evolution

Note that the quantities $\check{d}, \hat{d}, \check{d}^{(2)}, \hat{d}^{(2)}$ and $\check{R}(k), \hat{R}(k)$ can be easily calculated for the loss process modeled by MAP. Indeed,

$$\check{d} = E[\check{S}_n] = \int_0^{\infty}(\frac{M}{2\alpha}\wedge x)\mu dF(x)e = \mu\int_0^{\infty}(\frac{M}{2\alpha}\wedge x)\exp\{xC\}dxDe$$

$$= \mu\left(\int_0^{\frac{M}{2\alpha}} x\exp\{xC\}dxD + \int_{\frac{M}{2\alpha}}^{\infty}\frac{M}{2\alpha}\exp\{xC\}dxD\right)e$$

$$
\begin{aligned}
&= \mu \left(\frac{M}{2\alpha} \exp\{\frac{M}{2\alpha}C\}C^{-1}D - [\exp\{\frac{M}{2\alpha}C\} - I]C^{-2}D - \frac{M}{2\alpha}\exp\{\frac{M}{2\alpha}C\}C^{-1}D \right) e \\
&= \mu[I - \exp\{\frac{M}{2\alpha}C\}]C^{-2}De,
\end{aligned}
$$

$$
\hat{d} = E[\hat{S}_n] = \mu[I - \exp\{\frac{M}{\alpha}C\}]C^{-2}De,
$$

and, similarly,

$$
\begin{aligned}
\check{d}^{(2)} &= E[(\check{S}_n)^2] = \int_0^\infty (\frac{M}{2\alpha} \wedge x)^2 \mu \exp\{xC\} dx De \\
&= \mu \left(\int_0^{\frac{M}{2\alpha}} x^2 \exp\{xC\} dx D + \int_{\frac{M}{2\alpha}}^\infty (\frac{M}{2\alpha})^2 \exp\{xC\} dx D \right) e \\
&= \mu \left(-\frac{M}{\alpha} \exp\{\frac{M}{2\alpha}C\}C^{-2}D + 2[\exp\{\frac{M}{2\alpha}C\} - I]C^{-3}D \right) e,
\end{aligned}
$$

$$
\hat{d}^{(2)} = \mu \left(-2\frac{M}{\alpha} \exp\{\frac{M}{\alpha}C\}C^{-2}D + 2[\exp\{\frac{M}{\alpha}C\} - I]C^{-3}D \right) e.
$$

To compute $\check{R}_1(k)$ and $\check{R}_2(k)$, $k \geq 1$, we use the joint distribution (9).

$$
\begin{aligned}
\check{R}(k) &= E[\check{S}_{n-k}S_n] \\
&= \int_0^\infty \cdots \int_0^\infty (\frac{M}{2\alpha} \wedge x_0)x_k \; \mu \exp\{Cx_0\}D \cdots \exp\{Cx_k\}De \; dx_0 \cdots dx_k \\
&= \mu[I - \exp\{\frac{M}{2\alpha}C\}]C^{-2}DT^{k-1}C^{-2}De \\
\hat{R}(k) &= \mu[I - \exp\{\frac{M}{\alpha}C\}]C^{-2}DT^{k-1}C^{-2}De
\end{aligned}
$$

6 Modeling Timeouts

In this section we model losses due to both Timeouts as well as to Duplicate ACKs. We do not include in the analysis in this section the phenomenon of maximum window size already discussed in the previous section.

We shall assume that a fraction q_{TO} of the losses are caused by the expiration of a Timeout. Adopting the terminology of [30], we call these TO losses, as opposed to TD losses. We assume that the Timeout losses arrive also according to a stationary ergodic process, and their rate is thus

$$
\lambda_{TO} = q_{TO}\lambda.
$$

Assume that X_n is the window size when a TO loss occurs. Then the transmission is halted during a period of a constant duration T_{out}, after which the transmission rate grows exponentially fast by using the so called Slow Start phase [17], until the window reaches the size $X_n/2$. Then it grows linearly again till the next loss occurs. Since the Slow Start phase has a very fast growth rate (in comparison to the linear phase), we shall approximate this phase by an instantaneous jump of the window from its minimal value to the new value $X_n/2$. Moreover, we approximate the lowest value by zero (in practice it is 1).

As before, let T_n be, as before, the instant at which the nth loss occurs, and let $S_n = T_{n+1} - T_n$. If at time T_n there is a loss due to TO, then no losses can occur during the interval $(T_n, T_n + T_{out})$ since transmission is halted. We depict in Figure 3 a sample of our process.[1]

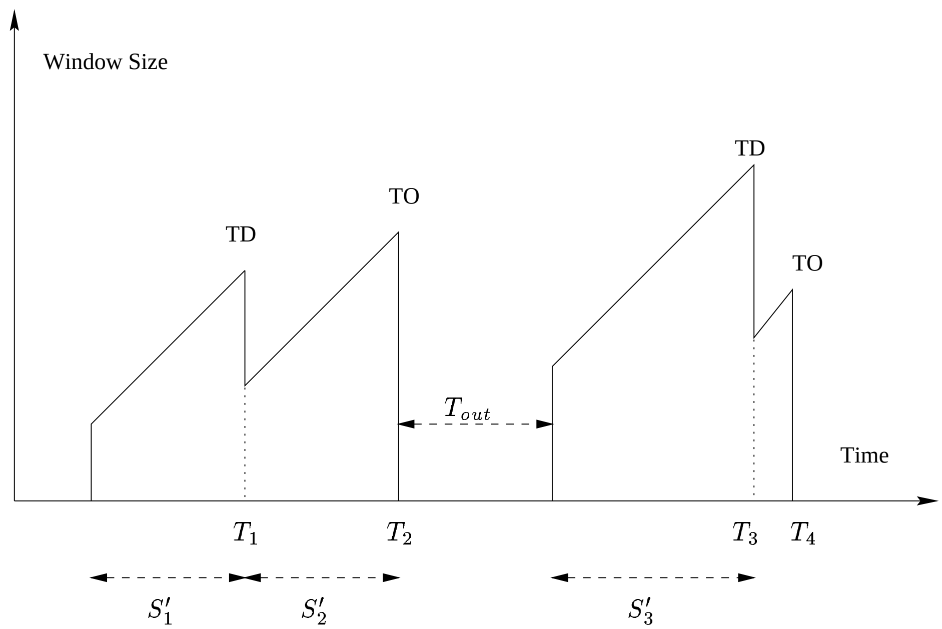

Figure 3: A model for TCP with TO and TD losses

[1]Note that we do not model here the backoff mechanism described [30] which occurs when losses occur very shortly after a Timeout. We believe that in connections without frequent TOs their impact can indeed be neglected.

Define now the sequence

$$S'_n = \begin{cases} S_n & \text{if the loss is } TD \\ S_n - T_{out} & \text{if the loss is } TO. \end{cases}$$

Proposition 9 *Assume that the process S'_n is stationary ergodic* [2] *and let $\lambda' = 1/E[S'_n]$ and $R'(k) = E[S'_n S'_{n+k}]$. Then the average throughput is given by*

$$\bar{X} = \frac{\lambda'\alpha}{2}[R'(0) + \frac{1}{\nu}\sum_{k=1}^{\infty}\nu^k R'(k)](1 - \lambda_{TO}T_{out}). \tag{30}$$

PROOF: Consider a new point process $\{T'_n\}$ whose inter-arrival times are $\{S'_n\}$, and consider the process describing the evolution of the window size X_n when we eliminate the periods T_{out}. This is shown in Figure 4. This is precisely the model described by equation (1), where S_n is replaced by S'_n and T_n replaced by T'_n.

According to Proposition 2, the throughput for the new model is given by

$$thp_1 = \frac{\lambda'\alpha}{2}[R'(0) + \frac{1}{\nu}\sum_{k=1}^{\infty}\nu^k R'(k)].$$

This equals to the number of transmission in the *original model* averaged only over the periods other than time-outs.

In our original model, the expected number of TO that occur during any interval $[0, t]$ is $\lambda_{TO}t$. Hence the fraction of time in which we are in a time-out interval T_{out} in the original model is given by $\lambda_{TO}T_{out}$. The average throughput in the original model is thus $(1 - \lambda_{TO}T_{out})thp_1 + \lambda_{TO}T_{out} \cdot 0$ which establishes (30). □

7 Justification of the assumptions and Concluding Remarks

We analyzed in this paper a simple flow control mechanism in which the send rate adapts to congestion signals by decreasing the transmission rate by a multiplicative factor each time a loss is detected. In absence of a loss, the rate increases linearly in time. We assumed that the loss process is a stationary point process.

[2] Note that the distribution of the time between the nth and the $(n+1)$st loss may depend on the type of nth loss (TD or TO). This however does not prevent the process S' of being stationary ergodic.

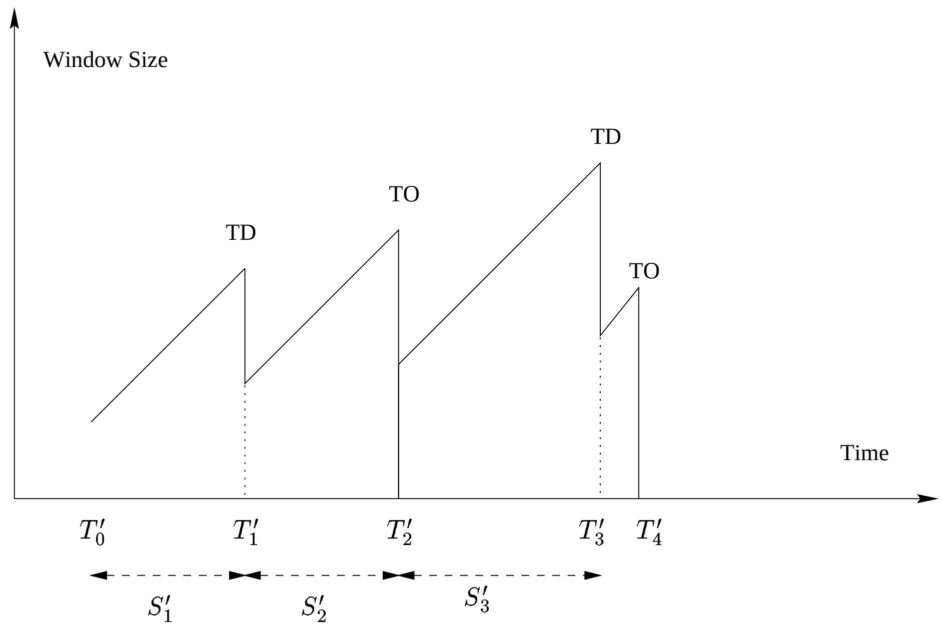

Figure 4: A model for TCP with TO and TD losses when eliminating the periods T_{out}

Our assumptions are quite adapted to the description of the TCP flow control used in the Internet, see [17].

In the actual TCP there are other phenomena which we do not take into account. The linear growth assumption holds in general when the round-trip time does not vary much; this is the case for example, if the buffers within the network along the path of a connection are small with respect to the delay-bandwidth product, i.e. when the queueing delay is negligible with respect to the round trip delay.

Equation (1) together with our probabilistic assumptions imply that the losses are governed by an exogenous process and do not depend on the current window size. In a network with a small number of connection this is not always a valid assumption. Indeed, if there are some bottleneck nodes along the path of the connection, then the chances of having a loss there *increases* as the size of the window increases. Thus behind the dynamics (1) lies an implicit assumption of a large network with many connections, so that the dependence between the loss events of a connection and its window size is negligible.

We believe that other mechanisms in TCP (such as the fast recovery) which we have not included in our model have a little influence on TCP performance. This assumption is in line with [30].

In [30] many empirical tests have been used to validate their model. In those tests losses were shown to be mainly of the TO type. We remark, however, that the examples used in their tests are of very long TCP connections. We have repeated tests on long TCP connections and obtained similar type of results. In particular, we obtained in such connections small covariance coefficients, which suggest that i.i.d. models for TCP can be used for such scenarios. As an example, in experimentation between clope.inria.fr in INRIA, Sophia-Antipolis, France and linus.levels.unisa.edu.au, South Australia, we obtained during 45 mins (in Nov. 1999) 277 losses due to time-outs and no losses due to TD. The average window size was very small: around 7 TCP packets. With small windows, indeed it is more likely that losses lead to time-outs (since the latter occurs when there are not sufficiently many acknowledgements to detect losses through TD). The average measured covariance coefficient of the inter losses time was 0.01.

However, we also made experimentations on local networks. Within different machines in INRIA we have not detected any losses during 15 mins periods. In other cases we obtained mostly TD losses. For example, we tested the connection between two machines in Sophia Antipolis (one in INRIA and one in ESSI - Université Nice Sophia-Atipolis; the two institutes are located around 1km apart). We obtained during a period of 15mins (in November 1999) 275 losses of which only 46 were TO. The average measured covariance coefficient of the inter losses time was -0.2. This suggests that the assumption of independence between losses is not appropriate for short distance TCP connections, and makes our model for correlated losses attractive. We would like to mention that the measured time-out durations in the short distance TCP connections were typically much smaller than the durations between losses. This validates the use of the more simplistic approach of Section 2 for short TCP connections, even without taking into account the TOs, instead of the model in Section 6.

Appendix

Let us show that the relations (8) hold. First, we note that the equation (7) is equivalent to

$$\pi C(I + C^{-1}D) = 0$$

or

$$\tilde{\mu}(-C^{-1}D) = \tilde{\mu}, \tag{31}$$

where $\tilde{\mu} = \pi C$. On the other hand, since μ is an eigenvector of the irreducible transition matrix $T = -C^{-1}D$, we have that $\tilde{\mu} = k\mu$. The latter implies that $\pi = \tilde{\mu}C^{-1} = k\mu C^{-1}$. Next we use the normalization condition $\pi e = 1$ to get the coefficient k. Namely,

$$k = \frac{1}{\mu C^{-1} e}$$

and hence

$$\pi = \tilde{\mu}C^{-1} = \frac{1}{\mu C^{-1} e}\mu C^{-1}.$$

Similarly, we have

$$\mu = \frac{1}{k}\pi C = -\frac{\pi C}{\pi D e} = \frac{\pi D}{\pi D e}.$$

In particular, by using the relations between π and μ, we can easily show that the fundamental arrival rate is equal to the arrival rate of the event-stationary version of MAP.

$$\lambda = \pi D e = \frac{1}{\mu C^{-1} e}\mu C^{-1} D e = -\frac{\mu e}{\mu C^{-1} e} = -\frac{1}{\mu C^{-1} e} = \frac{1}{E[S_n]}$$

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