Exponential Stability and Transfer Functions of a Heat Exchanger Network System

Cheng-Zhong Xu, Gauthier Sallet

INSTITUT NATIONAL DE RECHERCHE EN INFORMATIQUE ET EN AUTOMATIQUE

Exponential Stability and Transfer Functions of a Heat Exchanger Network System

Cheng-Zhong Xu and Gauthier Sallet

N° 3823

1999

THÈME 4

Rapport de recherche

ISSN 0249-6399 ISRN INRIA/RR--3823--FR+ENG

Exponential Stability and Transfer Functions of a Heat Exchanger Network System

Cheng-Zhong Xu and Gauthier Sallet

Thème 4 — Simulation et optimisation
de systèmes complexes
Projet Congé

Rapport de recherche n° 3823 — 1999 — 22 pages

Abstract: In this paper we study frequency and time domain behaviour of a heat exchanger network system. The system is governed by hyperbolic partial differential equations. Both the control operator and the observation operator are unbounded but admissible. Using the theory of symmetric hyperbolic systems, we prove exponential stability of the underlying semigroup for the heat exchanger network. Applying the recent representation theory of infinite-dimensional linear systems, we prove that the system is regular and derive various properties for the transfer functions, which are potentially useful for controller design.

Key-words: Heat exchangers, symmetric hyperbolic equations, exponential stability, regular systems, transfer functions.

(Résumé : tsvp)

INRIA-Lorraine (projet Congé) & URA CNRS 399 (MMAS)
ISGMP, Bât. A, Université de Metz, Ile de Saulcy
57045 Metz cedex 01
tél. : 03 87 31 54 14 – fax : 03 87 54 72 77
e-mail : xu@loria.fr

Etude de la stabilité exponentielle et des fonctions de transfert pour un réseau d'échangeurs thermiques

Résumé : Dans cet article on étudie la stabilité exponentielle et les fonctions de transfert pour un système de réseau d'échangeurs thermiques. Le système est décrit par une série d'équations différentielles aux dérivées partielles du type hyperbolique. Du point de vue de la théorie du contrôle l'opérateur de contrôle et celui d'observation sont tous non-borneés mais admissibles. En utilisant la théorie classique de systèmes symétriques hyperboliques (à deux variables indépendantes) on démontre que le semigroupe est exponentiellement stable pour le réseau d'échangeurs thermiques en considération. En même temps on propose une approche alternative, à savoir la méthode directe de Lyapunov pour l'étude de la stabilité pour une class de systèmes symétriques hyperboliques (non-limitée à deux variables indépendantes). Par la théorie de la représentation de systèmes linéaires de dimension infinie on déduit de nombreuses propriétés pour les fonctions de transfert du système telles que régularité, analyticité et bornitude dans le demi plan complex.

Mots-clé : Echangeurs thermiques, systèmes symétriques hyperboliques, stabilité exponentielle, système régulier, fonctions de transfert.

1 Introduction

In all chemical engineering processes which have heat, mass or momentum transfer, there must exist gradients in spatial directions. Some processes are specifically designed to take the advantage of gradients along the axis of flow. The tubular reactor, the shell and tube heat exchangers and the packed mass exchange columns all achieve their objectives in this way [14]. The dynamic of most of these processes, if not all, must be described by partial differential equations (PDE's), resulting in distributed parameter systems. On the other hand, there are hydraulic systems such as irrigation canals which are of distinctive features of distributed parameter dynamics, giving rise to delayed responses. The essential nature of distributed parameter dynamics cannot be ignored if we want to do control of these processes. The objective of our work is to develop a general framework useful for robust control of these distributed parameter processes. In this paper we study frequency and time domain behaviour of a class of hyperbolic systems represented by a counterflow heat exchanger network. As the reader will see this class of systems covers the classical double pipe heat exchangers [14], the packed mass exchange columns [14] [22] and the irrigation canals [23]. The approach of our study is essentially infinite dimensional : the analysis will be carried out based on PDE models in contrast to the approximation ideas, resulting in finite-dimensional models.

The network of counterflow heat exchangers to be considered is governed by two series of hyperbolic partial differential equations (PDE's) :

$$\begin{cases} \dfrac{\partial R_1(x,t)}{\partial t} = m_1 \dfrac{\partial R_1(x,t)}{\partial x} - K_1(R_1(x,t) - R_2(x,t)) + b_1(x)u(t) \\ \dfrac{\partial R_2(x,t)}{\partial t} = -m_2 \dfrac{\partial R_2(x,t)}{\partial x} + K_2(R_1(x,t) - R_2(x,t)) \end{cases} \tag{1.1}$$

for all $(x,t) \in]0, l_1[\times]0, +\infty[$, and

$$\begin{cases} \dfrac{\partial T_1(x,t)}{\partial t} = (1-\beta) m_1 \dfrac{\partial T_1(x,t)}{\partial x} - K_1(T_1(x,t) - T_2(x,t)) + b_2(x)u(t) \\ \dfrac{\partial T_2(x,t)}{\partial t} = -m_2 \dfrac{\partial T_2(x,t)}{\partial x} + K_2(T_1(x,t) - T_2(x,t)) \end{cases} \tag{1.2}$$

for all $(x,t) \in]l_1, l_2[\times]0, +\infty[$. The two series of PDE's are coupled by the boundary conditions :

$$\begin{cases} R_1(l_1,t) = (1-\beta)T_1(l_1,t) \\ R_2(l_1,t) = T_2(l_1,t) \\ T_1(l_2,t) = 0 \\ R_2(0,t) = d(t). \end{cases} \tag{1.3}$$

We consider the following output :

$$y(t) = T_2(l_2,t). \tag{1.4}$$

In the above equations m_i, K_i, l_i with $i = 1{,}2$ and β are positive constants such that $l_2 > l_1 > 0$ and $0 < \beta < 1$. The state variables $R_i(x,t)$ and $T_i(x,t)$, $i = 1,2$ are deviations of the temperatures of fluids from the steady state values in the coupled counterflow heat exchangers (see [14] or [15]). We assume that the functions $b_1(x)$ and $b_2(x)$ are continuously differentiable. They are determined uniquely from the steady state values (see [15]). The control variable $u(t)$, representing variation of the fluid flow rate, is internal and affects the PDE's in (1.1) and (1.2) themselves. The disturbance variable $d(t)$, representing variation of the fluid inlet temperature, comes through the boundary of the system and enters into the boundary condition (1.3). So, the equations (1.1)-(1.4) describe an infinite-dimensional linear system with boundary input and boundary output.

This heat exchanger network has been constructed in [15] in order to simulate the qualitative input-output behaviour of an industrial furnace. The reader is referred to [15] for the physical background of the problem. From [15] the system exhibits a phenomenon of non-minimum phase : the transfer function from u to y has zeros in the right half plane. Therefore, the H^∞ control theory developed in [7] and [8] has been applied in [15] to this system for minimizing the worst effect of the disturbance $d(t)$ on the output $y(t)$. The industrial interest of the designed H^∞ controller was also explained in more details in [15].

The present paper here is to show how the network system (1.1)-(1.3) is transformed into a dissipative symmetric hyperbolic system. Then, we prove that the associated semigroup is exponentially stable using the theorem of Rauch and Taylor [18].

As a secondary result, we propose a proof of the theorem of Rauch and Taylor by using the direct method of Lyapunov. Although we have not proved the theorem as generally as in [18] with the method of characteristics, the advantage of the direct method of Lyapunov is its simplicity and a priori not limited to one space variable. The direct method of Lyapunov allows us to attack symmetric hyperbolic systems in higher dimension and degenerate hyperbolic cases that the theory in [18] cannot treat (see our example 1 in Section 2). In particular, the heat exchanger network system (1.1)-(1.3) fits into our formulated framework.

We prove also that the system (1.1)-(1.4) is regular in the sense of Weiss [2]. The regularity of the controlled and observed system with exponential stability of the semigroup guarantees that the transfer functions $P(s)$ and $W(s)$ (corresponding to the input-output mappings $u \to y$ and $d \to y$, respectively) are in H^∞ (that is, analytic and bounded in the open right-hand half plane). This gives a positive answer to some of the open questions in [15]. The fact that the system under consideration is regular has useful consequences on the design of feedback controllers for the system. The cascade connection of two regular systems is again regular. The class of regular linear systems is closed under feedback. The most important consequence is that internal stability and external stability are equivalent for a regular system which is both stabilizable and detectable as proved in Rebarber [12].

Classical counterflow heat exchanger systems are usually supposed to be stable by process engineers without mathematical proof. The exponential stability of the classical heat exchangers has been proved in [17] by the direct method of Lyapunov when the diffusion term is taken into account in the flow. In the hyperbolic case without diffusion only strong stability has been obtained in [17] using the decomposition theorem of semigroups of contractions [1]. Then, exponential stability has been proved to be true with the help of Huang's theorem (see [28]). Here, using the theory of symmetric hyperbolic systems in [26] and [18] we prove exponential stability for a much wider class of heat exchanger networks. As mentioned above many dynamical processes such as tubular reactors, gas absorbers and irrigation canals could be put into the form of symmetric hyperbolic systems (see [19], [13], [14],[3] and [17]). Using the recent representation theory developed in Weiss [2] we are able to characterize the transfer functions in terms of the semigroup operator, input operators and the output operator. This characterization is useful for controller design purpose.

Our paper is organized as follows. In Section 2 the stability result of Rauch and Taylor is presented for a class of symmetric hyperbolic systems. There, we propose Lyapunov function candidates for proving exponential stability of some hyperbolic systems. In the first subsection of Section 3 we transform the equations (1.1)-(1.3) into the form of symmetric hyperbolic systems. Exponential stability is obtained for the system by applying the result presented in Section 1. The second subsection of Section 3 is devoted to characterizing the transfer functions of the system. Various properties are derived for the transfer functions. Section 4 contains our conclusions.

2 Symmetric Hyperbolic Systems

Consider a symmetric hyperbolic system of the form :

$$\begin{cases} \dfrac{\partial R(x,t)}{\partial t} = A(x)\dfrac{\partial R(x,t)}{\partial x} + B(x)R(x,t), \qquad (x,t) \in (0,1)\times \mathbb{R}^+ \\ R^-(0,t) = D_0 R^+(0,t) \\ R^+(1,t) = D_1 R^-(1,t) \\ R(x,0) = R_0(x), \end{cases} \tag{2.1}$$

where $R(x,t)$ is a $n\times 1$ vector function for $(x,t)\in(0,1)\times\mathbb{R}^+$, $B(x)$ is a real $n\times n$ matrix function and $A(x)$ is a diagonal matrix function for $x\in[0,1]$, and D_0 and D_1 are real constant matrices. The diagonal matrix $A(x)$ is partitioned as

$$A(x) = \begin{pmatrix} A^-(x) & 0 \\ 0 & A^+(x) \end{pmatrix}$$

with

$$A^-(x) = \mathrm{diag}(\lambda_1(x), \lambda_2(x), ..., \lambda_p(x))$$

and

$$A^+(x) = \mathrm{diag}(\lambda_{p+1}(x), \lambda_{p+2}(x), ..., \lambda_{p+q}(x))$$

$(p+q=n)$, and the vector $R(x,t)$ as

$$R(x,t) = \begin{pmatrix} R^-(x,t) \\ R^+(x,t) \end{pmatrix}$$

with $R^-(x,t)\in\mathbb{R}^p$ and $R^+(x,t)\in\mathbb{R}^q$, respectively. We denote by Λ^* the transposed matrix of Λ or the adjoint operator of Λ, as will be clear from the context, and by $A_x(x)$ the Jacobian of $A(x)$. To simplify the presentation, we assume that the hypotheses H.1—H.3 are satisfied for the system (2.1).

H.1 : $B(\cdot)\in C^0([0,1];\mathbb{R}^{n\times n})$ and $A(\cdot)\in C^1([0,1];\mathbb{R}^{n\times n})$.

H.2 : $\lambda_i(x) < 0,\ i = 1,2,...,p$ for any $x\in[0,1]$ and $\lambda_{p+i}(x) > 0,\ i=1,2,...,q$ for any $x\in[0,1]$.

H.3 : For each $w^- \in \mathbb{R}^p$ and $w^+ \in \mathbb{R}^q$ and $x \in [0,1]$,

$$\begin{pmatrix} w^- \\ w^+ \end{pmatrix}^* (B(x) + B^*(x) - A_x(x)) \begin{pmatrix} w^- \\ w^+ \end{pmatrix} \leq 0, \tag{2.2}$$

$$(w^-)^*(A^-(1) + D_1^* A^+(1) D_1) w^- \leq -r^- \|w^-\|^2_{\mathbb{R}^p} \tag{2.3}$$

and

$$(w^+)^*(A^+(0) + D_0^* A^-(0) D_0) w^+ \geq r^+ \|w^+\|^2_{\mathbb{R}^q}, \tag{2.4}$$

where $r^- \geq 0$ and $r^+ \geq 0$ are constant such that $r^- + r^+ > 0$.

Theorem 1 *(Rauch & Taylor [18]) : Let H.1-H.3 be satisfied for the system (2.1). Then, for each $R_0 \in (L^2[0,1])^n$, the system has a unique solution*

$$R(\cdot, t) \in C\left([0, +\infty), (L^2[0,1])^n\right).$$

The semigroup of bounded linear operators $U(t) : (L^2[0,1])^n \to (L^2[0,1])^n$ such that $R(\cdot,t) = U(t)R_0$ is exponentially stable :

$$\|U(t)\| \leq M e^{-\omega t}$$

for certain constants $M, \omega > 0$.

As the partial differential equations (2.1) are hyperbolic with two independent variables, Rauch and Taylor have proved Theorem 1 in [18] by means of the method of characteristics. Here, we consider a subset of the systems in (2.1) where the matrix function $B(x)$ is symmetric : $B^*(x) = B(x)$ for any $x \in [0,1]$. For this class of systems satisfying H.1-H.3 we construct a Lyapunov function to prove exponential stability. The advantage of the direct method of Lyapunov is its simplicity, and a priori not limited to one space variable. Using the method we will prove exponential stability of a symmetric hyperbolic system defined in a bounded domain in $\mathbb{R}^2$.

Proof of Theorem 1 : Suppose that $B(x)$ is symmetric in (2.1). Let us consider the following candidate of Lyapunov function :

$$V_\theta(R(\cdot,t)) = \int_0^1 R^*(x,t) \exp\left(\theta \int_0^x A(\xi) d\xi\right) R(x,t) dx,$$

with $\theta > 0$. For each $\theta > 0$, $\sqrt{V_\theta(f)}$ induces on H a norm of f which is equivalent to the usual norm on H. From (2.3) and (2.4) in H.3, we take $r^- > 0$ without loss of generality. Define the unbounded operator $\mathcal{A}$ associated with the system (2.1) by

$$\mathcal{D}(\mathcal{A}) = \left\{ f = \begin{pmatrix} f^- \\ f^+ \end{pmatrix} \in \left(H^1[0,1]\right)^{p+q} \;\middle|\; \begin{matrix} f^-(0) = D_0 f^+(0) \\ f^+(1) = D_1 f^-(1) \end{matrix} \right\} \tag{2.5}$$

and for all $f \in \mathcal{D}(\mathcal{A})$,

$$\mathcal{A}f(x) = A(x)\frac{\partial}{\partial x} f(x) + B(x) f(x). \tag{2.6}$$

From [18] and [19] it is a known fact that $\mathcal{A}$ is the generator of a contraction semigroup on H. For the convenience of the reader we give an elementary proof of it in the Appendix.

For each initial condition $R_0 \in \mathcal{D}(\mathcal{A})$ we compute the differential of $V_\theta(R(\cdot,t))$ following the trajectory of (2.1) :

$$\begin{array}{l} \dfrac{dV_\theta(R(\cdot,t))}{dt} = -\theta \int_0^1 R^*(x,t)A^2(x)\exp\left(\theta \int_0^x A(\xi)d\xi\right)R(x,t)dx \\ +R^*(1,t)\exp\left(\theta \int_0^1 A(\xi)d\xi\right)A(1)R(1,t) - R^*(0,t)A(0)R(0,t) \\ + \int_0^1 R^*(x,t)(2B(x) - A_x(x))\exp\left(\theta \int_0^x A(\xi)d\xi\right)R(x,t)dx. \end{array} \tag{2.7}$$

Examine each term in the above expression. From the second equation in (2.1) and (2.4), we obtain :

$$R^*(0,t)A(0)R(0,t) = (R^+(0,t))^*(A^+(0) + D_0^*A^-(0)D_0)R^+(0,t) \geq 0. \tag{2.8}$$

Each matrix being diagonal, using the second and third equations of (2.1) we can write

$$\begin{array}{l} R^*(1,t)\exp\left(\theta \int_0^1 A(\xi)d\xi\right)A(1)R(1,t) = \left\{\exp\left(\dfrac{\theta}{2}\displaystyle\int_0^1 A^-(\xi)d\xi\right)R^-(1,t)\right\}^* \\ \left\{A^-(1) + D_1^*A^+(1)D_1 + \left[\exp\left(-\dfrac{\theta}{2}\displaystyle\int_0^1 A^-(\xi)d\xi\right)D_1^*A^+(1)\exp\left(\theta\displaystyle\int_0^1 A^+(\xi)d\xi\right)D_1\right.\right. \\ \left.\left.\exp\left(-\dfrac{\theta}{2}\displaystyle\int_0^1 A^-(\xi)d\xi\right) - D_1^*A^+(1)D_1\right]\right\}\exp\left(\dfrac{\theta}{2}\displaystyle\int_0^1 A^-(\xi)d\xi\right)R^-(1,t). \end{array} \tag{2.9}$$

From (2.3), we have

$$A^-(1) + D_1^*A^+(1)D_1 \leq -r^-I. \tag{2.10}$$

In (2.9), the matrix in the square brackets is continuously differentiable and equal to zero for $\theta = 0$. Thus, (2.10) implies that there is a $\theta_1 > 0$ such that for any $0 < \theta < \theta_1$,

$$R^*(1,t)\exp\left(\theta\int_0^1 A(\xi)d\xi\right)A(1)R(1,t) \leq -\frac{r^-}{2}\left\|\exp\left(\frac{\theta}{2}\int_0^1 A^-(\xi)d\xi\right)R^-(1,t)\right\|^2_{\mathbb{R}^p}. \tag{2.11}$$

For the last term in (2.7), we can write

$$\begin{array}{l} R^*(x,t)(2B(x) - A_x(x))\exp\left(\theta\int_0^x A(\xi)d\xi\right)R(x,t) \\ = R^*(x,t)\exp\left(\frac{\theta}{2}\int_0^x A(\xi)d\xi\right)(2B(x) - A_x(x))\exp\left(\frac{\theta}{2}\int_0^x A(\xi)d\xi\right)R(x,t) \\ +R^*(x,t)\exp\left(\frac{\theta}{2}\int_0^x A(\xi)d\xi\right)G(\theta,x)\exp\left(\frac{\theta}{2}\int_0^x A(\xi)d\xi\right)R(x,t), \end{array} \tag{2.12}$$

where

$$\begin{array}{l} G(\theta,x) = \exp\left(-\frac{\theta}{2}\int_0^x A(\xi)d\xi\right)(2B(x) - A_x(x))\exp\left(\frac{\theta}{2}\int_0^x A(\xi)d\xi\right) \\ -(2B(x) - A_x(x)). \end{array} \tag{2.13}$$

Note that for any $w \in \mathbb{R}^n$, $w^*G(\theta,x)w$ is continuously differentiable with respect to θ, and

$$w^*G(0,x)w = 0, \tag{2.14}$$

$$w^*\frac{\partial}{\partial\theta}G(0,x)w = 0. \tag{2.15}$$

(We used the fact that $B(x)$ is symmetric in order to prove (2.15).) Using the last conditions and integrating by parts we can write

$$w^*G(\theta,x)w = \int_0^\theta(\theta - \eta)\frac{\partial^2}{\partial\eta^2}(w^*G(\eta,x)w)\,d\eta, \text{ for all } \quad w \in \mathbb{R}^n. \tag{2.16}$$

Then, for any $w \in \mathbb{R}^n$,

$$w^* G(\theta, x) w \leq \frac{\theta^2}{2} \|w\|^2_{\mathbb{R}^n} \sup_{\eta \in [0,\theta]} \left\| \frac{\partial^2}{\partial \eta^2} G(\eta, x) \right\|_{\mathcal{L}(\mathbb{R}^n)}. \tag{2.17}$$

From (2.13), there is a $K_1 > 0$ such that

$$\sup_{\eta \in [0,\theta]} \left\| \frac{\partial^2}{\partial \eta^2} G(\eta, x) \right\|_{\mathcal{L}(\mathbb{R}^n)} \leq K_1, \tag{2.18}$$

for any $x \in [0, 1]$ and $0 < \theta \leq \theta_1$. Substituting (2.2), (2.13), (2.17) and (2.18) into (2.12) allows to get

$$\begin{array}{l} R^*(x,t)(2B(x) - A_x(x)) \exp\left(\theta \int_0^x A(\xi) d\xi\right) R(x,t) \\ \leq K_1 \theta^2 \left\| \exp\left(\frac{\theta}{2} \int_0^x A(\xi) d\xi\right) R(x,t) \right\|^2_{\mathbb{R}^n}. \end{array} \tag{2.19}$$

Substituting (2.8), (2.11) and (2.19) into (2.7) gives us

$$\frac{dV_\theta(R(\cdot,t))}{dt} \leq -\theta(\lambda^2_{min} - K_1\theta) V_\theta(R(\cdot,t)), \tag{2.20}$$

where $\lambda^2_{min} = \min_{x \in [0,1], 1 \leq k \leq n} \lambda_i^2(x)$ which is positive by H.2. Choose $0 < \theta^* < \theta_1$ such that $\lambda^2_{min} - K_1 \theta^* \geq \lambda^2_{min}/2$. It follows from (2.20) that

$$\frac{dV_{\theta^*}(R(\cdot,t))}{dt} \leq -\left(\frac{\theta^* \lambda^2_{min}}{2}\right) V_{\theta^*}(R(\cdot,t)),$$

or,

$$V_{\theta^*}(R(\cdot,t)) \leq \exp\left(-t\theta^* \lambda^2_{min}/2\right) V_{\theta^*}(R_0). \tag{2.21}$$

The last estimate implies exponential stability of the semigroup from the fact that $\sqrt{V_{\theta^*}(\cdot)}$ defines an equivalent norm on $(L^2[0,1])^n$. ∎

Remark 1 Although we have not been able to prove the general Theorem 1 of Rauch and Taylor using our proposed Lyapunov function candidate as in [18] with the method of characteristics, the direct method of Lyapunov allows us to attack symmetric hyperbolic systems in higher dimension and degenerate hyperbolic cases that the theory in [18] cannot treat. In the following we present only an example to address the issue of possible applications. Because mathematical complexity goes beyond the scope of the paper, the stability problem of symmetric hyperbolic systems in higher dimension will be presented elsewhere. However, the essential ideas are presented in the following example.

Example 1 : We consider the open disc in $\mathbb{R}^2$:

$$\Omega = \left\{ (x,y) \in \mathbb{R}^2 \,\middle|\, x^2 + y^2 < 1 \right\}$$

and the boundary $\partial\Omega$:

$$\partial\Omega = \left\{ (x,y) \in \mathbb{R}^2 \,\middle|\, x^2 + y^2 = 1 \right\}.$$

Let $X = \left(L^2(\Omega)\right)^2$ equipped with the usual inner product : $< f, g >_X = \int_\Omega f^*(x,y) g(x,y) dx dy$. We set

$$N(x) = \begin{bmatrix} y+1 & x \\ x & -y+1 \end{bmatrix}, \qquad \tilde{A}_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \tilde{A}_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Define the unbounded operator $\tilde{\mathcal{A}}$ by

$$\mathcal{D}(\tilde{\mathcal{A}}) = \left\{(f_1, f_2)^* \in \left(H^1(\Omega)\right)^2 \mid N(x,y)(f_1(x,y), f_2(x,y))^* = 0 \ \forall \ (x,y) \in \partial\Omega\right\}$$

and for all $f = (f_1, f_2)' \in \mathcal{D}(\tilde{\mathcal{A}})$,

$$\tilde{\mathcal{A}} f(x,y) = \tilde{A}_1 \frac{\partial}{\partial x} f(x,y) + \tilde{A}_2 \frac{\partial}{\partial y} f(x,y).$$

The system to be considered is

$$\left\{ \begin{array}{l} \frac{\partial}{\partial t} u(x,y,t) = \tilde{\mathcal{A}} u(x,y,t) \\ u(x,y,0) = u_0(x,y). \end{array} \right. \tag{2.22}$$

The operator $\tilde{\mathcal{A}}$ is dissipative. The subspace $S(x,y) = \{(u,v)^* \in \mathbb{R}^2 \mid N(x,y)(u,v)^* = 0 \ \forall (x,y) \in \partial\Omega\}$ is maximal non-positive at each boundary point. From the result of Lax and Phillips [26] it is known that the range of $I - \tilde{\mathcal{A}}$ is X (see also [27]). So the operator $\tilde{\mathcal{A}}$ is the generator of a C_0-semigroup of contraction on X (see Theorem 4.3, p.14, [16]). For each $u_0 \in \mathcal{D}(\tilde{\mathcal{A}})$ the trajectory $u(\cdot,\cdot,t) \in \mathcal{D}(\tilde{\mathcal{A}})$ for $t \geq 0$.

Consider the following as candidate for Lyapunov function

$$V_\theta(u(\cdot,\cdot,t)) = \int_\Omega u^*(x,y,t) e^{\theta y \tilde{A}_2} u(x,y,t) dx dy, \ \ \theta > 0.$$

Using the boundary condition and the Green formula we obtain

$$\frac{d}{dt} V_\theta(u(\cdot,\cdot,t)) = -\theta V_\theta(u(\cdot,\cdot,t))$$

$$+ \int_0^{2\pi} u^*(\cos\eta, \sin\eta, t) e^{\theta \sin\eta \, \tilde{A}_2} \left[\tilde{A}_2 \sin\eta + \tilde{A}_1 \cos\eta\right] u(\cos\eta, \sin\eta, t) d\eta.$$

As the same as in the proof of Theorem 1 one can prove that there is a $\theta > 0$ such that

$$\frac{d}{dt} V_\theta(u(\cdot,\cdot,t)) \leq -\theta V_\theta(u(\cdot,\cdot,t)).$$

Thus, the system (2.22) is exponentially stable. ∎

3 Exponential Stability of the Heat Exchanger Network

In this section we show that the heat exchanger system can be transformed into the form (2.1). We prove exponential stability of the system by applying Theorem 1. Then, we show that the system is regular, the transfer functions $P(s)$ and $W(s)$ are in H^∞, $W^{-1}(s)$ is an entire function and $P(s)$ is strictly proper.

3.1 Exponential stability of the semigroup

Now, we show how to transform the heat exchanger system into the form (2.1). First, the following transformation allows to normalize the space variable to the interval $[0,1]$:

$$\phi(x,t) = \begin{pmatrix} \phi_1(x,t) \\ \phi_2(x,t) \\ \phi_3(x,t) \\ \phi_4(x,t) \end{pmatrix} = \begin{pmatrix} R_2(l_1 x, t) \\ T_1((l_1 - l_2)x + l_2, t) \\ R_1(l_1 x, t) \\ T_2((l_1 - l_2)x + l_2, t) \end{pmatrix}. \tag{3.1}$$

Then, the system (1.1)-(1.4) is equivalent to

$$\begin{cases} \dfrac{\partial \phi(x,t)}{\partial t} = A_1 \dfrac{\partial \phi(x,t)}{\partial x} + B_1 \phi(x,t) + b(x)u(t), \qquad (x,t) \in]0,1[\times \mathbb{R}^+ \\ \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}(0,t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} d(t) \\ \begin{pmatrix} \phi_3 \\ \phi_4 \end{pmatrix}(1,t) = \begin{pmatrix} 0 & 1-\beta \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}(1,t) \\ y(t) = \phi_4(0,t), \end{cases} \tag{3.2}$$

where

$$A_1 = \mathrm{diag}\left(-\frac{m_2}{l_1}, -\frac{(1-\beta)m_1}{l_2 - l_1}, \frac{m_1}{l_1}, \frac{m_2}{l_2 - l_1} \right)$$

$$B_1 = \begin{pmatrix} -K_2 & 0 & K_2 & 0 \\ 0 & -K_1 & 0 & K_1 \\ K_1 & 0 & -K_1 & 0 \\ 0 & K_2 & 0 & -K_2 \end{pmatrix} \tag{3.3}$$

and

$$b(x) = (0, b_2((l_1 - l_2)x + l_2), b_1(l_1 x), 0)^*.$$

Note that B_1 is not symmetric in (3.3). Secondly, the following linear (diagonal) transformation Λ exerted on (3.2) makes B_1 symmetric by keeping A_1 unchanged :

$$\Lambda(\phi_1, \phi_2, \phi_3, \phi_4) = \left(\sqrt{\frac{l_1}{(l_2 - l_1)(1-r)}}\phi_1, \sqrt{\frac{K_2}{K_1}}\phi_2, \sqrt{\frac{l_1}{(l_2 - l_1)(1-r)}}\sqrt{\frac{K_2}{K_1}}\phi_3, \phi_4 \right), \tag{3.4}$$

where $r \in [0, \beta]$ is an arbitrary constant such that the boundary condition becomes dissipative (see (2.3) and (3.9) below). Applying the transformation (3.4) on (3.2) and keeping the same notations we can write (3.2) under the form :

$$\begin{cases} \dfrac{\partial \phi(x,t)}{\partial t} = A_1 \dfrac{\partial \phi(x,t)}{\partial x} + B_2 \phi(x,t) + \tilde{b}(x)u(t), \qquad (x,t) \in]0,1[\times \mathbb{R}^+ \\ \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}(0,t) = \sqrt{\dfrac{l_1}{(l_2 - l_1)(1-r)}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} d(t) \\ \begin{pmatrix} \phi_3 \\ \phi_4 \end{pmatrix}(1,t) = \tilde{D}_1 \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}(1,t) \\ y(t) = \phi_4(0,t), \end{cases} \tag{3.5}$$

where

$$B_2 = \Lambda B_1 \Lambda^{-1} = \begin{pmatrix} -K_2 & 0 & \sqrt{K_1 K_2} & 0 \\ 0 & -K_1 & 0 & \sqrt{K_1 K_2} \\ \sqrt{K_1 K_2} & 0 & -K_1 & 0 \\ 0 & \sqrt{K_1 K_2} & 0 & -K_2 \end{pmatrix} \tag{3.6}$$

$$\tilde{D}_1 = \begin{pmatrix} 0 & (1-\beta)\sqrt{\dfrac{l_1}{(l_2 - l_1)(1-r)}} \\ \sqrt{\dfrac{(l_2 - l_1)(1-r)}{l_1}} & 0 \end{pmatrix}$$

and

$$\tilde{b}(x) = \left(0, \sqrt{K_2/K_1} b_2((l_1 - l_2)x + l_2), \sqrt{l_1 K_2/[(l_2 - l_1)(1-r)K_1]} b_1(l_1 x), 0\right)^* . \qquad (3.7)$$

We consider the system (3.5) on the state space $H = (L^2[0,1])^4$ equipped with the inner product

$$< f, g >_H = \int_0^1 \sum_{k=1}^4 f_k(x) g_k(x) dx.$$

Let $\mathcal{A}$ be the unbounded operator defined by

$$\mathcal{D}(\mathcal{A}) = \left\{ f \in (H^1([0,1]))^4 \;\middle|\; \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}(0) = 0, \begin{pmatrix} f_3 \\ f_4 \end{pmatrix}(1) = \tilde{D}_1 \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}(1) \right\}$$

and for each $f \in \mathcal{D}(\mathcal{A})$,

$$\mathcal{A} f(x) = A_1 \frac{\partial f(x)}{\partial x} + B_2 f(x). \qquad (3.8)$$

Theorem 2 *The operator $\mathcal{A}$ is the generator of a C_0-semigroup (noted as $e^{t\mathcal{A}}$) of contractions on H which is exponentially stable.*

Proof of Theorem 2 : The conditions H.1, H.2, (2.2) and (2.4) in H.3 are satisfied with $r^+ > 0$ because B_2 is dissipative and $D_0 = 0$. The condition (2.3) is also satisfied for all $0 \le r \le \beta$:

$$A_1^- + \tilde{D}_1^* A_1^+ \tilde{D}_1 = \begin{pmatrix} -\frac{r m_2}{l_1} & 0 \\ 0 & -\frac{(1-\beta) m_1 (\beta - r)}{(l_2 - l_1)(1-r)} \end{pmatrix} \le 0. \qquad (3.9)$$

The Theorem 2 is true from Theorem 1. ∎

Remark 2 From Theorem 2, zero is in the resolvent set $\rho(\mathcal{A})$ and the growth bound $\omega_0(\mathcal{A})$ of the considered semigroup is negative : $\omega_0(\mathcal{A}) = \lim_{t \to +\infty} t^{-1} \ln \|e^{t\mathcal{A}}\|$.

Note that the system (3.5) has two inputs one of which is boundary and that the observation is also boundary. We need to consider the adjoint operator $\mathcal{A}^*$ of $\mathcal{A}$ defined by

$$\mathcal{D}(\mathcal{A}^*) = \left\{ f \in (H^1([0,1]))^4 \;\middle|\; \begin{pmatrix} f_3 \\ f_4 \end{pmatrix}(0) = 0, \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}(1) = D_{\text{adj}} \begin{pmatrix} f_3 \\ f_4 \end{pmatrix}(1) \right\} \qquad (3.10)$$

with

$$D_{\text{adj}} = \begin{pmatrix} 0 & \sqrt{\frac{l_1(1-r)}{l_2 - l_1}} \\ \sqrt{\frac{l_2 - l_1}{l_1(1-r)}} & 0 \end{pmatrix} \qquad (3.11)$$

and for each $f \in \mathcal{D}(\mathcal{A}^*)$,

$$\mathcal{A}^* f(x) = -A_1 \frac{\partial f(x)}{\partial x} + B_2 f(x). \qquad (3.12)$$

Define the continuous linear form $C_{\text{adj}} : (H^1[0,1])^4 \to \mathbb{R}$ such that $C_{\text{adj}} f = f_1(0)$. Therefore, the linear form $C_{\text{adj}}(\mathcal{A}^*)^{-1}$ is continuous from H to $\mathbb{R}$. From the representation theorem of Riesz there is a unique element $\xi_{\text{adj}} \in H$ such that $C_{\text{adj}}(\mathcal{A}^*)^{-1} f = < \xi_{\text{adj}}, f >_H$ for any $f \in H$. We claim that $\xi_{\text{adj}} \in (H^1[0,1])^4$. The interested reader is referred to the Appendix for a proof. Of course the adjoint semigroup $e^{t\mathcal{A}^*}$ is exponentially stable.

3.2 Characterization of regularity and transfer functions

We need to define the Hilbert spaces H_1 and H_{-1} as follows : H_1 is $\mathcal{D}(\mathcal{A})$ with the norm $\|f\|_1 = \|\mathcal{A}f\|_H$ and H_{-1} is the completion with respect to the norm $\|f\|_{-1} = \|\mathcal{A}^{-1}f\|_H$. We have $H_1 \subset H \subset H_{-1}$, densely and with continuous embeddings. The operator $\mathcal{A}$ has a unique extension on the whole space H because it is defined on a dense set $\mathcal{D}(\mathcal{A})$ in H and continuous from H to H_{-1}. The semigroup $e^{t\mathcal{A}}$ can be extended to a C_0-semigroup on H_{-1} whose generator is nothing else than the extended operator $\mathcal{A} \in \mathcal{L}(H, H_{-1})$.

We define the duality product on $H_{-1} \times \mathcal{D}(\mathcal{A}^*)$ by continuous extension of the inner product on H : For all $h \in H$ and all $g \in \mathcal{D}(\mathcal{A}^*)$, $< h, g >_{H_{-1},\mathcal{D}(\mathcal{A}^*)} = < h, g >_H$. For each $h \in H_{-1}$, by taking $h_n \in H$ such that $\lim_{n\to+\infty} \|h - h_n\|_{-1} = 0$ we set

$$< h, g >_{H_{-1},\mathcal{D}(\mathcal{A}^*)} = \lim_{n\to+\infty} < h_n, g >_H, \quad \forall\, g \in \mathcal{D}(\mathcal{A}^*).$$

For each $h \in H_{-1}$, $g \to < h, g >_{H_{-1},\mathcal{D}(\mathcal{A}^*)}$ is a continuous linear form on $\mathcal{D}(\mathcal{A}^*)$. Conversely, given a continuous linear form φ on $\mathcal{D}(\mathcal{A}^*)$ there is a unique $h_\varphi \in H_{-1}$ such that

$$\varphi(f) = < h_\varphi, f >_{H_{-1},\mathcal{D}(\mathcal{A}^*)}, \quad \forall f \in \mathcal{D}(\mathcal{A}^*).$$

In other words, the mapping $J \ : H_{-1} \to \mathcal{D}'(\mathcal{A}^*)$ (topological dual of $\mathcal{D}(\mathcal{A}^*)$) such that $Jh(f) = < h, f >_{H_{-1},\mathcal{D}(\mathcal{A}^*)}$ is an isomorphism. Moreover, it is easy to see that for all $f \in H$ and all $g \in \mathcal{D}(\mathcal{A}^*)$,

$$< e^{t\mathcal{A}}\mathcal{A}f, g >_{H_{-1},\mathcal{D}(\mathcal{A}^*)} = < f, e^{t\mathcal{A}^*}\mathcal{A}^*g >_H, \quad \forall\, t \geq 0. \tag{3.13}$$

For the previously defined C_{adj} there is a unique $B_d \in H_{-1}$ such that

$$< B_d, f >_{H_{-1},\mathcal{D}(\mathcal{A}^*)} = C_{\text{adj}}(f) = f_1(0). \tag{3.14}$$

Define the continuous linear form $C : (H^1[0,1])^4 \to \mathbb{R}$ such that $Cf = f_4(0)$. The restriction of C to H_1 is noted by the same letter which is also continuous from H_1 to $\mathbb{R}$. Then, from (3.5) the output is written as

$$y(t) = C\phi(\cdot, t). \tag{3.15}$$

Lemma 1 *For all $d \in C_0^\infty(]0, +\infty[)$, $\phi(\cdot, 0) = 0$ and $u = 0$, the system (3.5) has a unique solution $\phi(\cdot,\cdot) \in C([0, +\infty[, H)$:*

$$\phi(\cdot, t) = k \int_0^t e^{(t-\tau)\mathcal{A}} B_d d(\tau)\, d\tau, \;\; \forall\, t \geq 0, \tag{3.16}$$

where B_d was defined in (3.14) and $k = m_2 \left[l_1(l_2 - l_1)(1 - r)\right]^{\frac{1}{2}}$. Moreover, the input-output mapping $d \to y$ is continuous from $L^2(]0, \infty[)$ to $L^2(]0, \infty[)$.

Proof of Lemma 1 : From the system (3.5) the uniqueness of solution in $C([0, +\infty[, H)$ is obvious. It is sufficient to prove that $\phi(\cdot, t)$ given in (3.16) is continuous from $[0, +\infty[$ to H and satisfies the first three equations in (3.5).

First, let us show how to find the expression in (3.16). Consider the dual system corresponding to (3.5) :

$$\begin{cases} \dfrac{\partial g(x,t)}{\partial t} = -\left[-A_1\dfrac{\partial g(x,t)}{\partial x} + B_2 g(x,t)\right], \qquad t \in [0,T[\\ g_3(0,t) = g_4(0,t) = 0 \\ \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}(1,t) = D_{\text{adj}}\begin{pmatrix} g_3 \\ g_4 \end{pmatrix}(1,t) \\ g(x,T) = g_0 \in \mathcal{D}(\mathcal{A}) \\ y_{\text{adj}}(t) = g_1(0,t), \end{cases} \tag{3.17}$$

Suppose that there is a smooth $\phi(x,t)$ satisfying (3.5). Taking the inner product with $g(x,t)$ in the first equation in (3.5) and integrating by parts we obtain

$$< \phi(\cdot,T), g_0 >_H = \int_0^T k y_{\text{adj}}(t) d(t)\, dt, \;\; \forall\, t \geq 0, \; \forall\, g_0 \in \mathcal{D}(\mathcal{A}^*),$$

where $y_{\text{adj}}(t) = C_{\text{adj}} e^{\mathcal{A}^*(T-t)} g_0$. It follows that

$$\phi(\cdot,t) = -kd(t)\xi_{\text{adj}} + k\int_0^t e^{\mathcal{A}(t-\tau)}\xi_{\text{adj}} d'(\tau)\, d\tau, \tag{3.18}$$

where $d'(t)$ denotes the derivative of $d(t)$. One can verify that this $\phi(\cdot,t)$ is a classical solution of (3.5). By direct computation from (3.18) and (3.13) we have

$$< \phi(\cdot,t), g >_{H_{-1},\mathcal{D}(\mathcal{A}^*)} = < \phi(\cdot,t), g >_H = k\int_0^t \left\langle e^{(t-\tau)\mathcal{A}}\mathcal{A}\xi_{\text{adj}}, g\right\rangle_{H_{-1},\mathcal{D}(\mathcal{A}^*)} d(\tau)d\tau, \quad \forall\, g \in \mathcal{D}(\mathcal{A}^*).$$

It is implied that

$$\phi(\cdot,t) = k\int_0^t e^{(t-\tau)\mathcal{A}} B_d d(\tau) d\tau.$$

Recall that in (3.18), the first term $\xi_{\text{adj}} \in \left(H^1[0,1]\right)^4$ and the second term is in $\mathcal{D}(\mathcal{A})$ (see p.107, [16]). Therefore, the output function is well defined :

$$y(t) = -kC\xi_{\text{adj}} d(t) + C\int_0^t k e^{(t-\tau)\mathcal{A}}\xi_{\text{adj}} d'(\tau) d\tau. \tag{3.19}$$

To prove that the mapping defined in (3.19) is continuous from $L^2(\mathbb{R}^+)$ to $L^2(\mathbb{R}^+)$ we differentiate the functional $\|\phi(\cdot,t)\|_H^2$ following the trajectory of (3.5). Using the boundary condition in (3.5) we get

$$\frac{d}{dt}\|\phi(\cdot,t)\|_H^2 \leq \left[\frac{m_2}{(l_2-l_1)(1-r)}\right] d^2(t) - \left[\frac{m_2}{l_2-l_1}\right]\phi_4^2(0,t).$$

Then, it follows from integration that

$$\int_0^{+\infty} \phi_4^2(0,t)dt \leq \frac{1}{1-r}\int_0^{+\infty} d^2(t)dt.$$

Since $C_0^\infty(\mathbb{R}^+)$ is dense in $L^2(\mathbb{R}^+)$, the input-output mapping is continuous in $L^2(\mathbb{R}^+)$. ∎

The mapping $d \to y$ is a shift invariant bounded operator from $L^2([0,+\infty[,\mathbb{R})$ to $L^2([0,+\infty[,\mathbb{R})$. We note by $W(s)$ the associated transfer function. It is analytic and bounded in the open right-hand half complex plane (c.f. Theorem 3.1 in [2]). Taking the Laplace transform in (3.19) we get the following identity :

$$\hat{y}(s) = -kC\xi_{\text{adj}}\hat{d}(s) + kC(sI-\mathcal{A})^{-1}\xi_{\text{adj}} s\hat{d}(s) = kC(sI-\mathcal{A})^{-1}B_d\hat{d}(s).$$

Hence, $W(s) = kC(sI - \mathcal{A})^{-1}B_d$ is valid in the open right-hand half plane. Computing algebraically $W(s)$ is equivalent to solving the following differential equation with parameter $\Re e(s) > 0$:

$$\left\{\begin{array}{l} s\hat{\phi} = \left(A_1\dfrac{\partial}{\partial x} + B_2\right)\hat{\phi} \\ \begin{pmatrix} \hat{\phi}_1 \\ \hat{\phi}_2 \end{pmatrix}(0) = \sqrt{\dfrac{l_1}{(l_2-l_1)(1-r)}}\begin{pmatrix} 1 \\ 0 \end{pmatrix}\hat{d} \\ \begin{pmatrix} \hat{\phi}_3 \\ \hat{\phi}_4 \end{pmatrix}(1) = \tilde{D}_1\begin{pmatrix} \hat{\phi}_1 \\ \hat{\phi}_2 \end{pmatrix}(1) \\ \hat{y} = \hat{\phi}_4(0). \end{array}\right. \tag{3.20}$$

Theorem 3 *The inverse $W^{-1}(s)$ of $W(s)$ is analytic everywhere in the complex plane and*

$$\lim_{\substack{s \to +\infty \\ s \in \mathbb{R}}} W(s) = 0. \tag{3.21}$$

Poof of Theorem 3 : The differential system (3.20) is un-coupled as follows :

$$\left\{\begin{array}{l} \begin{pmatrix} \dfrac{\partial\hat{\phi}_1}{\partial x} \\ \dfrac{\partial\hat{\phi}_3}{\partial x} \end{pmatrix} = \begin{bmatrix} -\dfrac{l_1(K_2+s)}{m_2} & -\dfrac{l_1\sqrt{K_1K_2}}{m_2} \\ -\dfrac{l_1\sqrt{K_1K_2}}{m_1} & \dfrac{l_1(K_1+s)}{m_1} \end{bmatrix}\begin{pmatrix} \hat{\phi}_1 \\ \hat{\phi}_3 \end{pmatrix} \\ \begin{pmatrix} \dfrac{\partial\hat{\phi}_2}{\partial x} \\ \dfrac{\partial\hat{\phi}_4}{\partial x} \end{pmatrix} = \begin{bmatrix} -\dfrac{(l_2-l_1)(K_1+s)}{m_1(1-\beta)} & -\dfrac{(l_2-l_1)\sqrt{K_1K_2}}{m_1(1-\beta)} \\ -\dfrac{(l_2-l_1)\sqrt{K_1K_2}}{m_2} & \dfrac{(l_2-l_1)(K_2+s)}{m_2} \end{bmatrix}\begin{pmatrix} \hat{\phi}_2 \\ \hat{\phi}_4 \end{pmatrix} \end{array}\right. \tag{3.22}$$

satisfying the boundary conditions :

$$\left\{\begin{array}{l} \begin{pmatrix} \hat{\phi}_2(0) \\ \hat{\phi}_4(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}\hat{y}(s) \\ \begin{pmatrix} \hat{\phi}_1(1) \\ \hat{\phi}_3(1) \end{pmatrix} = \sqrt{\dfrac{l_1}{(l_2-l_1)(1-r)}}\begin{bmatrix} 0 & 1 \\ 1-\beta & 0 \end{bmatrix}\begin{pmatrix} \hat{\phi}_2(1) \\ \hat{\phi}_4(1) \end{pmatrix} \\ \hat{d}(s) = \sqrt{\dfrac{(l_2-l_1)(1-r)}{l_1}}\hat{\phi}_1(0) \end{array}\right. \tag{3.23}$$

The unique solution of (3.22) is

$$\hat{d}(s) = [1, 0]e^{M_1(s)}\begin{bmatrix} 0 & 1 \\ 1-\beta & 0 \end{bmatrix}e^{M_2(s)}\begin{bmatrix} 0 \\ 1 \end{bmatrix}\hat{y}(s), \tag{3.24}$$

where

$$\begin{array}{l} M_1(s) = \begin{bmatrix} \dfrac{l_1(K_2+s)}{m_2} & \dfrac{l_1\sqrt{K_1K_2}}{m_2} \\ \dfrac{l_1\sqrt{K_1K_2}}{m_1} & -\dfrac{l_1(K_1+s)}{m_1} \end{bmatrix}, \\ M_2(s) = \begin{bmatrix} -\dfrac{(l_2-l_1)(K_1+s)}{m_1(1-\beta)} & -\dfrac{(l_2-l_1)\sqrt{K_1K_2}}{m_1(1-\beta)} \\ -\dfrac{(l_2-l_1)\sqrt{K_1K_2}}{m_2} & \dfrac{(l_2-l_1)(K_2+s)}{m_2} \end{bmatrix}. \end{array} \tag{3.25}$$

So, $W^{-1}(s) = \hat{d}(s)/\hat{y}(s)$ is analytic in the complex plane because $M_1(s)$ and $M_2(s)$ are polynomial matrices of first order in s.

Note that for any $s > 0$, the matrices $M_1(s)$ and $M_2(s)$ have respectively two eigenvalues one of which is negative and the other is positive :

$$\begin{aligned}
\lambda_1^{M_1} &= \tfrac{l_1}{2}\left(-\frac{K_1+s}{m_1}+\frac{K_2+s}{m_2}+\sqrt{\Delta_1}\right) > 0\\
\lambda_2^{M_1} &= \tfrac{l_1}{2}\left(-\frac{K_1+s}{m_1}+\frac{K_2+s}{m_2}-\sqrt{\Delta_1}\right) < 0\\
\lambda_1^{M_2} &= \frac{l_2-l_1}{2}\left(-\frac{K_1+s}{m_1(1-\beta)}+\frac{K_2+s}{m_2}+\sqrt{\Delta_2}\right) > 0\\
\lambda_2^{M_2} &= \frac{l_2-l_1}{2}\left(-\frac{K_1+s}{m_1(1-\beta)}+\frac{K_2+s}{m_2}-\sqrt{\Delta_2}\right) < 0
\end{aligned} \tag{3.26}$$

where

$$\begin{aligned}
\Delta_1 &= \left(\frac{K_1+s}{m_1}+\frac{K_2+s}{m_2}\right)^2+\frac{K_1+s}{m_1}+\frac{4K_1K_2}{m_1m_2}\\
\Delta_2 &= \left(\frac{K_1+s}{m_1(1-\beta)}+\frac{K_2+s}{m_2}\right)^2+\frac{K_1+s}{m_1}+\frac{4K_1K_2}{m_1m_2(1-\beta)}.
\end{aligned} \tag{3.27}$$

The corresponding eigenvectors (found using *Mathematica*)

$$\begin{aligned}
e_1^{M_1} &= \left[1, -\frac{m_1}{2\sqrt{K_1K_2}}\left(\frac{K_1+s}{m_1}+\frac{K_2+s}{m_2}+\sqrt{\Delta_1}\right)\right]^*\\
e_2^{M_1} &= \left[1, -\frac{m_1}{2\sqrt{K_1K_2}}\left(\frac{K_1+s}{m_1}+\frac{K_2+s}{m_2}-\sqrt{\Delta_1}\right)\right]^*
\end{aligned}$$

and

$$\begin{aligned}
e_1^{M_2} &= \left[1, -\frac{m_1(1-\beta)}{2\sqrt{K_1K_2}}\left(\frac{K_1+s}{m_1(1-\beta)}+\frac{K_2+s}{m_2}+\sqrt{\Delta_2}\right)\right]^*\\
e_2^{M_2} &= \left[1, -\frac{m_1(1-\beta)}{2\sqrt{K_1K_2}}\left(\frac{K_1+s}{m_1(1-\beta)}+\frac{K_2+s}{m_2}-\sqrt{\Delta_2}\right)\right]^*.
\end{aligned}$$

form a basis in $\mathbb{R}^2$, respectively.

We write

$$e^{M_2(s)}[0,1]^* = c_1\exp\left(\lambda_1^{M_2}\right)e_1^{M_2} + c_2\exp\left(\lambda_2^{M_2}\right)e_2^{M_2},$$

$$\begin{bmatrix} 0 & 1 \\ 1-\beta & 0 \end{bmatrix} e_1^{M_2} = \hat{c}_{11}e_1^{M_1} + \hat{c}_{12}e_2^{M_1},$$

$$\begin{bmatrix} 0 & 1 \\ 1-\beta & 0 \end{bmatrix} e_2^{M_2} = \hat{c}_{21}e_1^{M_1} + \hat{c}_{22}e_2^{M_1}.$$

Substituting the last expressions in (3.24) gives us

$$\begin{aligned}
W^{-1}(s) = \exp\left(\lambda_1^{M_1}+\lambda_1^{M_2}\right)\Big\{&c_1\hat{c}_{11} + \exp\left(\lambda_2^{M_1}-\lambda_1^{M_1}\right)c_1\hat{c}_{12}\\
&+\exp\left(\lambda_2^{M_2}-\lambda_1^{M_2}\right)c_2\hat{c}_{21} + \exp\left(\lambda_2^{M_2}-\lambda_1^{M_2}\right)c_2\hat{c}_{22}\Big\}.
\end{aligned} \tag{3.28}$$

Note that

$$c_1\hat{c}_{11} = \frac{K_1K_2}{m_1^2\sqrt{\Delta_1\Delta_2}} \neq 0.$$

The other terms in the curly brackets in (3.28) tend to zero exponentially as s goes to $+\infty$. So, we have

$$\lim_{\substack{s\to+\infty\\ s\in\mathbb{R}^+}} W(s) = \lim_{\substack{s\to+\infty\\ s\in\mathbb{R}^+}} \exp\left(-\lambda_1^{M_1}-\lambda_1^{M_2}\right)\frac{m_1^2\sqrt{\Delta_1\Delta_2}}{K_1K_2} = 0.$$

∎

Remark 3 From (3.28), $W(s)$ can be written as $W(s) = W_O(s)\exp(-l_2 s/m_2)$, where $W_O(s)$ is an outer factor in H^∞ (analytic in $\Re e(s) > 0$ as well as its inverse) and the delay term corresponds to the necessary time for the disturbance to arrive at the output (see [15]).

Now, define the input operator $B_u \in \mathcal{L}(\mathbb{R}, H)$ by

$$B_u z = \tilde{b}(x) z \tag{3.29}$$

where $\tilde{b}(x)$ was defined in (3.7). Let C_L be the Lebesgue extension of $C \in (H_1, \mathbb{R})$ (defined in (3.15)) in the sense of Weiss [9]. Formally, the system (3.5) is written as

$$\begin{cases} \dot{\phi}(\cdot,t) = \mathcal{A}\phi(\cdot,t) + B_u u(t) + B_d d(t) \\ y(t) = C\phi(\cdot,t). \end{cases} \tag{3.30}$$

We denote by $P(s)$ the transfer function corresponding to the input-output mapping $u \to y$.

Theorem 4 *The system (3.30) represented by $(\mathcal{A}, [B_u,\ B_d], C_L)$ is a regular system satisfying*

$$\begin{aligned} P(s) &= C_L(sI - \mathcal{A})^{-1} B_u \\ W(s) &= C_L(sI - \mathcal{A})^{-1} B_d \end{aligned} \tag{3.31}$$

for all $\Re e(s) > \omega_0(\mathcal{A})$.

Proof of Theorem 4 : By differentiating $\|\phi(\cdot,t)\|_H^2$ along the trajectory of (3.5) corresponding to $\phi_0 \in \mathcal{D}(\mathcal{A})$ and $u = d = 0$ we can see that $Ce^{\cdot\mathcal{A}} : \mathcal{D}(\mathcal{A}) \to L^2([0,T], \mathbb{R})$ is continuous from H to $L^2([0,T], \mathbb{R})$. This proves that C is admissible respect to the semigroup $e^{t\mathcal{A}}$. The admissibility of B_d is true from the proof of Lemma 1. The control operator B_u being bounded is a fortiori admissible. From our Lemma 1 the transfer function $W(s)$ has a strong limit at $+\infty$ along the real axis. Therefore, applying Theorem 1.3 and Theorem 4.7 of Weiss [2] respectively, we assert that $(\mathcal{A}, B_d, C_L)$ is a regular triple and that for all $\Re e(s) > \omega_0(\mathcal{A})$,

$$W(s) = C_L(sI - \mathcal{A})^{-1} B_d.$$

Since B_u is bounded, for each $s \in \rho(\mathcal{A})$ we have $(sI - \mathcal{A})^{-1} B_u \in \mathcal{D}(C_L)$. Then, applying Theorem 5.8 of Weiss [2] assures that $(\mathcal{A}, B_u, C_L)$ is regular. From Theorem 4.7 of Weiss [2] we have

$$P(s) = C_L(sI - \mathcal{A})^{-1} B_u = C(sI - \mathcal{A})^{-1} B_u$$

for all $\Re e(s) > \omega_0(\mathcal{A})$. ∎

Remark 4 *From our Theorem 4 and the Theorem 5.8 of [2] we see that $P(s)$ and $W(s)$ are analytic and bounded in $\Re e(s) > \omega_0(\mathcal{A})$. Here, we claim that $P(s)$ is strictly proper :*

$$\lim_{\substack{|s| \to +\infty \\ \Re e(s) \geq 0}} P(s) = 0.$$

Indeed, $P(s)$ is the Laplace transform of $Ce^{t\mathcal{A}}B_u$:

$$P(s) = \int_0^{+\infty} e^{-ts}Ce^{t\mathcal{A}}B_u dt.$$

From $\omega_0(\mathcal{A}) < 0$ and the admissibility of C one can find a $0 < \epsilon < -\omega_0(\mathcal{A})$ such that $e^{\epsilon t}Ce^{t\mathcal{A}}B_u$ is in $L^2([0,+\infty[,\mathbb{R})$ (see [20]). By means of the Paley-Wiener theorem (see [10]), $P(s)$ is in the Hardy space $H^2(\Re e(s) > -\epsilon)$. Each function (in particular, our $P(s)$ here) in $H^2(\Re e(s) > -\epsilon)$ tends uniformly to zero as s goes to infinity inside the closed plane $\Re e(s) \geq 0$ (see p.125, [10]).

4 Conclusions

We have studied frequency and time domain dynamics of a heat exchanger network system. This process is representative for a class of hyperbolic systems typical to chemical engineering systems such as gas absorbers and irrigation canals. The paper has shown how this kind of systems are transformed into the form of classical symmetric hyperbolic systems allowing to prove exponential stability using the theorem of Rauch and Taylor. We have also proposed Lyapunov function candidates for proving exponential stability of some hyperbolic systems. The heat exchanger system has a unbounded input operator and a unbounded output operator. We have shown that the system is regular and that the transfer functions of the system are in H^∞. Although the paper is a case study, the theory and the method that we have used are general and could be used for analysis of other processes as well. The results presented here are essential for various controller design methods (H^∞-control [7] [8] [15] [6], output feedback controllers [12] [20] and P.I. controllers [4] [5]) to be applied for this class of chemical engineering systems.

5 Appendix

A.1. We claim that the operator $\mathcal{A}$ as defined in (2.5) and (2.6) is the generator of a C_0-semigroup of contractions on H. To prove the claim we prove that it has a dense domain in $H = (L^2[0,1])^{p+q}$ and that it is dissipative as well as its adjoint operator $\mathcal{A}^*$. Then, it follows from the Corollary 4.4 of [16] (p.15, [16]) that $\mathcal{A}$ is the generator of a semigroup of contractions on H.

Since $C_0^\infty([0,1])$ is dense in $L^2[0,1]$ and $(C_0^\infty([0,1]))^{p+q}$ is contained in the domain, thus the domain $\mathcal{D}(\mathcal{A})$ is dense. From the hypotheses H.1-H.3 and

$$< \mathcal{A}f, f >_H = \frac{1}{2}\left[f^*(x)A(x)f(x)\right]\Big|_0^1 + \frac{1}{2}\int_0^1 f^*(x)[B(x) + B^*(x) - A_x(x)]f(x)dx$$

for all $f \in \mathcal{D}(\mathcal{A})$, we see that $\mathcal{A}$ is dissipative.

Let us compute the adjoint operator $\mathcal{A}^*$. Because $B(x)$ is continuous, we have only to compute the adjoint operator A_1

$$A_1 = A(x)\frac{\partial}{\partial x}.$$

Take any $f \in \mathcal{D}(A_1)$ and any $g \in \mathcal{D}(A_1^*)$. Then, we have

$$< A_1 f, g >_H = \left\langle \frac{\partial}{\partial x}\begin{pmatrix} f^- \\ f^+ \end{pmatrix}, A(x)\begin{pmatrix} g_1 \\ g_2 \end{pmatrix}\right\rangle.$$

It is meant that $A(x)g(x)$ has a derivative in the sense of distribution, defined by

$$< A_1 f, g >_H = - < f, \frac{\partial}{\partial x}[A(x)g(x)] > .$$

Since $(C_0^\infty([0,1]))^{p+q}$ is dense in H, we obtain that for any $g \in \mathcal{D}(A_1^*)$,

$$A_1^* g = -\frac{\partial}{\partial x}[A(x)g(x)] \in H. \tag{5.1}$$

Because $A(x)$ is continuously differentiable, it follows that $\mathcal{D}(A_1^*) \subset \left(H^1[0,1]\right)^n$. Conversely, for any $f \in \mathcal{D}(A_1)$ and any $g \in \mathcal{D}(A_1^*)$, by integrating by parts we have

$$< A_1 f, g >_H = - < f, A_1^* g >_H + f^*(x)A(x)g(x)|_0^1 . \tag{5.2}$$

Thus, (5.1) and (5.2) imply that for all $f \in \mathcal{D}(A_1)$ and $g \in \mathcal{D}(A_1^*)$,

$$f^*(x)A(x)g(x)|_0^1 = 0, \tag{5.3}$$

or, from the definition of (2.5)

$$(f^-(1))^* \left[A^-(1)g^-(1) + D_1^* A^+(1)g^+(1)\right] - (f^+(0))^* \left[D_0^* A^-(0)g^-(0) + A^+(0)g_2(0)\right] = 0.$$

Because the above identity is true for any $f \in \mathcal{D}(\mathcal{A})$, it is meant that for all $g \in \mathcal{D}\left(A_1^*\right)$,

$$\begin{cases} g^-(1) = -\left(A^-(1)\right)^{-1} D_1^* A^+(1)g^+(1) \\ g^+(0) = -\left(A^+(0)\right)^{-1} D_0^* A^-(0)g^-(0) \end{cases}$$

Therefore, the adjoint operator A^* is defined by

$$\mathcal{D}(\mathcal{A}) = \left\{ g = \begin{pmatrix} g^- \\ g^+ \end{pmatrix} \in \left(H^1[0,1]\right)^{p+q} \;\middle|\; \begin{array}{l} g^-(1) = -\left(A^-(1)\right)^{-1} D_1^* A^+(1)g^+(1) \\ g^+(0) = -\left(A^+(0)\right)^{-1} D_0^* A^-(0)g^-(0) \end{array} \right\}$$

and for all $g \in \mathcal{D}(\mathcal{A}^*)$,

$$\mathcal{A}^* g = -\frac{\partial}{\partial x}\left[A(x)\begin{pmatrix} g^-(x) \\ g^+(x) \end{pmatrix}\right] + B^*(x)g(x).$$

We prove that $\mathcal{A}^*$ is dissipative. Indeed, it is easy to see that

$$\begin{aligned} < \mathcal{A}^* g, g >_H = & -\frac{1}{2}\left(g^+(1)\right)^* \left[A^+(1)D_1\left(A^-(1)\right)^{-1} D_1^* A^+(1) + A^+(1)\right] g^+(1) \\ & +\frac{1}{2}\left(g^-(0)\right)^* \left[A^-(0) + A^-(0)D_0\left(A^+(0)\right)^{-1} D_0^* A^-(0)\right] g^-(0) \\ & +\frac{1}{2} < (B + B^* - A_x)g, g > \\ \le & -\frac{1}{2}\left(g^+(1)\right)^* \left[A^+(1)D_1\left(A^-(1)\right)^{-1} D_1^* A^+(1) + A^+(1)\right] g^+(1) \\ & +\frac{1}{2}\left(g^-(0)\right)^* \left[A^-(0) + A^-(0)D_0\left(A^+(0)\right)^{-1} D_0^* A^-(0)\right] g^-(0). \end{aligned} \tag{5.4}$$

The hypothesis H.3 is equivalent to

$$D_1^* A^+(1)D_1 \le -A^-(1) - (r^-)I \tag{5.5}$$

or,

$$L^*L \leq I + r^-\left(A^-(1)\right)^{-1}, \tag{5.6}$$

where $L = \left(A^+(1)\right)^{1/2} D_1 \left(-A^-(1)\right)^{-1/2}$. From (5.5) we have

$$I + r^-(A^-(1))^{-1} \geq 0.$$

Because $r^- > 0$,

$$I + r^-(A^-(-1))^{-1} \leq \alpha I \tag{5.7}$$

for some $0 \leq \alpha < 1$. This together with (5.6) implies that for all $w \in \mathbb{R}^p$,

$$w^*L^*Lw \leq \alpha\|w\|^2,$$

that is, $\|L\|_{\mathcal{L}(\mathbb{R}^p,\mathbb{R}^q)} \leq \alpha < 1$. Since $\|L^*\| = \|L\|$ (see Theorem II.21, p.31, Brézis), we get

$$\|L^*y\|^2 \leq \alpha\|y\|^2,$$

that is,

$$LL^* \leq \alpha I,$$

or,

$$A^+(1) + A^+(1)D_1(A^-(1))^{-1}D_1^*A^+(1) \geq (1-\alpha)A^+(1) \geq 0. \tag{5.8}$$

Using the same reasoning we can prove that there exists a $0 \leq \beta < 1$ such that

$$A^-(0) + A^-(0)D_0\left(A^+(0)\right)^{-1} D_0^*A^-(0) \leq (1-\beta)A^-(0). \tag{5.9}$$

Substituting (5.8) and (5.9) in (5.4) leads to

$$< A^*g, g >_H \leq 0$$

for all $g \in \mathcal{D}(\mathcal{A}^*)$. Therefore, $\mathcal{A}^*$ is dissipative. Thus, we finish the proof of our claim. ∎

A.2 Computing ξ_{adj} is equivalent to solving the following differential equation

$$\begin{cases} \dfrac{\partial g(x)}{\partial x} = A_1^{-1}B_2 g(x) - A_1^{-1}f(x) \\ g_3(0) = g_4(0) = 0 \\ \begin{bmatrix} g_1(1) \\ g_2(1) \end{bmatrix} = D_{\text{adj}} \begin{bmatrix} g_3(1) \\ g_4(1) \end{bmatrix} \\ < \xi_{\text{adj}}, f >_H = g_1(0). \end{cases}$$

The problem has a unique solution $g \in \mathcal{D}(A^*)$ for each $f \in H$. By direct computation one may find that for all $f \in H$,

$$g_1(0) = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}^* \begin{bmatrix} I \\ 0 \end{bmatrix} \left\{ [I \mid -D_{\text{adj}}] e^{A_1^{-1}B_2} \begin{bmatrix} I \\ 0 \end{bmatrix} \right\}^{-1} [I \mid -D_{\text{adj}}] \int_0^1 e^{A_1^{-1}B_2(1-\tau)} A_1^{-1} f(\tau) d\tau$$

$$
- \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}^* \int_0^1 e^{A_1^{-1}B_2(1-\tau)} A_1^{-1} f(\tau) d\tau .
$$

Thus, we obtain

$$
\xi_{adj}^*(x) = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}^* \begin{bmatrix} I \\ 0 \end{bmatrix} \left\{ [I \mid -D_{\mathrm{adj}}] e^{A_1^{-1}B_2} \begin{bmatrix} I \\ 0 \end{bmatrix} \right\}^{-1} [I \mid -D_{\mathrm{adj}}] e^{A_1^{-1}B_2(1-x)} A_1^{-1}
$$

$$
- \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}^* e^{A_1^{-1}B_2(1-x)} A_1^{-1} .
$$

Obviously, $\xi_{\mathrm{adj}}(x)$ is an analytic function of x. ∎

References

[1] C.D. Benchimol, A note on weak stabilizability of contraction semigroups, *SIAM J. Contr. and Optim.*, 16, pp.373-379, 1978.

[2] G. Weiss, Transfer functions of regular linear systems. Part I : characterization of regularity, *Transaction of American Mathematical Society*, 342, pp.827-854, 1994.

[3] RIC'97 Proceeding, *Regulation of Irrigation Canals : State of the Art of Research and Applications*, International Workshop, April 22-24 1997, Marrakech, Marocco.

[4] S.A. Pohjolainen, Robust multivariable PI-controllers for infinite dimensional systems, *IEEE Trans. Autom. Contr.*, 27, pp.17-30, 1985.

[5] C.Z. Xu and H. Jerbi, A robust PI-controller for infinite dimensional systems, *Int. J. Control*, 61, pp.33-45, 1995.

[6] C.Z. Xu, *Commande des systèmes à paramètres distribués*, Habilitation à Diriger la Recherche, Université de Metz, Janvier, 1997.

[7] G. Zames and B.A. Francis, Feedback, minimax sensitivity, and optimal robustness, *IEEE Trans. Autom. Contr.*, 28, pp.585-600, 1983.

[8] B.A. Francis and G. Zames, On H^∞-optimal sensitivity theory for SISO feedback systems, *IEEE Trans. Autom. Contr.*, 29, pp.9-16, 1984.

[9] G. Weiss, Admissible observation operators for linear semigroups, *Israel J. Math.*, 65, pp.17-43, 1989.

[10] H. Hoffman, *Banach spaces of analytic functions*, Prentice-Hall Inc., Englewood Cliffs, 1962.

[11] D. Salamon, Realization theory in Hilbert space, *Math. Systems Theory*, 21, pp.147-164, 1989.

[12] R. Rebarber, Conditions for the equivalence of internal and external stability for distributed parameter systems, *IEEE Trans. Autom. Contr.*, 38, pp.994-998, 1993.

[13] C.Z. Xu, Exact observability and exponential stability of infinite dimensional bilinear systems, *Math. Control Signals Systems*, 9, pp.73-93, 1996.

[14] J.C. Friedly, *Dynamic behavior of processes*, Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1972.

[15] J.P. Gauthier and C.Z. Xu, H^{∞}-control of a distributed parameter system with non-minimum phase, *Int. J. Control,* 53, pp.45-79, 1991.

[16] A. Pazy, *Semigroups of linear operators and applications to partial differential equations.* Springer-Verlag, New York, 1983.

[17] C.Z. Xu et J.P. Gauthier, Analyse et commande d'un échangeur thermique à contre-courant, *RAIRO APII*, 25, pp.377-396, 1991.

[18] J. Rauch and M. Taylor, Exponential decay of solutions to hyperbolic equations in bounded domain, *Indiana University Mathematics Journal*, 24, pp.79-86, 1974.

[19] D.L. Russell, Controllability and stabilizability theory for linear partial differential equations : Recent progress and open questions, *SIAM Review*, 20, pp.639-739, 1978.

[20] R.F. Curtain, Equivalence of input-output stability and exponential stability for infinite-dimensional systems, *Math. Systems Theory*, 21, pp.19-48, 1988.

[21] V. Jurdjevic and J.P. Quinn, Controllability and stability, *J. of Differential equations*, 28, pp.281-289, 1978.

[22] C.Z. Xu, Exponential stability of a class of infinite dimensional time-varying linear systems, *Proceedings of the International Conference on Control and Information*, June 5-9, 1995, Hong Kong.

[23] C.Z. Xu and G. Sallet, Proportional and Integral regulation of irrigation canal systems governed by the St Venant equation, *14th IFAC World Congress*, July 5-9, 1999, Beijing, China.

[24] C.Z. Xu, Linear symmetric hyperbolic systems and application to boundary PI-regulation of St Venant equations, Preprint, August, 1997.

[25] K.O. Friedrichs, Symmetric positive linear differential equations, *Comm. Pure and Appl. Math.*, 11, pp.333-418, 1958.

[26] P.D. Lax and R.S. Phillips, Local boundary conditions for dissipative symmetric linear differential operators, *Comm. Pure and Appl. Math.*, 13, pp.427-455, 1960.

[27] S.X. Chen, *Introduction to partial differential equations*, People Education Press (in Chinese), 1981.

[28] C.Z. Xu, J.P. Gauthier and I. Kupka, Exponential stability of the heat exchanger equation, *The Proceedings of the European Control Conference*, Groningen, The Netherlands, pp.303-307, 1993.

Unité de recherche INRIA Lorraine, Technopôle de Nancy-Brabois, Campus scientifique,
615 rue du Jardin Botanique, BP 101, 54600 VILLERS LÈS NANCY
Unité de recherche INRIA Rennes, Irisa, Campus universitaire de Beaulieu, 35042 RENNES Cedex
Unité de recherche INRIA Rhône-Alpes, 655, avenue de l'Europe, 38330 MONTBONNOT ST MARTIN
Unité de recherche INRIA Rocquencourt, Domaine de Voluceau, Rocquencourt, BP 105, 78153 LE CHESNAY Cedex
Unité de recherche INRIA Sophia-Antipolis, 2004 route des Lucioles, BP 93, 06902 SOPHIA-ANTIPOLIS Cedex

Éditeur
INRIA, Domaine de Voluceau, Rocquencourt, BP 105, 78153 LE CHESNAY Cedex (France)
ISSN 0249-6399