Occupation Times in Markov Processes

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Abstract: We consider, in a homogeneous Markov process with finite state space, the occupation times that is, the times spent by the process in given subsets of the state space during a finite interval of time. We first derive the distribution of the occupation time of one subset and then we generalize this result to the joint distribution of occupation times of different subsets of the state space by the use of order statistics from the uniform distribution. Next, we consider the distribution of weighted sums of occupation times. We obtain the forward and backward equations describing the behavior of these weighted sums and we show how these equations lead to simple expressions of this distribution.

Key-words: Occupation times, Markov processes, order statistics, availability, performability.

(Résumé : tsvp)

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Temps d'occupation dans les processus de Markov

Résumé : On considère, dans un processus de Markov homogène à espace d'états fini, les temps d'occupation c'est-à-dire, les temps passés par le processus dans des sous-ensembles d'états donnés durant un intervalle de temps fini. On obtient dans un premier temps la distribution du temps d'occupation d'un sous-ensemble et on généralise ce résultat à la distribution jointe de temps d'occupation de différents sous-ensembles de l'espace d'états par l'utilisation de statistiques d'ordre de la loi uniforme. On considère ensuite la distribution de sommes pondérées de temps d'occupation. On obtient les équations avant et arrière décrivant le comportement de ces sommes pondérées et on montre comment ces équations conduisent à des expressions simples de cette distribution.

Mots-clé : Temps d'occupation, processus de Markov, statistiques d'ordre, disponibilité, performabilité.

1 Introduction

Let $X = \{X_u, u \geq 0\}$ be a homogeneous Markov process with a finite state space S. The occupation time of a subset $U \subset S$ over the interval $[0, t)$ is defined by the random variable W_t as

$$W_t = \int_0^t 1_{\{X_u \in U\}} du,$$

where $1_{\{c\}} = 1$ if condition c is true and 0 otherwise. This random variable received a considerable attention since it is also known as the interval availability in reliability and dependability theory. An expression of the distribution of W_t has been obtained in [2] using order statistics from the uniform distribution over $[0, t)$. This expression is very interesting from a computational point of view and various methods have been developed to compute it even in the case of denumerable state spaces (see [2], [10], [8], [9] and the references therein).

In this paper, we first obtain the joint distribution of the random variables (W_t, X_t) using the forward and backward equations associated to the uniformized Markov chain of the process X. We then generalize this result to the joint distribution of $W_t^1, \ldots, W_t^m, X_t$, where W_t^i is the occupation time of a subset B_i over the interval $[0, t)$. Finally, we consider a weighted sum of occupation times, that is the random variable Y_t defined by

$$Y_t = \int_0^t \rho(X_u) du,$$

where for each $i \in S$, $\rho(i)$ is a non negative constant. The study of this random variable Y_t is also known as the performability analysis in the reliability and dependability theory (see [3], [5] and the references therein). We derive here the backward and forward equations describing the behavior of the joint distribution of (Y_t, X_t). These equations, which are partial differential equations, are then solved and we show that they lead to simple expressions of the joint distribution of (Y_t, X_t).

The rest of paper is organized as follows. In the next section, we consider the joint distribution of order statistics from the uniform distribution and the joint conditional distribution of the jumps in a Poisson process and we show how they are related. In section 3, we consider the case $m = 1$. We obtain the distribution of occupation time for a discrete time Markov chain. This distribution combined with the results of section 2 lead to simple expressions of the joint distribution of the couple (W_t, X_t). The results of section 3 are then generalized in section 4 to the case where $m > 1$. Finally, section 5 deals with the distribution of the couple (Y_t, X_t).

2 Order Statistics

2.1 The Uniform Distribution

In this subsection, we consider order statistics from the uniform distribution over $[0,t)$, where t is a fixed positive real number. More formally, let X_1, X_2, …, X_n be n i.i.d. random variables with common distribution the uniform distribution over $[0,t)$. We thus have, for all $i = 1,\ldots,n$ and $x \in \mathbb{R}$,

$$\mathbb{P}\{X_i \le x\} = \begin{cases} 0 & \text{if} \quad x \le 0 \\ x/t & \text{if} \quad x \in (0,t) \\ 1 & \text{if} \quad x \ge t \end{cases}$$

If the random variables X_1, X_2, …, X_n are rearranged in ascending order of magnitude and then written as

$$X_{(1)} \le X_{(2)} \le \cdots \le X_{(n)}$$

we call $X_{(i)}$ the ith order statistic, $i = 1,2,\ldots,n$. Because of the inequality relations among them the $X_{(i)}$ are necessarily dependent random variables.

Let $F_r(x)$ be the distribution of $X_{(r)}$. We then have, for $x \in (0,t)$,

$$\begin{aligned} F_r(x) &= \mathbb{P}\{X_{(r)} \le x\} \\ &= \mathbb{P}\{\text{at least } r \text{ of the } X_k \text{ are less than or equal to } x\} \\ &= \sum_{i=r}^{n} \binom{n}{i} \left(\frac{x}{t}\right)^i \left(1-\frac{x}{t}\right)^{n-i} \end{aligned} \tag{1}$$

The density $f_r(x)$ of $X_{(r)}$ is thus given, for $x \in (0,t)$, by

$$f_r(x) = \frac{n!}{(r-1)!(n-r)!}\frac{1}{t}\left(\frac{x}{t}\right)^r \left(1-\frac{x}{t}\right)^{n-r}.$$

It is shown in [1] that the joint density function of $X_{(r_1)}, X_{(r_2)}, \ldots, X_{(r_k)}$, for $1 \le k \le n$, $1 \le r_1 < r_2 < \cdots < r_k \le n$ and $x_1 \le x_2 \le \cdots \le x_k$ ($x_i \in (0,t)$) is

$$f_{r_1,r_2,\ldots,r_k}(x_1,x_2,\ldots,x_k) =$$

$$\frac{n!\left(\frac{1}{t}\right)^k \left(\frac{x_1}{t}\right)^{r_1-1} \left(\frac{x_2-x_1}{t}\right)^{r_2-r_1-1} \cdots \left(\frac{x_k-x_{k-1}}{t}\right)^{r_k-r_{k-1}-1} \left(1-\frac{x_k}{t}\right)^{n-r_k}}{(r_1-1)!(r_2-r_1-1)!\cdots(r_k-r_{k-1}-1)!(n-r_k)!}$$

The joint density function of $X_{(l_1)}, X_{(l_1+l_2)}, \ldots, X_{(l_1+l_2+\cdots+l_k)}$ is thus given, for $1 \leq k \leq n$, $1 \leq l_1 + l_2 + \cdots + l_k \leq n$ $(l_i \geq 1)$ and $x_1 \leq x_2 \leq \cdots \leq x_k$ $(x_i \in (0,t))$ by

$$g_{l_1,l_2,\ldots,l_k}(x_1, x_2, \ldots, x_k) = \frac{n! \left(\frac{1}{t}\right)^k \left(\frac{x_1}{t}\right)^{l_1-1} \left(\frac{x_2 - x_1}{t}\right)^{l_2-1} \cdots \left(\frac{x_k - x_{k-1}}{t}\right)^{l_k-1} \left(1 - \frac{x_k}{t}\right)^{n-(l_1+l_2\cdots+l_k)}}{(l_1-1)!(l_2-1)!\cdots(l_k-1)!(n-(l_1+l_2\cdots+l_k))!}.$$

Furthermore, if we write $Y_{l_1} = X_{l_1}$ and for $i = 2, \ldots, k$, we define Y_{l_i} as $Y_{l_i} = X_{(l_1+l_2+\cdots+l_i)} - X_{(l_1+l_2+\cdots+l_{i-1})}$ then the joint density function of $Y_{l_1}, Y_{l_2}, \ldots, Y_{l_k}$ is given, for $1 \leq k \leq n$, $1 \leq l_1 + l_2 + \cdots + l_k \leq n$ $(l_i \geq 1)$ and $0 < s_1 + s_2 + \cdots + s_k < t$ $(s_i \in (0,t))$ by

$$h_{l_1,l_2,\ldots,l_k}(s_1, s_2, \ldots, s_k) = g_{l_1,l_2,\ldots,l_k}(s_1, s_1 + s_2, \ldots, s_1 + \cdots + s_k),$$

that is,

$$h_{l_1,l_2,\ldots,l_k}(s_1, s_2, \ldots, s_k) = \frac{n! \left(\frac{1}{t}\right)^k \left(\frac{s_1}{t}\right)^{l_1-1} \left(\frac{s_2}{t}\right)^{l_2-1} \cdots \left(\frac{s_k}{t}\right)^{l_k-1} \left(1 - \frac{s_1 + s_2 + \cdots + s_k}{t}\right)^{n-(l_1+l_2\cdots+l_k)}}{(l_1-1)!(l_2-1)!\cdots(l_k-1)!(n-(l_1+l_2\cdots+l_k))!}.$$

In particular, for $k = n$ we get the joint density function of the random variables $Y_1 = X_{(1)}, Y_2 = X_{(2)} - X_{(1)}, \ldots, Y_n = X_{(n)} - X_{(n-1)}$ denoted by $h(x_1, x_2, \ldots, x_n)$ by making $l_1 = l_2 = \cdots = l_n = 1$ in the previous expression, that is

$$h(x_1, x_2, \ldots, x_n) = \begin{cases} \dfrac{n!}{t^n} & \text{if } \displaystyle\sum_{i=1}^{n} x_i \leq t \\ 0 & \text{otherwise.} \end{cases}$$

By writing $Y_{n+1} = t - X_{(n)}$, this also determines the (degenerate) joint density function of $Y_1, Y_2, \ldots, Y_n, Y_{n+1}$ in the region

$$x_i \geq 0 \quad (i = 1, \ldots, n, n+1) \quad \sum_{i=1}^{n+1} x_i = t.$$

The joint distribution of $Y_{l_1}, Y_{l_2}, \ldots, Y_{l_k}$ is denoted by $H_{l_1,l_2,\ldots,l_k}(s_1, s_2, \ldots, s_k)$, that is

$$H_{l_1,l_2,\ldots,l_k}(s_1, s_2, \ldots, s_k) = \mathbb{P}\{Y_{l_1} \leq s_1, Y_{l_2} \leq s_2, \ldots, Y_{l_k} \leq s_k\}.$$

This distribution is given in the following lemma.

Lemma 2.1 *For $1 \leq k \leq n$, $1 \leq l_1 + l_2 + \cdots + l_k \leq n$ ($l_i \geq 1$) and $0 < s_1 + s_2 + \cdots + s_k < t$ ($s_i \in (0,t)$), we have*

$$H_{l_1,l_2,\ldots,l_k}(s_1, s_2, \ldots, s_k) =$$

$$\sum_{\substack{i_1 \geq l_1, i_2 \geq l_2, \ldots, i_k \geq l_k, \\ i_1 + i_2 + \cdots + i_k \leq n}} \frac{n! \left(\frac{s_1}{t}\right)^{i_1} \left(\frac{s_2}{t}\right)^{i_2} \cdots \left(\frac{s_k}{t}\right)^{i_k} \left(1 - \frac{s_1 + s_2 + \cdots + s_k}{t}\right)^{n-(i_1+i_2\cdots+i_k)}}{i_1! i_2! \cdots i_k! (n - (i_1 + i_2 \cdots + i_k))!}.$$

Proof. It suffices to show that

$$\frac{\partial^k H_{l_1,l_2,\ldots,l_k}(s_1, s_2, \ldots, s_k)}{\partial s_1 \partial s_2 \cdots \partial s_k} = h_{l_1,l_2,\ldots,l_k}(s_1, s_2, \ldots, s_k).$$

To simplify writing, we define

$$\theta^{t;s_1,s_2,\ldots,s_k}_{n;i_1,i_2,\ldots,i_k} = \frac{\left(\frac{s_1}{t}\right)^{i_1} \left(\frac{s_2}{t}\right)^{i_2} \cdots \left(\frac{s_k}{t}\right)^{i_k} \left(1 - \frac{s_1 + s_2 + \cdots + s_k}{t}\right)^{n-(i_1+i_2\cdots+i_k)}}{i_1! i_2! \cdots i_k! (n - (i_1 + i_2 \cdots + i_k))!}.$$

We thus have, from relation (3),

$$H_{l_1,l_2,\ldots,l_k}(s_1, s_2, \ldots, s_k) = n! \sum_{\substack{i_1 \geq l_1, i_2 \geq l_2, \ldots, i_k \geq l_k, \\ i_1 + i_2 + \cdots + i_k \leq n}} \theta^{t;s_1,s_2,\ldots,s_k}_{n;i_1,i_2,\ldots,i_k}.$$

We obtain

$$\begin{aligned}
\frac{\partial H_{l_1,l_2,\ldots,l_k}(s_1, s_2, \ldots, s_k)}{\partial s_k} &= \frac{n!}{t} \sum_{\substack{i_1 \geq l_1, i_2 \geq l_2, \ldots, i_k \geq l_k, \\ i_1 + i_2 + \cdots + i_k \leq n}} \theta^{t;s_1,s_2,\ldots,s_k}_{n;i_1,i_2,\ldots,i_{k-1},i_k-1} \\
&\quad - \frac{n!}{t} \sum_{\substack{i_1 \geq l_1, i_2 \geq l_2, \ldots, i_k \geq l_k, \\ i_1 + i_2 + \cdots + i_k \leq n-1}} \theta^{t;s_1,s_2,\ldots,s_k}_{n-1;i_1,i_2,\ldots,i_k} \\
&= \frac{n!}{t} \sum_{\substack{i_1 \geq l_1, i_2 \geq l_2, \ldots, i_{k-1} \geq l_{k-1}, i_k \geq l_k - 1, \\ i_1 + i_2 + \cdots + i_k \leq n-1}} \theta^{t;s_1,s_2,\ldots,s_k}_{n-1;i_1,i_2,\ldots,i_k}
\end{aligned}$$

$$
\begin{aligned}
& -\frac{n!}{t} \sum_{\substack{i_1 \geq l_1, i_2 \geq l_2, \ldots, i_k \geq l_k, \\ i_1+i_2+\cdots+i_k \leq n-1}} \theta^{t;s_1,s_2,\ldots,s_k}_{n-1;i_1,i_2,\ldots,i_k} \\
= \; & \frac{n!}{t} \sum_{\substack{i_1 \geq l_1, i_2 \geq l_2, \ldots, i_{k-1} \geq l_{k-1}, \\ i_1+i_2+\cdots+i_{k-1} \leq n-1}} \theta^{t;s_1,s_2,\ldots,s_k}_{n-1;i_1,i_2,\ldots,i_{k-1},l_k-1}
\end{aligned}
$$

where the second equality is obtained by using the variable change $i_k \longrightarrow i_k+1$. By iterating successively the same argument with respect to variables $s_{k-1}, \ldots, s_2$, we obtain

$$
\frac{\partial^{k-1} H_{l_1,l_2,\ldots,l_k}(s_1,s_2,\ldots,s_k)}{\partial s_k \partial s_{k-1}, \cdots, \partial s_2} = \frac{n!}{t^{k-1}} \sum_{\substack{i_1 \geq l_1, \\ i_1 \leq n-(l_2+\cdots+l_k)}} \theta^{t;s_1,s_2,\ldots,s_k}_{n-k+1;i_1,l_2-1,\ldots,l_{k-1}-1,l_k-1},
$$

and finally,

$$
\begin{aligned}
\frac{\partial^{k} H_{l_1,l_2,\ldots,l_k}(s_1,s_2,\ldots,s_k)}{\partial s_k, \cdots, \partial s_2, \partial s_1} \; = \; & \frac{n!}{t^{k}} \sum_{\substack{i_1 \geq l_1, \\ i_1 \leq n-(l_2+\cdots+l_k)}} \theta^{t;s_1,s_2,\ldots,s_k}_{n-k+1;i_1-1,l_2-1,\ldots,l_k-1} \\
& -\frac{n!}{t^{k}} \sum_{\substack{i_1 \geq l_1, \\ i_1 \leq n-(l_2+\cdots+l_k)-1}} \theta^{t;s_1,s_2,\ldots,s_k}_{n-k;i_1,l_2-1,\ldots,l_k-1} \\
= \; & \frac{n!}{t^{k}} \sum_{\substack{i_1 \geq l_1-1, \\ i_1 \leq n-(l_2+\cdots+l_k)-1}} \theta^{t;s_1,s_2,\ldots,s_k}_{n-k;i_1,l_2-1,\ldots,l_k-1} \\
& -\frac{n!}{t^{k}} \sum_{\substack{i_1 \geq l_1, \\ i_1 \leq n-(l_2+\cdots+l_k)-1}} \theta^{t;s_1,s_2,\ldots,s_k}_{n-k;i_1,l_2-1,\ldots,l_k-1} \\
= \; & \frac{n!}{t^{k}} \theta^{t;s_1,s_2,\ldots,s_k}_{n-k;l_1-1,l_2-1,\ldots,l_k-1} \\
= \; & h_{l_1,l_2,\ldots,l_k}(s_1,s_2,\ldots,s_k)
\end{aligned}
$$

where the second equality is obtained by using the variable change $i_1 \longrightarrow i_1+1$. ■

2.2 The Poisson Process

Let $\{N_t, t \in \mathbb{R}\}$ be a Poisson process with rate λ and let T_0, T_0+T_1, ..., $T_0+T_1+\cdots+T_{n-1}$ be the n first instants of jumps of $\{N_t\}$ over $[0,t)$. It is well-known, see [4] that the density of the conditional distribution of $T_0, T_1, \ldots, T_{n-1}$ given that $\{N_t = n\}$ is

$$f(x_0, x_1, \ldots, x_{n-1}) = \begin{cases} \dfrac{n!}{t^n} & \text{if } \displaystyle\sum_{i=0}^{n-1} x_i \leq t \\ 0 & \text{otherwise.} \end{cases}$$

This is also, as seen in the previous subsection, the joint density of the order statistics from the uniform distribution over $(0,t)$.

If we write $T_n = t - (T_0 + T_1 + \cdots + T_{n-1})$, this also determines the (degenerate) joint density function of $T_0, T_1, \ldots, T_{n-1}, T_n$ in the region

$$x_i \geq 0 \quad (i = 0, \ldots, n-1, n) \qquad \sum_{i=0}^{n} x_i = t.$$

The symmetric role played by the variables $x_0, x_1, \ldots, x_{n-1}, x_n$ shows that for any subset $\{i_1, i_2, \cdots, i_n\}$ of the set $\{0, 1, \ldots, n-1, n\}$, we have $f(x_{i_1}, x_{i_2}, \ldots, x_{i_n}) = f(x_0, x_1, \ldots, x_{n-1})$.

It follows from relation (1) that we have, for $1 \leq l \leq n$, $\{i_1, i_2, \cdots, i_l\} \subset \{0, 1, \ldots, n-1, n\}$ and $s \in (0,t)$,

$$\begin{aligned} \mathbb{P}\{T_{i_1} + \cdots + T_{i_l} \leq s \mid N_t = n\} &= \mathbb{P}\{T_0 + \cdots + T_{l-1} \leq s \mid N_t = n\} \\ &= \mathbb{P}\{X_{(l)} \leq s\} \\ &= \sum_{k=l}^{n} \binom{n}{k} \left(\frac{s}{t}\right)^k \left(1 - \frac{s}{t}\right)^{n-k}. \end{aligned} \tag{2}$$

More generally, let k be an integer such that $1 \leq k \leq n$ and $l_1, l_2, \ldots, l_k$ be k integers such that $1 \leq l_1 + l_2 + \cdots + l_k \leq n$ $(l_i \geq 1)$. For any subset $\{i_1, i_2, \ldots, i_{l_1+l_2+\cdots+l_k}\}$ of the $\{0, 1, \ldots, n-1, n\}$, the vectors

$$\left(\sum_{j=1}^{l_1} T_{i_j}, \sum_{j=l_1+1}^{l_1+l_2} T_{i_j}, \ldots, \sum_{j=l_1+l_2+\cdots+l_{k-1}+1}^{l_1+l_2+\cdots+l_k} T_{i_j} \right)$$

and

$$\left(\sum_{j=0}^{l_1-1} T_j, \sum_{j=l_1}^{l_1+l_2-1} T_j, \ldots, \sum_{j=l_1+l_2+\cdots+l_{k-1}}^{l_1+l_2+\cdots+l_k-1} T_j \right)$$

have the same conditional distribution given that $N_t = n$, given by lemma 2.1, that is, for $0 < s_1 + s_2 + \cdots + s_k < t$ $(s_i \in (0,t))$,

$$\mathbb{P}\{\sum_{j=1}^{l_1} T_j \le s_1, \sum_{j=l_1+1}^{l_1+l_2} T_j \le s_2, \ldots, \sum_{j=l_1+l_2+\cdots+l_{k-1}+1}^{l_1+l_2+\cdots+l_k} T_j \le s_k \mid N_t = n\} =$$

$$\sum_{\substack{i_1 \ge l_1, i_2 \ge l_2, \ldots, i_k \ge l_k, \\ i_1 + i_2 + \cdots + i_k \le n}} \frac{n! \left(\frac{s_1}{t}\right)^{i_1} \left(\frac{s_2}{t}\right)^{i_2} \cdots \left(\frac{s_k}{t}\right)^{i_k} \left(1 - \frac{s_1 + s_2 + \cdots + s_k}{t}\right)^{n-(i_1+i_2\cdots+i_k)}}{i_1! i_2! \cdots i_k! (n - (i_1 + i_2 \cdots + i_k))!}. \tag{3}$$

3 Distribution of Occupation Times

Let $X = \{X_u, u \ge 0\}$ be a homogeneous Markov process with a finite state space S. The process X is given by its infinitesimal generator A and by its initial probability distribution α. We denote by $Z = \{Z_n, n \ge 0\}$ the uniformized Markov chain [7] associated to the Markov process X, with the same initial distribution α. Its transition probability matrix P is related to the matrix A by the relation $P = I + A/\lambda$, where I is the identity matrix and λ satisfies the relation $\lambda \ge \max\{-A_{i,i},\ i \in S\}$. The rate λ is the rate of the Poisson process $\{N_u, u \ge 0\}$, independent of Z, which counts the number of transitions of process $\{Z_{N_u}, u \ge 0\}$ over $[0,t)$. It is well-known that the processes $\{Z_{N_u}\}$ and X are stochastically equivalent. We consider a partition $S = U \cup D$ $(U \cap D = \emptyset)$ of the state space S and we consider the occupation time of the subset U.

3.1 The Discrete Time Case

We consider the Markov chain $Z = \{Z_n, n \ge 0\}$ and we define the random variable V_n by the total number of states of U visited during the n first transitions of Z, that is

$$V_n = \sum_{k=0}^{n} 1_{\{Z_k \in U\}}.$$

This random variable represents the occupation time of subset U during the n first transitions of the process Z. The following theorem gives the backward equation for the behavior of the couple (V_n, Z_n).

Theorem 3.1 *For $n \geq 1$ and $1 \leq k \leq n$, we have*

$$\text{for } i \in U,\ \mathbb{P}\{V_n \leq k, Z_n = j \mid Z_0 = i\} = \sum_{l \in E} P_{i,l} \mathbb{P}\{V_{n-1} \leq k-1, Z_{n-1} = j \mid Z_0 = l\}$$
$$\text{for } i \in D,\ \mathbb{P}\{V_n \leq k, Z_n = j \mid Z_0 = i\} = \sum_{l \in E} P_{i,l} \mathbb{P}\{V_{n-1} \leq k, Z_{n-1} = j \mid Z_0 = l\}$$

Proof. We have

$$\begin{aligned}
&\mathbb{P}\{V_n \leq k, Z_n = j \mid Z_0 = i\} \\
&\qquad = \sum_{l \in E} P_{i,l} \mathbb{P}\{V_n \leq k, Z_n = j \mid Z_1 = l, Z_0 = i\} \\
&\qquad = \sum_{l \in E} P_{i,l} \mathbb{P}\{\sum_{h=1}^{n} 1_{\{Z_h \in U\}} \leq k - 1_{\{i \in U\}}, Z_n = j \mid Z_1 = l, Z_0 = i\} \\
&\qquad = \sum_{l \in E} P_{i,l} \mathbb{P}\{\sum_{h=1}^{n} 1_{\{Z_h \in U\}} \leq k - 1_{\{i \in U\}}, Z_n = j \mid Z_1 = l\} \\
&\qquad = \sum_{l \in E} P_{i,l} \mathbb{P}\{\sum_{h=0}^{n-1} 1_{\{Z_h \in U\}} \leq k - 1_{\{i \in U\}}, Z_{n-1} = j \mid Z_0 = l\} \\
&\qquad = \sum_{l \in E} P_{i,l} \mathbb{P}\{V_{n-1} \leq k - 1_{\{i \in U\}}, Z_{n-1} = j \mid Z_0 = l\}
\end{aligned}$$

The third equality follows from the Markov property of Z and the fourth follows from the homogeneity of Z. ■

The following theorem gives the forward equation for the behavior of the couple (V_n, Z_n).

Theorem 3.2 *For $n \geq 1$ and $1 \leq k \leq n$, we have*

$$\text{for } j \in U,\ \mathbb{P}\{V_n \leq k, Z_n = j \mid Z_0 = i\} = \sum_{l \in E} \mathbb{P}\{V_{n-1} \leq k-1, Z_{n-1} = l \mid Z_0 = i\} P_{l,j}$$
$$\text{for } j \in D,\ \mathbb{P}\{V_n \leq k, Z_n = j \mid Z_0 = i\} = \sum_{l \in E} \mathbb{P}\{V_{n-1} \leq k, Z_{n-1} = l \mid Z_0 = i\} P_{l,j}$$

Proof. We have

$$\begin{aligned}
&\mathbb{P}\{V_n \leq k, Z_n = j, Z_0 = i\} \\
&\quad = \mathbb{P}\{V_{n-1} \leq k - 1_{\{j \in U\}}, Z_n = j, Z_0 = i\} \\
&\quad = \sum_{l \in E} \mathbb{P}\{V_{n-1} \leq k - 1_{\{j \in U\}}, Z_n = j, Z_{n-1} = l, Z_0 = i\} \\
&\quad = \sum_{l \in E} \mathbb{P}\{V_{n-1} \leq k - 1_{\{j \in U\}}, Z_0 = i \mid Z_n = j, Z_{n-1} = l\} \mathbb{P}\{Z_n = j, Z_{n-1} = l\} \\
&\quad = \sum_{l \in E} \mathbb{P}\{V_{n-1} \leq k - 1_{\{j \in U\}}, Z_0 = i \mid Z_{n-1} = l\} \mathbb{P}\{Z_n = j, Z_{n-1} = l\} \\
&\quad = \sum_{l \in E} \mathbb{P}\{V_{n-1} \leq k - 1_{\{j \in U\}}, Z_{n-1} = l, Z_0 = i\} \mathbb{P}\{Z_n = j \mid Z_{n-1} = l\} \\
&\quad = \sum_{l \in E} \mathbb{P}\{V_{n-1} \leq k - 1_{\{j \in U\}}, Z_{n-1} = l, Z_0 = i\} P_{l,j}
\end{aligned}$$

The fourth equality follows from the Markov property of Z and the last one follows from the homogeneity of Z. We thus obtain the desired relation by conditioning with respect to Z_0. ∎

If we denote, for $n \geq 0$ and $k \geq 0$, by $F(n,k)$ the matrix whose entry (i,j) is defined by

$$F_{i,j}(n,k) = \mathbb{P}\{V_n \leq k, Z_n = j \mid Z_0 = i\},$$

the results of theorems 3.1 and 3.2 can be easily expressed in matrix notation. We first decompose the matrices P and $F(n,k)$ with respect to the partition $\{U, D\}$ of the state space S as

$$P = \begin{pmatrix} P_U & P_{UD} \\ P_{DU} & P_D \end{pmatrix} \quad \text{and} \quad F(n,k) = \begin{pmatrix} F_U(n,k) & F_{UD}(n,k) \\ F_{DU}(n,k) & F_D(n,k) \end{pmatrix}.$$

The result of theorem 3.1 can then be written as

$$\begin{aligned}
\begin{pmatrix} F_U(n,k) & F_{UD}(n,k) \end{pmatrix} &= \begin{pmatrix} P_U & P_{UD} \end{pmatrix} F(n-1, k-1) \\
\begin{pmatrix} F_{DU}(n,k) & F_D(n,k) \end{pmatrix} &= \begin{pmatrix} P_{DU} & P_D \end{pmatrix} F(n-1, k)
\end{aligned}$$

or also as

$$F(n,k) = \begin{pmatrix} P_U & P_{UD} \\ 0 & 0 \end{pmatrix} F(n-1, k-1) + \begin{pmatrix} 0 & 0 \\ P_{DU} & P_D \end{pmatrix} F(n-1, k).$$

In the same way, the result of theorem 3.2 can be written as

$$\begin{pmatrix} F_U(n,k) \\ F_{DU}(n,k) \end{pmatrix} = F(n-1,k-1) \begin{pmatrix} P_U \\ P_{DU} \end{pmatrix}$$
$$\begin{pmatrix} F_{UD}(n,k) \\ F_D(n,k) \end{pmatrix} = F(n-1,k) \begin{pmatrix} P_{UD} \\ P_D \end{pmatrix}$$

or also as

$$F(n,k) = F(n-1,k-1) \begin{pmatrix} P_U & 0 \\ P_{DU} & 0 \end{pmatrix} + F(n-1,k) \begin{pmatrix} 0 & P_{UD} \\ 0 & P_D \end{pmatrix}.$$

The initial conditions are given, for $n \geq 0$, by

$$F(n,0) = \begin{pmatrix} 0 & 0 \\ 0 & (P_D)^n \end{pmatrix},$$

Note also that we have for every $k \geq n+1$

$$F(n,k) = P^n.$$

3.2 The Continuous Time Case

We consider now the Markov process $X = \{X_t, t \geq 0\}$ and the occupation time W_t of the subset U over $[0,t)$, that is

$$W_t = \int_0^t 1_{\{X_u \in U\}}.$$

This random variable represents the time spent by the process X in the subset U during the interval $[0,t)$. The joint distribution of the couple (W_t, X_t) is given by the following theorem. The notation $\mathbb{P}_i$ denotes the conditional probability given that $X_0 = i$, that is $\mathbb{P}_i\{.\} = \mathbb{P}\{. \mid X_0 = i\}$.

Theorem 3.3 *For every $i, j \in S$, for $t > 0$ and $s \in [0,t)$, we have*

$$\mathbb{P}\{W_t \leq s, X_t = j \mid X_0 = i\}$$
$$= \sum_{n=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} \sum_{k=0}^{n} \binom{n}{k} \left(\frac{s}{t}\right)^k \left(1 - \frac{s}{t}\right)^{n-k} \mathbb{P}\{V_n \leq k, Z_n = j \mid Z_0 = i\} \quad (4)$$

Proof. We have, for $s < t$,

$$
\begin{aligned}
&\mathbb{P}\{W_t \le s, X_t = j \mid X_0 = i\} \\
&= \sum_{n=0}^{\infty} \mathbb{P}_i\{W_t \le s, N_t = n, X_t = j\} \\
&= \sum_{n=0}^{\infty} \mathbb{P}_i\{W_t \le s, N_t = n, Z_n = j\} \qquad \text{since } \{X_t\} \text{ and } \{Z_{N_t}\} \text{ are equivalent} \\
&= \sum_{n=0}^{\infty} \mathbb{P}_i\{N_t = n\}\mathbb{P}_i\{W_t \le s, Z_n = j \mid N_t = n\} \\
&= \sum_{n=0}^{\infty} \mathbb{P}\{N_t = n\}\mathbb{P}_i\{W_t \le s, Z_n = j \mid N_t = n\} \\
&= \sum_{n=0}^{\infty} \mathbb{P}\{N_t = n\} \sum_{l=0}^{n+1} \mathbb{P}_i\{W_t \le s, V_n = l, Z_n = j \mid N_t = n\} \\
&= \sum_{n=0}^{\infty} \mathbb{P}\{N_t = n\} \sum_{l=0}^{n+1} \mathbb{P}_i\{V_n = l, Z_n = j \mid N_t = n\}\mathbb{P}_i\{W_t \le s \mid V_n = l, Z_n = j, N_t = n\} \\
&= \sum_{n=0}^{\infty} \mathbb{P}\{N_t = n\} \sum_{l=0}^{n+1} \mathbb{P}_i\{V_n = l, Z_n = j\}\mathbb{P}_i\{W_t \le s \mid V_n = l, Z_n = j, N_t = n\} \\
&= \sum_{n=0}^{\infty} \mathbb{P}\{N_t = n\} \sum_{l=0}^{n} \mathbb{P}_i\{V_n = l, Z_n = j\}\mathbb{P}_i\{W_t \le s \mid V_n = l, Z_n = j, N_t = n\}
\end{aligned}
$$

Note that the 4th and the 7th equalities follow from the independence of the processes $\{Z_n\}$ and $\{N_t\}$ and the fact that $X_0 = Z_0$. The last equality follows from the fact if $l = n+1$, we trivially obtain that $V_n = n+1$ and $N_t = n$ imply that $W_t = t$ and so we get $\mathbb{P}\{W_t \le s \mid V_n = n+1, Z_n = j, N_t = n\} = 0$ since we have supposed that $s < t$.

Let us consider now the expression $\mathbb{P}_i\{W_t \le s \mid V_n = l, Z_n = j, N_t = n\}$.

For fixed $i, j \in S$ and $0 \le l \le n$, we define the set

$$G_{l,n}^{i,j} = \left\{ \widehat{z} = (i, z_1, \ldots, z_{n-1}, j) \in S^{n+1} \mid l \text{ entries of } \widehat{z} \text{ are in } U \text{ and } n+1-l \text{ are in } D \right\}$$

and we denote by $\widehat{Z}$ the random vector $(Z_0, \ldots, Z_n)$. We then have

$$\begin{aligned}
&\mathbb{P}_i\{W_t \le s \mid V_n = l, Z_n = j, N_t = n\} \\
&= \sum_{\widehat{z} \in G_{l,n}^{i,j}} \mathbb{P}_i\{W_t \le s \mid \widehat{Z} = \widehat{z}, V_n = l, Z_n = j, N_t = n\} \mathbb{P}_i\{\widehat{Z} = \widehat{z} \mid V_n = l, Z_n = j, N_t = n\} \\
&= \sum_{\widehat{z} \in G_{l,n}^{i,j}} \mathbb{P}\{W_t \le s \mid \widehat{Z} = \widehat{z}, V_n = l, N_t = n\} \mathbb{P}\{\widehat{Z} = \widehat{z} \mid V_n = l, Z_n = j, Z_0 = i, N_t = n\} \\
&= \sum_{\widehat{z} \in G_{l,n}^{i,j}} \mathbb{P}\{W_t \le s \mid \widehat{Z} = \widehat{z}, V_n = l, N_t = n\} \mathbb{P}\{\widehat{Z} = \widehat{z} \mid V_n = l, Z_n = j, Z_0 = i\}
\end{aligned}$$

where the last equality follows from the independence of the processes $\{Z_n\}$ and $\{N_t\}$.

We denote by T_0, $T_0 + T_1$, ..., $T_0 + T_1 + \cdots + T_{n-1}$ the n first instants of jumps of the Poisson process $\{N_t\}$ over $[0, t)$ and we define $T_n = t - (T_0 + T_1 + \cdots + T_{n-1})$. We then have

$$\begin{aligned}
\mathbb{P}\{W_t \le s \mid \widehat{Z} = \widehat{z}, V_n = l, N_t = n\} &= \mathbb{P}\{T_{i_1} + \cdots + T_{i_l} \le s \mid \widehat{Z} = \widehat{z}, V_n = l, N_t = n\} \\
&= \mathbb{P}\{T_{i_1} + \cdots + T_{i_l} \le s \mid N_t = n\},
\end{aligned}$$

where the distinct indices $i_1, \ldots, i_l \in \{0, 1, \ldots, n\}$ correspond to the l entries of $\widehat{z}$ that are in U and the last equality is due to the independence of the processes $\{Z_n\}$ and $\{N_t\}$. Note that for $l = 0$, we obtain the correct result, which is equal to 1 by using the convention $\sum_a^b (\ldots) = 0$ if $a > b$.

From relation (2) we get, for $l = 0, \ldots, n$,

$$\begin{aligned}
\mathbb{P}\{T_{i_1} + \cdots + T_{i_l} \le s \mid N_t = n\} &= \mathbb{P}\{T_0 + \cdots + T_{l-1} \le s \mid N_t = n\} \\
&= \sum_{k=l}^{n} \binom{n}{k} \left(\frac{s}{t}\right)^k \left(1 - \frac{s}{t}\right)^{n-k}.
\end{aligned}$$

Again, the use of the convention $\sum_a^b (\ldots) = 0$ if $a > b$ allows us to consider both cases $l = 0$ and $l = n + 1$ as normal ones. Finally, we obtain

$$\begin{aligned}
&\mathbb{P}_i\{W_t \le s \mid V_n = l, Z_n = j, N_t = n\} \\
&\qquad = \sum_{\widehat{z} \in G_{l,n}^{i,j}} \sum_{k=l}^{n} \binom{n}{k} \left(\frac{s}{t}\right)^k \left(1 - \frac{s}{t}\right)^{n-k} \mathbb{P}\{\widehat{Z} = \widehat{z} \mid V_n = l, Z_n = j, Z_0 = i\} \\
&\qquad = \sum_{k=l}^{n} \binom{n}{k} \left(\frac{s}{t}\right)^k \left(1 - \frac{s}{t}\right)^{n-k}
\end{aligned}$$

That is, since $\mathbb{P}\{N_t = n\} = e^{-\lambda t}\dfrac{(\lambda t)^n}{n!}$,

$$
\begin{aligned}
&\mathbb{P}\{W_t \le s, X_t = j \mid X_0 = i\} \\
&\quad = \sum_{n=0}^{\infty} e^{-\lambda t}\frac{(\lambda t)^n}{n!} \sum_{l=0}^{n}\sum_{k=l}^{n} \binom{n}{k} \left(\frac{s}{t}\right)^k \left(1-\frac{s}{t}\right)^{n-k} \mathbb{P}\{V_n = l, Z_n = j \mid Z_0 = i\} \\
&\quad = \sum_{n=0}^{\infty} e^{-\lambda t}\frac{(\lambda t)^n}{n!} \sum_{k=0}^{n} \binom{n}{k} \left(\frac{s}{t}\right)^k \left(1-\frac{s}{t}\right)^{n-k} \sum_{l=0}^{k} \mathbb{P}\{V_n = l, Z_n = j \mid Z_0 = i\} \\
&\quad = \sum_{n=0}^{\infty} e^{-\lambda t}\frac{(\lambda t)^n}{n!} \sum_{k=0}^{n} \binom{n}{k} \left(\frac{s}{t}\right)^k \left(1-\frac{s}{t}\right)^{n-k} \mathbb{P}\{V_n \le k, Z_n = j \mid Z_0 = i\}
\end{aligned}
$$

and the proof is completed. ■

Corollary 3.4 *For $t > 0$ and $s \in [0, t)$, we have*

$$
\mathbb{P}\{W_t \le s\} = \sum_{n=0}^{\infty} e^{-\lambda t}\frac{(\lambda t)^n}{n!} \sum_{k=0}^{n} \binom{n}{k} \left(\frac{s}{t}\right)^k \left(1-\frac{s}{t}\right)^{n-k} \mathbb{P}\{V_n \le k\} \tag{5}
$$

Proof. We have, for $t > 0$ and $s \in [0, t)$,

$$
\mathbb{P}\{W_t \le s\} = \sum_{i \in S} \mathbb{P}\{X_0 = i\} \sum_{j \in S} \mathbb{P}\{W_t \le s, X_t = j \mid X_0 = i\}
$$

and for $n \ge 0$ and $0 \le k \le n$,

$$
\mathbb{P}\{V_n \le k\} = \sum_{i \in S} \mathbb{P}\{Z_0 = i\} \sum_{j \in S} \mathbb{P}\{V_n \le k, Z_n = j \mid Z_0 = i\}.
$$

The result easily follows from theorem 3.3 and because $X_0 = Z_0$. ■

From relation (4), for $t > 0$, the distribution $\mathbb{P}\{W_t \le s, X_t = j \mid X_0 = i\}$ is differentiable with respect to s for $s \in (0, t)$ and its derivative is given by the following corollary.

Corollary 3.5 *For $t > 0$ and $s \in (0, t)$,*

$$\frac{d\mathbb{P}\{W_t \leq s, X_t = j \mid X_0 = i\}}{ds}$$

$$= \lambda \sum_{n=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} \sum_{k=0}^{n} \binom{n}{k} \left(\frac{s}{t}\right)^k \left(1 - \frac{s}{t}\right)^{n-k} \mathbb{P}\{V_{n+1} = k+1, Z_{n+1} = j \mid Z_0 = i\} \qquad (6)$$

Proof. In order to simplify writing let $G_{i,j}(t,s) = \mathbb{P}\{W_t \leq s, X_t = j \mid X_0 = i\}$ and $F_{i,j}(n,k) = \mathbb{P}\{V_n \leq k, Z_n = j \mid Z_0 = i\}$ and $f_{i,j}(n,k) = \mathbb{P}\{V_n = k, Z_n = j \mid Z_0 = i\}$. From relation (4), for $t > 0$ and $s \in (0,t)$ we have

$$\begin{aligned}
\frac{dG_{i,j}(t,s)}{ds} &= \frac{1}{t} \sum_{n=1}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} \sum_{k=1}^{n} \frac{n!}{(k-1)!(n-k)!} \left(\frac{s}{t}\right)^{k-1} \left(1 - \frac{s}{t}\right)^{n-k} F_{i,j}(n,k) \\
&\quad - \frac{1}{t} \sum_{n=1}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} \sum_{k=0}^{n-1} \frac{n!}{k!(n-k-1)!} \left(\frac{s}{t}\right)^{k} \left(1 - \frac{s}{t}\right)^{n-k-1} F_{i,j}(n,k) \\
&= \frac{1}{t} \sum_{n=1}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} \sum_{k=0}^{n-1} \frac{n!}{k!(n-k-1)!} \left(\frac{s}{t}\right)^{k} \left(1 - \frac{s}{t}\right)^{n-k-1} F_{i,j}(n,k+1) \\
&\quad - \frac{1}{t} \sum_{n=1}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} \sum_{k=0}^{n-1} \frac{n!}{k!(n-k-1)!} \left(\frac{s}{t}\right)^{k} \left(1 - \frac{s}{t}\right)^{n-k-1} F_{i,j}(n,k) \\
&= \frac{1}{t} \sum_{n=1}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} \sum_{k=0}^{n-1} \frac{n!}{k!(n-k-1)!} \left(\frac{s}{t}\right)^{k} \left(1 - \frac{s}{t}\right)^{n-k-1} f_{i,j}(n,k+1) \\
&= \lambda \sum_{n=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} \sum_{k=0}^{n} \binom{n}{k} \left(\frac{s}{t}\right)^{k} \left(1 - \frac{s}{t}\right)^{n-k} f_{i,j}(n+1,k+1),
\end{aligned}$$

which completes the proof. ■

4 Joint Distribution of Occupation Times

We consider now a partition of the state space S in $m+1$ non empty subsets B_0, B_1, ..., B_m. We thus have $B_i \cap B_j = \emptyset$ for $i \neq j$ and $S = B_0 \cup B_1 \cup \cdots \cup B_m$.

4.1 The Discrete Time Case

We consider the random variables V_n^i defined by

$$V_n^i = \sum_{k=0}^{n} 1_{\{Z_k \in B_i\}}.$$

The following theorem gives the backward equation for the joint distribution of the V_n^i and Z_n.

Theorem 4.1 *For $r = 1, \ldots, m$ and $n \geq 1$ and $1 \leq k_1, \ldots, k_m \leq n$ we have*

$$\text{for } i \in B_r,\ \mathbb{P}\{V_n^1 \leq k_1, \ldots, V_n^r \leq k_r, \ldots, V_n^m \leq k_m, Z_n = j \mid Z_0 = i\} =$$

$$\sum_{l \in E} P_{i,l} \mathbb{P}\{V_{n-1}^1 \leq k_1, \ldots, V_{n-1}^r \leq k_r - 1, \ldots, V_{n-1}^m \leq k_m, Z_{n-1} = j \mid Z_0 = l\}$$

$$\text{for } i \in B_0,\ \mathbb{P}\{V_n^1 \leq k_1, \ldots, V_n^m \leq k_m, Z_n = j \mid Z_0 = i\} =$$

$$\sum_{l \in E} P_{i,l} \mathbb{P}\{V_{n-1}^1 \leq k_1, \ldots, V_{n-1}^m \leq k_m, Z_{n-1} = j \mid Z_0 = l\}$$

Proof. We denote by $\widehat{V_n}$ and $\widehat{k}$ the vectors $(V_n^1, \ldots, V_n^m)$ and $(k_1, \ldots, k_m)$ respectively and by e_i, $i = 1, \ldots, m$, the row vector of dimension m whose ith entry is 1 and the others 0. The proof can be done using the same steps used in the proof of theorem 3.1.

We have

$$\begin{aligned} \mathbb{P}\{\widehat{V_n} \leq \widehat{k}, Z_n = j \mid Z_0 = i\} &= \sum_{l \in E} P_{i,l} \mathbb{P}\{\widehat{V_n} \leq \widehat{k}, Z_n = j \mid Z_1 = l, Z_0 = i\} \\ &= \sum_{l \in E} P_{i,l} \mathbb{P}\{\widehat{V_{n-1}} \leq \widehat{k} - e_r 1_{\{i \in B_r\}}, Z_{n-1} = j \mid Z_0 = l\} \end{aligned}$$

■

The following theorem gives the forward equation for the joint distribution of the V_n^i and Z_n.

Theorem 4.2 *For $r = 1, \ldots, m$ and $n \geq 1$ and $1 \leq k_1, \ldots, k_m \leq n$ we have* *for* $j \in B_r$, $\mathbb{P}\{V_n^1 \leq k_1, \ldots, V_n^r \leq k_r, \ldots, V_n^m \leq k_m, Z_n = j \mid Z_0 = i\} =$

$$\sum_{l \in E} \mathbb{P}\{V_{n-1}^1 \leq k_1, \ldots, V_{n-1}^r \leq k_r - 1, \ldots, V_{n-1}^m \leq k_m, Z_{n-1} = l \mid Z_0 = i\} P_{l,j}$$

for $j \in B_0$, $\mathbb{P}\{V_n^1 \leq k_1, \ldots, V_n^m \leq k_m, Z_n = j \mid Z_0 = i\} =$

$$\sum_{l \in E} \mathbb{P}\{V_{n-1}^1 \leq k_1, \ldots, V_{n-1}^m \leq k_m, Z_{n-1} = l \mid Z_0 = i\} P_{l,j}$$

Proof. With the notation of the proof of theorem 4.1, we follow the same steps used in the proof of theorem 3.2.

We have

$$\begin{aligned}
&\mathbb{P}\{\widehat{V_n} \leq \widehat{k}, Z_n = j, Z_0 = i\} \\
&\quad = \mathbb{P}\{\widehat{V_{n-1}} \leq \widehat{k} - e_r 1_{\{j \in B_r\}}, Z_n = j, Z_0 = i\} \\
&\quad = \sum_{l \in E} \mathbb{P}\{\widehat{V_{n-1}} \leq \widehat{k} - e_r 1_{\{j \in B_r\}}, Z_n = j, Z_{n-1} = l, Z_0 = i\} \\
&\quad = \sum_{l \in E} \mathbb{P}\{\widehat{V_{n-1}} \leq \widehat{k} - e_r 1_{\{j \in B_r\}}, Z_0 = i \mid Z_n = j, Z_{n-1} = l\} \mathbb{P}\{Z_n = j, Z_{n-1} = l\} \\
&\quad = \sum_{l \in E} \mathbb{P}\{\widehat{V_{n-1}} \leq \widehat{k} - e_r 1_{\{j \in B_r\}}, Z_0 = i \mid Z_{n-1} = l\} \mathbb{P}\{Z_n = j, Z_{n-1} = l\} \\
&\quad = \sum_{l \in E} \mathbb{P}\{\widehat{V_{n-1}} \leq \widehat{k} - e_r 1_{\{j \in B_r\}}, Z_{n-1} = l, Z_0 = i\} \mathbb{P}\{Z_n = j \mid Z_{n-1} = l\} \\
&\quad = \sum_{l \in E} \mathbb{P}\{\widehat{V_{n-1}} \leq \widehat{k} - e_r 1_{\{j \in B_r\}}, Z_{n-1} = l, Z_0 = i\} P_{l,j}
\end{aligned}$$

The desired relation is then obtained by conditioning with respect to Z_0. ■

If we denote, for $n \geq 0$ and $k \geq 0$, by $F(n, k_1, \ldots, k_m)$ the matrix whose entry (i, j) is defined by

$$F_{i,j}(n, k_1, \ldots, k_m) = \mathbb{P}\{V_n^1 \leq k_1, \ldots, V_n^m \leq k_m, Z_n = j \mid Z_0 = i\},$$

the results of theorems 4.1 and 4.2 can be easily expressed in matrix notation. We first decompose the matrices P and $F(n, k_1, \ldots, k_m)$ with respect to the partition $\{B_0, B_1, \ldots, B_m\}$ of the state space S as

$$P = (P_{B_r B_h})_{0 \leq r,h \leq m} \quad \text{and} \quad F(n, k_1, \ldots, k_m) = (F_{B_r B_h}(n, k_1, \ldots, k_m))_{0 \leq r,h \leq m}.$$

The result of theorem 4.1 can then be written as

$$F_{B_rB_h}(n,k_1,\dots,k_m)=\sum_{l=0}^{m}P_{B_rB_l}F_{B_lB_h}(n-1,k_1,\dots,k_r-1_{\{r\neq 0\}},\dots,k_m).$$

In the same way, the result of theorem 3.2 can be written as

$$F_{B_rB_h}(n,k_1,\dots,k_m)=\sum_{l=0}^{m}F_{B_rB_l}(n-1,k_1,\dots,k_h-1_{\{h\neq 0\}},\dots,k_m)P_{B_lB_h}.$$

The initial conditions are given, for $n \geq 0$, by

$$F(n,0,\dots,0)=\begin{pmatrix} 0 & 0 \\ 0 & (P_{B_0B_0})^n \end{pmatrix},$$

Note that in the case where $k_1+\cdots+k_m \geq n+1$, with $k_i \leq n$ for $i=1,\dots,m$, the m dimensional joint distribution of $V_n^1,\dots,V_n^m$ can be expressed as a combination of h dimensional joint distribution of the V_n^i for $h=1,\dots,m-1$. This observation is based on the following general result.

For any random variables $U_1,\dots,U_m$ and any event A, we have

$$\begin{aligned}\mathbb{P}\{U_1\leq x_1,\dots,U_m\leq x_m,A\} &= \sum_{E\subset\{1,\dots,m\}}(-1)^{m-|E|+1}\mathbb{P}\{U_i\leq x_i; i\in E, A\}\\ &\quad +(-1)^m\mathbb{P}\{U_1>x_1,\dots,U_m>x_m,A\}, \end{aligned}\tag{7}$$

where the inclusion is strict, that is $E \neq \{1,\dots,m\}$, and where we define for convenience $\mathbb{P}\{U_i\leq x_i; i\in\emptyset, A\}=\mathbb{P}\{A\}$.

For what concerns our random variables $V_n^1,\dots,V_n^m$, we have trivially

$$\mathbb{P}\{V_n^1>k_1,\dots,V_n^m>k_m,X_n=j \mid Z_0=i\}=0 \qquad \text{if} \qquad k_1+\cdots+k_m\geq n+1,$$

so we get in this case the desired result, that is

$$\mathbb{P}_i\{V_n^1\leq k_1,\dots,V_n^m\leq k_m,Z_n=j\}=\sum_{E\subset\{1,\dots,m\}}(-1)^{m-|E|+1}\mathbb{P}_i\{V_n^i\leq k_i; i\in E, Z_n=j\}\tag{8}$$

4.2 The Continuous Time Case

We consider the random variables W_t^i, $i = 1, \ldots, m$, defined by

$$W_t^i = \int_0^t 1_{\{X_u \in B_i\}} du.$$

The joint distribution of the W_t^i and X_t is given in the next theorem.

Theorem 4.3 *For every $i, j \in S$, for every $t > 0$ and $s_1, \ldots, s_m \in [0, t)$ such that $s_1 + s_2 + \cdots + s_m < t$, we have*
$\mathbb{P}\{W_t^1 \leq s_1, \ldots, W_t^m \leq s_m, X_t = j \mid X_0 = i\} =$

$$\sum_{n=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} \sum_{\substack{k_1 \geq 0, k_2 \geq 0, \ldots, k_m \geq 0, \\ k_1 + k_2 + \cdots + k_m \leq n}} n! \theta_{n;k_1,k_2,\ldots,k_m}^{t;s_1,s_2,\ldots,s_m} \mathbb{P}\{V_n^1 \leq k_1, \ldots, V_n^m \leq k_m, Z_n = j \mid Z_0 = i\} \tag{9}$$

where

$$\theta_{n;k_1,k_2,\ldots,k_m}^{t;s_1,s_2,\ldots,s_m} = \frac{\left(\frac{s_1}{t}\right)^{k_1} \left(\frac{s_2}{t}\right)^{k_2} \cdots \left(\frac{s_k}{t}\right)^{k_m} \left(1 - \frac{s_1 + s_2 + \cdots + s_m}{t}\right)^{n-(k_1+k_2\cdots+k_m)}}{k_1!k_2!\cdots k_m!(n-(k_1+k_2\cdots+k_m))!}.$$

Proof. We denote by $\widehat{W_t}$, $\widehat{V_n}$, $\widehat{s}$ and $\widehat{l}$ the vectors $(W_t^1, \ldots, W_t^m)$, $(V_n^1, \ldots, V_n^m)$, $(s_1, \ldots, s_m)$ and $(l_1, \ldots, l_m)$ respectively. An inequality between two such vectors means the inequality for all their entries. For $n \geq 0$ we define the set E_n as

$$E_n = \{\widehat{l} = (l_1, l_2, \ldots, l_m) \in \mathbb{N}^n \mid l_1 + l_2 + \cdots + l_m \leq n\}.$$

We then have

$$\begin{aligned} & \mathbb{P}\{\widehat{W_t} \leq \widehat{s}, X_t = j \mid X_0 = i\} \\ = & \sum_{n=0}^{\infty} \mathbb{P}_i\{\widehat{W_t} \leq \widehat{s}, N_t = n, X_t = j\} \end{aligned}$$

$$
\begin{aligned}
&= \sum_{n=0}^{\infty} \mathbb{P}_i\{\widehat{W_t} \le \widehat{s}, N_t = n, Z_n = j\} \qquad \text{since } \{X_t\} \text{ and } \{Z_{N_t}\} \text{ are equivalent} \\
&= \sum_{n=0}^{\infty} \mathbb{P}_i\{N_t = n\}\mathbb{P}_i\{\widehat{W_t} \le \widehat{s}, Z_n = j \mid N_t = n\} \\
&= \sum_{n=0}^{\infty} \mathbb{P}\{N_t = n\}\mathbb{P}_i\{\widehat{W_t} \le \widehat{s}, Z_n = j \mid N_t = n\} \\
&= \sum_{n=0}^{\infty} \mathbb{P}\{N_t = n\} \sum_{\widehat{l}\in E_n} \mathbb{P}_i\{\widehat{W_t} \le \widehat{s}, \widehat{V_n} = \widehat{l}, Z_n = j \mid N_t = n\} \\
&= \sum_{n=0}^{\infty} \mathbb{P}\{N_t = n\} \sum_{\widehat{l}\in E_n} \mathbb{P}_i\{\widehat{V_n} = \widehat{l}, Z_n = j \mid N_t = n\}\mathbb{P}_i\{\widehat{W_t} \le \widehat{s} \mid \widehat{V_n} = \widehat{l}, Z_n = j, N_t = n\} \\
&= \sum_{n=0}^{\infty} \mathbb{P}\{N_t = n\} \sum_{\widehat{l}\in E_n} \mathbb{P}_i\{\widehat{V_n} = \widehat{l}, Z_n = j\}\mathbb{P}_i\{\widehat{W_t} \le \widehat{s} \mid \widehat{V_n} = \widehat{l}, Z_n = j, N_t = n\}
\end{aligned}
$$

Note that the 4th and the last equalities follow from the independence of the processes $\{Z_n\}$ and $\{N_t\}$ and the fact that $X_0 = Z_0$. Note also that in the 5th equality, the summation over $\widehat{l}$ should be for $\widehat{l} \in E_{n+1}$, but it can be done for $\widehat{l} \in E_n$ because if $l_1+l_2+\cdots+l_m = n+1$, we obtain that $\widehat{V_n} = \widehat{l}$ and $N_t = n$ imply that $V_n^1 + \cdots + V_n^m = n+1$ and so that $W_t^1+\cdots+W_t^m = t$, which gives us $\mathbb{P}\{\widehat{W_t} \le \widehat{s} \mid \widehat{V_n} = \widehat{l}, N_t = n\} = 0$ since we have supposed that $s_1 + \cdots + s_m < t$.

Let us consider now the expression $\mathbb{P}_i\{\widehat{W_t} \le \widehat{s} \mid \widehat{V_n} = \widehat{l}, Z_n = j, N_t = n\}$.

For $\widehat{l} = (l_1, l_2, \ldots, l_m) \in E_n$ and $i, j \in S$, we define the set

$$
G_{\widehat{l},n}^{i,j} = \left\{ \widehat{z} = (i, z_1, \ldots, z_{n-1}, j) \in S^{n+1} \;\middle|\; \begin{array}{l} l_1 \text{ entries of } \widehat{z} \text{ are in } B_1, \\ l_2 \text{ entries of } \widehat{z} \text{ are in } B_2, \\ \ldots, \\ l_m \text{ entries of } \widehat{z} \text{ are in } B_m \text{ and} \\ n+1-(l_1+l_2+\cdots+l_m) \text{ are in } B_0 \end{array} \right\}
$$

and we denote by $\widehat{Z}$ the random vector $(Z_0, \ldots, Z_n)$. We then have

$$
\begin{aligned}
&\mathbb{P}_i\{\widehat{W_t} \le \widehat{s} \mid \widehat{V_n} = \widehat{l}, Z_n = j, N_t = n\} \\
&= \sum_{\widehat{z}\in G_{\widehat{l},n}^{i,j}} \mathbb{P}_i\{\widehat{W_t} \le \widehat{s} \mid \widehat{Z} = \widehat{z}, \widehat{V_n} = \widehat{l}, Z_n = j, N_t = n\}\mathbb{P}_i\{\widehat{Z} = \widehat{z} \mid \widehat{V_n} = \widehat{l}, Z_n = j, N_t = n\}
\end{aligned}
$$

$$= \sum_{\widehat{z} \in G^{i,j}_{\widehat{l},n}} \mathbb{P}\{\widehat{W_t} \leq \widehat{s} \mid \widehat{Z} = \widehat{z}, \widehat{V_n} = \widehat{l}, N_t = n\} \mathbb{P}\{\widehat{Z} = \widehat{z} \mid \widehat{V_n} = \widehat{l}, Z_n = j, Z_0 = i, N_t = n\}$$

$$= \sum_{\widehat{z} \in G^{i,j}_{\widehat{l},n}} \mathbb{P}\{\widehat{W_t} \leq \widehat{s} \mid \widehat{Z} = \widehat{z}, \widehat{V_n} = \widehat{l}, N_t = n\} \mathbb{P}\{\widehat{Z} = \widehat{z} \mid \widehat{V_n} = \widehat{l}, Z_n = j, Z_0 = i\}$$

where the last equality follows from the independence of the processes $\{Z_n\}$ and $\{N_t\}$.

We have

$$\mathbb{P}\{\widehat{W_t} \leq \widehat{s} \mid \widehat{Z} = \widehat{z}, \widehat{V_n} = \widehat{l}, N_t = n\} = \mathbb{P}\{\widehat{T}(\widehat{l}) \leq \widehat{s} \mid \widehat{Z} = \widehat{z}, \widehat{V_n} = \widehat{l}, N_t = n\},$$

where

$$\widehat{T}(\widehat{l}) = \left(\sum_{j=1}^{l_1} T_{i_j}, \sum_{j=l_1+1}^{l_1+l_2} T_{i_j}, \ldots, \sum_{j=l_1+l_2+\cdots+l_{m-1}+1}^{l_1+l_2+\cdots+l_m} T_{i_j} \right)$$

So, using again the independence between $\{Z_n\}$ and $\{N_t\}$ and relation (3), we get

$$\begin{aligned} \mathbb{P}\{\widehat{T}(\widehat{l}) \leq \widehat{s} \mid \widehat{Z} = \widehat{z}, \widehat{V_n} = \widehat{l}, N_t = n\} &= \mathbb{P}\{\widehat{T}(\widehat{l}) \leq \widehat{s} \mid N_t = n\} \\ &= \sum_{\substack{k_1 \geq l_1, k_2 \geq l_2, \ldots, k_m \geq l_m, \\ k_1 + k_2 + \cdots + k_m \leq n}} n! \theta^{t; s_1, s_2, \ldots, s_m}_{n; k_1, k_2, \ldots, k_m}, \end{aligned}$$

Note that if one of the l_i's is equal to 0, the corresponding entry of the vector $\widehat{T}(\widehat{l})$ becomes 0 and the preceding formula is still valid. Indeed, suppose for simplicity that $l_m = 0$. We obtain

$$\sum_{\substack{k_1 \geq l_1, \ldots, k_m \geq 0, \\ k_1 + \cdots + k_m \leq n}} n! \theta^{t; s_1, s_2, \ldots, s_m}_{n; k_1, k_2, \ldots, k_m} = \sum_{\substack{k_1 \geq l_1, \ldots, k_{m-1} \geq l_{m-1}, \\ k_1 + \cdots + k_{m-1} \leq n}} \sum_{k_m=0}^{n-(k_1+\cdots+k_{m-1})} n! \theta^{t; s_1, \ldots, s_m}_{n; k_1, \ldots, k_m}$$

$$= \sum_{\substack{k_1 \geq l_1, \ldots, k_{m-1} \geq l_{m-1}, \\ k_1 + \cdots + k_{m-1} \leq n}} n! \theta^{t; s_1, \ldots, s_{m-1}}_{n; k_1, \ldots, k_{m-1}} \sum_{k_m=0}^{n-(k_1+\cdots+k_{m-1})} \frac{(n - (k_1 + \cdots + k_{m-1}))! \theta^{t; s_m}_{n-(k_1+\cdots+k_{m-1}); k_m}}{\left(1 - \frac{s_1 + \cdots + s_{m-1}}{t}\right)^{n-(k_1+\cdots+k_{m-1})}}$$

$$= \sum_{\substack{k_1 \geq l_1, \ldots, k_{m-1} \geq l_{m-1}, \\ k_1 + \cdots + k_{m-1} \leq n}} n! \theta^{t; s_1, \ldots, s_{m-1}}_{n; k_1, \ldots, k_{m-1}}.$$

Note also that if all the l_i's are equal to 0, then all the entries of $\widehat{T}(\widehat{l})$ are equal to 0 and the formula is still valid since

$$\sum_{\substack{k_1 \geq 0, k_2 \geq 0, \ldots, k_m \geq 0, \\ k_1 + k_2 + \cdots + k_m \leq n}} n!\theta^{t;s_1,s_2,\ldots,s_m}_{n;k_1,k_2,\ldots,k_m} = 1.$$

Putting together these results, we obtain

$$\begin{aligned}
&\mathbb{P}_i\{\widehat{W_t} \leq \widehat{s} \mid \widehat{V_n} = \widehat{l}, Z_n = j, N_t = n\} \\
&\quad = \sum_{\widehat{z} \in G^{i,j}_{\widehat{l},n}} \sum_{\substack{k_1 \geq l_1, k_2 \geq l_2, \ldots, k_m \geq l_m, \\ k_1 + k_2 + \cdots + k_m \leq n}} n!\theta^{t;s_1,s_2,\ldots,s_m}_{n;k_1,k_2,\ldots,k_m} \mathbb{P}\{\widehat{Z} = \widehat{z} \mid \widehat{V_n} = \widehat{l}, Z_n = j, Z_0 = i\} \\
&\quad = \sum_{\substack{k_1 \geq l_1, k_2 \geq l_2, \ldots, k_m \geq l_m, \\ k_1 + k_2 + \cdots + k_m \leq n}} n!\theta^{t;s_1,s_2,\ldots,s_m}_{n;k_1,k_2,\ldots,k_m} \sum_{\widehat{z} \in G^{i,j}_{\widehat{l},n}} \mathbb{P}\{\widehat{Z} = \widehat{z} \mid \widehat{V_n} = \widehat{l}, Z_n = j, Z_0 = i\} \\
&\quad = \sum_{\substack{k_1 \geq l_1, k_2 \geq l_2, \ldots, k_m \geq l_m, \\ k_1 + k_2 + \cdots + k_m \leq n}} n!\theta^{t;s_1,s_2,\ldots,s_m}_{n;k_1,k_2,\ldots,k_m},
\end{aligned}$$

and so,

$$\begin{aligned}
&\mathbb{P}\{\widehat{W_t} \leq \widehat{s}, X_t = j \mid X_0 = i\} \\
&= \sum_{n=0}^{\infty} e^{-\lambda t}\frac{(\lambda t)^n}{n!} \sum_{\widehat{l} \in E_n} \sum_{\substack{k_1 \geq l_1, k_2 \geq l_2, \ldots, k_m \geq l_m, \\ k_1 + k_2 + \cdots + k_m \leq n}} n!\theta^{t;s_1,s_2,\ldots,s_m}_{n;k_1,k_2,\ldots,k_m} \mathbb{P}\{\widehat{V_n} = \widehat{l}, Z_n = j \mid Z_0 = i\} \\
&= \sum_{n=0}^{\infty} e^{-\lambda t}\frac{(\lambda t)^n}{n!} \sum_{\widehat{k} \in E_n} n!\theta^{t;s_1,s_2,\ldots,s_m}_{n;k_1,k_2,\ldots,k_m} \sum_{\substack{l_1 \geq k_1, l_2 \geq k_2, \ldots, l_m \geq k_m, \\ l_1 + l_2 + \cdots + l_m \leq n}} \mathbb{P}\{\widehat{V_n} = \widehat{l}, Z_n = j \mid Z_0 = i\} \\
&= \sum_{n=0}^{\infty} e^{-\lambda t}\frac{(\lambda t)^n}{n!} \sum_{\widehat{k} \in E_n} n!\theta^{t;s_1,s_2,\ldots,s_m}_{n;k_1,k_2,\ldots,k_m} \mathbb{P}\{\widehat{V_n} \leq \widehat{l}\} \\
&= \sum_{n=0}^{\infty} e^{-\lambda t}\frac{(\lambda t)^n}{n!} \sum_{\substack{k_1 \geq 0, k_2 \geq 0, \ldots, k_m \geq 0, \\ k_1 + k_2 + \cdots + k_m \leq n}} n!\theta^{t;s_1,s_2,\ldots,s_m}_{n;k_1,k_2,\ldots,k_m} \mathbb{P}\{\widehat{V_n} \leq \widehat{l}, Z_n = j \mid Z_0 = i\}.
\end{aligned}$$

Thus, the proof is completed. ■

From relation (9), for $t > 0$, the distribution $\mathbb{P}\{W_t^1 \le s_1, \ldots, W_t^m \le s_m, X_t = j \mid X_0 = i\}$ is differentiable with respect to t and also with respect to $s_1, s_2, \ldots, s_m$ for $s_1, \ldots, s_m \in (0,t)$ and $s_1 + \cdots + s_m \in (0,t)$. These derivatives are given by the following corollary.

Corollary 4.4 *For $i, j \in S$, for $t > 0$ and $s_1, \ldots, s_m \in (0,t)$ such that $s_1 + \cdots + s_m < t$, we have*

$$\frac{\partial \mathbb{P}\{W_t^1 \le s_1, \ldots, W_t^m \le s_m, X_t = j \mid X_0 = i\}}{\partial s_1} =$$

$$\lambda \sum_{n=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} \sum_{\substack{k_1 \ge 0, \ldots, k_m \ge 0, \\ k_1 + \cdots + k_m \le n}} n! \theta_{n;k_1,k_2,\ldots,k_m}^{t;s_1,s_2,\ldots,s_m} \mathbb{P}_i\{V_{n+1}^1 = k_1 + 1, V_{n+1}^2 \le k_2, \ldots, V_{n+1}^m \le k_m, Z_{n+1} = j\} \tag{10}$$

and

$$\frac{\partial^m \mathbb{P}\{W_t^1 \le s_1, \ldots, W_t^m \le s_m, X_t = j \mid X_0 = i\}}{\partial s_1 \partial s_2 \cdots \partial s_m} =$$

$$\lambda^m \sum_{n=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} \sum_{\substack{k_1 \ge 0, \ldots, k_m \ge 0, \\ k_1 + \cdots + k_m \le n}} n! \theta_{n;k_1,k_2,\ldots,k_m}^{t;s_1,s_2,\ldots,s_m} \mathbb{P}_i\{V_{n+m}^1 = k_1 + 1, \ldots, V_{n+m}^m = k_m + 1, Z_{n+m} = j\} \tag{11}$$

and

$$\frac{\partial \mathbb{P}\{W_t^1 \le s_1, \ldots, W_t^m \le s_m, X_t = j \mid X_0 = i\}}{\partial t} =$$

$$\lambda \sum_{n=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} \sum_{\substack{k_1 \ge 0, \ldots, k_m \ge 0, \\ k_1 + \cdots + k_m \le n}} n! \theta_{n;k_1,k_2,\ldots,k_m}^{t;s_1,s_2,\ldots,s_m} \Big[\mathbb{P}_i\{V_{n+1}^1 \le k_1, \ldots, V_{n+1}^m \le k_m, Z_{n+1} = j\} - \mathbb{P}_i\{V_n^1 \le k_1, \ldots, V_n^m \le k_m, Z_n = j\}\Big] \tag{12}$$

Proof. In order to simplify notation, we define

$$F_{i,j}(n, k_1, k_2, \ldots, k_m) = \mathbb{P}\{V_n^1 \le k_1, V_n^2 \le k_2, \ldots, V_n^m \le k_m, Z_n = j \mid Z_0 = i\},$$

and

$$f_{i,j}(n,k_1,k_2,\ldots,k_m) = \mathbb{P}\{V_n^1 = k_1, V_n^2 \leq k_2, \ldots, V_n^m \leq k_m, Z_n = j \mid Z_0 = i\}.$$

We thus have the relation

$$F_{i,j}(n,k_1+1,k_2,\ldots,k_m) - F_{i,j}(n,k_1,k_2,\ldots,k_m) = f_{i,j}(n,k_1+1,k_2,\ldots,k_m).$$

For $t > 0$ and $s_1,\ldots,s_m \in (0,t)$ such that $s_1 + \cdots + s_m < t$, we have from relation (9)

$$\begin{aligned}
&\frac{\partial \mathbb{P}\{W_t^1 \leq s_1,\ldots,W_t^m \leq s_m, X_t = j \mid X_0 = i\}}{\partial s_1} \\
&= \frac{1}{t}\sum_{n=1}^{\infty} e^{-\lambda t}\frac{(\lambda t)^n}{n!} \sum_{\substack{k_1 \geq 1, k_2 \geq 0,\ldots,k_m \geq 0, \\ k_1+\cdots+k_m \leq n}} n!\theta^{t;s_1,s_2,\ldots,s_m}_{n;k_1-1,k_2,\ldots,k_m} F_{i,j}(n,k_1,k_2,\ldots,k_m) \\
&\quad -\frac{1}{t}\sum_{n=1}^{\infty} e^{-\lambda t}\frac{(\lambda t)^n}{n!} \sum_{\substack{k_1 \geq 0, k_2 \geq 0,\ldots,k_m \geq 0, \\ k_1+\cdots+k_m \leq n-1}} n!\theta^{t;s_1,\ldots,s_m}_{n-1;k_1,\ldots,k_m} F_{i,j}(n,k_1,k_2,\ldots,k_m) \\
&= \frac{1}{t}\sum_{n=1}^{\infty} e^{-\lambda t}\frac{(\lambda t)^n}{n!} \sum_{\substack{k_1 \geq 0,\ldots,k_m \geq 0, \\ k_1+\cdots+k_m \leq n-1}} n!\theta^{t;s_1,\ldots,s_m}_{n-1;k_1,\ldots,k_m} F_{i,j}(n,k_1+1,k_2,\ldots,k_m) \\
&\quad -\frac{1}{t}\sum_{n=1}^{\infty} e^{-\lambda t}\frac{(\lambda t)^n}{n!} \sum_{\substack{k_1 \geq 0,\ldots,k_m \geq 0, \\ k_1+\cdots+k_m \leq n-1}} n!\theta^{t;s_1,\ldots,s_m}_{n-1;k_1,\ldots,k_m} F_{i,j}(n,k_1,k_2,\ldots,k_m) \\
&= \frac{1}{t}\sum_{n=1}^{\infty} e^{-\lambda t}\frac{(\lambda t)^n}{n!} \sum_{\substack{k_1 \geq 0,\ldots,k_m \geq 0, \\ k_1+\cdots+k_m \leq n-1}} n!\theta^{t;s_1,\ldots,s_m}_{n-1;k_1,\ldots,k_m} f_{i,j}(n,k_1+1,k_2,\ldots,k_m) \\
&= \lambda\sum_{n=0}^{\infty} e^{-\lambda t}\frac{(\lambda t)^n}{n!} \sum_{\substack{k_1 \geq 0,\ldots,k_m \geq 0, \\ k_1+\cdots+k_m \leq n}} n!\theta^{t;s_1,\ldots,s_m}_{n;k_1,\ldots,k_m} f_{i,j}(n+1,k_1+1,k_2,\ldots,k_m).
\end{aligned}$$

The second relation follows easily using the same argument. For the third relation, since $t > 0$, we write

$$\mathbb{P}\{W_t^1 \leq s_1, \ldots, W_t^m \leq s_m, X_t = j \mid X_0 = i\}$$

$$= e^{-\lambda t} \sum_{n=0}^{\infty} \lambda^n \sum_{\substack{k_1 \geq 0, \ldots, k_m \geq 0, \\ k_1 + \cdots + k_m \leq n}} t^n \theta_{n;k_1,\ldots,k_m}^{t;s_1,\ldots,s_m} F_{i,j}(n, k_1, k_2, \ldots, k_m)$$

and so,

$$\frac{\partial \mathbb{P}\{W_t^1 \leq s_1, \ldots, W_t^m \leq s_m, X_t = j \mid X_0 = i\}}{\partial t}$$

$$\begin{aligned}
= \; & -\lambda e^{-\lambda t} \sum_{n=0}^{\infty} \lambda^n \sum_{\substack{k_1 \geq 0, \ldots, k_m \geq 0, \\ k_1 + \cdots + k_m \leq n}} t^n \theta_{n;k_1,\ldots,k_m}^{t;s_1,\ldots,s_m} F_{i,j}(n, k_1, k_2, \ldots, k_m) \\
& + e^{-\lambda t} \sum_{n=1}^{\infty} \lambda^n \sum_{\substack{k_1 \geq 0, \ldots, k_m \geq 0, \\ k_1 + \cdots + k_m \leq n-1}} t^{n-1} \theta_{n-1;k_1,\ldots,k_m}^{t;s_1,\ldots,s_m} F_{i,j}(n, k_1, k_2, \ldots, k_m) \\
= \; & -\lambda e^{-\lambda t} \sum_{n=0}^{\infty} \lambda^n \sum_{\substack{k_1 \geq 0, \ldots, k_m \geq 0, \\ k_1 + \cdots + k_m \leq n}} t^n \theta_{n;k_1,\ldots,k_m}^{t;s_1,\ldots,s_m} F_{i,j}(n, k_1, k_2, \ldots, k_m) \\
& + e^{-\lambda t} \sum_{n=0}^{\infty} \lambda^{n+1} \sum_{\substack{k_1 \geq 0, \ldots, k_m \geq 0, \\ k_1 + \cdots + k_m \leq n}} t^n \theta_{n;k_1,\ldots,k_m}^{t;s_1,\ldots,s_m} F_{i,j}(n+1, k_1, k_2, \ldots, k_m) \\
= \; & \lambda \sum_{n=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} \sum_{\substack{k_1 \geq 0, \ldots, k_m \geq 0, \\ k_1 + \cdots + k_m \leq n}} n! \theta_{n;k_1,\ldots,k_m}^{t;s_1,\ldots,s_m} \left[F_{i,j}(n+1, k_1, k_2, \ldots, k_m) - F_{i,j}(n, k_1, k_2, \ldots, k_m)\right]
\end{aligned}$$

■

Note that if $s_1 + s_2 + \cdots + s_m \geq t$, then we trivially have

$$\mathbb{P}\{W_t^1 > s_1, \ldots, W_t^m > s_m, X_t = j \mid X_0 = i\} = 0,$$

so relation (8) applies by replacing the V_n^l and the k_l by the W_t^l and the s_l respectively.

5 Distribution of Weighted Sums of Occupation Times

A constant performance level or reward rate $\rho(i)$ is associated with each state i of S and we consider the random variable Y_t defined by

$$Y_t = \int_0^t \rho(X_u)du.$$

We denote by $m+1$ the number of distinct rewards and we denote their values by

$$r_0 < r_1 < \cdots < r_{m-1} < r_m.$$

We then have $Y_t \in [r_0 t, r_m t]$ with probability one. Without loss of generality, we suppose that $r_0 = 0$. This can be easily done by considering the random variable $Y_t - r_0 t$ instead of Y_t and the reward rates $r_i - r_0$ instead of r_i.

The state space S can be then divided into disjoint subsets $B_m, B_{m-1}, \ldots, B_0$ where B_l is composed by the states of S having as reward rate r_l, that is $B_l = \{i \in S / \rho(i) = r_l\}$.

With this notation, we have

$$Y_t = \sum_{l=1}^{m} r_l \int_0^t 1_{\{X_u \in B_l\}} du = \sum_{l=1}^{m} r_l W_t^l. \tag{13}$$

Since the distribution of each W_t^l has at most two jumps at points 0 and t, we conclude that the distribution of Y_t has at most $m+1$ jumps at points $r_0 t = 0$, $r_1 t$, $\ldots$, $r_m t$. For $t > 0$, the jump at point $x = r_l t$ is equal to the probability that the process X, starting in subset B_l, stays during the whole interval $[0, t)$ in the subset B_l, that is

$$\mathbb{P}\{Y_t = r_l t\} = \alpha_{B_l} e^{A_{B_l B_l} t} \mathbb{1}_{B_l} \text{ for } t > 0,$$

where $\mathbb{1}_{B_l}$ is the column vector of dimension $|B_l|$ with all entries equal to 1. For every $i, j \in S$ and $t > 0$, we define the functions $F_{i,j}(t,x)$ by

$$F_{i,j}(t,x) = \mathbb{P}\{Y_t > x, X_t = j \mid X_0 = i\},$$

and we denote by $F(t,x)$ the matrix containing the terms $F_{i,j}(t,x)$ for $i, j \in S$. The matrices A, P and $F(t,x)$ can then be written using the partition $B_m, B_{m-1}, \ldots, B_0$ of S as

$$A = (A_{B_u B_v})_{0 \le u,v \le m} \;;\; P = (P_{B_u B_v})_{0 \le u,v \le m} \;;\; F(t,x) = (F_{B_u B_v}(t,x))_{0 \le u,v \le m}.$$

Note that we then have, for $t > 0$ and $l = 0, 1, \ldots, m$,

$$\mathbb{P}\{Y_t = r_l t, X_t = j \mid X_0 = i\} = \begin{cases} (e^{A_{B_l B_l} t})_{i,j} & \text{if } i, j \in B_l \\ 0 & \text{otherwise,} \end{cases},$$

that is

$$\mathbb{P}\{Y_t = r_l t, X_t = j \mid X_0 = i\} = \sum_{n=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} (P^n_{B_l B_l})_{i,j} 1_{\{i,j \in B_l\}}. \tag{14}$$

The distribution $F_{i,j}(t, x)$ can be obtained from relation (13), using the joint distribution of the W_t^l obtained in the previous section. From corollary 4.4, $F_{i,j}(t, x)$ is differentiable with respect to x and t in the domain

$$E = \{(t, x) \; ; \; t > 0 \text{ and } x \in \bigcup_{l=1}^{m} (r_{l-1} t, r_l t)\}.$$

The initial conditions are given, for $t > 0$, by

$$F_{i,j}(t, 0) = \mathbb{P}\{X_t = j \mid X_0 = i\} - \mathbb{P}\{Y_t = 0, X_t = j \mid X_0 = i\},$$

that is, in matrix notation,

$$F_{B_u B_v}(t, 0) = (e^{At})_{B_u B_v} - e^{A_{B_0 B_0} t} 1_{\{u=v=0\}},$$

which can also be written as

$$F_{B_u B_v}(t, 0) = \sum_{n=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} \left[(P^n)_{B_u B_v} - P^n_{B_0 B_0} 1_{\{u=v=0\}} \right]. \tag{15}$$

5.1 Backward and Forward Equations

In the following, we derive the backward and forward equations satisfied by the distribution of the couple (Y_t, X_t) and we give expressions of the solution of these equations.

Let us first state some useful results in the following lemma. Recall that $\{N(t)\}$ is a Poisson process with rate λ and that it is independent of the uniformized Markov chain Z. We denote by $N(t, t+s)$ the number of transitions during the interval $[t, t+s)$.

Lemma 5.1

$$\mathbb{P}\{N(t, t+s) = 0 \mid X_t = j\} = e^{-\lambda s} \tag{16}$$

$$\mathbb{P}\{X_{t+s} = j, N(t, t+s) = 1 \mid X_t = i\} = P_{i,j} \lambda s e^{-\lambda s} \tag{17}$$

$$\mathbb{P}\{N(t, t+s) \geq 2 \mid X_0 = i\} = o(s). \tag{18}$$

Proof. Recall that the processes $\{X_t\}$ and $\{Z_{N(t)}\}$ are equivalent and $Z_0 = X_0$. For relation (16), by homogeneity, we have

$$\begin{aligned}
\mathbb{P}\{N(t,t+s)=0 \mid X_t=j\} &= \mathbb{P}\{N(s)=0 \mid X_0=j\} \\
&= \mathbb{P}\{N(s)=0 \mid Z_0=j\} \\
&= \mathbb{P}\{N(s)=0\} \\
&= e^{-\lambda s}
\end{aligned}$$

In the same way, for relation (17), we have

$$\begin{aligned}
\mathbb{P}\{X_{t+s}=j, N(t,t+s)=1 \mid X_t=i\} &= \mathbb{P}\{X_s=j, N(s)=1 \mid X_0=i\} \\
&= \mathbb{P}\{Z_1=j, N(s)=1 \mid Z_0=i\} \\
&= \mathbb{P}\{N(s)=1 \mid Z_0=i, Z_1=j\}P_{i,j} \\
&= \mathbb{P}\{N(s)=1\}P_{i,j} \\
&= P_{i,j}\lambda s e^{-\lambda s}
\end{aligned}$$

For relation (18), we have

$$\begin{aligned}
\mathbb{P}\{N(t,t+s)\geq 2 \mid X_t=i\} &= \mathbb{P}\{N(s)\geq 2 \mid X_0=i\} \\
&= \mathbb{P}\{N(s)\geq 2\} \\
&= 1-e^{-\lambda s}-\lambda s e^{-\lambda s} \\
&= o(s)
\end{aligned}$$

This completes the proof. ■

The following theorem states the forward equation describing the behavior of the pair (Y_t, X_t).

Theorem 5.2 *For $t > 0$, $i, j \in S$, $1 \leq h \leq m$ and $x \in (r_{h-1}t, r_h t)$ we have*

$$\frac{\partial F_{i,j}(t,x)}{\partial t} = -\rho(j)\frac{\partial F_{i,j}(t,x)}{\partial x} + \sum_{k\in S} F_{i,k}(t,x)A_{k,j}. \tag{19}$$

Proof. By conditioning on the number of transitions in the interval $[t, t+s)$, we have

$$\begin{aligned}
\mathbb{P}_i\{Y_{t+s}>x, X_{t+s}=j\} &= \mathbb{P}_i\{Y_{t+s}>x, X_{t+s}=j, N(t,t+s)=0\} \\
&\quad + \mathbb{P}_i\{Y_{t+s}>x, X_{t+s}=j, N(t,t+s)=1\} \\
&\quad + \mathbb{P}_i\{Y_{t+s}>x, X_{t+s}=j, N(t,t+s)\geq 2\}
\end{aligned}$$

We consider the three terms of the right hand side separately.

For the first term, since $X_{t+s} = j$ and $N(t, t+s) = 0$ is equivalent to $X_t = j$ and $N(t, t+s) = 0$, we have

$$\mathbb{P}_i\{Y_{t+s} > x, X_{t+s} = j, N(t, t+s) = 0\} = \mathbb{P}_i\{Y_{t+s} > x, X_t = j, N(t, t+s) = 0\}$$

$$\begin{aligned}
&= \mathbb{P}_i\{Y_{t+s} > x \mid X_t = j, N(t, t+s) = 0\}\mathbb{P}_i\{X_t = j, N(t, t+s) = 0\} \\
&= \mathbb{P}_i\{Y_t > x - \rho(j)s \mid X_t = j, N(t, t+s) = 0\}\mathbb{P}_i\{X_t = j, N(t, t+s) = 0\} \\
&= \mathbb{P}_i\{Y_t > x - \rho(j)s \mid X_t = j\}\mathbb{P}_i\{X_t = j, N(t, t+s) = 0\} \\
&= \mathbb{P}_i\{Y_t > x - \rho(j)s, X_t = j\}\mathbb{P}_i\{N(t, t+s) = 0 \mid X_t = j\} \\
&= \mathbb{P}\{N(t, t+s) = 0 \mid X_t = j\}F_{i,j}(t, x - \rho(j)s) \\
&= e^{-\lambda s}F_{i,j}(t, x - \rho(j)s) \\
&= (1 - \lambda s)F_{i,j}(t, x - \rho(j)s) + o(s)
\end{aligned}$$

where the second equality follows from the fact that if $X_t = j$ and $N(t, t+s) = 0$ then we have $Y_{t+s} = Y_t + \int_t^{t+s} \rho(X_u)du = Y_t + \rho(j)s$, the third one and the fifth one follow from the Markov property, and the sixth follows from relation (16).

For the second term that we denote by $G(s)$, we define

$$G_k(s) = \mathbb{P}_i\{Y_{t+s} > x \mid X_t = k, X_{t+s} = j, N(t, t+s) = 1\}.$$

We then have

$$\begin{aligned}
G(s) &= \mathbb{P}_i\{Y_{t+s} > x, X_{t+s} = j, N(t, t+s) = 1\} \\
&= \sum_{k \in S} G_k(s)\mathbb{P}_i\{X_t = k, X_{t+s} = j, N(t, t+s) = 1\}
\end{aligned}$$

Let us define $\rho_{\min} = \min\{\rho_i\}$ and $\rho_{\max} = \max\{\rho_i\}$.

Since $Y_t + \rho_{\min}s \le Y_{t+s} \le Y_t + \rho_{\max}s$, we get

$$\mathbb{P}_i\{Y_t > x - \rho_{\min}s \mid X_t = k, X_{t+s} = j, N(t, t+s) = 1\} \le G_k(s)$$

and

$$G_k(s) \le \mathbb{P}_i\{Y_t > x - \rho_{\max}s \mid X_t = k, X_{t+s} = j, N(t, t+s) = 1\}.$$

Using the Markov property,

$$\mathbb{P}_i\{Y_t > x - \rho_{\min} s \mid X_t = k\} \leq G_k(s) \leq \mathbb{P}_i\{Y_t > x - \rho_{\max} s \mid X_t = k\},$$

We thus obtain

$$\sum_{k \in S} F_{i,k}(t, x - \rho_{\min} s) U_{k,j}(s) \leq G(s) \leq \sum_{k \in S} F_{i,k}(t, x - \rho_{\max} s) U_{k,j}(s),$$

where $U_{k,j}(s)$ is defined by

$$U_{k,j}(s) = \mathbb{P}\{X_{t+s} = j, N(t, t+s) = 1 \mid X_t = k\}.$$

From relation (17), we have

$$\lim_{s \longrightarrow 0} \frac{U_{k,j}(s)}{s} = \lambda P_{k,j},$$

so we obtain

$$\lim_{s \longrightarrow 0} \frac{G(s)}{s} = \lambda \sum_{k \in S} F_{i,k}(t, x) P_{k,j}.$$

For the third term, we have from relation (18),

$$\mathbb{P}_i\{Y_{t+s} > x, X_{t+s} = j, N(t, t+s) \geq 2\} \leq \mathbb{P}_i\{N(t, t+s) \geq 2\} = o(s).$$

Putting together the three terms, we obtain

$$\begin{aligned}
\frac{F_{i,j}(t+s, x) - F_{i,j}(t, x)}{s} &= \frac{(1 - \lambda s) F_{i,j}(t, x - \rho(j)s) - F_{i,j}(t, x)}{s} + \frac{G(s)}{s} + \frac{o(s)}{s} \\
&= \frac{F_{i,j}(t, x - \rho(j)s) - F_{i,j}(t, x)}{s} - \lambda F_{i,j}(t, x - \rho(j)s) + \frac{G(s)}{s} + \frac{o(s)}{s}
\end{aligned}$$

If we let now s tend to 0, we get

$$\frac{\partial F_{i,j}(t, x)}{\partial t} = -\rho(j) \frac{\partial F_{i,j}(t, x)}{\partial x} - \lambda F_{i,j}(t, x) + \lambda \sum_{k \in S} F_{i,k}(t, x) P_{k,j}.$$

Since $P = I + A/\lambda$, we obtain

$$\frac{\partial F_{i,j}(t, x)}{\partial t} = -\rho(j) \frac{\partial F_{i,j}(t, x)}{\partial x} + \sum_{k \in S} F_{i,k}(t, x) A_{k,j}.$$

The proof is thus complete. ■

Corollary 5.3 *For $t > 0$, $0 \leq p \leq m$, $i \in B_p$, $j \in S$, $1 \leq h \leq m$ and $x \in (r_{h-1}t, r_h t)$ we have*

$$F_{i,j}(t,x) = \sum_{k \in S} \int_0^t F_{i,k}(t-u, x-\rho(j)u)\lambda e^{-\lambda u} du P_{k,j} + e^{-\lambda t} 1_{\{h \leq p\}} 1_{\{i=j\}} \tag{20}$$

Proof. Consider equation (19) and the functions $\varphi_{i,j}$ defined by

$$\varphi_{i,j}(u) = F_{i,j}(t-u, x-\rho(j)u)e^{-\lambda u}.$$

Differentiating with respect to u, we get

$$\varphi'_{i,j}(u) = e^{-\lambda u} \left[-\frac{\partial F_{i,j}}{\partial t} - \rho(j) \frac{\partial F_{i,j}}{\partial x} \right] (t-u, x-\rho(j)u) - F_{i,j}(t-u, x-\rho(j)u)\lambda e^{-\lambda u}.$$

This gives using equation (19) and the relation $A = -\lambda(I-P)$,

$$\begin{aligned} \varphi'_{i,j}(u) &= -\sum_{k \in S} F_{i,k}(t-u, x-\rho(j)u) A_{k,j} e^{-\lambda u} - F_{i,j}(t-u, x-\rho(j)u)\lambda e^{-\lambda u} \\ &= -\sum_{k \in S} F_{i,k}(t-u, x-\rho(j)u)\lambda e^{-\lambda u} P_{k,j} \end{aligned}$$

Integrating now this expression between 0 and t, we obtain

$$\varphi_{i,j}(t) - \varphi_{i,j}(0) = -\sum_{k \in S} \int_0^t F_{i,k}(t-u, x-\rho(j)u)\lambda e^{-\lambda u} du P_{k,j}.$$

Finally, we have $\varphi_{i,j}(0) = F_{i,j}(t,x)$ and

$$\varphi_{i,j}(t) = F_{i,j}(0, x-\rho(j)t)e^{-\lambda t} = e^{-\lambda t} 1_{\{x-\rho(j)t<0\}} 1_{\{i=j\}} = e^{-\lambda t} 1_{\{h \leq p\}} 1_{\{i=j\}},$$

which completes the proof. ■

We now derive the backward equation for the evolution of the pair (Y_t, X_t).

Theorem 5.4 *For $t > 0$, $0 \leq p \leq m$, $i \in B_p$, $j \in S$, $1 \leq h \leq m$ and $x \in (r_{h-1}t, r_h t)$, we have*

$$F_{i,j}(t,x) = \sum_{k \in S} P_{i,k} \int_0^t F_{k,j}(t-u, x-\rho(i)u)\lambda e^{-\lambda u} du + e^{-\lambda t} 1_{\{h \leq p\}} 1_{\{i=j\}} \tag{21}$$

Proof. Let T_1 be the first sojourn time in the initial state. We have

$$F_{i,j}(t,x) = \int_0^\infty \mathbb{P}\{Y_t > x, X_t = j \mid T_1 = u, X_0 = i\}\lambda e^{-\lambda u} du.$$

If $u \geq t$ and $X_0 = i$ we have $Y_t = \rho(i)t = r_p t$ and $\mathbb{P}\{X_t = j \mid T_1 = u, X_0 = i\} = 1$ if $i = j$ and 0 otherwise. Moreover, since $r_p t > x$ is equivalent to $r_p t \geq r_h t$, that is $h \leq p$, we obtain

$$F_{i,j}(t,x) = \int_0^t \mathbb{P}\{Y_t > x, X_t = j \mid T_1 = u, X_0 = i\}\lambda e^{-\lambda u} du + e^{-\lambda t} 1_{\{h \leq p\}} 1_{\{i=j\}}.$$

Now,

$$\begin{aligned}
&\mathbb{P}\{Y_t > x, X_t = j \mid T_1 = u, X_0 = i\} \\
&\quad = \sum_{k \in S} \mathbb{P}\{Y_t > x, X_t = j \mid X_u = k, T_1 = u, X_0 = i\}\mathbb{P}\{X_u = k \mid T_1 = u, X_0 = i\}.
\end{aligned}$$

For the second term in the sum, we have

$$\begin{aligned}
\mathbb{P}\{X_u = k \mid T_1 = u, X_0 = i\} &= \mathbb{P}\{X_{T_1} = k \mid T_1 = u, X_0 = i\} \\
&= \mathbb{P}\{Z_1 = k \mid T_1 = u, Z_0 = i\} \\
&= P_{i,k},
\end{aligned}$$

For the first one, $T_1 = u$ and $X_0 = i$ implies that $Y_u = \rho(i)u$, so
$\mathbb{P}\{Y_t > x, X_t = j \mid X_u = k, T_1 = u, X_0 = i\}$

$$\begin{aligned}
&= \mathbb{P}\{\int_u^t \rho(X_v)dv > x - \rho(i)u, X_t = j \mid X_u = k, T_1 = u, X_0 = i\} \\
&= \mathbb{P}\{\int_u^t \rho(X_v)dv > x - \rho(i)u, X_t = j \mid X_u = k\} \\
&= \mathbb{P}\{Y_{t-u} > x - \rho(i)u, X_{t-u} = j \mid X_0 = k\} \\
&= F_{k,j}(t-u, x-\rho(i)u)
\end{aligned}$$

where the second equality follows from the Markov property and the third follows from homogeneity. Combining these results, we obtain relation (21). ■

Corollary 5.5 *For $t > 0$, $i, j \in S$, $1 \leq h \leq m$ and $x \in (r_{h-1}t, r_h t)$ we have*

$$\frac{\partial F_{i,j}(t,x)}{\partial t} = -\rho(i)\frac{\partial F_{i,j}(t,x)}{\partial x} + \sum_{k \in S} A_{i,k} F_{k,j}(t,x). \tag{22}$$

Proof. Consider equation (21) with $i \in B_p$, $0 \le p \le m$ and $j \in S$.
Differentiating $F_{i,j}(t,x)$ with respect to variable t, we get

$$\begin{aligned}\frac{\partial F_{i,j}(t,x)}{\partial t} &= \sum_{k \in S} P_{i,k} \left[\int_0^t \frac{\partial F_{k,j}(t-u, x-\rho(i)u)}{\partial t} \lambda e^{-\lambda u} du + F_{k,j}(0, x-\rho(i)t)\lambda e^{-\lambda t} \right] \\ &- \lambda e^{-\lambda t} 1_{\{h \le p\}} 1_{\{i=j\}}\end{aligned}$$

Differentiating now $F_{i,j}(t,x)$ with respect to variable x, we get

$$\frac{\partial F_{i,j}(t,x)}{\partial x} = \sum_{k \in S} P_{i,k} \int_0^t \frac{\partial F_{k,j}(t-u, x-\rho(i)u)}{\partial x} \lambda e^{-\lambda u} du.$$

Let us consider the functions $\psi_{k,j}$ and $\varphi_{k,j}$ defined by

$$\psi_{k,j}(u) = F_{k,j}(t-u, x-\rho(i)u) \quad \text{and} \quad \varphi_{k,j}(u) = \psi_{k,j}(u) e^{-\lambda u}.$$

Note that we have $\psi_{i,j}(t) = 1_{\{h \le p\}} 1_{\{i=j\}}$, so equation (21) can also be written as

$$\varphi_{i,j}(0) = \lambda \sum_{k \in S} P_{i,k} \int_0^t \varphi_{k,j}(u) du + \varphi_{i,j}(t) \tag{23}$$

Differentiating $\psi_{k,j}$ with respect to u, we get

$$\psi'_{k,j}(u) = -\frac{\partial F_{k,j}(t-u, x-\rho(i)u)}{\partial t} - \rho(i) \frac{\partial F_{k,j}(t-u, x-\rho(i)u)}{\partial x}.$$

We thus obtain

$$\begin{aligned}\frac{\partial F_{i,j}(t,x)}{\partial t} + \rho(i)\frac{\partial F_{i,j}(t,x)}{\partial x} &= -\lambda \sum_{k \in S} P_{i,k} \int_0^t \psi'_{k,j}(u) e^{-\lambda u} du + \lambda \sum_{k \in S} P_{i,k} \varphi_{k,j}(t) - \lambda \varphi_{i,j}(t) \\ &= \lambda \sum_{k \in S} P_{i,k} \varphi_{k,j}(0) - \lambda^2 \int_0^t \varphi_{k,j}(u) du - \lambda \varphi_{i,j}(t) \\ &= \lambda \sum_{k \in S} P_{i,k} \varphi_{k,j}(0) - \lambda \varphi_{i,j}(0) \\ &= \lambda \sum_{k \in S} P_{i,k} F_{k,j}(t,x) - \lambda F_{i,j}(t,x) \\ &= \sum_{k \in S} A_{i,k} F_{k,j}(t,x)\end{aligned}$$

where the second equality is obtained by integration by parts, the third equality follows from relation (23) and the last one from the relation $A = -\lambda(I - P)$. ■

Let D be the diagonal matrix containing the reward rates $\rho(i)$ and $F(t,x)$ the matrix containing the $F_{i,j}(t,x)$. The forward and backward equations (19) and (22) become respectively in matrix notation

$$\frac{\partial F(t,x)}{\partial t} = -\frac{\partial F(t,x)}{\partial x} D + F(t,x)A. \tag{24}$$

and

$$\frac{\partial F(t,x)}{\partial t} = -D\frac{\partial F(t,x)}{\partial x} + AF(t,x). \tag{25}$$

These equations are hyperbolic partial differential equations having a unique solution on the domain E with the initial condition given by relation (15), see for instance [6]. A method for obtaining these solutions is presented in the following subsection.

5.2 Solutions

The solution of the forward partial differential equation (24) is given by the following theorem.

Theorem 5.6 *For every $t > 0$ and $x \in [r_{h-1}t, r_h t)$ for $h = 1, 2, \ldots, m$, we have*

$$F(t,x) = \sum_{n=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} \sum_{k=0}^{n} \binom{n}{k} x_h^k (1 - x_h)^{n-k} C^{(h)}(n,k), \tag{26}$$

where $x_h = \dfrac{x - r_{h-1}t}{(r_h - r_{h-1})t}$ *and the matrices* $C^{(h)}(n,k) = \left(C^{(h)}_{B_u B_v}(n,k)\right)_{0 \le u,v \le m}$ *are given by the following recursive expressions*

for $0 \le u \le m$ and $h \le v \le m$:

for $n \ge 0$: $\quad C^{(1)}_{B_u B_v}(n,0) = (P^n)_{B_u B_v}$ *and* $C^{(h)}_{B_u B_v}(n,0) = C^{(h-1)}_{B_u B_v}(n,n)$ *for $h > 1$*

for $1 \le k \le n$:

$$C^{(h)}_{B_u B_v}(n,k) = \frac{r_v - r_h}{r_v - r_{h-1}} C^{(h)}_{B_u B_v}(n,k-1) + \frac{r_h - r_{h-1}}{r_v - r_{h-1}} \sum_{w=0}^{m} C^{(h)}_{B_u B_w}(n-1,k-1) P_{B_w B_v}, \tag{27}$$

for $0 \le u \le m$ *and* $0 \le v \le h-1$ *:*

for $n \ge 0$: $\quad C^{(m)}_{B_u B_v}(n,n) = 0_{B_u B_v}$ *and* $C^{(h)}_{B_u B_v}(n,n) = C^{(h+1)}_{B_u B_v}(n,0)$ *for* $h < m$

for $0 \le k \le n-1$ *:*

$$C^{(h)}_{B_u B_v}(n,k) = \frac{r_{h-1} - r_v}{r_h - r_v} C^{(h)}_{B_u B_v}(n,k+1) + \frac{r_h - r_{h-1}}{r_h - r_v} \sum_{w=0}^{m} C^{(h)}_{B_u B_w}(n-1,k) P_{B_w B_v}. \tag{28}$$

Proof. For $t > 0$ and $x \in (r_{h-1}t, r_h t)$ for $h = 1, 2, \ldots, m$, we write the solution of the forward equation (24) as

$$F(t,x) = \sum_{n=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} \sum_{k=0}^{n} \binom{n}{k} x_h^k (1 - x_k)^{n-k} C^{(h)}(n,k),$$

and we determine the relations that must be satisfied by the matrices $C^{(h)}(n,k)$. We have

$$\begin{aligned}\frac{\partial F(t,x)}{\partial t} = -\lambda F(t,x) + \frac{\lambda}{r_h - r_{h-1}} \sum_{n=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} \sum_{k=0}^{n} \binom{n}{k} x_h^k (1 - x_h)^{n-k} \\ \times \left[r_h C^{(h)}(n+1,k) - r_{h-1} C^{(h)}(n+1,k+1) \right],\end{aligned}$$

and

$$\begin{aligned}\frac{\partial F(t,x)}{\partial x} = \frac{\lambda}{r_h - r_{h-1}} \sum_{n=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} \sum_{k=0}^{n} \binom{n}{k} x_h^k (1 - x_h)^{n-k} \\ \times \left[C^{(h)}(n+1,k+1) - C^{(h)}(n+1,k) \right].\end{aligned}$$

Since $A = -\lambda(I - P)$, we obtain

$$F(t,x)A = -\lambda F(t,x) + \lambda F(t,x)P,$$

that is,

$$F(t,x)A = -\lambda F(t,x) + \lambda \sum_{n=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} \sum_{k=0}^{n} \binom{n}{k} x_h^k (1 - x_h)^{n-k} C^{(h)}(n,k) P.$$

It follows that if the matrices $C^{(h)}(n,k)$ are such that

$$C^{(h)}(n+1,k+1)[D - r_{h-1}I] = C^{(h)}(n+1,k)[D - r_h I] + (r_h - r_{h-1}) C^{(h)}(n,k) P \tag{29}$$

then equation (24) is satisfied. The recurrence relation (29) can also be written as follows, for every $h = 1, \ldots, m$ and $u = 0, 1, \ldots, m$

If $v = h, h+1, \ldots, m$ then

$$C^{(h)}_{B_u B_v}(n,k) = \frac{r_v - r_h}{r_v - r_{h-1}} C^{(h)}_{B_u B_v}(n,k-1) + \frac{r_h - r_{h-1}}{r_v - r_{h-1}} \sum_{w=0}^{m} C^{(h)}_{B_u B_w}(n-1,k-1) P_{B_w B_v},$$

and if $v = 0, 1, \ldots, h-1$ then

$$C^{(h)}_{B_u B_v}(n,k) = \frac{r_{h-1} - r_v}{r_h - r_v} C^{(h)}_{B_u B_v}(n,k+1) + \frac{r_h - r_{h-1}}{r_h - r_v} \sum_{w=0}^{m} C^{(h)}_{B_u B_w}(n-1,k) P_{B_w B_v}.$$

To get the initial conditions for the $C^{(h)}(n,k)$, we consider the jumps of $F(t,x)$. We consider first the jump at point $x = r_0 t = 0$. For $t > 0$ we have at point $x = 0$, that is for $h = 1$, from relation (26)

$$F(t,0) = \sum_{n=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} C^{(1)}(n,0),$$

It follows from relation (15) that for $u, v = 0, 1, \ldots, m$,

$$C^{(1)}_{B_u B_v}(n,0) = (P^n)_{B_u B_v} - P^n_{B_0 B_0} 1_{\{u=v=0\}}. \tag{30}$$

This implies in particular that for every $0 \le u \le m$

$$C^{(1)}_{B_u B_v}(n,0) = (P^n)_{B_u B_v} \text{ for } 1 \le v \le m.$$

We consider now the jumps at points $x = r_h t$ for $1 \le h \le m-1$. For $t > 0$ and $1 \le h \le m-1$ and $i, j \in S$, we have

$$F_{i,j}(t, r_h t) = \lim_{x \overset{<}{\longrightarrow} r_h t} F_{i,j}(t,x) - \mathbb{P}\{Y_t = r_h t, X_t = j \mid X_0 = i\}.$$

From relation (26) and relation (14), we get

$$C^{(h+1)}_{B_u B_v}(n,0) = C^{(h)}_{B_u B_v}(n,n) + P^n_{B_h B_h} 1_{\{u=v=h\}}. \tag{31}$$

This implies in particular that for every $0 \le u \le m$

$$C^{(h)}_{B_u B_v}(n,0) = C^{(h-1)}_{B_u B_v}(n,n) \text{ for } 1 < h \le v \le m,$$

and

$$C^{(h)}_{B_u B_v}(n,n) = C^{(h+1)}_{B_u B_v}(n,0) \text{ for } 0 \le v \le h-1 < m-1.$$

We consider finally the jump at point $x = r_m t$, that is for $h = m$. For $t > 0$ we have

$$0 = F_{i,j}(t, r_m t) = \lim_{x \overset{<}{\longrightarrow} r_m t} F_{i,j}(t,x) - \mathbb{P}\{Y_t = r_h t, X_t = j \mid X_0 = i\},$$

which leads as in the previous case to the relation

$$C^{(m)}_{B_u B_v}(n,n) = P^n_{B_m B_m} 1_{\{u=v=m\}}. \tag{32}$$

This implies in particular that for every $0 \le u \le m$

$$C^{(m)}_{B_u B_v}(n,n) = 0 \text{ for } 0 \le v \le m-1.$$

The proof is now complete. ■

Corollary 5.7 *For $h = 1, 2, \ldots, m$, $n \ge 0$ and $0 \le k \le n$, the matrices $C^{(h)}(n,k) = \left(C^{(h)}_{B_u B_v}(n,k)\right)_{0 \le u,v \le m}$ satisfy the following recursive expressions*

for $h \le u \le m$ and $0 \le v \le m$:

for $n \ge 0$: $\quad C^{(1)}_{B_u B_v}(n,0) = (P^n)_{B_u B_v}$ *and* $C^{(h)}_{B_u B_v}(n,0) = C^{(h-1)}_{B_u B_v}(n,n)$ *for $h > 1$*

for $1 \le k \le n$:

$$C^{(h)}_{B_u B_v}(n,k) = \frac{r_u - r_h}{r_u - r_{h-1}} C^{(h)}_{B_u B_v}(n,k-1) + \frac{r_h - r_{h-1}}{r_u - r_{h-1}} \sum_{w=0}^{m} P_{B_u B_w} C^{(h)}_{B_w B_v}(n-1,k-1),$$

for $0 \le u \le h-1$ and $0 \le v \le m$:

for $n \ge 0$: $\quad C^{(m)}_{B_u B_v}(n,n) = 0_{B_u B_v}$ *and* $C^{(h)}_{B_u B_v}(n,n) = C^{(h+1)}_{B_u B_v}(n,0)$ *for $h < m$*

for $0 \le k \le n-1$:

$$C^{(h)}_{B_u B_v}(n,k) = \frac{r_{h-1} - r_u}{r_h - r_u} C^{(h)}_{B_u B_v}(n,k+1) + \frac{r_h - r_{h-1}}{r_h - r_u} \sum_{w=0}^{m} P_{B_u B_w} C^{(h)}_{B_w B_v}(n-1,k).$$

Proof. The proof is the same as the proof of theorem 5.6 using the backward equation (25) and the expression (26). We thus obtain that the matrices $C^{(h)}(n,k)$ satisfy the relation

$$[D - r_{h-1}I]C^{(h)}(n+1,k+1) = [D - r_h I]C^{(h)}(n+1,k) + (r_h - r_{h-1})PC^{(h)}(n,k). \quad (33)$$

The recurrence relation (33) can also be written as follows, for every $h = 1, \ldots, m$ and $v = 0, 1, \ldots, m$

If $u = h, h+1, \ldots, m$ then

$$C^{(h)}_{B_u B_v}(n,k) = \frac{r_u - r_h}{r_u - r_{h-1}} C^{(h)}_{B_u B_v}(n,k-1) + \frac{r_h - r_{h-1}}{r_u - r_{h-1}} \sum_{w=0}^{m} P_{B_u B_w} C^{(h)}_{B_w B_v}(n-1,k-1),$$

and if $u = 0, 1, \ldots, h-1$ then

$$C^{(h)}_{B_u B_v}(n,k) = \frac{r_{h-1} - r_u}{r_h - r_u} C^{(h)}_{B_u B_v}(n,k+1) + \frac{r_h - r_{h-1}}{r_h - r_u} \sum_{w=0}^{m} P_{B_u B_w} C^{(h)}_{B_w B_v}(n-1,k).$$

As for the proof of theorem 5.6, we consider the jumps of $F(t,x)$.

Relation (30) implies that for every $0 \leq v \leq m$

$$C^{(1)}_{B_u B_v}(n,0) = (P^n)_{B_u B_v} \text{ for } 1 \leq u \leq m.$$

Relation (31) implies that for every $0 \leq v \leq m$

$$C^{(h)}_{B_u B_v}(n,0) = C^{(h-1)}_{B_u B_v}(n,n) \text{ for } 1 < h \leq u \leq m,$$

and

$$C^{(h)}_{B_u B_v}(n,n) = C^{(h+1)}_{B_u B_v}(n,0) \text{ for } 0 \leq u \leq h-1 < m-1.$$

Finally, relation (32) implies that for every $0 \leq v \leq m$

$$C^{(m)}_{B_u B_v}(n,n) = 0 \text{ for } 0 \leq u \leq m-1.$$

This completes the proof. ■

The following corollary gives an upper bound for the matrices $C^{(h)}(n,k)$. If M and K are two square matrices with the same dimension, the notation $M \leq K$ means that for every i, j, we have $M_{i,j} \leq K_{i,j}$.

Corollary 5.8 *For every $n \geq 0$, $0 \leq k \leq n$ and $1 \leq h \leq m$, we have*

$$0 \leq C^{(h)}(n,k) \leq P^n$$

Proof. The proof is easily made by a two level induction; first over the integer n and then, for fixed n, over the integer k, using the recurrence relation described in theorem 5.6 or equivalently in corollary 5.7. The result is evidently true for $n = 0$. Note that in relation (27), that is for $h \leq v$, we have

$$0 \leq \frac{r_v - r_h}{r_v - r_{h-1}} = 1 - \frac{r_h - r_{h-1}}{r_v - r_{h-1}} \leq 1,$$

and in relation (28), that is for $v \leq h-1$, we have

$$0 \leq \frac{r_{h-1} - r_v}{r_h - r_v} = 1 - \frac{r_h - r_{h-1}}{r_h - r_v} \leq 1.$$

Consider first the case $v \leq h-1$. The result is true for the couple (n,n), since in this case we have $C^{(m)}_{B_u B_v}(n,n) = 0$. Suppose the result is true for integer $n-1$ and for the couple $(n, k+1)$, then using relation (28), we get $C^{(h)}_{B_u B_v}(n,k) \geq 0$ and

$$\begin{aligned}
C^{(h)}_{B_u B_v}(n,k) &= \frac{r_{h-1} - r_v}{r_h - r_v} C^{(h)}_{B_u B_v}(n,k+1) + \frac{r_h - r_{h-1}}{r_h - r_v} \sum_{w=0}^{m} C^{(h)}_{B_u B_w}(n-1,k) P_{B_w B_v} \\
&\leq \frac{r_{h-1} - r_v}{r_h - r_v} (P^n)_{B_u B_v} + \frac{r_h - r_{h-1}}{r_h - r_v} \sum_{w=0}^{m} (P^{n-1})_{B_u B_w} P_{B_w B_v} \\
&= \frac{r_{h-1} - r_v}{r_h - r_v} (P^n)_{B_u B_v} + \frac{r_h - r_{h-1}}{r_h - r_v} (P^n)_{B_u B_v} \\
&= (P^n)_{B_u B_v}.
\end{aligned}$$

The same argument is used in the case where $h \leq v$ from relation (27). Moreover, the relation

$$C^{(h)}_{B_u B_v}(n,0) = C^{(h-1)}_{B_u B_v}(n,n) \text{ for } 1 < h \leq v \leq m,$$

and

$$C^{(h)}_{B_u B_v}(n,n) = C^{(h+1)}_{B_u B_v}(n,0) \text{ for } 0 \leq v \leq h-1 < m-1,$$

are used in the recurrence to take into account the evolution of the integer h in both cases $v \leq h-1$ and $h \leq v$. ■

This result is particularly interesting form a computational point of view to avoid overflow problems.

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