Topological Derivative for Nucleation of Non-Circular Voids

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Topological derivative for nucleation of non-circular voids

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Résumé : The hitherto existing literature concerning the topological derivative of shape functionals concerned perturbations of domains caused by introduction of circular or ball openings. In the present study the notion of the topological derivative is generalized to the case of openings of arbitrary shape. The report is concerned with energy functionals related to the Neumann problem and the $2D$ elasticity system.

Mots-clé : Shape optimization, shape derivative, topological derivative, compound asymptotics, Polya-Szegö mass matrix, Neumann problem.

(Abstract: pto)

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La dérivée topologique dans le cas d'ouvertures de forme arbitraire.

Abstract: Jusqu'à maintenant, la littérature existante concernant la dérivée topologique de fonctionnelles de forme s'intéressait aux perturbations de domaines dues à l'introduction d'ouvertures circulaires ou sphériques. On généralise, dans ce rapport, la notion de dérivée topologique au cas d'ouvertures de forme arbitraire. On s'intéresse aux fonctionnelles d'énergie relatives au problème de Neumann et d'élasticité $2D$.

Key-words: optimisation de forme, la dérivée par rapport au domaine, la dérivée topologique, la matrice de masse, problème de Neumann.

1. Introduction

One of the most challenging problems of optimization of solids and structures are the shape optimization problems which in the two-dimensional case are formulated as follows. Let a body be loaded on the boundary of a certain domain $\Omega \subset R^2$ that should be filled up with a given material. Assume that the amount of the material is fixed, which means that the area of the domain $\Omega \setminus K$ occupied by the material is given: $|\Omega \setminus K| = A$, A being given. The domain K is said to be empty. Let us focus our attention on minimization of a certain shape functional $J : \Omega \setminus K \to \mathbb{R}$, under the condition $|\Omega \setminus K| = A$.

In the case of J being the functional of compliance and within the framework of linear elasticity, the problem is ill-posed. To make it correctly posed one should let the domains K and $\Omega \setminus K$ be perfectly mixed, which gives rise to appearing perforated micro-structures, cf. Allaire and Kohn (1993), Allaire et al. (1997). One can proceed further in two ways.

One strategy is to accept the perforated microstructures. The other is to seek suboptimal solutions by: (i) imposing isoperimetric constraints, as suggested by Ambrosio and Buttazzo (1993); (ii) admitting openings in those subdomains where removing material does not cause essential changes of the merit function, cf. Eschenauer et al. (1994).

The perforated microstructures are not uniquely determined. The microstructures which assure correct relaxation for all regimes of strain invariants have a hierarchic property; they constitute n-rank laminates (n = 1,2,3), the anisotropic materials of more mathematical than physical nature and rather impossible to be manufactured. The only known optimal and non-hierarchic microstructures are those discovered by Vigdergauz (1994). These are, however, optimal only in some regimes of strain values. The aforementioned reasons prompts one to looking for suboptimal solutions by penalizing the domains occupied by the perforated microstructures, see Allaire et al. (1997) and Bendsøe (1995). The results should coincide with those found by the "bubble" method of Eschenauer et al. (1994).

In the present report we put forward a new technique of finding a topological derivative of energy functional in the plane problems: of Neumann and of linear elasticity. A proof is given that the topological derivative is a quadratic form of the gradient of the solution of the original problem (without opening) measured at the point of nucleation of a small opening of arbitrary shape. This quadratic form represents the so-called characteristic function of the "bubble" method. The minima of this characteristic function show the places, where the material can be removed.

The notion of the topological derivative has been put forward in the papers by Sokołowski and Żochowski (1997, 1999) under the condition of the openings being balls in $\mathbb{R}^n$. The methods of compound asymptotics developed in Maz'ya and Nazarov (1987) and Maz'ya et al. (1991) make it possible to generalize the notion of the topological derivative to the case of nucleation of openings of arbitrary shape.

We show in the present report that the topological derivative of the energy functional of the Neumann's problem is a positive definite quadratic form (with matrix $\boldsymbol{G} = (G^{\alpha\beta})$) in the case of nucleation of star-shaped openings. To this end we apply the sensitivity analysis methods of Sokołowski and Zolesio (1992). A stronger result has been reported by Maz'ya and Nazarov (1997) who proved that the energy increment due to appearing of an opening is a positive definite quadratic form, of the domain being star-shaped or not. The proof is based on the mass matrix $\boldsymbol{m}$ of Polya-Szegö (see Schiffer and Szegö (1949)) being negative definite. In the present report we prove that the matrices $\boldsymbol{m}$ and $\boldsymbol{G}$ are identical if the opening is an ellipse in $\mathbb{R}^2$.

A generalization of the notion of the mass matrix to the isotropic elasticity is due to Maz'ya et al. (1991). In the present report we generalize the mass matrix concept to the anisotropic elasticity. Moreover, we show that the generalized mass matrix $\boldsymbol{M} = (M^{\alpha\beta\lambda\nu})$ of isotropic elasticity coincides with the matrix $\boldsymbol{G} = (G^{\alpha\beta\lambda\nu})$ (that occurs in the expression for the topological derivative of energy of the elastic medium) in the case when the opening is circular. Further generalization of this result to the three-dimensional elasticity problem and to the ball-shaped openings is straightforward.

In Appendix A the main properties of the mass matrix for the Neumann problem are recalled. In Appendix B the mass matrix for the two-dimensional elasticity is introduced and examined. In our approach the mass matrix components form a fourth rank tensor. Appendix C recalls the derivation by Maz'ya and Nazarov (1987) of the formula for the energy increment due to appearing of an opening in the Neumann problem.

The summation convention is adopted. It refers to small Greek indices α, β, λ, μ, κ, $\delta, \ldots$ (except for ε) taking values 1 or 2.

2. The notion of topological derivative

Let $\Omega \subset \mathbb{R}^2$ be an open domain. This domain is parametrized by the cartesian coordinate system (x_1,x_2). It is assumed that $\boldsymbol{O} = (0,0) \in \Omega$. Let us form a family of domains ω_ε around $\boldsymbol{O}$ in the following way. Let $\boldsymbol{O} \in \omega_\varepsilon$ and

$$\omega_\varepsilon = \left\{ x \,\middle|\, \frac{x}{\varepsilon} \in \omega \right\} \ , \tag{2.1}$$

where $x = (x_1,x_2)$, ω is an open domain in $\mathbb{R}^2$; ε is a positive, small parameter. The domain ω will be parametrized by the cartesian coordinate system (y_1,y_2); its central point $\boldsymbol{O} \in \omega$, see Fig. 2.1.

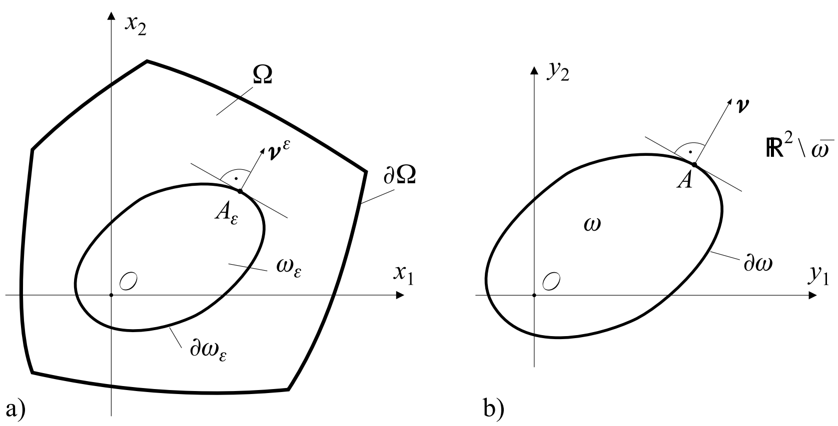

Fig. 2.1. a) Domains Ω and ω_ε; b) The rescaled domain ω

Let $A = (y_1,y_2) \in \partial\omega$. Then $A_\varepsilon := (\varepsilon y_1,\varepsilon y_2) \in \partial\omega_\varepsilon$. Let us direct the normal vectors: $\boldsymbol{\nu} = (\nu_1,\nu_2)$ at A and $\boldsymbol{\nu}^\varepsilon = (\nu_1^\varepsilon,\nu_2^\varepsilon)$ at A_ε outward to the domains ω and ω_ε, respectively. Let us note that $\nu_\alpha^\varepsilon = \nu_\alpha$ for every $\varepsilon > O$. The areas of ω_ε and ω are linked by: $|\omega_\varepsilon| = \varepsilon^2|\omega|$. Let $K \subset \overline{\Omega}$ be a compact subset. Assume that a shape functional:

$$J : \Omega \setminus K \to \mathbb{R} \tag{2.2}$$

is given. Assume that the following limit exists

$$\mathfrak{T}_\omega(\boldsymbol{O}) = \lim_{\varepsilon \searrow 0} \frac{J(\Omega \setminus \overline{\omega}_\varepsilon) - J(\Omega)}{|\overline{\omega}_\varepsilon|} \ . \tag{2.3}$$

This limit will be called the topological derivative of the functional J, computed at the point $\boldsymbol{O} \in \Omega$, for nucleation of voids of the shape ω. In an obvious manner one can define the value $\mathfrak{T}_\omega(x_0)$ at other point $x_0 = (x_{01},x_{02}) \in \Omega$. Note that

$$\mathfrak{T}_\omega(\boldsymbol{O}) = |\omega|^{-1}\widetilde{\mathfrak{T}}_\omega(\boldsymbol{O}) \ , \tag{2.4}$$

where

$$\widetilde{\mathfrak{T}}_\omega(\boldsymbol{O}) = \lim_{\varepsilon \searrow 0} \frac{J(\Omega \setminus \overline{\omega}_\varepsilon) - J(\Omega)}{\varepsilon^2} \ . \tag{2.5}$$

In the case when ω is a circle the index ω in (2.3) and (2.5) will be omitted. The topological derivatives $\mathfrak{T}$ and $\widetilde{\mathfrak{T}}$ defined in this way were introduced by Sokołowski and Żochowski (1997) for the shape functionals in $\mathbb{R}^n$; ω is assumed there to be an open ball in $\mathbb{R}^n$.

3. The Neumann problem in the plane domain $\Omega \setminus \overline{\omega}_\varepsilon$. Asymptotic expansion of the solution u_ε

The subject of our consideration is an asymptotic analysis of the solution u_ε of the problem:

(P_ε) find $u_\varepsilon \in H^{2+k}(\Omega_\varepsilon)$ satisfying the Laplace equation

$$\mu \Delta u_\varepsilon = 0 \quad \text{in} \quad \Omega_\varepsilon = \Omega \setminus \overline{\omega}_\varepsilon \ , \tag{3.1}$$

where $\Delta = \dfrac{\partial^2}{\partial (x_1)^2} + \dfrac{\partial^2}{\partial (x_2)^2}$, $k \geq 2$, and the boundary conditions

$$\mu \frac{\partial u_\varepsilon}{\partial \nu^\varepsilon} = 0 \quad \text{on} \quad \partial \omega_\varepsilon \tag{3.2}$$

$$\mu \frac{\partial u_\varepsilon}{\partial n} = p \quad \text{on} \quad \partial \Omega \ . \tag{3.3}$$

The given loading $p \in H^{\frac{1}{2}+k}(\partial\Omega)$ satisfies the condition

$$\int_{\partial\Omega} p \, ds = 0 \tag{3.4}$$

assuring the solution u_ε being determined up to an additive constant.

We assume that the domains Ω_ε and ω_ε satisfy the known conditions of regularity such that u_ε exists and is of class C^2.

The modulus μ is assumed to be constant.

Let us formulate the problem on the domain Ω (or without an opening):

(P_0) find $v \in C^2(\overline{\Omega})$ such that

$$\mu \Delta v = 0 \quad \text{in} \quad \Omega \ , \tag{3.5}$$

$$\mu \frac{\partial v}{\partial n} = p \quad \text{on} \quad \partial \Omega \ . \tag{3.6}$$

The asymptotic expansion of the solution u_ε can be found in section 4.4 of the paper by Maz'ya and Nazarov (1987). It has the form

$$u_\varepsilon(x) = v(x) + \varepsilon w\left(\frac{x}{\varepsilon}\right) + \varepsilon^2 z(x) + \varepsilon^2 O\left(\left\|\frac{x}{\varepsilon}\right\|^{-1}\right) \ . \tag{3.7}$$

The function v is the solution of (P_0).

The function w is defined on $\mathbb{R}^2 \setminus \overline{\omega}$, see (2.1). By virtue of the formula

$$\Delta u_\varepsilon(x) = \Delta v(x) + \frac{1}{\varepsilon} \, \Delta_y w(y)\big|_{y=\frac{x}{\varepsilon}} + \varepsilon^2 \Delta z + \dots \ , \tag{3.8}$$

where

$$\Delta_y = \frac{\partial^2}{\partial (y_1)^2} + \frac{\partial^2}{\partial (y_2)^2} \ , \tag{3.9}$$

we find

$$\Delta_y w = 0 \ , \qquad w = w(y) \ , \qquad y \in \mathbb{R}^2 \setminus \overline{\omega} \ . \tag{3.10}$$

In the vicinity of $x=(0{,}0)$ we have

$$v(x) = v(\boldsymbol{O}) + \epsilon^0_\beta x_\beta + O\left(\|x\|^2\right) \ , \tag{3.11}$$

where $\epsilon^0_\beta = \dfrac{\partial v}{\partial x_\beta}(\boldsymbol{O})$. The boundary condition (3.2) imposes

$$\frac{\partial u_\varepsilon}{\partial \nu^\varepsilon} = \epsilon^0_\beta \nu_\beta + \left.\frac{\partial w}{\partial \nu}\right|_{y=\frac{x}{\varepsilon}} + O(\varepsilon) = 0 \qquad \text{on} \quad \partial\omega_\varepsilon \ . \tag{3.12}$$

Hence we have

$$\frac{\partial w}{\partial \nu} = -\epsilon^0_\beta \nu_\beta \ . \tag{3.13}$$

We conclude that the function w satisfies the following conditions

$$(P_\omega) \quad \left| \begin{array}{lll} \Delta_y w = 0 & \text{in} & \mathbb{R}^2 \setminus \overline{\omega} \qquad (3.14) \\ \dfrac{\partial w}{\partial \nu} = -\epsilon^0_\beta \nu_\beta & \text{on} & \partial\omega \qquad (3.15) \\ w(y) \to 0 \ , & \text{if} & \|y\| \to \infty \qquad (3.16) \end{array} \right.$$

Here $\|y\| = \left((y_1)^2 + (y_2)^2\right)^{\frac{1}{2}}$.

Let us encompass the domain ω by a circle $B_R = \{y \,|\, \|y\| \le R\}; \partial B_R = \Gamma_R$. The condition of function w being harmonic implies the variational equation

$$\int\limits_{B_R \setminus \overline{\omega}} \frac{\partial w}{\partial y_\lambda}\frac{\partial \varphi}{\partial y_\lambda}\, dy = \int\limits_{\Gamma_R} \frac{\partial w}{\partial n}\varphi\, ds - \int\limits_{\partial\omega} \frac{\partial w}{\partial \nu}\varphi\, ds \tag{3.17}$$

valid for each $\varphi \in H^2(B_R)$.

The function $w = w(y)$ is harmonic, singular for $y = \boldsymbol{O}$ and vanishes at infinity. Thus it is a linear combination of the functions

$$\frac{\partial}{\partial y_\alpha}(\ln\|y\|) \ , \qquad \frac{\partial^2}{\partial y_\alpha \partial y_\beta}(\ln\|y\|) \ , \ldots \ .$$

Thus the following expansion holds

$$w(y) = C_\alpha \frac{y_\alpha}{\|y\|} + O\left(\frac{1}{\|y\|^2}\right) \ . \tag{3.18}$$

Along Γ_R we have: $\dfrac{\partial w}{\partial n} = \dfrac{\partial w}{\partial \|y\|}$. There we have $\dfrac{\partial w}{\partial n} = O\left(\dfrac{1}{R^2}\right)$ and $\dfrac{\partial w}{\partial n} ds = O\left(\dfrac{1}{R}\right) d\theta$. Thus the first integral at the right-hand side of (3.17) tends to zero if $R \to \infty$. Now, the substitution $\varphi = \text{const}$ implies the identity:

$$\int\limits_{\partial\omega} \frac{\partial w}{\partial n}\, ds = \epsilon^0_\beta \int\limits_{\partial\omega} \nu_\beta\, ds = 0 \ . \tag{3.19}$$

The condition (3.16) eliminates the candidate: $w = \text{const}$ for the solution of (P_ω). We conclude that the problem (P_ω) is well-posed and its solution w is unique. This solution can be decomposed as follows

$$w(y) = \epsilon^0_\beta w_\beta(y) \ , \tag{3.20}$$

the functions w_β being solutions to the problems:

$$
(P^\beta_\omega)\quad
\left|\begin{array}{lll}
\Delta_y w_\beta = 0 & \text{in} \quad \mathbb{R}^2 \setminus \overline{\omega}\ , & (3.21)\\
\dfrac{\partial w_\beta}{\partial \nu} = -\nu_\beta & \text{on} \quad \partial\omega\ , & (3.22)\\
w_\beta \to 0 & \text{in infinity}\ . & (3.23)
\end{array}\right.
$$

Let us denote by $G(x-y)$ the Green's function for the Laplace equation in $\mathbb{R}^2$. The function $S(y) = G(y-\boldsymbol{O})$ satisfies the equation

$$\mu\Delta_y S + \delta(y-\boldsymbol{O}) = 0 \tag{3.24}$$

and the condition: $\|y\|\nabla S = O(1)$ in infinity, which makes this function unique. Here $\delta(\cdot)$ represents the Dirac measure. The function S has the known form:

$$S(y) = -\frac{1}{2\pi\mu}\ln\|y\|\ . \tag{3.25}$$

The solutions to the (P^β_ω) problems are singular for $y = \boldsymbol{O}$. It turns out that the terms of the weakest singularities will play a crucial role further. Let us disclose them as follows, see Schiffer and Szegö (1949)

$$w_\beta(y) = m_{\beta\alpha}\frac{\partial S}{\partial y_\alpha} + O\left(\frac{1}{\|y\|^2}\right)\ . \tag{3.26}$$

The coefficients $m_{\beta\alpha}$ will be called the components of the mass matrix $\boldsymbol{m}$, as suggested by G. Polya in the papers cited in Schiffer and Szegö (1949). It turns out that the matrix $\boldsymbol{m}$ is symmetric and its components can be computed by the following formula

$$m_{\alpha\beta} = -\mu\delta_{\alpha\beta}|\omega| - \frac{1}{2}\mu\int_{\partial\omega}(\nu_\alpha w_\beta + \nu_\beta w_\alpha)\,ds\ , \tag{3.27}$$

see Appendix A. Thus we have now a better insight into the structure of the functions w_β. Let us come back to the expansion (3.7), taking into account the decomposition (3.20):

$$u_\varepsilon(x) = v(x) + \varepsilon\epsilon^0_\beta w_\beta\left(\frac{x}{\varepsilon}\right) + \varepsilon^2 z(x) + \varepsilon^2 O\left(\left\|\frac{x}{\varepsilon}\right\|^{-1}\right)\ . \tag{3.28}$$

The function $z(x)$ should be determined by the condition (3.3), see Appendix B. Nevertheless, this function will not play any role in this and the next sections.

In the neighbourhood of $\partial\omega_\varepsilon$ the expansion (3.28) can be written in the form

$$u_\varepsilon(x) = v(\boldsymbol{O}) + \varepsilon\epsilon^0_\beta W_\beta\left(\frac{x}{\varepsilon}\right) + \varepsilon^2 z(x) + \varepsilon^2 O\left(\left\|\frac{x}{\varepsilon}\right\|^{-1}\right)\ , \tag{3.29}$$

where

$$W_\beta(y) = y_\beta + w_\beta(y)\ . \tag{3.30}$$

The functions W_β are solutions to the following problems

$$
\left(\tilde{P}^\beta_\omega\right)\quad
\left|\begin{array}{lll}
\Delta_y W_\beta = 0 & \text{in} \quad \mathbb{R}^2 \setminus \overline{\omega}\ , & (3.31)\\
\dfrac{\partial W_\beta}{\partial \nu} = 0 & \text{on} \quad \partial\omega\ , & (3.32)\\
W_\beta \to y_\beta & \text{if} \quad \|y\| \to \infty\ . & (3.33)
\end{array}\right.
$$

The first derivatives of the solution u_ε can be expressed as follows

$$\frac{\partial u_\varepsilon}{\partial x_\alpha} = \epsilon^0_\beta \left.\frac{\partial W_\beta}{\partial y_\alpha}\right|_{y=\frac{x}{\varepsilon}} + \varepsilon^2\frac{\partial z}{\partial x_\alpha} + o(\varepsilon)\ . \tag{3.34}$$

4. Topological derivative of the energy functional

The subject of our consideration will be the shape functional

$$\mathfrak{S}(\Omega_\varepsilon) = J_\varepsilon(u_\varepsilon) \ , \tag{4.1}$$

where

$$J_\varepsilon(\tilde{v}) = \frac{1}{2}\int\limits_{\Omega_\varepsilon} \mu \frac{\partial \tilde{v}}{\partial x_\alpha}\frac{\partial \tilde{v}}{\partial x_\alpha} dx - \int\limits_{\partial\Omega} p\tilde{v}\, ds \tag{4.2}$$

and u_ε is the solution to the problem (3.1) – (3.4). Let $j(\varepsilon) := \mathfrak{S}(\Omega_\varepsilon)$. We shall prove

Theorem 4.1.
Under the conditions concerning the domains Ω and ω and the loading p on $\partial\Omega$, introduced in Sec. 3, the function $j(\varepsilon)$ can be expanded as follows

$$j(\varepsilon_0 + \varepsilon) = j(\varepsilon_0) + j'(\varepsilon_0)\varepsilon + \frac{1}{2}j''(\varepsilon_0)\varepsilon^2 + o\left((\varepsilon_0+\varepsilon)^2\right) \ , \tag{4.3}$$

where

$$j'(\varepsilon_0) \to j'(0^+) = 0 \ , \quad \text{if} \quad \varepsilon_0 \searrow 0^+ \ , \tag{4.4}$$

$$j''(\varepsilon_0) \to j''(0^+) = \epsilon^0_\beta G_{\beta\lambda}\epsilon^0_\lambda, \quad \text{if} \quad \varepsilon_0 \searrow 0^+ \tag{4.5}$$

and the matrix $(G_{\beta\lambda})$ is expressed by

$$G_{\beta\lambda} = -\mu[\delta_{\beta\lambda}|\omega| + \int\limits_{\partial\omega} \gamma_{\lambda\beta}(\boldsymbol{w}) y_\alpha \nu_\alpha\, ds] \ . \tag{4.6}$$

Here $\boldsymbol{w} = (w_1, w_2)$ and

$$\gamma_{\lambda\mu}(\boldsymbol{w}) = \frac{1}{2}\left(\frac{\partial w_\lambda}{\partial y_\mu} + \frac{\partial w_\mu}{\partial y_\lambda} + \frac{\partial w_\lambda}{\partial y_\sigma}\frac{\partial w_\mu}{\partial y_\sigma}\right) \ . \tag{4.7}$$

The functions w_β are solutions to the problems (P^β_ω). The quantity $\frac{1}{2}j''(0^+)$ determines the value of the topological derivative $\tilde{\mathfrak{T}}_\omega(\boldsymbol{O})$ for the functional $\mathfrak{S}(\Omega_\varepsilon)$.

Proof.
We shall draw upon the following lemma, concerning sensitivity of the solutions of the Neumann problem. Consider the following family of Neumann's problem in the domain Ω_t of boundaries γ_t and Γ, see Fig. 4.1:

$$(P_t) \quad \left| \begin{array}{l} \text{find } u_t \in H_1(\Omega_t)/\mathbb{R} \text{ such that} \\ \displaystyle\int\limits_{\Omega_t} \frac{\partial u_t}{\partial x_\alpha}\frac{\partial \varphi}{\partial x_\alpha}\, dx = \int\limits_{\Gamma} p\varphi\, ds \quad \forall \varphi \in H^1(\Omega_t) \end{array} \right. \tag{4.8}$$

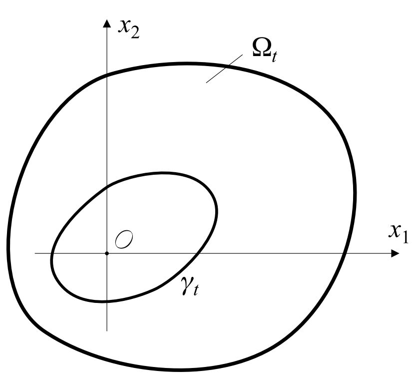

Fig. 4.1. Domain Ω_t with the varying internal boundary γ_t

The domain Ω_t is an image of the domain Ω

$$\Omega_t = T_t(\boldsymbol{V})(\Omega) \ . \tag{4.9}$$

The mapping T_t depends on the velocity field $\boldsymbol{V}$ such that supp $\boldsymbol{V} \cap \Gamma = \emptyset$ and

$$\gamma_t = T_t(\boldsymbol{V})(\gamma) \ . \tag{4.10}$$

Then the shape derivative of u_t, denoted by u_t' (cf. Sokołowski and Zolesio (1992)) is given by the unique solution to the following problem

$$(\overline{P}_t) \quad \left| \begin{array}{l} \text{find } u_t' \in H^1(\Omega_t)\backslash\mathbb{R} \text{ such that} \\ \displaystyle\int\limits_{\Omega_t} \frac{\partial u_t'}{\partial x_\alpha}\frac{\partial \varphi}{\partial x_\alpha}\, dx + \int\limits_{\gamma_t} \left(\frac{\partial u_t}{\partial x_\alpha}\frac{\partial \varphi}{\partial x_\alpha}\right) \boldsymbol{V}\cdot\boldsymbol{\nu}_t\, ds_t = 0 \quad \forall \varphi \in H^2(\Omega_t) \ . \end{array} \right. \tag{4.11}$$

The latter problem can be reformulated as follows, cf. Sokołowski and Zolesio (1992, Prop. 3.2, p. 120)

$$\left| \begin{array}{ll} \Delta u_t' = 0 & \text{in } \Omega_t \\ \dfrac{\partial u_t'}{\partial x_\alpha}\nu_{t\alpha} = \mathrm{div}_{\gamma_t}\left(\boldsymbol{V}\cdot\boldsymbol{\nu}_t \nabla_{\gamma_t} u_t\right) & \text{on } \gamma_t \ . \end{array} \right.$$

Using this result it is easy to see that the derivative of the energy functional associated with the Neumann problem equals,

$$\frac{d\mathfrak{S}(\Omega_t)}{dt} = -\frac{1}{2}\int\limits_{\Gamma} p u_t'\, ds = -\frac{1}{2}\int\limits_{\Omega_t} \frac{\partial u_t'}{\partial x_\alpha}\frac{\partial u_t}{\partial x_\alpha}\, dx$$

hence

$$\frac{d\mathfrak{S}(\Omega_t)}{dt} = \frac{1}{2}\int\limits_{\gamma_t} |\nabla u_t|^2 \boldsymbol{V}\cdot\boldsymbol{\nu}_t\, ds_t \tag{4.12}$$

for

$$\mathfrak{S}(\Omega_t) = \frac{1}{2}\int\limits_{\Omega_t} \frac{\partial u_t}{\partial x_\alpha}\frac{\partial u_t}{\partial x_\alpha}\, dx - \int\limits_{\Gamma} p u_t\, ds = -\frac{1}{2}\int\limits_{\Gamma} p u_t\, ds \ . \tag{4.13}$$

To analyse the behaviour of the function $j(\varepsilon)$ in the point 0^+ we introduce two small parameters: $\varepsilon_0 > 0$ and ε being sufficiently small. The latter parameter will play the role of the parameter t. The following perturbation of the domain will be considered, cf. Fig. 4.2

$$\Omega_{\varepsilon_0+\varepsilon} = \Omega \setminus \overline{\omega}_{\varepsilon_0+\varepsilon} \ . \tag{4.14}$$

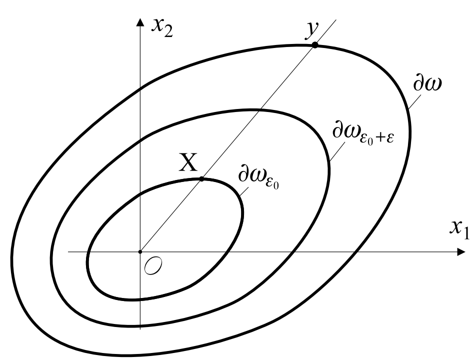

Fig. 4.2. Variation of the boundary: from $\partial\omega_{\varepsilon_0}$ to $\partial\omega_{\varepsilon_0+\varepsilon}$

For any $\varepsilon_0 > 0$, ε_0 small enough and $|\varepsilon| < \dfrac{\varepsilon_0}{2}$, we define the one-to-one mapping

$$\tilde{T}_\varepsilon \; : \; \partial\omega_{\varepsilon_0} \to \partial\omega_{\varepsilon_0+\varepsilon} \tag{4.15}$$

of the form

$$\tilde{x}(\varepsilon) = \tilde{T}_\varepsilon(X) = \left(1 + \frac{\varepsilon}{\varepsilon_0}\right) X \; , \qquad X \in \partial\omega_{\varepsilon_0} \; . \tag{4.16}$$

The mapping $\tilde{T}_\varepsilon$ is extended to a small neighbourhood of $\partial\omega_{\varepsilon_0}$. Let $\varepsilon_0 > 0$ be fixed and let $\eta \in C_0^\infty(\Omega)$,$0 \leq \eta(x) \leq 1$ be a function such that $\eta(x) = 0$ for $|x| < r_0$ and $|x| > r_1$, for some $r_1 > r_0 > 0$ which depend on ε_0, $\eta(x) = 1$ for $x \in \omega_{\varepsilon_0+\varepsilon}$ and for all $\varepsilon \in [0,\varepsilon_0/4]$. Introduce the vector field

$$\boldsymbol{V}(\varepsilon,x) = \eta(x)\frac{x}{\varepsilon_0 + \varepsilon} \; , \quad x \in \Omega, \; |\varepsilon| < \frac{\varepsilon_0}{2} \tag{4.17}$$

and denote by $x(\varepsilon) = T_\varepsilon(\boldsymbol{V})(X)$ a solution to the system

$$\frac{dx(\varepsilon)}{d\varepsilon} = \boldsymbol{V}(\varepsilon,x(\varepsilon)) \tag{4.18}$$

$$x(0) = X \; . \tag{4.19}$$

Then $T_\varepsilon \; : \; \Omega \to \Omega$ is an extension of $\tilde{T}_\varepsilon$ which equals to the identity in the ball $B_{r_0}(0)$ and in the exterior of the ball $B_{r_1}(0)$. Here we assume that r_1 is chosen in such a way that $\partial\omega_{\varepsilon_0+\varepsilon} \subset B_{r_1}(0)$ for all $\varepsilon \in [0,\varepsilon_0/4]$. Using the field $\boldsymbol{V}$ and the material derivative method we can evaluate the shape derivatives with respect to the parameter ε, at $\varepsilon = 0$, of the energy functional $j(\varepsilon_0 + \varepsilon) = \mathfrak{S}(\Omega_{\varepsilon_0+\varepsilon})$ defined below.

The energy functional is given by

$$\mathfrak{S}(\Omega_{\varepsilon_0+\varepsilon}) = \mathfrak{S}(\Omega \backslash \overline{\omega}_{\varepsilon_0+\varepsilon}) = \frac{1}{2} \int\limits_{\Omega_{\varepsilon_0+\varepsilon}} \frac{\partial u_{\varepsilon_0+\varepsilon}}{\partial x_\alpha}\frac{\partial u_{\varepsilon_0+\varepsilon}}{\partial x_\alpha}\, dx - \int\limits_{\partial\Omega} p u_{\varepsilon_0+\varepsilon}\, ds \; . \tag{4.20}$$

Now we can apply the general formula (4.12) to find the topological derivative. For ε sufficiently small we find

$$\frac{d\mathfrak{S}(\Omega_{\varepsilon_0+\varepsilon})}{d\varepsilon} = j'(\varepsilon_0 + \varepsilon) \tag{4.21}$$

and

$$j'(\varepsilon_0 + \varepsilon) = -\frac{1}{2}\mu \int\limits_{\partial\omega_{\varepsilon_0+\varepsilon}} \left(\frac{\partial u_{\varepsilon_0+\varepsilon}}{\partial x_\alpha}\frac{\partial u_{\varepsilon_0+\varepsilon}}{\partial x_\alpha}\right) \frac{x_\sigma(\varepsilon)}{\varepsilon_0 + \varepsilon}\nu_\sigma(x(\varepsilon))\, ds_{\varepsilon_0+\varepsilon} \; . \tag{4.22}$$

According to (3.34) we have

$$\frac{\partial u_{\varepsilon_0+\varepsilon}}{\partial x_\alpha}\frac{\partial u_{\varepsilon_0+\varepsilon}}{\partial x_\alpha} = \epsilon_\beta^0 \left(\left.\frac{\partial W_\beta}{\partial y_\alpha}\right|_{y=\frac{x}{\varepsilon_0+\varepsilon}}\right)\left(\left.\frac{\partial W_\lambda}{\partial y_\alpha}\right|_{y=\frac{x}{\varepsilon_0+\varepsilon}}\right)\epsilon_\lambda^0 + O(\varepsilon_0 + \varepsilon) \; . \tag{4.23}$$

We change the integration domain from $\partial\omega_{\varepsilon_0+\varepsilon}$ to $\partial\omega$. Using (4.18) and the formula: $ds_{\varepsilon_0+\varepsilon} = (\varepsilon_0+\varepsilon)\, ds$ we find

$$j'(\varepsilon_0 + \varepsilon) = -\frac{1}{2}(\varepsilon_0 + \varepsilon)\mu \left(\int\limits_{\partial\omega} \frac{\partial W_\beta}{\partial y_\alpha}\frac{\partial W_\lambda}{\partial y_\alpha} y_\sigma \nu_\sigma\, ds\right)\epsilon_\beta^0\epsilon_\lambda^0 + O\left((\varepsilon_0 + \varepsilon)^2\right) \; . \tag{4.24}$$

Let us note that $j'(0^+) = 0$.

Direct differentiation of (4.24) gives

$$j''(\varepsilon_0) = -\frac{1}{2}\mu \left(\int\limits_{\partial\omega} \frac{\partial W_\beta}{\partial y_\alpha}\frac{\partial W_\lambda}{\partial y_\alpha} y_\sigma \nu_\sigma\, ds\right)\epsilon_\beta^0\epsilon_\lambda^0 + o(\varepsilon_0) \tag{4.25}$$

hence one arrives at

$$j''(0^+) = \epsilon^0_\beta G_{\beta\lambda}\epsilon^0_\lambda \ , \tag{4.26}$$

with

$$G_{\beta\lambda} = -\frac{1}{2}\mu \int\limits_{\partial\omega} \frac{\partial W_\beta}{\partial y_\alpha}\frac{\partial W_\lambda}{\partial y_\alpha} y_\sigma \nu_\sigma \, ds \ . \tag{4.27}$$

To rearrange the matrix $(G_{\beta\lambda})$ to the form (4.6) one should make use of (3.30) and of the formula:

$$\delta_{\lambda\mu}|\omega| = \int\limits_{\omega} \frac{\partial y_\lambda}{\partial y_\alpha}\frac{\partial y_\beta}{\partial y_\alpha}\, dy = \int\limits_{\partial\omega} y_\beta \frac{\partial y_\lambda}{\partial \nu}\, ds = \int\limits_{\partial\omega} y_\beta \nu_\lambda \, ds \ . \tag{4.28}$$

Hence

$$\int\limits_{\partial\omega} y_\sigma \nu_\sigma \, ds = \delta_{\sigma\sigma}|\omega| = 2|\omega| \ , \tag{4.29}$$

which confirms the equation (4.6).

Definition 4.1.
Let the boundary $\partial\omega$ be piece-wise smooth. The domain ω will be called star-shaped if one can place the origin $\boldsymbol{O}$ of the coordinate system (y_1,y_2) such that $\boldsymbol{O} \in \omega$ and

$$y_\alpha(s)\nu_\alpha(s) > 0 \qquad \text{a.e.} \quad \text{on } \partial\omega \ , \tag{4.30}$$

where s is a natural parameter of $\partial\omega$.

The convex domains are star-shaped. One can easy draw star-shaped domains which are not convex, as well as domains which are not star-shaped.

Let us consider the functional

$$E(\Omega_\varepsilon) = \int\limits_{\Omega_\varepsilon} \mu \frac{\partial u_\varepsilon}{\partial x_\alpha}\frac{\partial u_\varepsilon}{\partial x_\alpha}\, dx \tag{4.31}$$

or $E(\Omega_\varepsilon) = -2\mathfrak{S}(\Omega_\varepsilon)$. Passing to the right-hand limit: $\varepsilon_0 \to 0^+$ in the expansion (4.3) gives

$$E(\Omega_\varepsilon) = E(\Omega) - \varepsilon^2 \epsilon^0_\beta G_{\beta\lambda}\epsilon^0_\lambda + o(\varepsilon^2) \ , \tag{4.32}$$

where

$$E(\Omega) = \int\limits_{\Omega} \mu \frac{\partial v}{\partial x_\alpha}\frac{\partial v}{\partial x_\alpha}\, dx \ . \tag{4.33}$$

We shall prove

Theorem 4.2.
Assume that the domain ω is star-shaped. Then, nucleation of a small opening ω_ε increases the energy:

$$E(\Omega_\varepsilon) > E(\Omega) \ . \tag{4.34}$$

Proof.
We have

$$E(\Omega_\varepsilon) - E(\Omega) = \frac{1}{2}\varepsilon^2 \mu \int\limits_{\partial\omega} \frac{\partial W}{\partial y_\alpha}\frac{\partial W}{\partial y_\alpha} y_\sigma \nu_\sigma \, ds + o(\varepsilon^2) \ , \tag{4.35}$$

where $W = \epsilon^0_\beta W_\beta$. Taking into account the boundary condition (3.32) one finds

$$\frac{\partial W_\beta}{\partial y_\alpha} = \frac{\partial W_\beta}{\partial \nu}\nu_\alpha + \frac{\partial W_\beta}{\partial s}\tau_\alpha = \frac{\partial W_\beta}{\partial s}\tau_\alpha \ , \tag{4.36}$$

where (τ_α) are components of the unit vector tangent to $\partial\omega$. Thus we have

$$\frac{\partial W}{\partial y_\alpha}\frac{\partial W}{\partial y_\alpha} = \left(\frac{\partial W}{\partial s}\right)^2 \tag{4.37}$$

and

$$E(\Omega_\varepsilon) - E(\Omega) = \frac{1}{2}\varepsilon^2\mu \int_{\partial\omega} \left(\frac{\partial W}{\partial s}\right)^2 y_\alpha \nu_\alpha \, ds + o(\varepsilon^2) \ . \tag{4.38}$$

The constant functions do not satisfy the condition (3.33) in the problem $(\widetilde{P}^\beta_\omega)$. Thus $W \neq$ const and $\partial W/\partial s$ is not identically zero along $\partial\omega$. The condition (4.30) implies the inequality (4.34). ■

By (4.36) and (4.37) the formula (4.27) can be put in the form

$$G_{\beta\lambda} = -\frac{1}{2}\mu \int_{\partial\omega} \frac{\partial W_\beta}{\partial s}\frac{\partial W_\lambda}{\partial s} y_\sigma \nu_\sigma \, ds \ . \tag{4.39}$$

Let us introduce the "force" fields

$$\sigma^\alpha(u) = \mu \frac{\partial u}{\partial y_\alpha} \tag{4.40}$$

and the boundary forces: $\sigma_\nu = \sigma^\alpha \nu_\alpha$ and $\sigma_\tau = \sigma^\alpha \tau_\alpha$. The boundary condition (3.32) implies $\sigma_\nu(W_\beta) = 0$. Thus the formula (4.39) can be expressed in terms of the "force" fields as follows

$$G_{\beta\lambda} = -\frac{1}{2\mu} \int_{\partial\omega} \sigma_\tau(W_\beta)\sigma_\tau(W_\lambda) y_\sigma \nu_\sigma \, ds \tag{4.41}$$

and the energy increment is given by

$$E(\Omega_\varepsilon) - E(\Omega) = \frac{1}{2\mu}\varepsilon^2 \int_{\partial\omega} [\sigma_\tau(W)]^2 \, y_\sigma \nu_\sigma \, ds + o(\varepsilon^2) \ . \tag{4.42}$$

The result (4.32) has its counterpart in the literature. By applying the method of compound asymptotics Maz'ya and Nazarov (1987) found the expansion of the form

$$E(\Omega_\varepsilon) = E(\Omega) - \varepsilon^2 \epsilon^0_\beta m_{\beta\lambda} \epsilon^0_\lambda + o(\varepsilon^{2+0}) \ , \tag{4.43}$$

where $m_{\beta\lambda}$ are coefficients in the representation (3.26) and can be computed by (3.27), cf. Appendix A. The derivation of the formula (4.43) can be found in Appendix B.

Let us consider the relations between the matrices $\boldsymbol{G}$ and $\boldsymbol{m}$. By Theorem 4.2 the star-shaped property implies positive definiteness of the matrix $(-\boldsymbol{G})$. No obvious reason is seen why this matrix should always be positive definite. On the other hand, the matrix $(-\boldsymbol{m})$ is always positive definite, see Appendix A. Thus result (4.43) of Maz'ya and Nazarov (1987) means that drilling an opening always results in the increase of the energy, i.e. irrespective of its shape. On the other hand the Theorem 4.2 states that drilling a star-shaped opening increases the energy, and not more.

A natural question arises, whether the matrices $\boldsymbol{G}$ and $\boldsymbol{m}$ coincide for some shapes of ω. The answer is formulated as

Theorem 4.3.
In the case when ω is an ellipse the matrices $\boldsymbol{G}$ and $\boldsymbol{m}$ coincide.

The proof is divided into small steps.

Step 1. Formulae for W_1,W_2

Let us consider problems ($\tilde{P}^\beta_\omega$;3.31-3.33). We assume that the function of complex variable

$$\wp(\zeta) = \ell\left(\zeta + \frac{c}{\zeta} + \frac{c_2}{\zeta^2} + \frac{c_3}{\zeta^3} + ...\right) \tag{4.44}$$

depending on real parameters ℓ,c,c_i ,$i = 2$,3,..., transforms $\mathbb{R} \setminus \omega$ onto the exterior of the unit circle in the complex plane.

The assumption requires some explanations. With the choice of $\ell \cdot \exp(i\lambda)$ instead of ℓ it follows that the latter transformation works for the domain ω rotated by the angle λ. On the other hand, we can rotate the coordinate system (y_1,y_2) before the transformation since we deal with the only one domain ω. Therefore, without any loss of the generality, we can assume that $\lambda = 0$. The choice of $c \in \mathbb{R}$ is a standard assumption while selecting in a more general way c_i with complex values does not really change the formulae obtained in the sequel.

For $c_i = 0$,$i = 2$,3,..., and $c \in (0,1)$, ω is an ellipse with the major and minor axes coinciding with the coordinate system in the plane. In particular, we have

$$\ell = \frac{1}{2}(a+b) \ , \quad c = \frac{a-b}{a+b}, \quad a > b,$$

where a,b are the half-lengths of major and minor axis of the ellipse ω. For $c = 0$, $a = b$ and the ellipse ω becomes a circle. On the other hand, the ellipse tends to a cut (crack) with the limit passage $c \to 1$.

Schiffer and Szegö (1949) have shown that for $\ell \in \mathbb{R}$ the solutions W_1 ,W_2 of problems ($\tilde{P}^1_\omega$), ($\tilde{P}^2_\omega$) take the following form

$$W_1 = \ell\Re\left[\zeta + \frac{1}{\zeta}\right], \quad W_2 = -\ell\Re\left[i\left(\zeta - \frac{1}{\zeta}\right)\right] \ , \zeta = \wp^{-1}(z) \tag{4.45}$$

Step 2. $G_{\alpha\beta}$ for an arbitrary opening ω

The formulae for $G_{\alpha\beta}$ are given by (4.27), that is

$$G_{\alpha\beta} = \int_{\partial\omega} \nabla W_\alpha \cdot \nabla W_\beta \, \boldsymbol{y} \cdot \boldsymbol{\nu} ds \ . \tag{4.46}$$

We have

$$\boldsymbol{y} \cdot \boldsymbol{\nu} = \Re\left(z \cdot \exp(-i\alpha)\right) \ , \tag{4.47}$$

where $z = \wp(\zeta)$. We make use of the following formula (see Muskhelishvili N.I. (1975), formula (49.3))

$$\exp(-i\alpha) = \frac{1}{\rho} \cdot \frac{1}{|\wp'(\zeta)|} \cdot \overline{\zeta} \cdot \overline{\wp'(\zeta)} \ , \tag{4.48}$$

where $\rho = |\zeta|$. Therefore,

$$\boldsymbol{y} \cdot \boldsymbol{\nu} = \frac{1}{\rho} \cdot \frac{1}{|\wp'(\zeta)|} \cdot \Re\left[\overline{\zeta} \cdot \overline{\wp'(\zeta)} \cdot \wp(\zeta)\right] \tag{4.49}$$

On the contour $\partial\omega$ we have $\rho = 1$, and ds takes the form

$$ds = \sqrt{(dy_1)^2 + (dy_2)^2} = |dz| = |\wp'(\zeta)d\zeta| = |\wp'(\zeta)|\rho d\vartheta \ , \tag{4.50}$$

because $|d\zeta| = \rho d\vartheta$, and we evaluate

$$\boldsymbol{y} \cdot \boldsymbol{\nu} ds = \Re\left[\overline{\zeta} \cdot \overline{\wp'(\zeta)} \cdot \wp(\zeta)\right]_{\rho=1} d\vartheta \ . \tag{4.51}$$

Next step is to evaluate $\|\nabla W_1\|^2$,$\nabla W_1 \cdot \nabla W_2$,$\|\nabla W_2\|^2$ on the contour $\rho = 1$.

For the function $f(z) = \ell\left(\zeta + \frac{1}{\zeta}\right) = W_1(y_1,y_2) + iV(y_1,y_2)$, $\zeta = \wp^{-1}(z)$, compare with (4.45), taking into account that $f(z)$ is a holomorphic function, it follows that

$$f'(z) = \frac{\partial W_1}{\partial y_1} - i\frac{\partial W_1}{\partial y_2}$$

hence

$$\|\nabla W_1\|^2 = |f'(z)|^2 = |f'(\zeta)|^2 \frac{1}{|\wp'(\zeta)|^2} ,$$

where $f'(\zeta) = \ell\left(1 - \frac{1}{\zeta^2}\right)$. Thus

$$\|\nabla W_1\|_{\rho=1} = \left\{ \frac{4\ell^2 \sin^2\vartheta}{|\wp'(\zeta)|^2} \right\}_{|\rho=1} \tag{4.52}$$

In the similar way we evaluate

$$\{\nabla W_1 \cdot \nabla W_2\}_{\rho=1} = \left\{ -\frac{2\ell^2 \sin 2\vartheta}{|\wp'(\zeta)|^2} \right\}_{|\rho=1} \tag{4.53}$$

and

$$\|\nabla W_2\|_{\rho=1} = \left\{ \frac{4\ell^2 \cos^2\vartheta}{|\wp'(\zeta)|^2} \right\}_{|\rho=1} \tag{4.54}$$

Using (4.51)-(4.54) in (4.46) we find

$$G_{11} = -2\mu\ell^2 \int_0^{2\pi} \sin^2\vartheta \; h(\vartheta)d\vartheta \ , \tag{4.55}$$

$$G_{12} = 2\mu\ell^2 \int_0^{2\pi} \sin\vartheta\cos\vartheta \; h(\vartheta)d\vartheta \ , \tag{4.56}$$

$$G_{22} = -2\mu\ell^2 \int_0^{2\pi} \cos^2\vartheta \; h(\vartheta)d\vartheta \ , \tag{4.57}$$

where

$$h(\vartheta) = \Re\left[\exp(-i\vartheta)\frac{\wp(\zeta)}{\wp'(\zeta)}\right]_{\rho=1}$$

Step 3. $G_{\alpha\beta}$ for an ellipse

For an ellipse

$$h(\vartheta) = \Re\left[\exp(-i\vartheta)\frac{\exp(i\vartheta) + c\exp(-i\vartheta)}{\exp(i\vartheta) - c\exp(-i\vartheta)}\right] = \frac{\beta}{\beta^2\cos^2\vartheta + \sin^2\vartheta} ,$$

where we denote $\beta = \frac{1-c}{1+c}$. Therefore, simple calculations show that

$$G_{11} = -2\pi\mu\ell^2(1-c), \quad G_{12} = 0, \quad G_{22} = -2\pi\mu\ell^2(1+c) \ . \tag{4.58}$$

The same formulae can be obtained for the mass matrix for an ellipse (Schiffer, Szegö 1949) which confirms the equality: $m_{\alpha\beta} = G_{\alpha\beta}$ for the ellipse.

In general, however, $\boldsymbol{G} \neq \boldsymbol{m}$ in the case of ω not being any ellipse since $\boldsymbol{G}$ depends on ℓ,c and all the coefficients c_k ,$k \geq 2$. The mass matrix $\boldsymbol{m}$ depends only on ℓ,c and is independent of of the coefficients c_k ,$k \geq 2$.

5. The plane problem of linear elasticity in the domain $\Omega\backslash\overline{\omega}_\varepsilon$. Asymptotic expansion of the solution $\boldsymbol{u}^\varepsilon$

5.1. Formulation of the problem

We consider a plane elasticity problem in the domain $\Omega\backslash\overline{\omega}_\varepsilon$, see. Fig. 2.1. Both plane stress and plane strain cases will be dealt with. Our main unknown is the displacement field $\boldsymbol{u}^\varepsilon(x) = (u_1^\varepsilon(x), u_2^\varepsilon(x))$; $u_\alpha^\varepsilon(x)$ represents the displacement of point x in the x_α direction. Let $\tilde{\boldsymbol{v}} = (\tilde{v}_1, \tilde{v}_2)$ be a certain vector field. Let us define the strains associated with this field

$$\epsilon_{\alpha\beta}(\tilde{\boldsymbol{v}}) = \frac{1}{2}\left(\frac{\partial\tilde{v}_\alpha}{\partial x_\beta} + \frac{\partial\tilde{v}_\beta}{\partial x_\alpha}\right) . \tag{5.1}$$

The stresses associated with strains (5.1) are given by the constitutive relationships

$$\sigma^{\alpha\beta}(\tilde{\boldsymbol{v}}) = A^{\alpha\beta\lambda\mu}\epsilon_{\lambda\mu}(\tilde{\boldsymbol{v}}) , \tag{5.2}$$

where $(A^{\alpha\beta\lambda\mu})$ form the tensor $\boldsymbol{A}$ of reduced moduli of elasticity. The term: *reduced* concerns only the case of plane stress.

The material occupying the domain $\Omega\backslash\overline{\omega}_\varepsilon$ is assumed to be homogeneous, the components $A^{\alpha\beta\lambda\mu}$ are constants. They obey the following symmetry rules

$$A^{\alpha\beta\lambda\mu} = A^{\lambda\mu\alpha\beta} , \qquad A^{\alpha\beta\lambda\mu} = A^{\beta\alpha\lambda\mu} = A^{\beta\alpha\mu\lambda} \tag{5.3}$$

and satisfy the following conditions of positive definiteness

$$A^{\alpha\beta\lambda\mu}\kappa_{\alpha\beta}\kappa_{\lambda\mu} \geq c\kappa_{\alpha\beta}\kappa_{\alpha\beta} \qquad \forall\boldsymbol{\kappa} \in \mathbb{M}_s^2 , \tag{5.4}$$

where

$$\mathbb{M}_s^2 = \{\boldsymbol{\kappa}|\ \boldsymbol{\kappa} = (\kappa_{\alpha\beta}) , \kappa_{\alpha\beta} = \kappa_{\beta\alpha}\}$$

and $c > 0$.

The stresses associated with the unknown displacement field $\boldsymbol{u}^\varepsilon$ satisfy the homogeneous equations of equilibrium

$$\frac{\partial\sigma^{\alpha\beta}(\boldsymbol{u}^\varepsilon)}{\partial x_\beta} = 0 . \tag{5.5}$$

The body forces are omitted. The loading is applied to the boundary $\partial\Omega$

$$\sigma^{\alpha\beta}(\boldsymbol{u}^\varepsilon)n_\beta = p^\alpha(x) , \qquad x \in \partial\Omega ; \tag{5.6}$$

here p^α represent intensities of the boundary. The boundary of the opening is unloaded:

$$\sigma^{\alpha\beta}(\boldsymbol{u}^\varepsilon)\nu_\beta = 0 \qquad \text{on } \partial\omega_\varepsilon ; \tag{5.7}$$

note that the vector $\boldsymbol{\nu}^\varepsilon = (\nu_1, \nu_2)$ is outward normal to $\partial\omega_\varepsilon$, see Fig. 2.1.

The strong formulation of the equilibrium problem has the form

(P_ε) | find $\boldsymbol{u}^\varepsilon \in H^{2+k}(\Omega_\varepsilon, \mathbb{R}^2)$ that satisfies: the equilibrium equations (5.5), the constitutive relations (5.2) for $\tilde{\boldsymbol{v}} = \boldsymbol{u}^\varepsilon$ and the boundary conditions (5.6), (5.7); $k \geq 2$.

The variational form of the equations (5.5)

$$\int_{\Omega_\varepsilon} \sigma^{\alpha\beta}(\boldsymbol{u}^\varepsilon)\epsilon_{\alpha\beta}(\tilde{\boldsymbol{v}})\,dx = \int_{\partial\Omega} p^\alpha\tilde{\boldsymbol{v}}_\alpha\,ds \quad \forall\tilde{\boldsymbol{v}} \in H^{2+k}(\Omega_\varepsilon; \mathbb{R}^2) \tag{5.8}$$

implies the solvability conditions of (P_ε)

$$\int_{\partial\Omega} p^\alpha \tilde{v}_\alpha \, ds = 0 \qquad \forall \tilde{\boldsymbol{v}} \in \mathcal{R} \tag{5.9}$$

where

$$\mathcal{R} = \{\tilde{\boldsymbol{v}} | \; \epsilon_{\alpha\beta}(\tilde{\boldsymbol{v}}) = 0\} \ . \tag{5.10}$$

The set $\mathcal{R}$ consists of translations and infinitesimal rotations, cf. Nečas and Hlavaček (1981):

$$\mathcal{R} = \left\{\tilde{\boldsymbol{v}} | \; \tilde{v}_\alpha = v^0_\alpha + \phi \, e_{\alpha\beta} \, x_\beta \right\} \ , \tag{5.11}$$

where v^0_α, ϕ are constants and $(e_{\alpha\beta})$ represents the Ricci tensor of the components

$$e_{11} = e_{22} = 0 \ , \qquad e_{12} = -e_{21} = 1 \ . \tag{5.12}$$

Substitution of the constitutive relations

$$\sigma^{\alpha\beta}(\boldsymbol{u}^\varepsilon) = A^{\alpha\beta\lambda\mu} \epsilon_{\lambda\mu}(\boldsymbol{u}^\varepsilon) \tag{5.13}$$

into (5.8) gives the weak formulation of the problem (P_ε). The solution $\boldsymbol{u}^\varepsilon$ is determined up to additive terms of the set $\mathcal{R}$, while the fields $\epsilon_{\lambda\mu}(\boldsymbol{u}^\varepsilon)$ and $\sigma_{\lambda\mu}(\boldsymbol{u}^\varepsilon)$ are uniquely determined provided that Ω and ω_ε satisfy appropriate regularity assumptions, see Duvaut and Lions (1972).

5.2. Asymptotic expansion of $\boldsymbol{u}^\varepsilon$

We shall apply the method of compound asymptotics to the problem (5.8), see Maz'ya et al. (1991). Let us consider first the problem for the domain without an opening. The unknown displacement field $\boldsymbol{v}(x)$ is the solution to the problem

(P_0)

find $\boldsymbol{v} \in H^{2+k}(\Omega; \mathbb{R}^2)$ such that the stresses

$$\sigma^{\alpha\beta}(\boldsymbol{v}) = A^{\alpha\beta\lambda\mu} \epsilon_{\lambda\mu}(\boldsymbol{v}) \tag{5.14}$$

satisfy the equlibrium equations

$$\frac{\partial \sigma^{\alpha\beta}(\boldsymbol{v})}{\partial x_\beta} = 0 \tag{5.15}$$

and the boundary conditions

$$\sigma^{\alpha\beta}(\boldsymbol{v}) n_\beta = p^\alpha(x) \ , \qquad x \in \partial\Omega \ . \tag{5.16}$$

The asymptotic expansion of the solution $\boldsymbol{u}^\varepsilon$ of the (P_ε) problem reads

$$\boldsymbol{u}^\varepsilon(x) = \boldsymbol{v}(x) + \varepsilon \boldsymbol{w}\left(\frac{x}{\varepsilon}\right) + \varepsilon^2 \boldsymbol{z}(x) + \varepsilon^2 O\left(\left|\left|\frac{x}{\varepsilon}\right|\right|^{-1}\right) \ , \tag{5.17}$$

where $\boldsymbol{v}$ is the solution to the problem (P_0). The field $\boldsymbol{v}$ does not satisfy the condition (5.7). By adding the term $\varepsilon \boldsymbol{w}\left(\dfrac{x}{\varepsilon}\right)$ we satisfy this condition. The strains associated with $\boldsymbol{u}^\varepsilon$ can be expanded as follows

$$\epsilon_{\lambda\mu}(\boldsymbol{u}^\varepsilon(x)) = \epsilon_{\lambda\mu}(\boldsymbol{v}(x)) + \epsilon_{\lambda\mu}(\boldsymbol{w})\left(\frac{x}{\varepsilon}\right) + O(\varepsilon) \ . \tag{5.18}$$

Consequently, the stresses can be represented by

$$\sigma^{\alpha\beta}(\boldsymbol{u}^\varepsilon(x)) = A^{\alpha\beta\lambda\mu}\left[\epsilon_{\lambda\mu}(\boldsymbol{v}(x)) + \epsilon_{\lambda\mu}(\boldsymbol{w})\left(\frac{x}{\varepsilon}\right)\right] + O(\varepsilon) \ . \tag{5.19}$$

the fields $\epsilon_{\lambda\mu}(\boldsymbol{w})$ being defined by

$$\epsilon_{\lambda\mu}(\boldsymbol{w}) = \frac{1}{2}\left(\frac{\partial w_\lambda}{\partial y_\mu} + \frac{\partial w_\mu}{\partial y_\lambda}\right) \tag{5.20}$$

for the fields $\boldsymbol{w} = (w_\lambda(y_1,y_2))$ given in the domain $\mathbb{R}^2\backslash\overline{\omega}$ parametrized by the coordinates y_1, y_2. Along the boundary $\partial\omega_\varepsilon$ we have

$$\epsilon_{\lambda\mu}(\boldsymbol{v}) = \epsilon_{\lambda\mu}(\boldsymbol{v})(\boldsymbol{O}) + O(\varepsilon) \ .$$

Thus the condition (5.7) is approximated by

$$A^{\alpha\beta\lambda\mu}\epsilon_{\lambda\mu}(\boldsymbol{w})\nu_\beta = -A^{\alpha\beta\lambda\mu}\epsilon_{\lambda\mu}(\boldsymbol{v})(\boldsymbol{O})\nu_\beta \tag{5.21}$$

for $\dfrac{x}{\varepsilon} \in \partial\omega_\varepsilon$ or $y \in \partial\omega$.
The equilibrium equations (5.5) imply

$$\frac{\partial\sigma^{\alpha\beta}(\boldsymbol{v})}{\partial x_\beta} + \frac{1}{\varepsilon}\frac{\partial}{\partial y_\beta}\left[A^{\alpha\beta\lambda\mu}\epsilon_{\lambda\mu}(\boldsymbol{w})\right]\Big|_{y=\frac{x}{\varepsilon}} + O(\varepsilon) = 0 \ . \tag{5.22}$$

Thus we confirm the equations (5.15) and find the governing equations for $\boldsymbol{w}$

$$\frac{\partial}{\partial y_\beta}\left[A^{\alpha\beta\lambda\mu}\epsilon_{\lambda\mu}(\boldsymbol{w})\right] = 0 \qquad \text{in} \quad \mathbb{R}^2\backslash\overline{\omega} \ . \tag{5.23}$$

The field $\boldsymbol{w}$ is the solution of the problem on the rescaled domain $\mathbb{R}^2\backslash\overline{\omega}$:

(P_ω)	find $\boldsymbol{w}$ defined in $\mathbb{R}^2\backslash\overline{\omega}$ such that $\lvert\boldsymbol{w}\rvert \to 0$ if $\lVert y\rVert \to \infty$ and $\boldsymbol{w}$ satisfies the equations (5.23) along with the boundary conditions (5.21).

The field $\boldsymbol{w}$ depends in the linear way on the values of strains $\epsilon_{\lambda\mu}(\boldsymbol{v})$ at point $x = \boldsymbol{O}$. Let us introduce notation

$$\epsilon^0_{\lambda\mu} = \epsilon_{\lambda\mu}(\boldsymbol{v})(\boldsymbol{O}) \tag{5.24}$$

to shorten further formulae. The solution $\boldsymbol{w}$ of the problem (P_ω) can be decomposed according to the formula

$$\boldsymbol{w}(y) = \epsilon^0_{\lambda\mu}\boldsymbol{\chi}^{(\lambda\mu)}(y) \ , \tag{5.25}$$

the functions $\boldsymbol{\chi}^{(\lambda\mu)}$ being solutions to the problems

$$\left(P^{(\lambda\mu)}_\omega\right) \quad \left|\ \begin{array}{lll} \text{find } \boldsymbol{\chi}^{(\lambda\mu)} \text{ defined on } \mathbb{R}^2\backslash\omega \text{ such that} & & \\ \dfrac{\partial}{\partial y_\beta}\left[A^{\alpha\beta\gamma\delta}\epsilon_{\gamma\delta}(\boldsymbol{\chi}^{(\lambda\mu)})\right] = 0 & \text{in } \mathbb{R}^2\backslash\overline{\omega} \ , & (5.26) \\ A^{\alpha\beta\gamma\delta}\epsilon_{\gamma\delta}(\boldsymbol{\chi}^{(\lambda\mu)})\nu_\beta = -A^{\alpha\beta\lambda\mu}\nu_\beta & \text{on } \partial\omega \ , & (5.27) \\ \boldsymbol{\chi}^{(\lambda\mu)} \to \boldsymbol{O} & \text{if } \lVert y\rVert \to \infty \ . & (5.28) \end{array}\right.$$

The variational formulation and the existence and uniqueness results of the exterior problems of linear elasticity are given by Giroire (1976) and Bonnemay (1979). In the case of tractions prescribed on $\partial\omega$ the variational formulation can be found in Bonnemay (1979, Ch I, Sec. 3, page 8). The space of solutions is defined by

$$V(\mathbb{R}^2\backslash\overline{\omega}) = (W^1_0(\mathbb{R}^2\backslash\overline{\omega})/\mathbb{R})^2 \ ,$$

where

$$W_0^1(\mathbb{R}^2\backslash\overline{\omega}) = \{v \in L^2_{loc}(\mathbb{R}^2\backslash\overline{\omega}) | \frac{v}{(1+\|y\|^2)^{1/2}\ln(2+\|y\|^2)} \in L^2(\mathbb{R}^2\backslash\overline{\omega}) \text{ and } \frac{\partial v}{\partial y_\alpha} \in L^2(\mathbb{R}^2\backslash\overline{\omega})\} ,$$

with the norm

$$\|v\|^2_{1,0,\mathbb{R}^2\backslash\overline{\omega}} = \left\| \frac{v}{(1+\|y\|^2)^{1/2}\ln(2+\|y\|^2)} \right\|^2_{0,\mathbb{R}^2\backslash\overline{\omega}} + \|\nabla v\|^2_{0,\mathbb{R}^2\backslash\overline{\omega}} .$$

Here $\|\cdot\|_{0,\mathbb{R}^2\backslash\overline{\omega}}$ means the $L^2(\mathbb{R}^2\backslash\overline{\omega})$ norm.
The following expansion holds, see Appendix B

$$\chi_\alpha^{(\lambda\mu)} = C_{\alpha\beta}^{\lambda\mu}\frac{y_\beta}{\|y\|} + O\left(\frac{1}{\|y\|^2}\right) , \tag{5.29}$$

where $C_{\alpha\beta}^{\lambda\mu}$ are certain constants. Let us write down the weak formulation of $(P_\omega^{(\lambda\mu)})$. Let the circle B_R encompass the domain ω, as in Sec. 3. The following variational equation holds

$$A^{\alpha\beta\lambda\mu}\int\limits_{B_R\backslash\overline{\omega}} \epsilon_{\lambda\mu}(\boldsymbol{\chi}^{(\kappa\delta)})\epsilon_{\alpha\beta}(\widetilde{\boldsymbol{v}})\,dy = A^{\alpha\beta\lambda\mu}\int\limits_{\Gamma_R} \underline{\epsilon_{\lambda\mu}(\boldsymbol{\chi}^{(\kappa\delta)})}n_\beta\widetilde{v}_\alpha\,ds + A^{\alpha\beta\kappa\delta}\int\limits_{\partial\omega}\widetilde{v}_\alpha\nu_\beta\,ds \tag{5.30}$$

The term underscored in (5.30) is of order $O(\|y\|^{-2})$. Thus the integral over $\Gamma_R = \partial B_R$ vanishes if $R\to\infty$ and the fields $\widetilde{v}_\alpha$ are sufficiently smooth and of compact support.
Passing to infinity: $R\to\infty$ gives the variational equation

$$A^{\alpha\beta\lambda\mu}\int\limits_{\mathbb{R}^2\backslash\overline{\omega}} \epsilon_{\lambda\mu}\left(\boldsymbol{\chi}^{(\kappa\delta)}\right)\epsilon_{\alpha\beta}(\widetilde{\boldsymbol{v}})\,dy = A^{\alpha\beta\kappa\delta}\int\limits_{\partial\omega}\widetilde{v}_\alpha\nu_\beta\,ds \tag{5.31}$$

valid for $\widetilde{v}_\alpha \in V(\mathbb{R}^2\backslash\overline{\omega})$.

Let us prove that the problems $(P_\omega^{(\lambda\mu)})$ are uniquely solvable. It is sufficient to show that the right-hand side of (5.31) vanishes for $\widetilde{\boldsymbol{v}} \in \mathcal{R}$. Let us put first $\widetilde{v}_\alpha = v_\alpha^0 = \text{const}$. Then

$$\int\limits_{\partial\omega} v_\alpha^0\nu_\beta\,ds = v_\alpha^0\int\limits_\omega \frac{\partial}{\partial y_\beta}(1)\,dy = 0 . \tag{5.32}$$

Let us take now $\widetilde{v}_\alpha = e_{\alpha\sigma}y_\sigma$ or $\widetilde{v}_1 = y_2$, $\widetilde{v}_2 = -y_1$. The right-hand side of (5.31) equals, see (4.28)

$$A^{\alpha\beta\kappa\delta}e_{\alpha\sigma}\int\limits_{\partial\omega} y_\sigma\nu_\beta\,ds = A^{\alpha\beta\kappa\delta}e_{\alpha\sigma}\delta_{\sigma\beta}|\omega| = A^{\alpha\beta\kappa\delta}e_{\alpha\beta} = 0 ,$$

due to the symmetry property (5.3).

Let us reformulate the problems $(P_\omega^{(\lambda\mu)})$. Assume that we know the fields $\boldsymbol{E}^{(\lambda\mu)}$ defined on $\mathbb{R}^2$ such that

$$\epsilon_{\alpha\beta}(\boldsymbol{E}^{(\lambda\mu)}) = \delta_{\alpha\beta}^{\lambda\mu} , \tag{5.33}$$

where the components

$$\delta_{\alpha\beta}^{\lambda\mu} = \frac{1}{2}\left(\delta_\alpha^\lambda\delta_\beta^\mu + \delta_\alpha^\mu\delta_\beta^\lambda\right) \tag{5.34}$$

form a unit tensor in the space of tensors of fourth rank, of symmetry properties (5.3). Then the condition (5.27) can be rewritten in the form

$$A^{\alpha\beta\lambda\mu}\epsilon_{\lambda\mu}\left(\boldsymbol{\chi}^{(\kappa\delta)} + \boldsymbol{E}^{(\kappa\delta)}\right)\nu_\beta = 0 . \tag{5.35}$$

Let $\boldsymbol{e}_1$, $\boldsymbol{e}_2$ are basis vectors of the coordinate system (y_1,y_2). The fields $\boldsymbol{E}^{(\lambda\mu)}$ of property (5.33) are given by

$$\boldsymbol{E}^{(\lambda\mu)}(y) = \frac{1}{2}(y_\lambda \boldsymbol{e}_\mu + y_\mu \boldsymbol{e}_\lambda) \tag{5.36}$$

or

$$\boldsymbol{E}^{(\lambda\mu)}(y) = E^{(\lambda\mu)}_\alpha(y)\boldsymbol{e}_\alpha \tag{5.37}$$

with

$$E^{(\lambda\mu)}_\alpha(y) = \frac{1}{2}(y_\lambda \delta_{\mu\alpha} + y_\mu \delta_{\lambda\alpha}) \ . \tag{5.38}$$

Let us define new fields

$$\boldsymbol{\psi}^{(\alpha\beta)}(y) = \boldsymbol{E}^{(\alpha\beta)}(y) + \boldsymbol{\chi}^{(\alpha\beta)}(y) \ . \tag{5.39}$$

They are solutions to the following problems

$$\left(\widetilde{P}^{(\lambda\mu)}_\omega\right) \quad \left| \begin{array}{lll} \text{find } \boldsymbol{\psi}^{(\kappa\delta)} \text{ defined on } \mathbb{R}^2\backslash\overline{\omega} \text{ such that} & & \\ \dfrac{\partial}{\partial y_\beta}\left[A^{\alpha\beta\lambda\mu}\epsilon_{\lambda\mu}(\boldsymbol{\psi}^{(\kappa\delta)})\right] = 0 & \text{in } \mathbb{R}^2\backslash\overline{\omega} & (5.40) \\ A^{\alpha\beta\lambda\mu}\epsilon_{\lambda\mu}(\boldsymbol{\psi}^{(\kappa\delta)})\nu_\beta = 0 & \text{on } \partial\omega & (5.41) \\ \boldsymbol{\psi}^{(\kappa\delta)} \to \boldsymbol{E}^{(\kappa\delta)} \ , & \text{if } \|y\| \to \infty \ . & (5.42) \end{array} \right.$$

Well posedness of the problems $(P^{(\lambda\mu)}_\omega)$ imply that the functions $\boldsymbol{\psi}^{(\kappa\delta)}$ are uniquely determined.

Substitution of (5.25) into the asymptotic formula (5.17) gives

$$\boldsymbol{u}^\varepsilon(x) = \boldsymbol{v}(x) + \varepsilon\epsilon^0_{\lambda\mu}\boldsymbol{\chi}^{(\lambda\mu)}\left(\frac{x}{\varepsilon}\right) + \varepsilon^2\boldsymbol{z}(x) + \dots \ . \tag{5.43}$$

In the vicinity of the domain ω_ε the function $\boldsymbol{v}(x)$ can be expanded by the Taylor series

$$\boldsymbol{v}(x) = \underline{\boldsymbol{v}(\boldsymbol{O})} + \epsilon^0_{\lambda\mu}\boldsymbol{E}^{(\lambda\mu)}(x) + \underline{\phi\boldsymbol{r}(x)} + O(\varepsilon^2) \ , \tag{5.44}$$

where ϕ represents a rigid rotation of the neighbourhood of $\boldsymbol{O}$ and $\boldsymbol{r}(x) = \frac{1}{2}(-x_2\boldsymbol{e}_1 + x_1\boldsymbol{e}_2)$. The terms underscored in (5.44) belong to the set $\mathcal{R}$. Hence we have

$$\epsilon_{\alpha\beta}(\widetilde{\boldsymbol{v}}) = \epsilon^0_{\alpha\beta} + O(\varepsilon) \ . \tag{5.45}$$

Thus the strains associated with the field $\boldsymbol{u}^\varepsilon$ can be expanded as follows

$$\epsilon_{\alpha\beta}(\boldsymbol{u}^\varepsilon) = \epsilon^0_{\alpha\beta} + \epsilon^0_{\lambda\mu}\varepsilon_{\alpha\beta}(\boldsymbol{\chi}^{(\lambda\mu)})|_{y=\frac{x}{\varepsilon}} + O(\varepsilon) \tag{5.46}$$

or, alternatively

$$\epsilon_{\alpha\beta}(\boldsymbol{u}^\varepsilon) = \epsilon^0_{\lambda\mu}\varepsilon_{\alpha\beta}(\boldsymbol{\psi}^{(\lambda\mu)})|_{y=\frac{x}{\varepsilon}} + O(\varepsilon) \ . \tag{5.47}$$

6. Topological derivative of the energy functional in the two-dimensional elasticity problem

Let us examine the shape functional

$$\mathcal{J}(\Omega_\varepsilon) = J_\varepsilon(\boldsymbol{u}^\varepsilon) \ , \tag{6.1}$$

where

$$J_\varepsilon(\widetilde{\boldsymbol{v}}) = \frac{1}{2}\int_{\Omega_\varepsilon} A^{\alpha\beta\lambda\mu}\epsilon_{\alpha\beta}(\widetilde{\boldsymbol{v}})\epsilon_{\lambda\mu}(\widetilde{\boldsymbol{v}})\,dx - \int_{\partial\Omega} p_\alpha \widetilde{v}_\alpha\,dx \ . \tag{6.2}$$

The field $\boldsymbol{u}^{\varepsilon}$ is the solution of the problem (P_{ε}) formulated in Sec. 5.1. Let us introduce the function: $j(\varepsilon) = \mathcal{J}(\Omega_{\varepsilon})$.

Theorem 6.1.
Let the hitherto adopted assumptions concerning Ω, ω and p_{α} hold here. Then for $\varepsilon_0 > 0$, $|\varepsilon| < \varepsilon_0/2$, the function $\varepsilon \to j(\varepsilon_0 + \varepsilon)$ has the following expansion

$$j(\varepsilon_0 + \varepsilon) = j(\varepsilon_0) + j'(\varepsilon_0)\varepsilon + \frac{1}{2}\varepsilon^2 j''(\varepsilon_0) + o\left((\varepsilon_0 + \varepsilon)^2\right) \ , \tag{6.3}$$

where

$$j'(\varepsilon_0) \to j'(0^+) = 0 \quad \text{if} \quad \varepsilon_0 \searrow 0^+ \ , \tag{6.4}$$

$$j''(\varepsilon_0) \to j''(0^+) \ , \quad \text{if} \quad \varepsilon_0 \searrow 0^+ \ , \tag{6.5}$$

$$j''(0^+) = \epsilon^0_{\alpha\beta}\, G^{\alpha\beta\lambda\mu} \epsilon^0_{\lambda\mu} \ . \tag{6.6}$$

The matrix $\boldsymbol{G} = (G^{\iota\varrho\gamma\delta})$ is defined as follows

$$G^{\iota\varrho\gamma\delta} = -A^{\alpha\beta\lambda\mu}\left[\delta^{\iota\varrho}_{\alpha\beta}\, \delta^{\gamma\delta}_{\lambda\mu}\, |\omega| + \int\limits_{\partial\omega} \gamma^{(\iota\varrho)(\gamma\delta)}_{\alpha\beta\lambda\mu}(\boldsymbol{\chi}) y_{\sigma}\nu_{\sigma}\, ds\right] \ , \tag{6.7a}$$

where

$$\gamma^{(\iota\varrho)(\gamma\delta)}_{\alpha\beta\lambda\mu}(\boldsymbol{\chi}) = \frac{1}{2}\left[\delta^{\iota\varrho}_{\alpha\beta}\epsilon_{\lambda\mu}(\boldsymbol{\chi}^{(\gamma\delta)}) + \delta^{\gamma\delta}_{\lambda\mu}\epsilon_{\alpha\beta}(\boldsymbol{\chi}^{(\iota\varrho)}) + \epsilon_{\alpha\beta}(\boldsymbol{\chi}^{(\iota\varrho)})\epsilon_{\lambda\mu}(\boldsymbol{\chi}^{(\gamma\delta)})\right] \ . \tag{6.7b}$$

The functions $\boldsymbol{\chi}^{(\alpha\beta)}$ are solutions to the problems $(P^{(\alpha\beta)}_{\omega})$ formulated in Sec. 5. The quantity $\frac{1}{2}j''(0^+)$ determines the value of the topological derivative $\widetilde{\mathfrak{T}}_{\omega}(\boldsymbol{O})$ of the functional $\mathcal{J}(\Omega_{\varepsilon})$.

Proof.
We proceed similarly to the lines of the proof of Theorem 4.1. We consider the domain Ω_{ε_0} and its perturbation $\Omega_{\varepsilon_0+\varepsilon}$. Next we determine the transformation T_{ε} and the velocity field $\boldsymbol{V}(\varepsilon,x(\varepsilon))$. The equations (5.14)-(5.19) hold good. We shall make use of the formula for the shape derivative of the solution of the equations of linear elasticity, given in the book of Sokołowski and Zolesio (1992, Theorem 3.11, page 140).

Thus we find

$$j'(\varepsilon_0 + \varepsilon) = J'(\boldsymbol{u}^{\varepsilon_0+\varepsilon}) \ , \tag{6.8}$$

with

$$J'(\tilde{\boldsymbol{v}}) = -\frac{1}{2}A^{\alpha\beta\lambda\mu}\int\limits_{\partial\omega_{\varepsilon_0+\varepsilon}} \epsilon_{\alpha\beta}(\tilde{\boldsymbol{v}})\epsilon_{\lambda\mu}(\tilde{\boldsymbol{v}})\boldsymbol{V}(\varepsilon,x(\varepsilon)) \cdot \boldsymbol{\nu}(x(\varepsilon))\, ds_{\varepsilon_0+\varepsilon} \ . \tag{6.9}$$

Substitution of (4.15) gives

$$j'(\varepsilon_0 + \varepsilon) = -\frac{1}{2}A^{\alpha\beta\lambda\mu}\int\limits_{\partial\omega_{\varepsilon_0+\varepsilon}} \epsilon_{\alpha\beta}(\boldsymbol{u}^{\varepsilon_0+\varepsilon})\epsilon_{\lambda\mu}(\boldsymbol{u}^{\varepsilon_0+\varepsilon})\frac{x_{\sigma}(\varepsilon)}{\varepsilon_0 + \varepsilon}\nu_{\sigma}(x(\varepsilon))\, ds_{\varepsilon_0+\varepsilon} \ . \tag{6.10}$$

Now we substitute (5.47) and change the integration domain from $\partial\omega_{\varepsilon_0+\varepsilon}$ to $\partial\omega$, which results in

$$j'(\varepsilon_0 + \varepsilon) = -\frac{1}{2}(\varepsilon_0 + \varepsilon)A^{\alpha\beta\lambda\mu}\epsilon^0_{\iota\varrho}\epsilon^0_{\gamma\delta}\int\limits_{\partial\omega} \epsilon_{\alpha\beta}(\boldsymbol{\psi}^{(\iota\varrho)})\epsilon_{\lambda\mu}(\boldsymbol{\psi}^{(\gamma\delta)})y_{\sigma}\nu_{\sigma}\, ds + (\varepsilon_0 + \varepsilon)O(\varepsilon) \ . \tag{6.11}$$

Hence $j'(0^+) = 0$. To find $j''(0^+)$ we differentiate (6.11) with respect to ε and then put $\varepsilon = 0$. Passing to zero with ε_0 gives

$$j''(0^+) = \epsilon^0_{\iota\varrho}G^{\iota\varrho\gamma\delta}\epsilon^0_{\gamma\delta} \ , \tag{6.12}$$

where

$$G^{\iota\varrho\gamma\delta} = -\frac{1}{2}A^{\alpha\beta\lambda\mu}\int\limits_{\partial\omega}\epsilon_{\alpha\beta}(\boldsymbol{\psi}^{(\iota\varrho)})\epsilon_{\lambda\mu}(\boldsymbol{\psi}^{(\gamma\delta)})y_\sigma\nu_\sigma\, ds\ . \tag{6.13}$$

Substitution of (5.39) and making use of (4.28) results in the formula (6.7). This ends the proof. ■

Let us examine now the functional

$$E(\Omega_\varepsilon) = \int\limits_{\Omega_\varepsilon} A^{\alpha\beta\lambda\mu}\epsilon_{\alpha\beta}(\boldsymbol{u}^\varepsilon)\epsilon_{\lambda\mu}(\boldsymbol{u}^\varepsilon)\, dx \tag{6.14}$$

linked with $\mathcal{J}(\Omega_\varepsilon)$ by the formula: $E(\Omega_\varepsilon) = -2\mathcal{J}(\Omega_\varepsilon)$. Let us rewrite the expansion (6.3) in the form

$$E(\Omega_\varepsilon) = E(\Omega) - \varepsilon^2\epsilon^0_{\alpha\beta}G^{\alpha\beta\lambda\mu}\epsilon^0_{\lambda\mu} + o(\varepsilon^2)\ . \tag{6.15}$$

Now we are ready to formulate

Theorem 6.2.
Consider an elastic two-dimensional body loaded on its boundary. Its energy increases if in its interior a star-shaped domain appears.

Proof.
Assume that ω is star-shaped. We shall prove that

$$E(\Omega_\varepsilon) > E(\Omega) \tag{6.16}$$

if $\varepsilon > 0$.

Let us define the function $\boldsymbol{W} = \epsilon^0_{\iota\varrho}\boldsymbol{\psi}^{(\iota\varrho)}$. Then we can write

$$E(\Omega_\varepsilon) - E(\Omega) = \frac{1}{2}\varepsilon^2 A^{\alpha\beta\lambda\mu}\int\limits_{\partial\omega}\epsilon_{\alpha\beta}(\boldsymbol{W})\epsilon_{\lambda\mu}(\boldsymbol{W})y_\sigma\nu_\sigma\, ds\ . \tag{6.17}$$

By virtue of the positive definiteness property (5.4) one can estimate

$$E(\Omega_\varepsilon) - E(\Omega) \geq \frac{c}{2}\varepsilon^2\int\limits_{\partial\omega}\epsilon_{\alpha\beta}(\boldsymbol{W})\epsilon_{\alpha\beta}(\boldsymbol{W})y_\sigma\nu_\sigma\, ds\ . \tag{6.18}$$

Note that $\boldsymbol{\psi}^{(\iota\varrho)} \notin \mathcal{R}$, hence $\boldsymbol{W} \notin \mathcal{R}$. We conclude that $\epsilon_{\alpha\beta}(\boldsymbol{W})\epsilon_{\alpha\beta}(\boldsymbol{W})$ is not identically zero along $\partial\omega$. Since the condition (4.30) holds the inequality (6.16) follows. Thus nucleation of a star-shaped domain increases the energy. ■

It turns out that the components of the matrix $\boldsymbol{G}$ are determined by the values of the hoop stresses $\sigma_{\tau\tau}(\boldsymbol{\psi}^{(\iota\varrho)})$ around $\partial\omega$. To show this property it is sufficient to prove that

$$G^{\iota\varrho\gamma\delta} = -\frac{1}{2}\int\limits_{\partial\omega} C_{\tau\tau\tau\tau}(s)\sigma^{\tau\tau}(\boldsymbol{\psi}^{(\iota\varrho)})\sigma^{\tau\tau}(\boldsymbol{\psi}^{(\gamma\delta)})y_\sigma\nu_\sigma\, ds\ , \tag{6.19}$$

where the quantity $C_{\tau\tau\tau\tau}$ follows from the decomposition of the tensor $\boldsymbol{C} = \boldsymbol{A}^{-1}$ in the basis $\widetilde{\boldsymbol{e}}_1 = \boldsymbol{\tau}$, $\widetilde{\boldsymbol{e}}_2 = \boldsymbol{\nu}$:

$$\boldsymbol{C} = \widetilde{C}_{\alpha\beta\lambda\mu}(s)\widetilde{\boldsymbol{e}}_\alpha(s)\otimes\widetilde{\boldsymbol{e}}_\beta(s)\otimes\widetilde{\boldsymbol{e}}_\lambda(s)\otimes\widetilde{\boldsymbol{e}}_\mu(s)\ . \tag{6.20}$$

Hence $C_{\tau\tau\tau\tau} = \widetilde{C}_{1111}$. Similarly, the quantity $\sigma^{\tau\tau}(s)$ follows from the representation

$$\boldsymbol{\sigma} = \widetilde{\sigma}^{\alpha\beta}(s)\widetilde{\boldsymbol{e}}_\alpha \otimes \widetilde{\boldsymbol{e}}_\beta \;, \tag{6.21}$$

or $\widetilde{\sigma}^{11} = \sigma^{\tau\tau}$; above $\widetilde{\boldsymbol{e}}_\alpha = \widetilde{\boldsymbol{e}}_\alpha(s)$.

The derivation of (6.19) is as follows. Let us recall the boundary condition (5.41)

$$\sigma^{\alpha\beta}(\boldsymbol{\psi}^{(\kappa\delta)})\nu_\beta = 0 \qquad \text{on} \quad \partial\omega \;. \tag{6.22}$$

According to (6.21) we can write

$$\sigma^{\alpha\beta} = \sigma^{\tau\tau}\tau_\alpha\tau_\beta + \sigma^{\nu\tau}(\nu_\alpha\tau_\beta + \nu_\beta\tau_\alpha) + \sigma^{\nu\nu}\nu_\alpha\nu_\beta \;.$$

The condition (6.22) implies

$$\sigma^{\nu\tau}(\boldsymbol{\psi}^{(\kappa\delta)}) = 0 \;, \qquad \sigma^{\nu\nu}(\boldsymbol{\psi}^{(\kappa\delta)}) = 0 \qquad \text{on} \quad \partial\omega \;. \tag{6.23}$$

Thus along $\partial\omega$ we have

$$\sigma^{\alpha\beta} = \sigma^{\tau\tau}\tau_\alpha\tau_\beta \;. \tag{6.24}$$

Let us invert the relation

$$\sigma^{\alpha\beta} = A^{\alpha\beta\lambda\mu}\epsilon_{\lambda\mu} \tag{6.25}$$

to the form

$$\epsilon_{\lambda\mu} = C_{\lambda\mu\alpha\beta}\sigma^{\alpha\beta} \tag{6.26}$$

involving the flexibility matrix $\boldsymbol{C}$. One can write $\boldsymbol{C} = \boldsymbol{A}^{-1}$, having in mind (6.25) and (6.26). Now we can express $G^{\iota\varrho\gamma\delta}$ in terms of stresses

$$G^{\iota\varrho\gamma\delta} = -\frac{1}{2}\int\limits_{\partial\omega} C_{\alpha\beta\lambda\mu}\sigma^{\alpha\beta}(\boldsymbol{\psi}^{(\iota\varrho)})\sigma^{\lambda\mu}(\boldsymbol{\psi}^{(\gamma\delta)})y_\sigma\nu_\sigma \, ds \;. \tag{6.27}$$

We make use of the invariance property

$$C_{\alpha\beta\lambda\mu}\sigma^{\alpha\beta}_{(1)}(s)\sigma^{\lambda\mu}_{(2)}(s) = \widetilde{C}_{\alpha\beta\lambda\mu}(s)\widetilde{\sigma}^{\alpha\beta}_{(1)}(s)\widetilde{\sigma}^{\lambda\mu}_{(2)}(s) \tag{6.28}$$

and the representation (6.24). We find

$$C_{\alpha\beta\lambda\mu}\sigma^{\alpha\beta}(\boldsymbol{\psi}^{(\iota\varrho)})\sigma^{\lambda\mu}(\boldsymbol{\psi}^{(\gamma\delta)}) = C_{\tau\tau\tau\tau}(s)\sigma^{\tau\tau}(\boldsymbol{\psi}^{(\iota\varrho)})\sigma^{\tau\tau}(\boldsymbol{\psi}^{(\gamma\delta)}) \;, \tag{6.29}$$

which ends the proof of (6.19).

Let us note that the formula (6.15) can be expressed as follows

$$E(\Omega_\varepsilon) - E(\Omega) = \frac{1}{2}\varepsilon^2 \int\limits_{\partial\omega} C_{\tau\tau\tau\tau}(s)\,[\sigma^{\tau\tau}(\boldsymbol{W})]^2\, y_\sigma\nu_\sigma \, ds + o(\varepsilon^2) \;. \tag{6.30}$$

It is seen that the energy increment is determined by the values of the hoop stress $\sigma^{\tau\tau}(\boldsymbol{W})$ around the boundary of the rescaled opening.

Remark.

The compound asymptotics method applied in the Neumann problem have made it possible to derive the formula (4.43) for the energy of a body with a small opening. A sketch of its derivation is given in the Appendix C. In a similar way one can substantiate the expansion

$$E(\Omega_\varepsilon) - E(\Omega) = -\varepsilon^2\epsilon^0_{\alpha\beta}M^{\alpha\beta\lambda\mu}\epsilon^0_{\lambda\mu} + o(\varepsilon^2) \tag{6.31}$$

for the plane elasticity problem, cf. Movchan and Movchan (1995, Sec. 5.1.3, page 256). The components of the matrix $\boldsymbol{M}$ are given by the formula

$$M^{\kappa\delta\sigma\gamma} = -A^{\alpha\beta\kappa\delta} \int\limits_{\partial\omega} \psi_\alpha^{(\sigma\gamma)} \nu_\beta \, ds \ , \tag{6.32}$$

see Appendix B.

According to Theorem 6.2 the energy increases if the body is weakened by an opening of star-shape. On the other hand the formula (6.31) implies a stronger conclusion: the energy increases always, irrespective of the shape of the opening. This follows from the matrix $\boldsymbol{M}$ being negative definite, see Appendix B.

At the present stage of the best authors' knowledge we can formulate

Theorem 6.3.
Assume that the body is isotropic, i.e. tensor $\boldsymbol{A}$ can be represented by

$$\boldsymbol{A} = 2k\boldsymbol{\Lambda}_1 + 2\mu\boldsymbol{\Lambda}_2 \ , \tag{6.33}$$

where k represents the Kelvin modulus and μ is the Kirchhoff modulus. The tensors $\boldsymbol{\Lambda}_\alpha$ are given by

$$\Lambda_1^{\alpha\beta\lambda\mu} = \frac{1}{2}\delta^{\alpha\beta}\delta^{\lambda\mu} \ , \tag{6.34}$$

$$\Lambda_2^{\alpha\beta\lambda\mu} = \frac{1}{2}\left(\delta^{\alpha\lambda}\delta^{\beta\mu} + \delta^{\alpha\mu}\delta^{\beta\lambda} - \delta^{\alpha\beta}\delta^{\lambda\mu}\right) \ . \tag{6.35}$$

Moreover, assume that $\overline{\omega}$ is a circle: $\|y\| \le R$.
Then

$$M^{\alpha\beta\lambda\mu} = G^{\alpha\beta\lambda\mu} \ . \tag{6.36}$$

Proof.
The proof is constructive; we shall find the components of the tensor $\boldsymbol{G}$.

The tensor $\boldsymbol{C} = \boldsymbol{A}^{-1}$ is easy constructed from (6.33)

$$\boldsymbol{C} = \frac{1}{2k}\boldsymbol{\Lambda}_1 + \frac{1}{2\mu}\boldsymbol{\Lambda}_2 \ . \tag{6.37}$$

Due to isotropy: $C_{\tau\tau\tau\tau} = C_{1111}$ and hence $C_{\tau\tau\tau\tau} = \frac{1}{2}\left(\frac{1}{2k} + \frac{1}{2\mu}\right) = \frac{k+\mu}{4k\mu}$. Note that $C_{\tau\tau\tau\tau}$ is independent of s. In the polar system (r,θ) the component $\sigma^{\tau\tau}$ equals $\sigma_{\theta\theta}$ on $\partial\omega$. Thus the formula (6.19) reduces to

$$G^{\iota\varrho\gamma\delta} = -\frac{1}{8}\frac{k+\mu}{k\mu}R^2 \int\limits_0^{2\pi} \sigma_{\theta\theta}(\boldsymbol{\psi}^{(\iota\varrho)})\sigma_{\theta\theta}(\boldsymbol{\psi}^{(\gamma\delta)}) \, d\theta \ . \tag{6.38}$$

Due to symmetries of the problem the tensor $\boldsymbol{G}$ must be isotropic. Thus there exist constants g_α such that

$$\boldsymbol{G} = 2g_1\boldsymbol{\Lambda}_1 + 2g_2\boldsymbol{\Lambda}_2 \ . \tag{6.39}$$

To find g_α it is sufficient to find two components, e.g. G^{1111} and G^{1122}.

Let us note that the following stresses applied at infinity:

$$\sigma_\infty^{11} = k + \mu \ , \qquad \sigma_\infty^{22} = k - \mu \ , \qquad \sigma_\infty^{12} = 0 \tag{6.40}$$

are equivalent to the following strain values at infinity

$$\epsilon_{11}^\infty = 1 \ , \qquad \epsilon_{22}^\infty = 0 \ , \qquad \epsilon_{12}^\infty = 0 \ . \tag{6.41}$$

This state of strain is associated with the field $\boldsymbol{\psi}^{(11)}$.

The hoop stress $\sigma_{\theta\theta}(R,\theta)$ caused by stresses $(\sigma_\infty^{11}, \sigma_\infty^{22}, \sigma_\infty^{12} = 0)$ applied at infinity are independent of the elastic moduli and are given by, see Sokołowski and Żochowski (1997, Eq. $(54)_2$)

$$\sigma_{\theta\theta}(R,\theta) = (\sigma_\infty^{11} + \sigma_\infty^{22}) - 2(\sigma_\infty^{11} - \sigma_\infty^{22}) \cos 2\theta \ . \tag{6.42}$$

Substitution of (6.40) gives

$$\sigma_{\theta\theta}\left(\boldsymbol{\psi}^{(11)}\right)(R,\theta) = 2k - 4\mu \cos 2\theta \ . \tag{6.43}$$

On the other hand, the following stresses applied at infinity

$$\sigma_\infty^{11} = k - \mu \ , \qquad \sigma_\infty^{22} = k + \mu \ , \qquad \sigma_\infty^{12} = 0 \tag{6.44}$$

cause there the following strains

$$\epsilon_\infty^{11} = 0 \ , \qquad \epsilon_\infty^{22} = 1 \ , \qquad \epsilon_\infty^{12} = 0 \tag{6.45}$$

which are associated with the displacement field $\boldsymbol{\psi}^{(22)}$. Thus we find

$$\sigma_{\theta\theta}\left(\boldsymbol{\psi}^{(22)}\right)(R,\theta) = 2k + 4\mu \cos 2\theta \ . \tag{6.46}$$

Substitution of (6.43), (6.46) into (6.38) and integration over θ gives

$$\begin{aligned} G^{1111} &= -\frac{\pi R^2(k^2 + 2\mu^2)(k+\mu)}{k\mu} \ , \\ G^{1122} &= -\frac{\pi R^2(k^2 - 2\mu^2)(k+\mu)}{k\mu} \ . \end{aligned} \tag{6.47}$$

From the equations

$$G^{1111} = g_1 + g_2 \ , \qquad G^{1122} = g_1 - g_2 \tag{6.48}$$

one finds

$$g_1 = -\pi R^2 \frac{(k+\mu)k}{\mu} \ , \qquad g_2 = -2\pi R^2 \frac{(k+\mu)\mu}{k} \ . \tag{6.49}$$

Other components are given by (6.39). They are as follows

$$\begin{gathered} G^{2222} = G^{1111} \ , \qquad G^{1212} = g_2 \ , \qquad G^{2211} = G^{1122} \ , \\ G^{2121} = G^{2112} = G^{1221} = g_2 \ . \end{gathered} \tag{6.50}$$

The components of the mass matrix $\boldsymbol{M}$ for the circle can be found from the formulae found in Movchan and Movchan (1995, Eq. 5.1.22) for the elliptical opening. One can easy check that the matrix $\boldsymbol{M}$ found in this way is equal to the matrix $\boldsymbol{G}$ found above. To this end it is sufficient to note that $\lambda = k - \mu$ and

$$M^{1111} = m_{11} \ , \qquad M^{1122} = m_{12} \ , \qquad M^{2222} = m_{22} \ , \qquad M^{1212} = \frac{1}{2} m_{33} \ , \tag{6.51}$$

where the notation λ and m_{ij} is used by Movchan and Movchan (1995).

Thus the theorem is proved. ■

In the case of isotropy and the opening ω being of circular shape the quadratic form $j''(0^+) = \epsilon_{\alpha\beta}^0 G^{\alpha\beta\lambda\mu} \epsilon_{\lambda\mu}^0$ is expressed as follows

$$j''(0^+) = g_1(\epsilon_{11}^0 + \epsilon_{22}^0)^2 + g_2\left[(\epsilon_{11}^0 + \epsilon_{22}^0)^2 + 4(\epsilon_{12}^0)^2\right] \ . \tag{6.52}$$

Let us decompose $\boldsymbol{\epsilon}^0$ into the spherical and deviatoric part

$$\boldsymbol{\epsilon}^0 = \frac{1}{2} tr \boldsymbol{\epsilon}^0 \boldsymbol{I} + \boldsymbol{e}^0 \ , \tag{6.53}$$

where $\boldsymbol{I}$ represents the unit tensor in $\mathbb{M}^2_s$. The norm of the deviator $\boldsymbol{e}^0$ equals $\|\boldsymbol{e}^0\| = (e^0_{\alpha\beta} e^0_{\alpha\beta})^{1/2}$ or

$$\|\boldsymbol{e}^0\|^2 = \frac{1}{2}\left[(\epsilon^0_{11} - \epsilon^0_{22})^2 + 4(\epsilon^0_{12})^2\right] \ . \tag{6.54}$$

The formula (6.52) can be written in the form

$$j''(0^+) = g_1 (tr \boldsymbol{\epsilon}^0)^2 + 2 g_2 \|\boldsymbol{e}^0\|^2 \ . \tag{6.55}$$

Let us recall the constitutive relations

$$tr \boldsymbol{\epsilon}^0 = \frac{1}{2k} tr \boldsymbol{\sigma}^0 \ , \qquad \boldsymbol{e}^0 = \frac{1}{2\mu} \boldsymbol{s}^0 \ , \tag{6.56}$$

with $\boldsymbol{s}^0 = \boldsymbol{\sigma}^0 - \frac{1}{2} tr \boldsymbol{\sigma}^0 \boldsymbol{I}$ being the deviator of stress. Substitution of (6.56) into (6.55) gives

$$j''(0^+) = \frac{1}{4}\left[\frac{g_1}{k^2}(tr \boldsymbol{\sigma}^0)^2 + 2\frac{g_2}{\mu^2}\|\boldsymbol{s}^0\|^2\right] \tag{6.57}$$

an by taking account of (6.49) one finds

$$j''(0^+) = -\frac{\pi R^2 (k+\mu)}{4k\mu}\left[(tr \boldsymbol{\sigma}^0)^2 + 4\|\boldsymbol{s}^0\|^2\right] \ . \tag{6.58}$$

Note that the expression in the square brackets does not depend on the ratio μ/k.
Let us rearrange the formula (6.58) to the form expressed in terms of the principal stresses σ^0_I, σ^0_{II}:

$$j''(0^+) = -\frac{\pi R^2 (k+\mu)}{4k\mu}\left[(\sigma^0_I + \sigma^0_{II})^2 + 2(\sigma^0_I - \sigma^0_{II})^2\right] \ . \tag{6.59}$$

This formula has already been found by Sokołowski and Żochowski (1997). It differs in a factor from the characteristic function of the bubble method, cf. Eschenauer et al. (1994, Eq. (25)) and Schumacher (1995).

Let us come back to the general case of a non-circular opening and discuss the expression (6.6) for the topological derivative of the energy functional. This formula is expressed in terms of strains $\epsilon^0_{\alpha\beta}$ of the virgin body, measured at a point of possible nucleation of a void ω_ε. One can rearrange this formula such that it would depend on the stresses $\sigma^0_{\alpha\beta} = A^{\alpha\beta\lambda\mu}\epsilon^0_{\lambda\mu}$. This would be not a plain formality. Let us discuss it below in a greater detail.

Let us introduce new vector fields

$$\boldsymbol{\Phi}_{(\sigma\gamma)}(y) = C_{\sigma\gamma\iota\varrho} \boldsymbol{\psi}^{(\iota\varrho)}(y) \ . \tag{6.60}$$

They are solutions to the following problems

$$\left(\widehat{P}^{\omega}_{(\sigma\gamma)}\right) \quad \left|\begin{array}{lll} \text{find } \boldsymbol{\Phi}_{(\sigma\gamma)} \text{ defined in } \mathbb{R}^2 \backslash \overline{\omega} \text{ such that} & & \\ \dfrac{\partial}{\partial y_\beta}\left[A^{\alpha\beta\lambda\mu}\epsilon_{\lambda\mu}(\boldsymbol{\Phi}_{(\sigma\gamma)})\right] = 0 & \text{in } \mathbb{R}^2 \backslash \overline{\omega} \ , & (6.61) \\ A^{\alpha\beta\lambda\mu}\epsilon_{\lambda\mu}(\boldsymbol{\Phi}_{(\sigma\gamma)})\nu_\beta = 0 & \text{on } \partial\omega \ , & (6.62) \\ \sigma^{\alpha\beta}(\boldsymbol{\Phi}_{(\sigma\gamma)}) \to \delta^{\alpha\beta}_{\sigma\gamma} & \text{if } \|y\| \to \infty \ . & (6.63) \end{array}\right.$$

It is sufficient to prove (6.63). We compute

$$\sigma^{\alpha\beta}(\boldsymbol{\Phi}_{(\sigma\gamma)}) = A^{\alpha\beta\lambda\mu}\epsilon_{\lambda\mu}(\boldsymbol{\Phi}_{(\sigma\gamma)}) = A^{\alpha\beta\lambda\mu}C_{\sigma\gamma\iota\varrho}\epsilon_{\lambda\mu}(\boldsymbol{\psi}^{(\iota\varrho)}) \ .$$

According to (5.42)

$$\epsilon_{\lambda\mu}(\boldsymbol{\psi}^{(\iota\varrho)}) \to \delta^{\iota\varrho}_{\lambda\mu} \quad \text{if} \quad \|y\| \to \infty \ .$$

Hence

$$\sigma^{\alpha\beta}(\boldsymbol{\Phi}_{(\sigma\gamma)}) \to A^{\alpha\beta\lambda\mu}C_{\sigma\gamma\iota\varrho}\delta^{\iota\varrho}_{\lambda\mu} = \delta^{\alpha\beta}_{\sigma\gamma}$$

by definition of $\boldsymbol{C}$ and of the object $(\delta^{\alpha\beta}_{\sigma\gamma})$.

The function $\boldsymbol{W} = \epsilon^0_{\iota\varrho}\boldsymbol{\psi}^{(\iota\varrho)}$ can be expressed by $\boldsymbol{W} = \overset{\circ}{\sigma}{}^{\sigma\gamma}\boldsymbol{\Phi}_{(\sigma\gamma)}$ and substitution of this formula into (6.30) gives

$$E(\Omega_\varepsilon) - E(\Omega) = \varepsilon^2\overset{\circ}{\sigma}{}^{\sigma\gamma}H_{\sigma\gamma\iota\varrho}\overset{\circ}{\sigma}{}^{\iota\varrho} + o(\varepsilon^2) \ , \tag{6.64}$$

where

$$H_{\sigma\gamma\iota\varrho} = \frac{1}{2}\int\limits_{\partial\omega} C_{\tau\tau\tau\tau}(s)\sigma^{\tau\tau}(\boldsymbol{\Phi}_{(\sigma\gamma)})\sigma^{\tau\tau}(\boldsymbol{\Phi}_{(\iota\varrho)})y_\alpha\nu_\alpha \, ds \ . \tag{6.65}$$

Note that in the case of isotropy the stress fields $\sigma^{\alpha\beta}(\boldsymbol{\Phi}_{(\lambda\mu)})$ are independent of the elastic moduli k and μ, see e.g. Hahn (1985, Sec. 8.4.2.3), which simplifies the formulae (6.65). This formula is highly useful since just the stress fields $\sigma^{\alpha\beta}(\boldsymbol{\Phi}_{(\sigma\gamma)})$, associated with constant stresses at infinity, are displayed in the available monographs and handbooks on stress concentration around holes.

7. Final remarks

The method of compound asymptotics has made it possible to generalize the notion of the topological derivative for the case of nucleation of non-circular openings in the plane problems of Neumann and linear elasticity. Proceeding similarly one can generalize the topological derivative to the case of nucleation of an inclusion made from an isotropic material of properties different than the anisotropic properties of the virgin body. Moreover, in both the cases a passage to the three-dimensional setting is straightforward.

Optimization of shells seems to be a challenging task. Nucleation of circular openings makes sense only in the problems of optimal formation of shells on a sphere, see Lewiński and Sokołowski (1998). It seems that using the compound asymptotics method will make it possible to develop effective algorithms for finding the topological derivatives of various shape functionals, describing the overall behaviour of various shells in which curvilinear openings can nucleate.

The formulae for the topological derivative of the energy functional for the case of an opening of arbitrary shape can be a starting point for further optimal design. In a natural way the following problem arises: for given $(\epsilon^0_{\alpha\beta})$ and given area $|\omega|$ find the optimal shape of ω such that the quality $j''(0^+)$ assumes maximum. In the context of an infinite isotropic body loaded at infinity, and with the help of the formula (6.31), this problem has been partly solved by Cherkaev et al. (1998). In the context of the present report such optimization means optimizing the shape of the bubbles depending on the actual deformation state. It seems that a lot can be done in this direction.

APPENDIX A. The mass matrix in the Neumann problem

The components $m_{\alpha\beta}$, entering the representation (3.26) of the solutions of problems (P^β_ω) are expressed by the following integral formula

$$m_{\alpha\beta} = -\mu\int\limits_{\partial\omega} \nu_\alpha W_\beta \, ds \tag{A.1}$$

or

$$m_{\alpha\beta} = -\mu \left[\delta_{\alpha\beta}|\omega| + \int\limits_{\partial\omega} \nu_\alpha w_\beta \, ds \right] \tag{A.2}$$

or

$$m_{\alpha\beta} = -\mu\delta_{\alpha\beta}|\omega| - \mathcal{M}_{\alpha\beta} \ , \tag{A.3}$$

where

$$\mathcal{M}_{\alpha\beta} = \int\limits_{\mathbb{R}^2\setminus\overline{\omega}} \mu \frac{\partial w_\alpha}{\partial y_\lambda} \frac{\partial w_\beta}{\partial y_\lambda} \, dy \ . \tag{A.4}$$

The formulae (A.3), (A.4) imply the symmetry property: $m_{\alpha\beta} = m_{\beta\alpha}$. Symmetrization of (A.2) gives (3.27).

Proof of the formula (A.1).
Let the domain ω be encompassed by a circle B_R of boundary $\Gamma_R = \{y|\ \|y\| = R\}$; $R \gg \mathrm{diam}(\omega)$. The functions w_β and W_β are harmonic, hence the following variational equation hold

$$\mu \int\limits_{B_R\setminus\overline{\omega}} \frac{\partial w_\alpha}{\partial y_\lambda} \frac{\partial \widetilde{\varphi}}{\partial y_\lambda} \, dy = \mu \int\limits_{\Gamma_R} \frac{\partial w_\alpha}{\partial n} \widetilde{\varphi} \, ds - \mu \int\limits_{\partial\omega} \widetilde{\varphi} \frac{\partial w_\alpha}{\partial \nu} \, ds \ , \tag{A.5}$$

$$\mu \int\limits_{B_R\setminus\overline{\omega}} \frac{\partial W_\beta}{\partial y_\lambda} \frac{\partial \widehat{\varphi}}{\partial y_\lambda} \, dy = \mu \int\limits_{\Gamma_R} \widehat{\varphi} \frac{\partial W_\beta}{\partial n} \, ds - \mu \int\limits_{\partial\omega} \widehat{\varphi} \frac{\partial W_\beta}{\partial \nu} \, ds \ , \tag{A.6}$$

for $\widetilde{\varphi}$ and $\widehat{\varphi}$ sufficiently regular. Let us put $\widetilde{\varphi} = W_\beta$ and $\widehat{\varphi} = w_\alpha$. We find

$$J_{\beta\alpha} = \mu \int\limits_{\partial\omega} \left(W_\beta \frac{\partial w_\alpha}{\partial \nu} - w_\alpha \frac{\partial W_\beta}{\partial \nu} \right) ds \ , \tag{A.7}$$

where

$$J_{\beta\alpha} = \mu \int\limits_{\Gamma_R} \left(W_\beta \frac{\partial w_\alpha}{\partial n} - w_\alpha \frac{\partial W_\beta}{\partial n} \right) ds \ . \tag{A.8}$$

We shall prove that $J_{\beta\alpha} = m_{\alpha\beta}$. To this end let us recall the formula (3.26). We find

$$W_\beta = y_\beta + m_{\beta\alpha} \frac{\partial S}{\partial y_\alpha} + O\left(\frac{1}{\|y\|^2} \right) \ , \tag{A.9}$$

$$\frac{\partial w_\alpha}{\partial n} = m_{\alpha\sigma} \frac{\partial^2 S}{\partial y_\sigma \partial n} + O\left(\frac{1}{\|y\|^3} \right) \ , \tag{A.10}$$

$$w_\alpha = m_{\alpha\sigma} \frac{\partial S}{\partial y_\sigma} + O\left(\frac{1}{\|y\|^2} \right) \ , \tag{A.11}$$

$$\frac{\partial W_\beta}{\partial n} = n_\beta + m_{\beta\sigma} \frac{\partial^2 S}{\partial y_\sigma \partial n} + O\left(\frac{1}{\|y\|^3} \right) \ . \tag{A.12}$$

We substitute (A.9) – (A.12) into (A.8) and disregard the terms of order $O\left(\frac{1}{\|y\|^n}\right)$, $n \geq 2$. We find

$$J_{\beta\alpha} = m_{\alpha\sigma} \int\limits_{\Gamma_R} \left[y_\beta \frac{\partial^2 \overline{S}}{\partial y_\sigma \partial n} - \frac{\partial \overline{S}}{\partial y_\sigma} \frac{\partial y_\beta}{\partial n} \right] ds + O\left(\frac{1}{R} \right) \ , \tag{A.13}$$

where $\overline{S} = \mu S$.

The weak formulation of the equation (3.24) reads

$$\int_{B_R} \frac{\partial \overline{S}}{\partial y_\alpha} \frac{\partial \varphi}{\partial y_\alpha}\, dy = \int_{\Gamma_R} \frac{\partial \overline{S}}{\partial n} \varphi\, ds + \int_{B_R} \varphi \delta(y)\, dy \tag{A.14}$$

for sufficiently regular φ. The function $\dfrac{\partial \overline{S}}{\partial y_\sigma}$ satisfies the equation

$$\Delta_y \left(\frac{\partial \overline{S}}{\partial y_\sigma} \right) + \frac{\partial \delta(y)}{\partial y_\sigma} = 0\ . \tag{A.15}$$

Let us consider the mollifier function defined by

$$\phi_h(y) = h^{-2} \phi\left(\frac{y}{h}\right)\ ,$$

where h is a small positive number and

$$\phi(x) = \begin{cases} C \exp\left[-1/(1-\|x\|^2)\right] & \text{if } \|x\| < 1 \\ 0 & \text{if } \|x\| \geq 1 \end{cases}$$

and C is chosen such that

$$\int_{\mathbb{R}^2} \phi(x)\, dx = 1\ .$$

Then

$$\int_{\mathbb{R}^2} \phi_h(y)\, dy = 1\ .$$

Let $\overline{S}_h(y)$ be a solution to the equation (A.14) with $\delta(y)$ replaced by $\phi_h(y)$. Since $\overline{S}(x-y)$ is the Green function, one can represent $\overline{S}_h(y)$ as follows

$$\overline{S}_h(y) = \int_{\mathbb{R}^2} \overline{S}(y-x) \phi_h(x)\, dx\ .$$

By symmetry: $\overline{S}_h(y) = \overline{S}_h(-y)$, $\phi_h(x) = \phi_h(-x)$ we can rearrange the formula above to the form

$$\overline{S}_h(y) = \int_{\mathbb{R}^2} \phi_h(y-z) \overline{S}(z)\, dz\ .$$

According to Theorem 1.6.1 and Lemma 2.1.3 reported in Ziemer (1989) the functions $\overline{S}_h(y)$ tend to $\overline{S}(y)$ pointwise, as well as in Sobolev spaces, when $h \searrow 0$. We shall make use of this property in the sequel.

Note that the function $\dfrac{\partial \overline{S}_h}{\partial y_\sigma}$ satisfies the variational equation

$$\int_{B_R} \frac{\partial^2 \overline{S}_h}{\partial y_\sigma \partial y_\alpha} \frac{\partial \varphi}{\partial y_\alpha}\, dy = \int_{\Gamma_R} \frac{\partial^2 \overline{S}_h}{\partial n \partial y_\sigma} \varphi\, ds + \int_{B_R} \varphi \frac{\partial \phi_h(y)}{\partial y_\sigma}\, dy \tag{A.16}$$

for sufficiently regular φ.

The function y_β is harmonic, hence

$$\int_{B_R} \frac{\partial y_\beta}{\partial y_\alpha} \frac{\partial \widetilde{\varphi}}{\partial y_\alpha}\, dy = \int_{\Gamma_R} \frac{\partial y_\beta}{\partial n} \widetilde{\varphi}\, ds \tag{A.17}$$

for sufficiently regular $\widetilde{\varphi}$.

Now we put $\varphi = y_\beta$ in (A.16) and $\widetilde{\varphi} = \dfrac{\partial \overline{S}_h}{\partial y_\sigma}$ in (A.17) to make their left-hand sides identical. By equating the right-hand sides we find

$$K^h_{\sigma\beta} = L^h_{\sigma\beta} \ ,$$

where

$$K^h_{\sigma\beta} = -\int\limits_{B_R} y_\beta \frac{\partial \phi_h(y)}{\partial y_\sigma}\, dy \tag{A.18a}$$

and

$$L^h_{\sigma\beta} = \int\limits_{\Gamma_R} \left(\frac{\partial^2 \overline{S}_h}{\partial n \partial y_\sigma} y_\beta - \frac{\partial y_\beta}{\partial n} \frac{\partial \overline{S}_h}{\partial y_\sigma} \right) ds \ . \tag{A.18b}$$

Note that $K^h_{\alpha\beta} = \delta_{\alpha\beta}$ for h sufficiently small and

$$\lim_{h \searrow 0} K^h_{\sigma\beta} = \int\limits_{\Gamma_R} \left(\frac{\partial^2 \overline{S}}{\partial n \partial y_\sigma} y_\beta - \frac{\partial y_\beta}{\partial n} \frac{\partial \overline{S}}{\partial y_\sigma} \right) ds$$

since $\dfrac{\partial \overline{S}_h}{\partial y_\sigma}$ tend to $\dfrac{\partial \overline{S}}{\partial y_\sigma}$.

We conclude that

$$\int\limits_{\Gamma_R} \left(\frac{\partial^2 \overline{S}}{\partial n \partial y_\sigma} y_\beta - \frac{\partial y_\beta}{\partial n} \frac{\partial \overline{S}}{\partial y_\sigma} \right) ds = \delta_{\sigma\beta} \ .$$

By (A.13) we find

$$J_{\beta\alpha} = \lim_{R \searrow 0} J_{\beta\alpha} = m_{\alpha\sigma}\delta_{\sigma\beta} = m_{\alpha\beta} \ , \tag{A.19}$$

since $J_{\beta\alpha}$ does not depend on R, cf. (A.8). Thus we have

$$m_{\alpha\beta} = \mu \int\limits_{\partial\omega} \left(W_\beta \frac{\partial w_\alpha}{\partial \nu} - w_\alpha \frac{\partial W_\beta}{\partial \nu} \right) ds \ . \tag{A.20}$$

By introducing (3.22) and (3.32) one finds (A.1).

To prove (A.2) we substitute $W_\alpha = y_\alpha + w_\alpha$ and make use of the identity

$$\int\limits_{\omega} \frac{\partial y_\alpha}{\partial y_\lambda} \frac{\partial y_\beta}{\partial y_\lambda}\, dy = \int\limits_{\partial\omega} y_\beta \frac{\partial y_\alpha}{\partial \nu}\, ds \tag{A.21}$$

from which we have

$$\delta_{\alpha\beta}|\omega| = \int\limits_{\partial\omega} y_\beta \nu_\alpha\, ds \ , \tag{A.22}$$

where $|\omega|$ represents the area of ω.

Let us prove now (A.3). At first we show that

$$\mathcal{M}_{\alpha\beta} = \mu \int\limits_{\partial\omega} w_\beta \nu_\alpha\, ds \ . \tag{A.23}$$

Let us put $\widetilde{\varphi} = w_\beta$ in (A.5). Let us note that

$$\lim_{R \to \infty} \int\limits_{\Gamma_R} w_\beta \frac{\partial w_\alpha}{\partial n}\, ds \to 0 \ , \tag{A.24}$$

since $w_\beta = O\left(\frac{1}{\|y\|}\right)$, $\frac{\partial w_\alpha}{\partial n} = O\left(\frac{1}{\|y\|^2}\right)$, $ds = O(\|y\|)$. Thus we arrive at

$$\mu \int\limits_{\mathbb{R}^2\setminus\overline{\omega}} \frac{\partial w_\alpha}{\partial y_\lambda}\frac{\partial w_\beta}{\partial y_\lambda}\, dy = -\mu \int\limits_{\partial\omega} w_\beta \frac{\partial w_\alpha}{\partial \nu}\, ds\ . \tag{A.25}$$

Substitution of $\partial w_\alpha/\partial\nu = -\nu_\alpha$ on $\partial\omega$ gives (A.23), from which we find (A.3).

Let note that the quadratic form

$$-q^\alpha m_{\alpha\beta} q^\beta = \mu|\omega|\,|q|^2 + \mu \int\limits_{\mathbb{R}^2\setminus\overline{\omega}} |\nabla w|^2\, dy\ , \tag{A.26}$$

where $w = q^\alpha w_\alpha$, is positive definite. We conclude that the matrix $(m_{\alpha\beta})$ is negative definite.

The present Appendix has been written with the help of the papers by: Schiffer and Szegö (1949), Maz'ya et al. (1991) and Movchan and Movchan (1995).

APPENDIX B.

The mass matrix in the plane problems of anisotropic elasticity.

B1. The Somigliana tensor

Let us recall the fundamental solutions (Somigliana's solutions or Kelvin's solutions, according to other authors) in the plane problem of linear elasticity. The body is considered as homogeneous and anisotropic, of elastic moduli $A^{\alpha\beta\lambda\mu}$, extended to the whole plane $\mathbb{R}^2$, parametrized by the cartesian coordinate system (y_1,y_2) with the basis $(\boldsymbol{e}_1,\boldsymbol{e}_2)$.

Assume that in point $\boldsymbol{O} = (0,0)$ a force of value 1 is applied in the direction of the σth axis. This force is represented as a concentrated body force of intensity

$$\boldsymbol{b}_{(\sigma)} = \delta(y - \boldsymbol{O})\boldsymbol{e}_\sigma \tag{B.1}$$

with the components: $(\boldsymbol{b}_{(\sigma)})_\alpha = \delta_{\alpha\sigma}\delta(y-\boldsymbol{O})$. This force causes the displacement field $\boldsymbol{T}_{(\sigma)}$ of components $(\boldsymbol{T}_{(\sigma)})_1$ and $(\boldsymbol{T}_{(\sigma)})_2$. In each circular domain $B_R = \{y|\ \|y\| < R\}$ the following variational equation is satisfied

$$\int\limits_{B_R} A^{\alpha\beta\iota\varrho}\epsilon_{\iota\varrho}(\boldsymbol{T}_{(\sigma)})\epsilon_{\alpha\beta}(\boldsymbol{\varphi})\, dy = \boldsymbol{\varphi}(\boldsymbol{O})\cdot\boldsymbol{e}_\sigma + \int\limits_{\Gamma_R} A^{\alpha\beta\iota\varrho}\epsilon_{\iota\varrho}(\boldsymbol{T}_{(\sigma)})n_\beta\varphi_\alpha\, ds \tag{B.2}$$

for all sufficiently regular test functions $\boldsymbol{\varphi} = (\varphi_\alpha)$.

The field $\boldsymbol{T}_{(\sigma)}$ can be found by making the Fourier transform of the equations

$$A^{\alpha\beta\iota\varrho}\frac{\partial^2(\boldsymbol{T}_{(\sigma)})_\iota}{\partial y_\varrho \partial y_\beta} + \delta_{\alpha\sigma}\delta(y-\boldsymbol{O}) = 0\ . \tag{B.3}$$

One can prove that the fields $(\boldsymbol{T}_{(\sigma)})_\alpha$ include singularities of order $O(\ln\|y\|)$, while $\epsilon_{\alpha\beta}(\boldsymbol{T}_{(\sigma)})$ include singularities of order $O(\|y\|^{-1})$. The fields $\boldsymbol{T}_{(\sigma)}$ are infinite at infinity, but $\epsilon_{\alpha\beta}(\boldsymbol{T}_{(\sigma)})$ and, consequently the stresses $\sigma^{\alpha\beta}(\boldsymbol{T}_{(\sigma)})$ vanish at infinity and thus they well describe the response of the body to the concentrated force (B.1).

In the case of general anisotropy the fields $(\boldsymbol{T}_{(\sigma)})$ cannot be put in closed forms. This is possible in some special types of anisotropy. The simplest formulae concern the isotropy case. Then the components $(\boldsymbol{T}_{(\sigma)})_\alpha$ are given by, cf. Bonnet (1995, Eq. (4.49)), Hahn (1985, page 274), Movchan and Movchan (1995, Eq. 5.1.18)

$$(\boldsymbol{T}_{(\sigma)})_\alpha = \frac{1}{4\pi\mu(k+\mu)}\left[-(k+2\mu)\delta_{\sigma\alpha}\ln\|y\| + k\frac{y_\sigma y_\alpha}{\|y\|^2}\right]\ . \tag{B.4}$$

B2. Definition of the mass matrix

The fields $\boldsymbol{\chi}^{(\kappa\delta)}$, or the solutions of the problems $(P_\omega^{(\kappa\delta)})$, are singular at $y = \boldsymbol{O}$ and are there of order $O(\|y\|^{-n})$, $n = 1,2,3,\ldots$. The subsequent singular terms are generated by the functions

$$\frac{\partial}{\partial y_\beta}\left((\boldsymbol{T}_{(\sigma)})_\omega\right) , \qquad \frac{\partial^2}{\partial y_\beta \partial y_\alpha}\left((\boldsymbol{T}_{(\sigma)})_\omega\right) , \ \ldots$$

describing the response of the body to the hyperforces: $\dfrac{\partial}{\partial y_\beta}\delta(y)$, $\dfrac{\partial^2}{\partial y_\beta \partial y_\alpha}\delta(y)$, …applied at $y = \boldsymbol{O}$. The most important are the terms of the weakest singularity of order $O(\|y\|^{-1})$. Let us disclose them as follows

$$\chi_\sigma^{(\kappa\delta)} = M^{\kappa\delta\lambda\mu}\epsilon_{\lambda\mu}(\boldsymbol{T}_{(\sigma)}) + O(\|y\|^{-2}) \ , \tag{B.5}$$

where $M^{\kappa\delta\lambda\mu}$ are certain constants depending on the shape of the ω domain.

Theorem B1.
The coefficients $M^{\kappa\delta\lambda\mu}$ that enter the expansion (B.5) are determined by the following integral formulae

(i) formula (6.32)

(ii)
$$M^{\kappa\delta\sigma\gamma} = -A^{\kappa\delta\sigma\gamma}|\omega| - \mathcal{M}^{\kappa\delta\sigma\gamma} \ , \tag{B.6}$$
where
$$\mathcal{M}^{\kappa\delta\sigma\gamma} = \int\limits_{\mathbb{R}^2\setminus\overline{\omega}} A^{\alpha\beta\lambda\mu}\epsilon_{\lambda\mu}(\boldsymbol{\chi}^{(\kappa\delta)})\epsilon_{\alpha\beta}(\boldsymbol{\chi}^{(\sigma\gamma)})\, dy \tag{B.7}$$
or
$$\mathcal{M}^{\kappa\delta\sigma\gamma} = A^{\alpha\beta\kappa\delta}\int\limits_{\partial\omega} \chi_\alpha^{(\sigma\gamma)}\nu_\beta\, ds \tag{B.8}$$

(iii) the following symmetry properties hold
$$\mathcal{M}^{\kappa\delta\sigma\gamma} = \mathcal{M}^{\sigma\gamma\kappa\delta} \ , \qquad \mathcal{M}^{\kappa\delta\sigma\gamma} = \mathcal{M}^{\delta\kappa\sigma\gamma} = \mathcal{M}^{\kappa\delta\gamma\sigma} \ . \tag{B.9}$$

(iv) the matrix $\boldsymbol{M}$ is negative definite.

Proof of (i).
Let us encompass the domain ω by the circle B_R, similarly to Sec. 5, where the variational equation (5.30) is derived. The fields $\boldsymbol{\psi}^{(\sigma\gamma)}$ or solutions to the problems $(\widetilde{P}_\omega^{(\sigma\gamma)})$, see Sec. 5, Eqs. (5.40) – (5.42), satisfy the variational equation of the form similar to (5.30):

$$A^{\alpha\beta\lambda\mu}\int\limits_{B_R\setminus\overline{\omega}} \epsilon_{\lambda\mu}\left(\boldsymbol{\psi}^{(\sigma\gamma)}\right)\epsilon_{\alpha\beta}(\widehat{\boldsymbol{v}})\, dy = A^{\alpha\beta\lambda\mu}\int\limits_{\Gamma_R} \epsilon_{\lambda\mu}\left(\boldsymbol{\psi}^{(\sigma\gamma)}\right) n_\beta \widehat{v}_\alpha\, ds \tag{B.10}$$

valid for sufficiently regular fields $\widehat{v}_\alpha$. The boundary conditions (5.41) have been taken into account.

Let us put $\widetilde{\boldsymbol{v}} = \boldsymbol{\psi}^{(\sigma\gamma)}$ in (5.30) and $\widehat{\boldsymbol{v}} = \boldsymbol{\chi}^{(\kappa\delta)}$ in (B.10). Thus we arrive at two identities of the same left-hand sides. Equating the right-hand sides gives the formula

$$N^{\kappa\delta\sigma\gamma} = -A^{\alpha\beta\kappa\delta}\int\limits_{\partial\omega} \psi_\alpha^{(\sigma\gamma)}\nu_\beta\, ds \ , \tag{B.11}$$

where

$$N^{\kappa\delta\sigma\gamma} := A^{\alpha\beta\lambda\mu}\int\limits_{\Gamma_R}\left[\epsilon_{\lambda\mu}\left(\boldsymbol{\chi}^{(\kappa\delta)}\right)\psi_\alpha^{(\sigma\gamma)} - \epsilon_{\lambda\mu}\left(\boldsymbol{\psi}^{(\sigma\gamma)}\right)\chi_\alpha^{(\kappa\delta)}\right] n_\beta\, ds \ . \tag{B.12}$$

Thus the matrix $\boldsymbol{N}$ does not depend on R. The coefficients $N^{\kappa\delta\sigma\gamma}$ will be found by passing with R to infinity in the formula (B.12). We substitute

$$\psi_\alpha^{(\sigma\gamma)}|_{\Gamma_R} = E_\alpha^{(\sigma\gamma)} + O\left(\frac{1}{R}\right) , \tag{B.13}$$

$$\epsilon_{\lambda\mu}\left(\boldsymbol{\psi}^{(\sigma\gamma)}\right)|_{\Gamma_R} = \delta_{\lambda\mu}^{\sigma\gamma} + O\left(\frac{1}{R^2}\right) , \tag{B.14}$$

into (B.12). By disregarding the terms that do not contribute to the final result we obtain

$$N^{\kappa\delta\sigma\gamma} = \lim_{R\to\infty} A^{\alpha\beta\lambda\mu} \int\limits_{\Gamma_R} \left[\epsilon_{\lambda\mu}\left(\boldsymbol{\chi}^{(\kappa\delta)}\right) E_\alpha^{(\sigma\gamma)} - \chi_\alpha^{(\kappa\delta)} \delta_{\lambda\mu}^{\sigma\gamma}\right] n_\beta \, ds , \tag{B.15}$$

and substitution of the representation (B.5) gives

$$N^{\kappa\delta\sigma\gamma} = M^{\kappa\delta\iota\varrho} \eta_{..\iota\varrho}^{\sigma\gamma} , \tag{B.16}$$

where

$$\eta_{..\iota\varrho}^{\sigma\gamma} := A^{\alpha\beta\lambda\mu} \lim_{R\to\infty} \int\limits_{\Gamma_R} \left[\epsilon_{\lambda\mu}\left(\epsilon_{\iota\varrho}(\boldsymbol{T})\right) E_\alpha^{(\sigma\gamma)} - \epsilon_{\iota\varrho}(\boldsymbol{T}_{(\alpha)}) \delta_{\lambda\mu}^{\sigma\gamma}\right] n_\beta \, ds . \tag{B.17}$$

We have introduced the following notation cf. (B.5)

$$\boldsymbol{\chi}^{(\kappa\delta)} = M^{\kappa\delta\lambda\mu} \epsilon_{\lambda\mu}(\boldsymbol{T}) + O\left(\|y\|^{-2}\right) , \tag{B.18}$$

with

$$\boldsymbol{\chi}^{(\kappa\delta)} = \left[\chi_1^{(\kappa\delta)}, \chi_2^{(\kappa\delta)}\right] , \qquad \boldsymbol{T} = \left[\boldsymbol{T}_{(1)}, \boldsymbol{T}_{(2)}\right] . \tag{B.19}$$

The matrix $\boldsymbol{T}$ is a representation of the Somigliana tensor.

Our aim now is to show that

$$\eta_{..\iota\varrho}^{\sigma\gamma} = \delta_{\iota\varrho}^{\sigma\gamma} . \tag{B.20}$$

The key idea of the proof is to make use of the Somigliana's equation (B.2).

Let the hyperforce

$$\boldsymbol{s}^{(\lambda\mu)} = \left[\epsilon_{\lambda\mu}(\boldsymbol{b}_{(1)}), \epsilon_{\lambda\mu}(\boldsymbol{b}_{(2)})\right] \tag{B.21a}$$

be applied at the point $\boldsymbol{O} = (0,0)$; λ and μ are viewed here as fixed. Its components read

$$s_\alpha^{(\lambda\mu)} = \epsilon_{\lambda\mu}(\boldsymbol{b}_{(\alpha)}) \tag{B.21b}$$

and are given by

$$s_\alpha^{(\lambda\mu)} = \frac{1}{2}\left[\frac{\partial}{\partial y_\mu}\left(\delta_{\alpha\lambda}\delta(y)\right) + \frac{\partial}{\partial y_\lambda}\left(\delta_{\alpha\mu}\delta(y)\right)\right] = \frac{1}{2}\left(\delta_{\alpha\lambda}\frac{\partial\delta}{\partial y_\mu} + \delta_{\alpha\mu}\frac{\partial\delta}{\partial y_\lambda}\right) . \tag{B.22}$$

The force (B.21a) causes the displacement field

$$\boldsymbol{U}^{(\lambda\mu)} = \left[\epsilon_{\lambda\mu}(\boldsymbol{T}_{(1)}), \epsilon_{\lambda\mu}(\boldsymbol{T}_{(2)})\right] \tag{B.23}$$

or

$$\boldsymbol{U}^{(\lambda\mu)} = \epsilon_{\lambda\mu}(\boldsymbol{T}) . \tag{B.24}$$

The displacement field $\boldsymbol{U}^{(\lambda\mu)}$ satisfies the variational equation (B.2) in which the force $\boldsymbol{b}_{(\sigma)}$ is now replaced with $\boldsymbol{s}^{(\lambda\mu)}$. It reads

$$\int\limits_{B_R} A^{\alpha\beta\iota\varrho} \epsilon_{\iota\varrho}\left(\epsilon_{\lambda\mu}(\boldsymbol{T})\right) \epsilon_{\alpha\beta}(\boldsymbol{\varphi}) \, dy = \left\langle \boldsymbol{s}^{(\lambda\mu)}, \boldsymbol{\varphi}\right\rangle + \int\limits_{\Gamma_R} A^{\alpha\beta\iota\varrho} \epsilon_{\iota\varrho}\left(\epsilon_{\lambda\mu}(\boldsymbol{T})\right) n_\beta \varphi_\alpha \, ds , \tag{B.25}$$

where φ_α denote the components of the vector test function $\boldsymbol{\varphi}$ which is assumed to be sufficiently regular.

Using the pseudoforces introduced below, we have

$$\boldsymbol{\varphi}\cdot\boldsymbol{s}_h^{(\lambda\mu)} = \varphi_\alpha s_{h\alpha}^{(\lambda\mu)} = \frac{1}{2}\left(\varphi_\lambda\frac{\partial\phi_h}{\partial y_\mu} + \varphi_\mu\frac{\partial\phi_h}{\partial y_\lambda}\right) = \epsilon_{\lambda\mu}(\phi_h\cdot\boldsymbol{\varphi}) - \epsilon_{\lambda\mu}(\boldsymbol{\varphi})\phi_h(y) \tag{B.26}$$

which leads to the expression, by the limit passage,

$$\left\langle \boldsymbol{s}^{(\lambda\mu)},\boldsymbol{\varphi}\right\rangle = \lim_{h\searrow 0}\left\langle \boldsymbol{s}_h^{(\lambda\mu)},\boldsymbol{\varphi}\right\rangle = -\epsilon_{\lambda\mu}(\boldsymbol{\varphi})(\boldsymbol{O})\ . \tag{B.27}$$

Let us introduce the pseudoforces $\boldsymbol{s}_h^{(\lambda\mu)}$ of components $\left(\boldsymbol{s}_h^{(\lambda\mu)}\right)_\alpha$ given by (B.22) with $\delta(y)$ replaced by $\phi_h(y)$. Here $\phi_h(y)$ is the mollifier defined in Appendix A. The pseudoforces $\boldsymbol{s}_h^{(\lambda\mu)}$ cause the displacement fields $\epsilon_{\lambda\mu}(\boldsymbol{T}^h)$ which satisfy the variational equation:

$$A^{\alpha\beta\iota\varrho}\int_{B_R}\epsilon_{\iota\varrho}\left(\epsilon_{\lambda\mu}(\boldsymbol{T}^h)\right)\epsilon_{\alpha\beta}(\boldsymbol{\varphi})\,dy = \int_{B_R}\boldsymbol{\varphi}\cdot\boldsymbol{s}_h^{(\lambda\mu)}\,dy + A^{\alpha\beta\iota\varrho}\int_{\Gamma_R}\epsilon_{\iota\varrho}\left(\epsilon_{\lambda\mu}(\boldsymbol{T}^h)\right)n_\beta\varphi_\alpha\,ds$$

for sufficiently regular $\boldsymbol{\varphi}$. One can substitute $\boldsymbol{\varphi} = \boldsymbol{E}^{(\kappa\delta)}$ to obtain

$$A^{\alpha\beta\iota\varrho}\int_{B_R}\epsilon_{\iota\varrho}\left(\epsilon_{\lambda\mu}(\boldsymbol{T}^h)\right)\,dy = \int_{B_R}\boldsymbol{E}^{(\kappa\delta)}\cdot\boldsymbol{s}_h^{(\lambda\mu)}\,dy + A^{\alpha\beta\iota\varrho}\int_{\Gamma_R}\epsilon_{\iota\varrho}\left(\epsilon_{\lambda\mu}(\boldsymbol{T}^h)\right)E_\alpha^{(\kappa\delta)}n_\beta\,ds\ . \tag{B.28}$$

Note that the stresses associated with the field $\boldsymbol{E}^{(\kappa\delta)}$ satisfy the homogeneous equations of equilibrium. Thus the following variational equation holds

$$A^{\alpha\beta\iota\varrho}\int_{B_R}\epsilon_{\iota\varrho}\left(\boldsymbol{E}^{(\kappa\delta)}\right)\epsilon_{\alpha\beta}(\widetilde{\boldsymbol{v}})\,dy = A^{\alpha\beta\iota\varrho}\int_{\Gamma_R}\epsilon_{\iota\varrho}\left(\boldsymbol{E}^{(\kappa\delta)}\right)n_\beta\widetilde{v}_\alpha\,ds \tag{B.29}$$

for sufficiently regular $\widetilde{v}_\alpha$. Let us substitute $\widetilde{\boldsymbol{v}} = \epsilon_{\lambda\mu}(\boldsymbol{T}^h)$ to obtain

$$A^{\alpha\beta\kappa\delta}\int_{B_R}\epsilon_{\alpha\beta}\left(\epsilon_{\lambda\mu}(\boldsymbol{T}^h)\right)\,dy = A^{\alpha\beta\kappa\delta}\int_{\Gamma_R}\epsilon_{\lambda\mu}(\boldsymbol{T}^h_{(\alpha)})n_\beta\,ds\ . \tag{B.30}$$

By equating the right-hand sides of the indentities (B.28) and (B.30) one finds

$$A^{\alpha\beta\iota\varrho}\int_{\Gamma_R}\left[\epsilon_{\iota\varrho}\left(\epsilon_{\lambda\mu}(\boldsymbol{T}^h)\right)E_\alpha^{(\kappa\delta)} - \epsilon_{\lambda\mu}(\boldsymbol{T}^h_{(\alpha)})\delta_{\iota\varrho}^{\kappa\delta}\right]n_\beta\,ds = -\int_{B_R}E_\alpha^{(\kappa\delta)}(s_h^{(\lambda\mu)})_\alpha\,dy\ .$$

Let us go with h to zero. The left-hand side of the above identity tends to $\eta^{\kappa\delta}_{..\lambda\mu}$ defined by (B.17). The right-hand side tends to $\epsilon_{\lambda\mu}(\boldsymbol{E}^{(\kappa\delta)}) = \delta^{\kappa\delta}_{\lambda\mu}$. This proves (B.20).

Substitution (B.20) into (B.16) gives $\boldsymbol{N} = \boldsymbol{M}$. The thesis (i) is proved.

Let us proceed to derive the formula (B.6). We substitute $\widetilde{\boldsymbol{v}} = \boldsymbol{\chi}^{(\sigma\gamma)}$ in the variational equation (5.30). We note that

$$\int_{\Gamma_R}\epsilon_{\lambda\mu}\left(\boldsymbol{\chi}^{(\kappa\delta)}\right)\chi_\alpha^{(\sigma\gamma)}n_\beta\,ds \to 0$$

it $R\to\infty$. Indeed, $\epsilon_{\lambda\mu}\left(\boldsymbol{\chi}^{(\kappa\delta)}\right) = O(R^{-2})$ and $\chi_\alpha^{(\sigma\gamma)} = O(R^{-1})$, $ds = R\,d\theta$. Thus the above result holds. Consequently, we find

$$A^{\alpha\beta\lambda\mu}\int_{\mathbb{R}^2\setminus\overline{\omega}}\left[\epsilon_{\lambda\mu}\left(\boldsymbol{\chi}^{(\kappa\delta)}\right)\epsilon_{\alpha\beta}\left(\boldsymbol{\chi}^{(\sigma\gamma)}\right)\right]\,dy = A^{\alpha\beta\kappa\delta}\int_{\partial\omega}\chi_\alpha^{(\sigma\gamma)}\nu_\beta\,ds\ , \tag{B.31}$$

which proves (B.8).

Let us write down (B.11) in the form

$$M^{\kappa\delta\sigma\gamma} = -A^{\alpha\beta\kappa\delta} \int\limits_{\partial\omega} \left[E_\alpha^{(\sigma\gamma)} + \chi_\alpha^{(\sigma\gamma)} \right] \nu_\beta \, ds \; . \tag{B.32}$$

By using the formula (A.22) we compute

$$\int\limits_{\partial\omega} E_\alpha^{(\sigma\gamma)} \nu_\beta \, ds = \frac{1}{2} \int\limits_{\partial\omega} (y_\gamma \delta_{\alpha\sigma} + y_\sigma \delta_{\alpha\gamma}) \, \nu_\beta \, ds = \frac{1}{2} (\delta_{\alpha\sigma}\delta_{\gamma\beta} + \delta_{\alpha\gamma}\delta_{\sigma\beta}) \, |\omega| = \delta_{\alpha\beta}^{\sigma\gamma} |\omega| \; . \tag{B.33}$$

Thus the formulae (B.22) and (B.8) imply (B.6), which proves the thesis (ii).

The symmetry properties (B.9) are the consequences of the formulae (B.6) – (B.7) and the symmetry properties (5.3) of the tensor $\boldsymbol{A}$.

Let us prove (iv). Consider the quadratic form

$$f(\boldsymbol{q}) = q_{\alpha\beta}(-M^{\alpha\beta\lambda\mu}) q_{\lambda\mu} \; .$$

By (B.6), (B.7) we have

$$f(\boldsymbol{q}) = q_{\kappa\delta} A^{\kappa\delta\sigma\gamma} q_{\sigma\gamma} |\omega| + \int\limits_{\mathbb{R}^2 \setminus \overline{\omega}} \epsilon_{\lambda\mu}(\boldsymbol{\phi}) A^{\lambda\mu\alpha\beta} \epsilon_{\alpha\beta}(\boldsymbol{\phi}) \, dy \; , \tag{B.34}$$

with $\boldsymbol{\phi} = q_{\alpha\beta} \boldsymbol{\chi}^{(\alpha\beta)}$. Since $\boldsymbol{\chi}^{(\alpha\beta)} \notin \mathcal{R}$ we know that $\epsilon_{\lambda\mu}(\boldsymbol{\phi}) \neq 0$. The estimation (5.4) implies $f(\boldsymbol{q}) > 0$ if $\boldsymbol{q} \neq \boldsymbol{0}$. The thesis (iv) is proved.

The present Appendix has been written with the help of the book by Maz'ya et al. (1991), where the notion of the mass matrix had been introduced in the case of isotropy of the tensor $\boldsymbol{A}$.

APPENDIX C.

The Maz'ya-Nazarov formula for the energy of a body with a small opening. The Neumann problem.

The aim of the present Appendix is to recall the derivation of the formula (4.43). This formula appeared in the literature for the first time in the paper by Maz'ya and Nazarov (1987). The derivation given below repeats the arguments of the original derivation.

The energy function $E(\Omega_\varepsilon)$ equals

$$E(\Omega_\varepsilon) = \int\limits_{\partial\Omega} p(x) u_\varepsilon(x) \, ds \; . \tag{C.1}$$

To use the formula above we must assure that the boundary condition (3.3) is fulfilled with accuracy $O(\varepsilon^2)$. Let us write down the expansion (3.28) with using (3.26). We write

$$u_\varepsilon(x) = v(x) - \varepsilon^2 m_{\beta\alpha} \frac{1}{2\pi\mu} \frac{x_\alpha}{\|x\|^2} \epsilon_\beta^0 + \varepsilon^2 z(x) + \varepsilon O\left(\left\| \frac{\varepsilon}{x} \right\|^2 \right) \; . \tag{C.2}$$

The condition (3.3) imposes

$$\mu \frac{\partial z}{\partial n} = \frac{1}{2\pi} m_{\beta\alpha} \epsilon_\beta^0 \frac{\partial}{\partial n} \left(\frac{x_\alpha}{\|x\|^2} \right) \qquad \text{on } \partial\Omega \; . \tag{C.3}$$

Thus the function z is harmonic and satisfies (C.3). Let us decompose this function

$$z(x) = m_{\beta\alpha} \epsilon_\beta^0 z_\alpha(x) \; , \tag{C.4}$$

where z_α are solutions to the problems

$$\left(\widehat{P}_\alpha\right) \quad \left| \begin{array}{lll} \text{find } z_\alpha \text{ defined in } \Omega \text{ such that} & & \\ \mu\Delta z_\alpha = 0 & \text{in } \Omega \ , & \text{(C.5)} \\ \mu\dfrac{\partial z_\alpha}{\partial n} = \dfrac{1}{2\pi}\dfrac{\partial}{\partial n}\left(\dfrac{x_\alpha}{\|x\|^2}\right) & \text{on } \partial\Omega \ . & \text{(C.6)} \end{array} \right.$$

Let us note that the functions $f_\alpha = \dfrac{1}{2\pi}\dfrac{x_\alpha}{\|x\|^2}$ do not satisfy (C.5) for $x = \boldsymbol{O} \in \Omega$.

Let

$$z_\alpha(x) = T_\alpha(x) - \frac{\partial S(x)}{\partial x_\alpha} \ , \tag{C.7}$$

where $S(x)$ satisfies (3.24). Thus we have

$$z_\alpha(x) = T_\alpha(x) + \frac{1}{2\pi\mu}\frac{x_\alpha}{\|x\|^2} \tag{C.8}$$

and

$$\mu\Delta z_\alpha = \mu\Delta T_\alpha - \mu\Delta\left(\frac{\partial S}{\partial x_\alpha}\right) = \mu\Delta T_\alpha + \frac{\partial\delta(x)}{\partial x_\alpha} \ . \tag{C.9}$$

Along the boundary $\partial\Omega$ we have

$$\mu\frac{\partial z_\alpha}{\partial n} = \mu\frac{\partial T_\alpha}{\partial n} + \frac{1}{2\pi}\frac{\partial}{\partial n}\left(\frac{x_\alpha}{\|x\|^2}\right) \ . \tag{C.10}$$

We see that the functions T_α satisfy:

$$\mu\Delta T_\alpha = -\frac{\partial\delta(x)}{\partial x_\alpha} \qquad \text{in } \Omega \tag{C.11}$$

$$\mu\frac{\partial T_\alpha}{\partial n} = 0 \qquad \text{on} \quad \partial\Omega \ . \tag{C.12}$$

Substitution of (C.8) into (C.2) gives

$$u_\varepsilon(x) = v(x) + \varepsilon^2 m_{\beta\alpha}\,\epsilon^0_\beta\,T_\alpha(x) + \varepsilon O\left(\left\|\frac{\varepsilon}{x}\right\|^2\right) \ . \tag{C.13}$$

We substitute now the above expression into (C.1) and find

$$E(\Omega_\varepsilon) = E(\Omega) + \varepsilon^2 m_{\beta\alpha}\epsilon^0_\beta \int\limits_{\partial\Omega} p(x) T_\alpha(x)\,ds + o\left(\varepsilon^{2+0}\right) \ . \tag{C.14}$$

Let us write down the weak formulation of the problems (C.11 – C.12) and (P_0) of Sec. 3:

$$\mu\int\limits_\Omega \frac{\partial T_\alpha}{\partial x_\beta}\frac{\partial\varphi}{\partial x_\beta}\,dx = \int\limits_{\partial\Omega}\frac{\partial\delta(x)}{\partial x_\alpha}\varphi\,ds \ , \tag{C.15}$$

$$\mu\int\limits_\Omega \frac{\partial v}{\partial x_\beta}\frac{\partial\widetilde{\varphi}}{\partial x_\beta}\,dx = \int\limits_{\partial\Omega} p\widetilde{\varphi}\,ds \ . \tag{C.16}$$

Substitution of $\varphi = v$ and $\widetilde{\varphi} = T_\alpha$ leads to the identity

$$\int\limits_{\partial\Omega} pT_\alpha\,ds = \int\limits_{\partial\Omega}\frac{\partial\delta(x)}{\partial x_\alpha}v(x)\,dx \ , \tag{C.17}$$

which gives

$$\int\limits_{\partial\Omega} pT_\alpha\,ds = -\int\limits_\Omega \delta(x)\frac{\partial v}{\partial x_\alpha}\,dx = -\frac{\partial v}{\partial x_\alpha}(\boldsymbol{O}) = -\epsilon^0_\alpha \ . \tag{C.18}$$

The expansion (4.43) is now confirmed.

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