# Preferential Entailments, Extensions and Reductions of The Vocabulary

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# Preferential entailments, extensions and reductions of the vocabulary

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**Abstract:** A preferential entailment is defined by a binary relation, or "preference relation", either among interpretations (or models) or among "states" which are "copies of interpretations". Firstly, we show how an extension of the vocabulary allows to express any preferential entailment as a preferential entailment without state. Secondly, by reducing the vocabulary, we show how to express some preferential entailments in a smaller language. This second method works only for particular preferential entailments, including the widely used circumscription. For our purpose, we need to make precise the operations of extension or reduction of the vocabulary, which may have applications in other domains. We study which properties of an inference operation are preserved by the reduction and extension of the vocabulary introduced in this text. We provide examples of applications of our results. These applications are all related to various kinds of circumscriptions, because this suffices to provide examples of useful and non trivial results. Moreover, our study shows that many preferential entailments of the two kinds may be easily expressed in terms of circumscription. All along the text, we take great care in providing constructive definitions, and to keep these constructions as simple and natural as possible.

**Key-words:** Circumscription, common sense reasoning, knowledge representation, non monotonic reasoning, preferential entailment, propositional logic.

*(Résumé : tsvp)*

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# Inférence préférentielle, extension et réduction du vocabulaire

**Résumé :** Ce papier décrit deux façons de simplifier différentes inférences non monotones, appelées inférences préférentielles. Une inférence préférentielle est définie par une relation binaire, “de préférence”, portant soit directement sur les interprétations (ou modèles) soit sur des objets appelés “états” qui sont des “copies d'interprétations”. On ne garde d'un ensemble de données fourni sous la forme d'une théorie logique que les formules vraies pour les modèles de la théorie qui sont minimaux pour cette relation, ou dont une copie est minimale. Une première façon consiste à augmenter le vocabulaire afin de transformer la variante la plus complexe, définie à l'aide d'états, en la variante la plus simple définie directement en termes de modèles. Cette méthode permet aussi parfois de simplifier l'étude d'une inférence préférentielle tout en restant dans le même type d'inférence. La seconde méthode permet de se contenter d'un vocabulaire réduit pour exprimer une inférence préférentielle donnée. Cette méthode ne convient que pour des inférences préférentielles particulières, mais qui sont très utilisées dans la littérature car la circonscription est concernée. Nous commençons par une description précise des opérations naturelles associées à la réduction ou à l'extension du vocabulaire, et cette partie peut s'appliquer à bien d'autres domaines de la logique propositionnelle que l'inférence préférentielle. Nous étudions ensuite en détail quelles sont les propriétés classiques des inférences préférentielles et des relations de préférence qui sont préservées par ces extensions et réductions du vocabulaire. De nombreux exemples d'utilisation de ces résultats sont donnés, qui fournissent des résultats nouveaux non triviaux. Tous ces exemples concernent des circonscriptions, mais les méthodes présentées peuvent s'appliquer à d'autres inférences préférentielles. Il faut d'ailleurs noter que notre étude montre comment de nombreuses inférences préférentielles des deux types peuvent être traduites de façon naturelle en termes de circonscription. Tout au long de ce texte, nous fournissons des méthodes constructives, nous attachant à garder ces constructions aussi simples et directes que possible, afin de permettre en particulier une utilisation de nos résultats pour le calcul automatique des inférences préférentielles concernées.

**Mots-clé :** Circonscription, inférence préférentielle, logique propositionnelle, raisonnement de sens commun, raisonnement non monotone, représentation des connaissances.

# Contents

# 1 Introduction

The notion of preferential entailment has shown to be very useful in knowledge representation, when dealing with some aspects of common sense reasoning such as implicit knowledge or rules with exceptions. Various more or less complex kinds of "preferential entailments" have appeared in the literature. The simplest versions use a "preference relation" directly between classical models of a theory. An intricate version adds an intermediate step, using a "preference relation" between "states", a state being a copy of a classical model. We show here how we can study this more complicated notion thanks to the simplest one, by extending the vocabulary. Then, we generalize this method and we give a few "preservation results": we examine which important properties of preferential entailments, or even more general inference operations, are preserved by this operation, when we start from the greater language and come back to the smaller original, or "useful", language. This study provides various important results, which are detailed below.

In the literature about non monotonic reasoning, another modification of the vocabulary has been introduced, using as an auxiliary language a language smaller than the "useful" language. We study also this method. Again, we provide various preservation results and important practical applications. We even get an example in which the two methods work.

Our study is important for at least two reasons: It gives various useful results which should help the automatic computation. It allows to get a better understanding of complex notions, by expressing very simply these notions in terms of easier and more natural ones.

Notice that we do not deal with implementation problems here. However, we give precise and detailed constructions in each case, in order to facilitate an eventual implementation of parts of this work. Thus, we provide precise methods for computing the "restriction of a formula to a smaller language" (that we call the trace of a formula). We describe the elementary preferential entailment obtained when we start from the more complex variant (with states) of a preferential entailment, and we take care to get the smallest possible extended vocabulary. We describe the various possible operations in the other way, when we come back from the auxiliary language to the original "useful" language. For what concerns X-mappings, where sets of formulas are used, we describe a method of computation of the new set of formulas from the original set. All these results are given for the two kinds of transformations examined in the present text.

In section 2 we introduce in great details the notations used in the text, in order to facilitate the reading of the rest of this text. In order to keep things reasonably simple, we restrict our attention to propositional logic. Indeed, this case is rich enough to allow to study the problems addressed here, and the predicate calculus case would considerably complicate this study, by introducing bewildering conditions of applicability for each of the properties studied here.

In section 3, we examine the passage from a language to a smaller one. There, we give various simple results of elementary logic which are not very well known, but which are very useful for our purpose. We think that this study alone is important in a context of knowledge representation, as such variations of the vocabulary are very frequent.

In section 4 we introduce two notions of "preferential entailments" widely used in the literature, and also a lesser known but promising related notion, X-mapping. We begin in subsection 4.1 by a reminder of the definitions and the results useful for our purpose. In subsection 4.2 we describe the method allowing to express simply any preferential entailment in the wide meaning (using the "states") in terms of the much simpler and natural notion of elementary preferential entailments (using directly classical models). We give the definitions of this first kind of transformation, called "reduction of the vocabulary", because we examine what happens when we start from the greater auxiliary language and come back to the original smaller language.

In section 5 we introduce the main "logical properties" of the kind of non monotonic inference studied here. In subsection 5.1, we list a menagerie of the main logical properties useful for understanding the behavior of a given logical formalism. These properties have immediate translations in terms of common sense, thus they are very important for a user of such inferences (*working with* these inferences). Moreover, they have important technical consequences, thus they are important also to any researcher who is *working on* these inferences. These properties generally come in various versions (theory version, formula version, ...).

In subsection 5.2, we introduce the main properties that a relation associated to a preferential entailment may possess, and we remind how these properties are related to the properties of the inference relation defined by the preferential entailment considered.

In section 6, we examine which of the properties introduced in the two preceding sections are preserved by the operation of "reduction of the vocabulary" introduced in subsection 4.2. We get two kinds of properties (those preserved, and those non necessarily preserved) and it happens that the preserved properties have received much more attention in the literature.

In section 7, we apply the preservation results obtained in section 6 to circumscription. This is not to say that these are the only possible applications, by far, but we think that this provides important and interesting applications of our preceding results. In particular, by elaborating from our preceding results and from a result recently published in the literature, we provide new characterizations of the notion of "finite cumulative preferential entailment (wide meaning, with states)" in terms of various kinds of circumscriptions. We show why, in the finite case, the two notions of ordinary circumscription and of cardinality-based circumscription are easily inter-definable. We show also how our preservation results suffice to describe the situation in the infinite case for that matter. We take care to provide constructive and relatively efficient methods for all the results given in this section.

In section 8, we examine another transformation used in the literature on the subject. The aim is to describe, again in simple and natural terms, some kinds of preferential entailments thanks to preferential entailments defined in a smaller vocabulary. This can be done for some important kinds of preferential entailments, including circumscriptions with varying propositions. In this case, the smaller language lacks the propositional symbols which are varying in the circumscription. Again, the main application should be improving the efficiency of automatic computations, as it is easier to compute a preferential entailment in a smaller language when moreover, and this is the case here, the preferential entailment in the smaller language is simpler. This method is called "extension of the vocabulary" because again, we examine what happens when we start from the auxiliary vocabulary (here, the smallest vocabulary), in order to come back to the original "useful" vocabulary. We give the preservation results concerning this new transformation, as we have done for the first transformation in section 6. We provide a few applications of these results, again about circumscriptions. We apply an old result about the suppression of the varying predicates for predicate circumscriptions to the propositional case. Then, we get the corresponding result, together with its precise conditions of applicability. These conditions are new, and, to our knowledge, the equivalent in the predicate calculus case is not known yet. Also, we give the corresponding result for cardinality-based circumscription. This provides a particular case in which we have the choice of using either the method of section 6 or the method of section 8. Indeed, we may either extend or reduce the vocabulary, exactly for the same purpose: suppressing the varying propositions in a finite cardinality-based circumscription.

In section 3, dealing with classical propositional logic, the proofs of propositions or other results, which range from practically automatic to straightforward, have been omitted for the sake of clarity and conciseness, being "referenced" as [pa] . For the same reasons, in section 5, the proofs of the known results about preferential entailments are not included, but references are given. For what concerns precisely the purpose of this text, the proofs are given (sections 4, 6, 7 and 8).

# 2 Notations

## 2.1 Classical propositional logic: theories and models

• $\mathbf{L}$, $V$, $\varphi$, $\mathcal{T}$:

We work in a propositional language $\mathbf{L}$, assimilated to the set of all the formulas in the language. $V(\mathbf{L})$ denotes the vocabulary of $\mathbf{L}$: it is a set of *propositional symbols*. Letters such as $\varphi, \psi, ...$ denote formulas in $\mathbf{L}$. A *formula* will be assimilated to its equivalence class ($\mathbf{L}$, the set of all the formulas, is the Lindenbaum algebra). Letters such as $\mathcal{T}$ denote sets of formulas. $V(\mathcal{T})$, or $V(\varphi)$, denotes the set of the elements of $V(\mathbf{L})$ appearing in some chosen writing of $\mathcal{T}$, or of $\varphi$. Letters $X, \Phi, \mathbf{p}$, etc.. will sometimes also denote sets of formulas, but only when the deductive closure by $Th$ is not intended.

• $\mathbf{M}$, $\mu$, $\mathcal{P}(E)$, $\mu \models \cdots$:

Letters such as $\mu, \nu, ....$ denote *interpretations for* $\mathbf{L}$, an interpretation being a mapping from $V(\mathbf{L})$ to $\{TRUE, FALSE\}$. An interpretation is assimilated to a subset of $V(\mathbf{L})$: $P \in \mu$ iff $\mu$ associates $TRUE$ to $P$. The writing $\mu \models \varphi$ is defined classically: e.g. if $P, Q, Z$ are in $V(\mathbf{L})$, $\mu \models P \wedge (\neg Q \vee Z)$ iff $P \in \mu$ and ($Q \notin \mu$ or $Z \in \mu$). The set of all the interpretations for $\mathbf{L}$, assimilated to the set $\mathcal{P}(V(\mathbf{L}))$, is denoted by $\mathbf{M}$. As usual, for any set $E$, $\mathcal{P}(E)$ denotes the set of all the subsets of $E$.

A *model* of $\mathcal{T}$ is an interpretation $\mu$ such that $\mu \models \varphi$ for any $\varphi \in \mathcal{T}$.

$\mathbf{M}(\mathcal{T})$ denotes the set of all the models of $\mathcal{T}$, $\mathbf{M}(\varphi)$ denotes the set of all the models of $\varphi$.

• $\mathbf{T}, Th(\mathcal{T}), \mathcal{T} \models \cdots$:

A *theory* $\mathcal{T}$ is a set of formulas closed for the deduction: $\mathcal{T} = Th(\mathcal{T})$, where $Th(\mathcal{T}) = \{\varphi \in \mathbf{L}, \mathcal{T} \models \varphi\}$. We denote by $\mathbf{T}$ the set of all the theories of $\mathbf{L}$. $\models$ denotes the semantical entailment: $\mathcal{T} \models \varphi$ iff any interpretation of $\mathcal{T}$ is an interpretation of $\varphi$, and $\mathcal{T} \models \mathcal{T}_1$ iff for any $\varphi_1 \in \mathcal{T}_1$ we have $\mathcal{T} \models \varphi_1$ (as we work in propositional logic, which is complete, we could also say that $\models = \vdash$ denotes the syntactical inference). When $\mathcal{T}_1$ and $\mathcal{T}_2$ are in $\mathbf{T}$, and $\varphi$ is in $\mathbf{L}$, we have $\mathcal{T}_1 \models \varphi$ iff $\varphi \in \mathcal{T}_1$ and $\mathcal{T}_1 \models \mathcal{T}_2$ iff $\mathcal{T}_2 \subseteq \mathcal{T}_1$.

• $\top, \bot, \sqcup$:

Two special formulas are denoted $\top$ (true formula, true in any interpretation) and $\bot$ (false formula, true in no interpretation), they are logical constants, belonging to any language $\mathbf{L}$ but not to $V(\mathbf{L})$. If $V(\mathbf{L}) = \emptyset$, then $\mathbf{L} = \{\top, \bot\}$. For any $\mathbf{L}$, $Th(\emptyset) = Th(\top) = \{\top\}$ and $Th(\bot) = \mathbf{L}$.

If $\mathcal{T}_1, \mathcal{T}_2$ are subsets of $\mathbf{L}$, and $\varphi$ is a formula in $\mathbf{L}$, we note $\mathcal{T}_1 \sqcup \mathcal{T}_2$ for the theory $Th(\mathcal{T}_1 \cup \mathcal{T}_2)$ and $\mathcal{T} \sqcup \varphi$ for the theory $Th(\mathcal{T} \cup \{\varphi\})$. Thus $\mathbf{M}(\mathcal{T}_1 \sqcup \mathcal{T}_2) = \mathbf{M}(\mathcal{T}_1 \cup \mathcal{T}_2) = \mathbf{M}(\mathcal{T}_1) \cap \mathbf{M}(\mathcal{T}_2)$.

## 2.2 Complete theories and interpretations

• $\mathbf{C}, \mathcal{C}, Th(\mu)$:

A theory is *complete* iff it has exactly one model iff it contains $\top$ and, for any $\varphi \in \mathbf{L}$, we have exactly one of the two possibilities $\varphi \in \mathcal{T}$ or $\neg\varphi \in \mathcal{T}$. We denote by $\mathbf{C}$ the set of all the complete theories of $\mathbf{L}$. Letter $\mathcal{C}$ will denote a complete theory.. $Th(\mu)$ denotes the set of all the formulas satisfied by $\mu$: $Th(\mu) = \{\varphi \in \mathbf{L}, \mu \models \varphi\}$. For any subset $\mathbf{M}_1$ of $\mathbf{M}$, $Th(\mathbf{M}_1) = \{\varphi \in \mathbf{L}, \mu \models \varphi \text{ for any } \mu \in \mathbf{M}_1) = \bigcap_{\mu \in \mathbf{M}_1} Th(\mu)$.

This ambiguous usage of $Th$ and of $\models$ (applied to sets of formulas, or to sets of interpretations) is classical in logic and should not provoke confusion.

For any $\mathcal{T} \in \mathbf{T}$, $\mathbf{C}(\mathcal{T})$ will denote the set of all the complete theories $\mathcal{C}$ which entail (i.e. contain) $\mathcal{T}$: $\mathbf{C}(\mathcal{T}) = \{\mathcal{C} \in \mathbf{C} \;/\; \mathcal{C} \models \mathcal{T}\}$. We know that we have $\mathcal{T} = \bigcap_{\mathcal{C} \in \mathbf{C}(\mathcal{T})} \mathcal{C}$.

$\mathbf{C}$ may be assimilated to $\mathbf{M}$: For any $\mu \in \mathbf{M}$ we have $Th(\mu) \in \mathbf{C}$ and for any $\mathcal{C} \in \mathbf{C}$, $\mathbf{M}(\mathcal{C})$ is a singleton $\{\mu\} \subseteq \mathbf{M}$. $\mathbf{C}(\mathcal{T})$ could be assimilated to $\mathbf{M}(\mathcal{T})$: $\mathcal{C} \in \mathbf{C}(\mathcal{T})$ iff $\mathcal{C} = Th(\mu)$ and $\mu \in \mathbf{M}(\mathcal{T})$.

As we assimilate a formula to its equivalence class, if $V(\mathbf{L})$ is finite with $n$ elements, $\mathbf{M}$ and $\mathbf{C}$ have $2^n$ elements while $\mathbf{L}$ and $\mathbf{T}$ have $2^{(2^n)}$ elements. If $V(\mathbf{L})$ is infinite, we still have $card(\mathbf{M}) = card(\mathbf{C})$ but we no longer have one-to-one mappings between $\mathcal{P}(\mathbf{M})$ and $\mathbf{T}$ and between $\mathbf{T}$ and $\mathbf{L}$: for any infinite cardinal $\lambda$ we have, if $card(V(\mathbf{L})) = \lambda$, $card(\mathbf{L}) = \lambda$, $card(\mathbf{M}) = card(\mathbf{C}) = card(\mathbf{T}) = 2^\lambda$ (we have still $\mathbf{C} \subset \mathbf{T}$) while $card(\mathcal{P}(\mathbf{M})) = 2^{(2^\lambda)}$ (see subsection 2.4 below).

## 2.3 Normal forms of formulas

If $\varphi$ is a formula in $\mathbf{L}$, we define as usual a *disjunctive normal form* of $\varphi$ as a writing of $\varphi$ as a disjunction of conjunctions of literals. $\varphi_1 \vee \cdots \vee \varphi_n$ is a *reduced disjunctive normal form* of $\varphi$ if each $\varphi_i$ is a conjunction of literals (in which two distinct literals never correspond to the same propositional symbol), the $\varphi_i$'s are all distinct and, if $\varphi'$ is a conjunction of literals such that $\varphi_i \models \varphi'$ and $\varphi' \models \varphi$, then $\varphi_i = \varphi'$. The *standard disjunctive normal form* of $\varphi$ is the (reduced) disjunctive normal form $\varphi = \varphi_1 \vee \cdots \vee \varphi_n$ where $\{\varphi_1, \cdots, \varphi_n\}$ is the set of all the conjunctions of literals (corresponding to distinct symbols) $\varphi_i$ such that $\varphi_i \models \varphi$ and, if $\varphi'$ is a conjunction of literals (corresponding to distinct symbols) such that $\varphi_i \models \varphi'$ and $\varphi' \models \varphi$, then $\varphi_i = \varphi'$.

For instance, $(P \wedge R) \vee (Q \wedge \neg R)$ and $(P \wedge R) \vee (Q \wedge \neg R) \vee (P \wedge Q)$ are two reduced disjunctive normal forms of the same formula, only the second one being the standard disjunctive normal form.

The *conjunctive normal forms*, *reduced conjunctive normal forms* and *standard conjunctive normal form* are defined similarly, by duality.

## 2.4 Topology among interpretations (or complete theories)

• $TC$:

$TC(\cdots)$ denotes the classical topological closure: for any set $\mathbf{C}_1 \subseteq \mathbf{C}$, we note $TC(\mathbf{C}_1)$ for the closure of the set $\mathbf{C}_1$, i.e. $TC(\mathbf{C}_1) = \mathbf{C}(\bigcap_{\mathcal{C}_1 \in \mathbf{C}_1} \mathcal{C}_1)$. Thanks to the correspondence between $\mathbf{C}$ and $\mathbf{M}$, for any set $\mathbf{M}_1 \subseteq \mathbf{M}$, we will also denote $TC(\mathbf{M}_1)$ for the closure of $\mathbf{M}_1$, i.e. $TC(\mathbf{M}_1) = \mathbf{M}(Th(\mathbf{M}_1))$.

A subset $\mathbf{M}_1$ of $\mathbf{M}$ is equal to $\mathbf{M}(\mathcal{T})$ for some theory $\mathcal{T}$ iff it is closed for the classical topology: $TC(\mathbf{M}_1) = \mathbf{M}_1$. A subset $\mathbf{M}_1$ of $\mathbf{M}$ is the set of all the models of some formula $\varphi$ in $\mathbf{L}$ (i.e. of some finitely axiomatizable theory $\mathcal{T}$ of $\mathbf{L}$) iff it is open and closed: $TC(\mathbf{M}_1) = \mathbf{M}_1$ and $TC(\mathbf{M} - \mathbf{M}_1) = \mathbf{M} - \mathbf{M}_1$. If $\mathbf{M}$ is finite (i.e. if $V(\mathbf{L})$ is finite), any subset of $\mathbf{M}$ is trivially closed (thus also open). The same things can be expressed in terms of complete theories: a subset $\mathbf{C}_1$ of $\mathbf{C}$ is equal to some $\mathbf{C}(\mathcal{T})$ iff it is closed, i.e. $TC(\mathbf{C}_1) = \mathbf{C}_1$, and $\mathbf{C}_1$ is some $\mathbf{C}(\varphi)$ iff it is open and closed.

## 2.5 Dealing with different languages

• $\mathbf{T}', \mathbf{C}', \mathbf{M}', Th', \models', \sqcup', TC'$ :

$\mathbf{T}, \mathbf{C}, \mathbf{M}, Th, \models, \sqcup, TC$ should be indexed by $\mathbf{L}$, or equivalently by $V(\mathbf{L})$. However, to keep the notations readable, we will denote two languages by say $\mathbf{L}$ and $\mathbf{L}'$, and all what concerns $\mathbf{L}$ will be denoted as above, while we will use $\mathbf{T}', \mathbf{C}', \mathbf{M}', Th', \models', \sqcup', TC'$ for what concerns $\mathbf{L}'$.

We have $V(\mathbf{L}) \subseteq V(\mathbf{L}')$ iff $\mathbf{L} \subseteq \mathbf{L}'$.

### 2.6 Substituting formulas to propositional symbols

• $\mathcal{F}, \mathcal{T}[\mathbf{p}]$:

Let $S$ be a set, we call $\mathcal{F}(S)$ the set of all the (indexed) sets of formulas $\{\varphi_s\}_{s\in S}$ where each $\varphi_s$ is in $\{\bot, \top\}$. Let $\mathbf{P}$ be a set of propositional symbols of a language $\mathbf{L}$ and $\mathcal{T}$ be a set of formulas in $\mathbf{L}$. We write $\mathcal{T} = \mathcal{T}[\mathbf{P}]$ to show that the elements of $\mathbf{P}$ may appear in $\mathcal{T}$, and, for any set $\mathbf{p} = \{\varphi_P\}_{P\in\mathbf{P}}$ of formulas of $\mathbf{L}$ indexed by $\mathbf{P}$ we denote by $\mathcal{T}[\mathbf{p}]$ the set of all the formulas of $\mathcal{T}$ in which each occurrence of each $P \in \mathbf{P}$ is replaced by the formula $\varphi_P$. As we are in propositional logic, we may restrict our attention to sets $\mathbf{p}$ (indexed by $\mathbf{P}$) which are in $\mathcal{F}(\mathbf{P})$ without loss of generality.

## 3 Traces in a smaller vocabulary

Very often in non monotonic reasoning it is useful to reduce or to extend the vocabulary. We give now a few useful results about this matter. As explained in the introduction, none of these results is difficult, thus for the sake of conciseness, we omit all the proofs, referencing these results as [pa] (notice that most of these results are not new anyway). Even if they are easy, these results are very important in our context, thus we try to list here all the results useful for us, as they are not very frequently seen in papers or books dealing with propositional logic.

**$\mathbf{L}$ and $\mathbf{L}'$ are two languages such that $V(\mathbf{L}) \subseteq V(\mathbf{L}')$ (i.e. $\mathbf{L} \subseteq \mathbf{L}'$).**

Any set $\mathcal{T}$ of formulas of $\mathbf{L}$ may be considered as a set of formulas in $\mathbf{L}'$. If $\mathcal{T}_1$ and $\mathcal{T}_2$ are sets of formulas in $\mathbf{L}$, we have $\mathcal{T}_1 \models \mathcal{T}_2$ iff $\mathcal{T}_1 \models' \mathcal{T}_2$, i.e. $\mathbf{M}(\mathcal{T}_1) \subseteq \mathbf{M}(\mathcal{T}_2)$ iff $\mathbf{M}'(\mathcal{T}_1) \subseteq \mathbf{M}'(\mathcal{T}_2)$ (remind the notations introduced in subsection 2.5).

However, there is a little subtlety hidden in these notations. If $\mathcal{T}$ is deductively closed for $\mathbf{L}$, i.e. if $\mathcal{T} \in \mathbf{T}$, then as soon as $\mathbf{L}'$ is different from $\mathbf{L}$ and $\mathcal{T}$ is different from $Th(\emptyset) = \{\top\}$, $\mathcal{T}$ is not deductively closed for $\mathbf{L}'$: $\mathcal{T} \notin \mathbf{T}'$. Indeed, let $\varphi$ be some formula in $\mathcal{T}$ different from $\top$ and $P'$ be a symbol in $V(\mathbf{L}') - V(\mathbf{L})$. The formula $\varphi \vee P'$ is not in $\mathbf{L}$ (which means that it does not exist any writing of $\varphi \vee P'$ using only symbols in $\mathbf{L}$), thus it is not in $\mathcal{T}$, while it is in $Th'(\mathcal{T})$, the deductive closure of $\mathcal{T}$ for $\mathbf{L}'$. We have, in these conditions, $Th'(\mathcal{T}) \supset \mathcal{T} = Th(\mathcal{T})$. This explains why, if $\mathcal{T}'$ is a theory in $\mathbf{T}'$ and $\mathcal{T}$ a theory in $\mathbf{T}$, we do not have $\mathcal{T} \models' \mathcal{T}'$ iff $\mathcal{T}' \subseteq \mathcal{T}$. All we have is $\mathcal{T} \models' \mathcal{T}'$ iff $\mathcal{T}' \subseteq Th'(\mathcal{T})$. Notice that $Th'$ defines a canonical injection from $\mathbf{T}$ to $\mathbf{T}'$.

**Definition 3.1** For any interpretation $\mu$ for $\mathbf{L}$, we define $\mathbf{M}'(\mu)$ as the set of all the interpretations $\mu'$ for $\mathbf{L}'$ such that $\mu' \cap V(\mathbf{L}) = \mu$:

$$\mathbf{M}'(\mu) \;=\; \{\mu' \subseteq V(\mathbf{L}') \;/\; \mu' \cap V(\mathbf{L}) = \mu\} \;=\; \{\mu \cup \mu'' \;/\; \mu'' \subseteq V(\mathbf{L}') - V(\mathbf{L})\}. \quad \square$$

**Remark 3.2** For any $\mu \subseteq V(\mathbf{L})$, we have: $\mathbf{M}'(\mu) = \mathbf{M}'(Th(\mu))$.
This justifies the name given to this set: $\mathbf{M}'(\mu)$ is a natural abbreviation for $\mathbf{M}'(Th(\mu))$. □

**Definitions 3.3** 1. For any interpretation $\mu'$ for $\mathbf{L}'$, we define $tr(\mu')$, which is the following interpretation for $\mathbf{L}$:

$$tr(\mu') = \mu' \cap V(\mathbf{L}).$$

2. For any $\mathbf{M}'_1 \subseteq \mathbf{M}'$, we define the subset $tr(\mathbf{M}'_1)$ of $\mathbf{M}$ as

$$tr(\mathbf{M}'_1) \;=\; \{tr(\mu') \;/\; \mu' \in \mathbf{M}'_1\} \;=\; \{\mu \subseteq V(\mathbf{L}) \;/\; \mathbf{M}'_1 \cap \mathbf{M}'(\mu) \neq \emptyset\}.$$

3. For any $\mathcal{T}' \subseteq \mathbf{L}'$, we denote by $tr(\mathcal{T}')$, and we call *trace of* $\mathcal{T}'$ *in* $\mathbf{L}$, the theory of $\mathbf{L}$ defined by:

$$tr(\mathcal{T}') = Th'(\mathcal{T}') \cap \mathbf{L}'.$$

4. For any $\varphi' \in \mathbf{L}'$, we denote by $tr(\varphi')$, and we call *trace of* $\varphi'$ *in* $\mathbf{L}$, the formula $tr(\varphi')$ of $\mathbf{L}$ defined by (see remark 3.4-2 below for a justification of this definition):

$$Th'(tr(\varphi')) = tr(Th'(\varphi')). \quad \square$$

**Remarks 3.4** [pa]

1. If $tr(\mathcal{T}')$ is considered as a theory of $\mathbf{L}'$, we have:
$$\begin{aligned}\mathbf{M}'(tr(\mathcal{T}')) &= \mathbf{M}'(Th'(tr(\mathcal{T}')))\\ &= \{\mu' \subseteq V(\mathbf{L}') \;/\; \mu' \cap V(\mathbf{L}) = \mu'' \cap V(\mathbf{L}) \text{ for some } \mu'' \in \mathbf{M}(\mathcal{T}')\}\\ &= \textstyle\bigcup_{\mu \in \mathbf{M}(tr(\mathcal{T}'))} \mathbf{M}'(\mu)\\ &= \textstyle\bigcup_{\mu' \in \mathbf{M}'(\mathcal{T}')} \mathbf{M}'(\mu' \cap V(\mathbf{L})).\end{aligned}$$

2. The mapping $tr$ from $\mathbf{M}'$ to $\mathbf{M}$ defined in point 1 of definition 3.3 is continuous. Thus, the set $tr(\mathbf{M}'(\mathcal{T}'))$ is closed for $TC$ (see property 3.5 below). Moreover, if instead of $\mathcal{T}'$ we start from a formula $\varphi'$ in $\mathbf{L}'$, the set $tr(\mathbf{M}'(\varphi'))$ is open and closed for $TC$[1], thus it corresponds to some formula in $\mathbf{L}$, which, again together with property 3.5 below, justifies definition 3.3-4[2].

3. Moreover, for any subset $\mathbf{M}'_1$ of $\mathbf{M}'$, we have: $TC(tr(\mathbf{M}'_1)) = tr(TC'(\mathbf{M}'_1))$[3]. □

Notice that, with definitions 3.3, we get four different kinds of $tr$. This is to keep, as far as possible, the notations easy to remember. The use of the same notation in points 1 and 2 is usual in mathematics, and similarly the use of the same notation in points 3 and 4 is justified by the following equality: $tr(\{\varphi'\}) = Th(tr(\varphi'))$. As for the use of the same notation in points 2 and 3, it is justified by the following property:

[1]The "converse" is false: $tr(\mathbf{M}'(\mathcal{T}'))$ may be open and closed even if $\mathbf{M}'(\mathcal{T}')$ is not open. Notice also that it is very frequent to have $tr(\mathbf{M}'(\varphi')) \cap tr(\mathbf{M}'(\neg\varphi')) \neq \emptyset$ for some $\varphi' \in \mathbf{L}'$.

[2]The best way to prove this result is to use property 3.8-1b below, where, in order to avoid cross references, any occurrence of $tr(\varphi')$ must be replaced by $tr(Th(\varphi'))$, with the modifications naturally induced.

[3]Notice that we may have $tr(\mathbf{M}'_1) = TC(tr(\mathbf{M}'_1))$ without having $\mathbf{M}'_1 = TC'(\mathbf{M}'_1)$.

**Property 3.5** [pa] For any $\mathcal{T}' \subseteq \mathbf{L}'$, we have

$$\mathbf{M}(tr(\mathcal{T}')) = tr(\mathbf{M}'(\mathcal{T}')). \quad \Box$$

Here are a few immediate consequences of these definitions and remarks:

**Property 3.6** [pa] $\mathcal{T} \subseteq \mathbf{L}$, $\mathcal{T}' \subseteq \mathbf{L}'$. We give here a few properties of the trace $tr$ of a set of formulas:

1. $\mathbf{M}'(\mathcal{T}') \subseteq \mathbf{M}'(tr(\mathcal{T}'))$, i.e. $\mathcal{T}' \models' tr(\mathcal{T}')$.
2. For any subsets $\mathcal{T}'_1, \mathcal{T}'_2$ of $\mathbf{L}'$, if $\mathcal{T}'_1 \models' \mathcal{T}'_2$ then $tr(\mathcal{T}'_1) \models tr(\mathcal{T}'_2)$ (converse false).
3. If $\mathcal{T}$ is a subset of $\mathbf{L}$, and if we consider $\mathcal{T}$ as a set of formulas in $\mathbf{L}'$, we have: $tr(\mathcal{T}) = Th(\mathcal{T})$.
4. If $V(\mathcal{T}') \cap V(\mathbf{L}) = \emptyset$, and if $Th'(\mathcal{T}') \neq Th'(\bot)$, then $tr(\mathcal{T}') = Th(\top) = \{\top\}$.
   $\bot \in Th'(\mathcal{T}')$ iff $\bot \in tr(\mathcal{T}')$.
5. $tr \circ Th'$ is the identity in $\mathbf{T}$.
6. $\mathcal{T}' \models' \mathcal{T}$ iff $tr(\mathcal{T}') \models \mathcal{T}$.
   If $\mathcal{T} \models' \mathcal{T}'$ then $\mathcal{T} \models tr(\mathcal{T}')$, the converse being false, in general. $\Box$

Here is a small observation, which has its interest in itself, and which makes the connection between one of our results and a result previously published (in [Lif85], see theorem 8.18 below).

**Remark 3.7** If $\mathbf{p}$ is an indexed set of formulas $\mathbf{p} = \{\varphi_P\}_{P \in V(\mathbf{L}') - V(\mathbf{L})}$, let us denote (see subsection 2.6) by $\mathcal{T}'[\mathbf{p}]$ the set of all the formulas in $\mathcal{T}'$ where each $P \in V(\mathbf{L}') - V(\mathbf{L})$ is replaced by $\varphi_P$. We get

$$tr(\mathcal{T}') = \bigcap_{\mathbf{p} \in \mathcal{F}(V(\mathbf{L}') - V(\mathbf{L}))} \mathcal{T}'[\mathbf{p}].$$

If $\mathcal{T}'$ is finite, we get thus (assimilating $\mathcal{T}'[\mathbf{p}]$ to the conjunction of its elements)

$$tr(\mathcal{T}') = Th(\bigvee_{\mathbf{p} \in \mathcal{F}(V(\mathcal{T}') - V(\mathbf{L}))} \mathcal{T}'[\mathbf{p}]).$$

If we allow the use of *quantified Boolean formulas* (see e.g. [Lad77]) and even more general quantifications ranging over sets of formulas, we may write $tr(\mathcal{T}') = \exists\, \mathbf{p}\ \mathcal{T}'[\mathbf{p}]$. Here, $\exists\, \mathbf{p}$ means $\exists p_1 \cdots \exists p_i, \cdots$ where $\{p_1, \cdots p_i, \cdots\}$ is a set of "propositional variables" in one-to-one mapping with the set $V(\mathbf{L}') - V(\mathbf{L})$, or at least with the set $V(\mathcal{T}') - V(\mathbf{L})$. $\exists p_i\ Exp[p_i]$ is to be understood as usual as: there exists some formula $\varphi'_i$ of $\mathbf{L}'$ to be substituted to each occurrence of $p_i$ in the expression $Exp[p_i]$ such that we have $Exp[\varphi'_i]$. As we are in the propositional case, there is no problem of *definability in* $\mathbf{L}'$, and we may even consider only the two formulas $\top$ and $\bot$ as our possible choice of $\varphi'_i$ (see subsection 2.6): if $Exp[p_i]$ is a formula, $\exists p_i\ Exp[p_i]$ is thus a notation for $Exp[\top] \vee Exp[\bot]$. This explains the presence of the set $\mathcal{F}(V(\mathcal{T}') - V(\mathbf{L}))$ in the expressions of $tr(\mathcal{T}')$ given above. $\Box$

Here are a few more "technical" results, easy to get and useful to remind.

**Property 3.8** [pa] $\mathcal{T}'$, $\mathcal{T}'_1$, $\mathcal{T}'_2$ are subsets of $\mathbf{L}'$, $\varphi'$, $\varphi'_1$, $\varphi'_2$ are formulas in $\mathbf{L}'$.

1. (a) $tr(\mathcal{T}') = \bigsqcup_{\mathcal{T}\subseteq\mathbf{L},\mathcal{T}'\models'\mathcal{T}} \mathcal{T}$,
   $tr(\mathcal{T}') = \bigsqcup_{\varphi\in\mathbf{L},\mathcal{T}'\models'\varphi} \varphi$.
   If $V(\mathbf{L})$ is finite, we have then $tr(\varphi') = \bigwedge_{\varphi\in\mathbf{L},\varphi'\models'\varphi} \varphi$.

   (b) If $\varphi'$ is in a disjunctive normal form, $tr(\varphi')$ is obtained by suppressing the symbols of $V(\mathbf{L}') - V(\mathbf{L})$. Notice that if a term (conjunction) disappears completely, it must be replaced as usual by $\top$ (the neutral element for $\wedge$), thus the result of the disjunction is $\top$.

   (c) If $\varphi'$ is in standard conjunctive normal form, $tr(\varphi')$ is obtained by suppressing the terms (disjunctions) containing a symbol in $V(\mathbf{L}') - V(\mathbf{L})$. If no term remains, the result is $\top$.

2. (a) $tr(\varphi'_1 \vee \varphi'_2) = tr(\varphi'_1) \vee tr(\varphi'_2)$.

   (b) $\mathcal{T}'_1$ and $\mathcal{T}'_2$ are theories of $\mathbf{L}'$: $tr(\mathcal{T}'_1 \cap \mathcal{T}'_2) = tr(\mathcal{T}'_1) \cap tr(\mathcal{T}'_2)$.

   (c) If $\mathcal{T}'_1 \subseteq \mathbf{L}', \mathcal{T}'_2 \subseteq \mathbf{L}'$, then $tr(\mathcal{T}'_1 \sqcup' \mathcal{T}'_2) \models tr(\mathcal{T}'_1) \sqcup tr(\mathcal{T}'_2)$.
   If $\mathcal{T} \subseteq \mathbf{L}$ and $\mathcal{T}' \subseteq \mathbf{L}'$, we have: $tr(\mathcal{T} \sqcup' \mathcal{T}') = tr(\mathcal{T}) \sqcup tr(\mathcal{T}') = \mathcal{T} \sqcup tr(\mathcal{T}')$.

3. $tr(\mathcal{T}') = \{tr(\varphi') \ / \ \varphi' \in Th'(\mathcal{T}')\}$. □

We will also occasionally need the dual notion of $tr$, that we introduce now.

**Definition 3.9** 1. For any formula $\varphi' \in \mathbf{L}'$, we define the *strong trace of* $\varphi'$, written $tr^*(\varphi')$, as the formula

$$tr^*(\varphi') = \neg tr(\neg\varphi').$$

2. For any $\mathcal{T}' \subseteq \mathbf{L}'$, we define the *strong trace of* $\mathcal{T}'$ *in* $\mathbf{L}$, written $tr^*(\mathcal{T}')$, as the subset of $\mathbf{L}$ defined by:

$$tr^*(\mathcal{T}') = \{tr^*(\varphi') \ / \ \mathcal{T}' \models' \varphi'\}.$$

3. For any $\mathbf{M}'_1 \subseteq \mathbf{M}'$, we define the subset $tr^*(\mathbf{M}'_1)$ of $\mathbf{M}$ by: $tr^*(\mathbf{M}'_1) = \{\mu \subseteq V(\mathbf{L}) \ / \ \mathbf{M}'(\mu) \subseteq \mathbf{M}'_1\}$. □

**Property 3.10** [pa] For any $\mathcal{T}' \subseteq \mathbf{L}'$, we get:

$$\mathbf{M}(tr^*(\mathcal{T}')) = tr^*(\mathbf{M}'(\mathcal{T}')).$$

Thus, if $\mathbf{M}'_1$ is a closed set, so is $tr^*(\mathbf{M}'_1)$. □

We obtain most of the properties of the strong trace by duality from the properties of the trace. Let us give a few examples now:

**Property 3.11** [pa]

1. For any $\mathcal{T}' \subseteq \mathbf{L}'$, we have $tr^*(\mathcal{T}') \in \mathbf{T}$, ie. $Th(tr^*(\mathcal{T}')) = tr^*(\mathcal{T}')$.
2. $\mathbf{M}'(tr^*(\mathcal{T}')) \subseteq \mathbf{M}'(\mathcal{T}')$, i.e. $tr^*(\mathcal{T}') \models' \mathcal{T}'$.
3. If $\mathcal{T}$ is a subset of $\mathbf{L}$, and if we consider $\mathcal{T}$ as a set of formulas in $\mathbf{L}'$, we have: $tr^*(\mathcal{T}) = Th(\mathcal{T})$.

   Conversely (see also property 3.6-3) if $\mathcal{T}'$ is a subset of $\mathbf{L}'$ and if $tr(\mathcal{T}') = tr^*(\mathcal{T}')$, then $\mathcal{T}'$ is equivalent to a subset $\mathcal{T}$ of $\mathbf{L}$, which means that we have $Th'(\mathcal{T}') = Th'(\mathcal{T})$ and $tr(\mathcal{T}') = tr^*(\mathcal{T}') = Th(\mathcal{T})$.
4. If $\mathcal{T}'$ is a set of formulas with all its symbols in $V(\mathbf{L}') - V(\mathbf{L})$, we have: $tr^*(\mathcal{T}') = Th(\bot) = \mathbf{L}$.

   Beware that we may have (see also property 3.6-3) $tr(\mathcal{T}') = Th(\top)$ and $tr^*(\mathcal{T}') = Th(\bot)$ for some $\mathcal{T}' \subseteq \mathbf{L}'$, even if $\mathcal{T}'$ is not equivalent to a set of formulas without any symbol in $V(\mathbf{L})$. Let us choose $V(\mathbf{L}) = \{A\}$, $V(\mathbf{L}') = \{A, B\}$ and $\varphi' = A \Leftrightarrow B$. Then we have $tr(\varphi') = \top$ and $tr^*(\varphi') = \bot$ while clearly $Th'(\varphi')$ cannot be equivalent to some $\mathcal{T}'$ such that $V(\mathcal{T}') \subseteq V(\mathbf{L}') - V(\mathbf{L}) = \{B\}$.
5. For any subsets $\mathcal{T}$ of $\mathbf{L}$ and $\mathcal{T}'$ of $\mathbf{L}'$ we have: $\mathcal{T} \models' \mathcal{T}'$ iff $\mathcal{T} \models tr^*(\mathcal{T}')$.
6. (a) If $V(\mathbf{L})$ is finite we get: $tr^*(\varphi') = \bigvee_{\varphi \in \mathbf{L},\, \varphi \models' \varphi'} \varphi$.
   (b) If $\varphi'$ is in a conjunctive normal form, $tr^*(\varphi')$ is obtained by suppressing the symbols of $V(\mathbf{L}') - V(\mathbf{L})$. Notice that if a term (disjunction) disappears completely, it must be replaced as usual by $\bot$ (the neutral element for $\vee$), thus the result of the conjunction is $\bot$.
   (c) If $\varphi'$ is in standard disjunctive normal form, $tr^*(\varphi')$ is obtained by keeping only the terms (conjunctions) which do not contain any symbol of $V(\mathbf{L}') - V(\mathbf{L})$. If no term remains, the result is $\bot$.
7. $tr^*(\mathcal{T}'_1 \sqcup' \mathcal{T}'_2) = tr^*(\mathcal{T}'_1) \sqcup tr^*(\mathcal{T}'_2)$. $\quad tr^*(\varphi'_1 \wedge \varphi'_2) = tr^*(\varphi'_1) \wedge tr^*(\varphi'_2)$. □

**Example 3.12** $V(\mathbf{L}) = \{P, Q\}$, $V(\mathbf{L}') = V(\mathbf{L}) \cup \{P'\}$.

• $\varphi'_1 = \neg Q \vee (P \wedge \neg P') = (\neg Q \vee P) \wedge (\neg Q \vee \neg P')$: these are the standard disjunctive (respectively conjunctive) normal forms of $\varphi'_1$.
$tr(\varphi'_1) = \neg Q \vee P$.
$tr^*(\varphi'_1) = \neg Q$: notice that $\neg Q = (\neg Q \vee P) \wedge \neg Q)$.

• $\varphi'_2 = (P \wedge \neg Q) \vee (\neg P \wedge \neg Q \wedge P') \vee (\neg P \wedge Q \wedge \neg P') =$
$(P \wedge \neg Q \wedge P') \vee (P \wedge \neg Q \wedge \neg P') \vee (\neg P \wedge \neg Q \wedge P') \vee (\neg P \wedge Q \wedge \neg P') =$
$(P \vee \neg Q \vee \neg P') \wedge (P \vee Q \vee P') \wedge (\neg P \vee \neg Q \vee P') \wedge (\neg P \vee \neg Q \vee \neg P') =$
$(P \vee \neg Q \vee \neg P') \wedge (P \vee Q \vee P') \wedge (\neg P \vee \neg Q)$.

The first two writings are disjunctive normal forms, the last two writings are conjunctive normal forms, the first writing and the last writing being the two standard normal forms of $\varphi'_2$.

$$\begin{aligned} tr(\varphi'_2) &= \neg P \vee \neg Q \text{ (only term without } P' \text{ in the last writing of } \varphi'_2\text{, cf property 3.8-1c))} \\ &= (P \wedge \neg Q) \vee (\neg P \wedge \neg Q) \vee (\neg P \wedge Q) \text{ (property 3.8-1b).} \\ tr^*(\varphi'_2) &= P \wedge \neg Q \text{ (first writing, property 3.11-6c).} \\ &= (P \vee \neg Q) \wedge (P \vee Q) \wedge (\neg P \vee \neg Q) \text{ (property 3.11-6b).} \end{aligned}$$

We have, as always, $tr^*(\varphi'_i) \models' \varphi'_i \models' tr(\varphi'_i)$ (properties 3.6-1 and 3.11-2). □

**Example 3.13** $V(\mathbf{L}) = \{P\}, V(\mathbf{L}') = \{P, P'\}$.
$\varphi'_1 = \neg P' \vee P$, $\varphi'_2 = P'$.
$\varphi'_1 \wedge \varphi'_2 = P' \wedge P$, $\varphi'_1 \vee \varphi'_2 = \top$,
$tr(\varphi'_1) = \top$, $tr(\varphi'_2) = \top$, $tr(\varphi'_1 \wedge \varphi'_2) = P$ and $P \neq \top \wedge \top$.
$tr^*(\varphi'_1) = P$, $tr^*(\varphi'_2) = \bot$, $tr^*(\varphi'_1 \vee \varphi'_2) = \top$ and $\top \neq P \vee \bot$.
Thus, we cannot exchange $tr$ and $tr^*$ in properties 3.8-2 and 3.11-7. However, we have, as given in these properties: $tr(\varphi'_1 \vee \varphi'_2) = \top$ and $\top = \top \vee \top$.

As for any formula in $\mathbf{L}$, we have $tr(P) = tr^*(P) = P$.
As for any formula in $\mathbf{L}' - \mathbf{L}$ we have $tr(P') = \top$ and $tr^*(P') = \bot$.
Thus we get: $tr^*(\{\varphi'_1, \varphi'_2\}) = tr^*(\{P, P'\}) = Th(tr^*(P \wedge P')) = Th(\bot) =$
$Th(tr^*(\{\varphi'_1\}) \cup tr^*(\{\varphi'_2\})) = Th(tr^*(\{P\}) \cup tr^*(\{P'\}))$.
However: $tr(\{\varphi'_1, \varphi'_2\}) = tr(\{P, P'\}) =$
$Th(tr(P \wedge P')) = Th(P) = Th(tr(\{P\}) \cup tr(\{P'\}))$
$\neq Th(tr(\{\varphi'_1\}) \cup tr(\{\varphi'_2\})) = Th(\top)$. □

# 4 Preferential entailments

## 4.1 Preferential entailments and multi preferential entailments

As there are many different uses of the expression "preferential entailment" in the literature (see e.g. [Boc99b] for a small discussion about this issue) we will make the definitions precise. For didactical reasons, we begin by the simplest definition, even if a mathematically oriented reader could prefer starting from the more complex definition, then applied to a particular case. Any "preferential entailment" is a particular case of an inference operation that we call pre-circumscription:

**Definition 4.1** A *pre-circumscription* $f$ (in $\mathbf{L}$) is an extensive (i.e. $f(\mathcal{T}) \supseteq \mathcal{T}$ for any $\mathcal{T}$) mapping from $\mathbf{T}$ to $\mathbf{T}$. For any subset $\mathcal{T}$ of $\mathbf{L}$ not in $\mathbf{T}$, we use the abbreviation

$f(\mathcal{T}) = f(Th(\mathcal{T}))$, assimilating a pre-circumscription to a (particular) extensive mapping from $\mathcal{P}(\mathbf{L})$ to itself[4]. We use also the notation $f(\varphi)$ for $f(\{\varphi\}) = f(Th(\varphi))$. $\square$

**Definitions 4.2** 1. A *preference relation* in $\mathbf{L}$ is a binary relation $\prec$ over $\mathbf{M}$. $\mathbf{M}_\prec(\mathcal{T})$ denotes the set of the models of $\mathcal{T}$ minimal for $\prec$: $\mathbf{M}_\prec(\mathcal{T}) = \{\mu \in \mathbf{M}(\mathcal{T})$ / for no $\nu \in \mathbf{M}(\mathcal{T})$ we have $\nu \prec \mu\}$.

Using the correspondence between $\mathbf{M}$ and $\mathbf{C}$, we may also consider $\prec$ as a relation over $\mathbf{C}$. We will indifferently write $Th(\mu) \prec Th(\nu)$ or $\mu \prec \nu$. We write $\mathbf{C}_\prec(\mathcal{T})$ for the set of the elements $\mathcal{C}$ of $\mathbf{C}(\mathcal{T})$ minimal for $\prec$: $\mathcal{C} \in \mathbf{C}(\mathcal{T})$ and no $\mathcal{C}' \in \mathbf{C}(\mathcal{T})$ is such that $\mathcal{C}' \prec \mathcal{C}$.

Thus, $\mathbf{M}_\prec(\mathcal{T})$ corresponds to $\mathbf{C}_\prec(\mathcal{T})$: $\mathbf{C}_\prec(\mathcal{T}) = \{Th(\mu) \mid \mu \in \mathbf{M}_\prec(\mathcal{T})\}$ and $\mathbf{M}_\prec(\mathcal{T}) = \{\mu \in \mathbf{M} \mid Th(\mu) \in \mathbf{C}_\prec(\mathcal{T})\}$.

2. The (classical) *preferential entailment* $f = f_\prec$ is the pre-circumscription in $\mathbf{L}$ defined by

$$f_\prec(\mathcal{T}) = Th(\mathbf{M}_\prec(\mathcal{T})).$$

Using the correspondence between $\mathbf{M}$ and $\mathbf{C}$, we can equivalently define

$$f_\prec(\mathcal{T}) = \bigcap_{\mathcal{C} \in \mathbf{C}_\prec(\mathcal{T})} \mathcal{C}. \ \square$$

**Remark 4.3** We have $\mathbf{M}(f_\prec(\mathcal{T})) = TC(\mathbf{M}_\prec(\mathcal{T}))$, or equivalently $\mathbf{C}(f_\prec(\mathcal{T})) = TC(\mathbf{C}_\prec(\mathcal{T}))$. $\square$

**Notation 4.4** For any $\mu \in \mathbf{M}$ and any preference relation $\prec'$ in the extended language $\mathbf{L}'$, we denote by $\mathbf{M}'_{\prec'}(\mu)$ the subset of $\mathbf{M}'(\mu)$ (see definition 3.1) consisting of all the elements which are minimal for $\prec'$ in $\mathbf{M}'(\mu)$: $\mathbf{M}'_{\prec'}(\mu) = \{\mu' \in \mathbf{M}'(\mu)$ / for no $\nu' \in \mathbf{M}'(\mu)$ we have $\nu' \prec' \mu'\}$. $\square$

Definition 4.2 is the classical definition of preferential entailments, originating in [Sho88], and applied to the propositional case, contrarily to the definitions in e.g. [KLM90] which uses a set $\mathbf{S}$ of states, mapped to $\mathbf{M}$ (which, as noted in [Sch97], is equivalent to allow copies of models). In the predicate calculus case, any complete theory has as many models as we want (we must even go outside the notion of set). Thus, in the predicate calculus case (not studied in the present text), we should split the notion of preferential entailments as given in definition 4.2 in two different notions: it is not the same thing to start from a relation among $\mathbf{C}$ and from a relation among $\mathbf{M}$. Now, if we want to simulate in the propositional case some aspects of the preferential entailments in the predicate calculus case when they are defined

[4]Thus, for a reader familiar with the terminology used in [KLM90], a pre-circumscription is an inference operation satisfying the full (or theory) versions of "reflexivity", "left logical equivalence LLE", "right weakening RW" and "AND".

from a relation among $\mathbf{M}$, we may use a more general notion introduced in [KLM90]. We give the precise definitions concerned now, but, to distinguish clearly this notion from the classical preferential entailment in the propositional calculus, we use another term, reflecting the nature of this operator: multi preferential entailment.

**Definitions 4.5** 1. $\mathbf{S}$ is some set of *copies of elements of* $\mathbf{M}$ (or of $\mathbf{C}$), also called *states*: there exists a mapping $l$ from $\mathbf{S}$ to $\mathbf{M}$, and for any $\mu \in \mathbf{M}$ we call the set $l^{-1}(\mu) = \{\mu_1, \mu_2, \cdots\}$ the set (possibly empty) of the *copies of the interpretation* $\mu$ in $\mathbf{S}$. If $\mathbf{M}$ is finite, a simple counting argument shows that we may assume without loss of generality that $\mathbf{S}$ is finite [KLM90]. $\mathbf{S}(\mathcal{T})$ is the subset of $\mathbf{S}$ defined by $\mathbf{S}(\mathcal{T}) = l^{-1}(\mathbf{M}(\mathcal{T}))$.

2. A *multi preference relation* in $\mathbf{L}$ is a binary relation $\prec_m$ over such a set $\mathbf{S}$. For any $\mathcal{T} \in \mathbf{T}$, we define the sets $\mathbf{S}_{\prec_m}(\mathcal{T}) = \{\mu_i \in \mathbf{S}(\mathcal{T}) /$ for no $\nu_j \in \mathbf{S}(\mathcal{T})$, we have $\nu_j \prec_m \mu_i\}$, $\mathbf{M}_{\prec_m}(\mathcal{T}) = l(\mathbf{S}_{\prec_m}(\mathcal{T}))$ and $\mathbf{C}_{\prec_m}(\mathcal{T}) = \{Th(\mu) \ / \ \mu \in \mathbf{M}_{\prec_m}(\mathcal{T})\}$.

3. A *multi preferential entailment* is a pre-circumscription defined by:

$$f_{\prec_m}(\mathcal{T}) = Th(\mathbf{M}_{\prec_m}(\mathcal{T})) \qquad \text{i.e.} \qquad f_{\prec_m}(\mathcal{T}) = \bigcap_{\mathcal{C} \in \mathbf{C}_{\prec_m}(\mathcal{T})} \mathcal{C}. \quad \square$$

This definition is in the lines of some definitions given in [KLM90] and followers. We may refer the reader to e.g. Definition 5.6 in [KLM90], except that we do not require a priori any special property for $\prec_m$. Notice that rigorously we should precise each time the set $\mathbf{S}$ and the mapping $l$, using a notation such as $(\mathbf{S}, l, \prec_m)$, however, to keep the notations simple enough, we try to avoid this kind of notation, and, when we will define a multi preference relation $\prec_m$, we will define also $\mathbf{S}$ and $l$. Clearly, any preferential entailment is a multi-preferential entailment (it suffices to take $\mathbf{S} = \mathbf{M}$ and the identity mapping for $l$).

**Remark 4.6** $\mathbf{M}_{\prec_m}(\mathcal{T}) = \{\mu \in \mathbf{M}(\mathcal{T}) /$ there exists some copy $\mu_i$ of $\mu$ in $\mathbf{S}$ such that for no copy $\nu_j$ of some $\nu \in \mathbf{M}(\mathcal{T})$, we have $\nu_j \prec_m \mu_i\}$.

As with preferential entailments, we get:

$$\mathbf{M}(f_{\prec_m}(\mathcal{T})) = TC(\mathbf{M}_{\prec_m}(\mathcal{T})) = \mathbf{M}(\bigcap_{\mu \in \mathbf{M}_{\prec_m}(\mathcal{T})} Th(\mu)),$$

thus, when $V(\mathbf{L})$ is finite: $\mathbf{M}(f_{\prec_m}(\mathcal{T})) = \mathbf{M}_{\prec_m}(\mathcal{T})$. $\square$

For the sake of exhaustivity, let us remind that a yet more general notion has sometimes been considered, where $l$ is a mapping from $\mathbf{S}$ to $\mathbf{T}$ (or equivalently here to $\mathcal{P}(\mathbf{M})$) instead of $\mathbf{C}$ (or $\mathbf{M}$): see e.g. Definition 3.2 in [Boc99b], which is Definition 3.13 in [KLM90] without any condition for $\prec_m$. The main difference is that multi preferential entailments satisfy (CR) (called Or in [KLM90]) and (CT) (called Cut in [KLM90]), while this more general notion satisfies still (CT), but not necessarily (CR).

As this notion is more complicated (being considered as “cumbersome” in [KLM90]), we postpone its study to [MR99].

To a great extend, the notion of preferential entailment suffices to study multi preferential entailments, as we will explain in subsection 4.2.

Let us define now another particular pre-circumscription, which is useful in studying some important cases of multi preferential entailments.

**Definitions 4.7** [SF96, MR98a, MR98b] Let $f$ be a pre-circumscription.

1. A formula $\varphi$ is *accessible for* $f$ iff there exists a theory $\mathcal{T}$ such that $\varphi \notin \mathcal{T}$ and $\varphi \in f(\mathcal{T})$. The set of the formulas *inaccessible for* $f$ is $I_f = \mathbf{L} - \bigcup_{\mathcal{T}\in\mathbf{T}}(f(\mathcal{T}) - \mathcal{T}) = \bigcap_{\mathcal{T}\in\mathbf{T}}(\mathbf{L} - (f(\mathcal{T}) - \mathcal{T}))$.
2. A pre-circumscription $f$ is an *X-mapping* iff there exists some subset $X$ of $\mathbf{L}$ such that $\varphi \in f(\mathcal{T})$ iff $(\mathcal{T} \sqcup \varphi) \cap X \subseteq Th(\mathcal{T})$. In this case, we will denote $f$ by $f_X$. □

## 4.2 Multi preferential entailments as traces of classical preferential entailments

As already written after definition 4.5, any preferential entailment is a multi-preferential entailment. However, we will show now that we have some kind of “converse”: we may study very easily the multi-preferential entailments from the more natural and much simpler notion of preferential entailments. We think this study is important for at least three reasons: 1) The notion of preferential entailment is much more natural. It is a good thing to “reduce” the more complex notion of multi preferential entailment to the simpler notion, for what concerns the efficient computation, and also for what concerns the knowledge representation point of view. 2) This makes clear the difference of nature between the properties which are preserved when we restrict the language, and the other properties. It appears that the other properties, which are at least as important from a knowledge representation perspective, have not been studied so thoroughly in the literature, and it is possible that this brittleness with respect to restriction of the language is one reason of this fact. 3) As we will see in section 7, this study gives rise to a number of interesting results in non monotonic reasoning.

**Theorem 4.8** If $f$ is a multi preferential entailment defined in $\mathbf{L}$, there exists a language $\mathbf{L}' \supseteq \mathbf{L}$ and a preferential entailment $f' = f'_{\prec'}$ defined in $\mathbf{L}'$ such that for any $\mathcal{T}$ in $\mathbf{L}$ we have $f(\mathcal{T}) = f'(\mathcal{T}) \cap \mathbf{L} = tr(f'(\mathcal{T}))$. If $V(\mathbf{L})$ is finite, so is $V(\mathbf{L}')$. □

<u>Proof</u> We choose some set $\mathbf{P}''$ of propositional symbols not in $V(\mathbf{L})$, which is large enough to satisfy the following condition: $l$ being the mapping associated to $\prec_m$ as defined in definition 4.5, for any $\mu \in \mathbf{M}$, there exists an injective mapping $m_\mu$ from $l^{-1}(\mu)$ to $\mathcal{P}(\mathbf{P}'')$, the set of all the subsets of $\mathbf{P}''$ (i.e. $card(\mathbf{P}'')$ is such that $2^{card(\mathbf{P}'')} \geq card(l^{-1}(\mu))$ for any $\mu \in \mathbf{M}$). $\mathbf{L}'$ is the language which has $V(\mathbf{L}) \cup \mathbf{P}''$ as vocabulary. As written in subsection 2.5, $\mathbf{M}'$

denotes the set of all the interpretations for $\mathbf{L}'$, and similarly for any such notation with a " $'$ ". To each element $\mu_i$ in $\mathbf{S}$ we associate the interpretation $m(\mu_i)$ for $\mathbf{L}'$ defined by $m(\mu_i) = l(\mu_i) \cup m_\mu(\mu_i)$, where $\mu = l(\mu_i) \subseteq V(\mathbf{L})$ and $m_\mu(\mu_i) \subseteq \mathbf{P}''$. In this way, we know that $m$ is an injective mapping from $\mathbf{S}$ to $\mathbf{M}'$. We define the preference relation $\prec'$ over $\mathbf{M}'$ as follows:

1) $\mu' \prec' \mu'$ if $\mu' \notin m(\mathbf{S})$, 2) $m(\mu_i) \prec' m(\nu_j)$ if $\mu_i \prec_m \nu_j$, and 3) nothing else.

**($\uparrow$Def:$\prec_m \rightsquigarrow \prec'$)**

From definitions 4.2 and 4.5 we see that we have, for any theory $\mathcal{T}$ in $\mathbf{L}$, $\mathbf{S}_{\prec_m}(\mathcal{T}) = m^{-1}(\mathbf{M}'_{\prec'}(Th'(\mathcal{T})))$, thus, from definition 4.5, $\mathbf{M}_{\prec_m}(\mathcal{T}) = l^{-1}(m^{-1}(\mathbf{M}'_{\prec'}(Th'(\mathcal{T}))))$. Now, from definition 3.3, we get that we have, for any subset $\mathbf{M}'_1$ of $\mathbf{M}'$, $l^{-1}(m^{-1}(\mathbf{M}'_1)) = tr(\mathbf{M}'_1)$. Thus we get, for any $\mathcal{T} \subseteq \mathbf{L}$, $\mathbf{M}_{\prec_m}(\mathcal{T}) = tr(\mathbf{M}'_{\prec'}(Th'(\mathcal{T}))) = tr(\mathbf{M}'_{\prec'}(Th'(\mathcal{T})) \cap m(\mathbf{S}))$.

Also, from remark 3.4, we know that we have: $TC(\mathbf{M}_{\prec_m}(\mathcal{T})) = tr(TC'(\mathbf{M}'_{\prec'}(Th'(\mathcal{T}))))$, i.e. $f_{\prec_m}(\mathcal{T}) = tr(f'_{\prec'}(\mathcal{T})) = f'_{\prec'}(\mathcal{T}) \cap \mathbf{L}$.

A consequence of this proof is that if $V(\mathbf{L})$ is finite, we may choose a finite set $\mathbf{S}$, making $\mathbf{P}''$ finite also with $card(\mathbf{P}'') = \lceil log_2(Max_{\mu \in \mathbf{M}}\, card(l^{-1}(\mu)))\rceil$. $\square$

Intuitively, we have now a language reach enough to make that any copy of interpretation of the original multi preferential entailment corresponds to a distinct interpretation in the new language, and the preference relation $\prec'$ defined in this new language corresponds to the original multi preference relation defined in the set $\mathbf{S}$ of copies of interpretations for $\mathbf{L}$.

**Remark 4.9** It may happen (if $\mathbf{L}$ is finite, it always happen) that the set $m(\mathbf{S}) \subseteq V(\mathbf{L}')$ is a closed set in $\mathbf{M}'$, which means that this is the set of the models of some theory $\mathcal{T}' \subseteq \mathbf{L}'$. In this case we have, for any $\mathcal{T} \subseteq \mathbf{L}$, $\mu \in \mathbf{M}_{\prec_m}(\mathcal{T})$ iff $\mu \in tr(\mathbf{M}'_{\prec'}(\mathcal{T} \cup \mathcal{T}'))$, thus $f(\mathcal{T}) = tr(f'(\mathcal{T} \cup \mathcal{T}')) = f'(\mathcal{T} \cup \mathcal{T}') \cap \mathbf{L}$. If moreover this set is open (again, this is always the case if $\mathbf{L}$ is finite) it corresponds to some formula $\beta' \in \mathbf{L}'$ and we get $f(\mathcal{T}) = tr(f'(\mathcal{T} \cup \{\beta'\})) = f'(\mathcal{T} \cup \{\beta'\}) \cap \mathbf{L}$.

The interest of introducing such a fixed set $\mathcal{T}'$, or a fixed formula $\beta'$, is that what happens outside the set $\mathbf{M}'(\mathcal{T}')$ for the preference relation $\prec'$ does not matter. Thus, we may get rid of the loops $\mu' \prec' \mu'$ given by the first term of the alternative in ($\uparrow$Def:$\prec_m \rightsquigarrow \prec'$). This is particularly useful when we start from some given $\prec_m$ and when we want some specific properties such as irreflexivity for the relation $\prec'$. We will sometimes be able to find a solution using some set $\mathcal{T}'$ while no solution exists with the basic case. We will give examples of this situation in section 7. $\square$

The natural question now is how far can we go in such a study: how easy is it to examine the properties of a multi preferential entailment $f$ from the properties of some classical preferential entailments $f'$ giving rise to $f$. This text provides a few clues, by examining which properties of $f'$ are preserved by the operations just described.

To make things precise, let us call (Def$\downarrow$), (Def$\downarrow_{\mathcal{T}'}$) and (Def$_{\beta'}$) respectively the basic case, the case where $m(\mathbf{S})$ is closed and the case where it is also open:

**Definition 4.10** Let $f'$ be some pre-circumscription in $\mathbf{L}'$ and $\mathcal{T}' \cup \{\beta'\}$ be some subset of $\mathbf{L}'$. We define successively three pre-circumscriptions $f$ in $\mathbf{L}$ as follows:

- $f(\mathcal{T}) = tr(f'(\mathcal{T})) = f'(\mathcal{T}) \cap \mathbf{L}$. **(Def$\downarrow$)**
- $f(\mathcal{T}) = tr(f'(\mathcal{T} \cup \mathcal{T}')) = f'(\mathcal{T} \cup \mathcal{T}') \cap \mathbf{L}$ **(Def$\downarrow_{\mathcal{T}'}$)**
- $f(\mathcal{T}) = tr(f'(\mathcal{T} \cup \{\beta'\})) = f'(\mathcal{T} \cup \{\beta'\}) \cap \mathbf{L}$ **(Def$\downarrow_{\beta'}$)**

□

In the next two properties, we suppose that $f' = f_{\prec'}$ is a preferential entailment in $\mathbf{L}'$, and we examine successively the three cases (Def$\downarrow$), (Def$\downarrow_{\mathcal{T}'}$) and (Def$\downarrow_{\beta'}$), getting the three corresponding definitions of $\prec_m$ or of $\prec$ in terms of $\prec'$.

The first property is an immediate consequence of the arguments given in the proof of theorem 4.8, and is given here only in order to make the correspondence just defined precise.

**Property 4.11** We consider here a preference relation $\prec'$ in $\mathbf{M}'$ and the preferential entailment $f' = f_{\prec'}$ in $\mathbf{L}'$. We suppose that $f$ is defined from $f'$ in terms of (Def$\downarrow$), (Def$\downarrow_{\mathcal{T}'}$) and (Def$\downarrow_{\beta'}$) respectively.

$f$ is then a multi preferential entailment $f = f_{\prec_m}$ in $\mathbf{L}$ defined as follows:

(Def$\downarrow$) $\mathbf{S} = \mathbf{M}'$, for any $\mu' \in \mathbf{S}$, $l(\mu') = \mu' \cap V(\mathbf{L})$, and the preference relation $\prec'$ in $\mathbf{L}'$ is taken as our multi preference relation $\prec_m$. **(Def$\downarrow$:$\prec' \rightsquigarrow \prec_m$)**

(Def$\downarrow_{\mathcal{T}'}$) $\mathbf{S} = \mathbf{M}'(\mathcal{T}')$, for any $\mu' \in \mathbf{S}$, $l(\mu') = \mu' \cap V(\mathbf{L})$, and the restriction to $\mathbf{M}'(\mathcal{T}')$ of the preference relation $\prec'$ in $\mathbf{L}'$ is our multi preference relation $\prec_m$. **(Def$\downarrow_{\mathcal{T}'}$:$\prec' \rightsquigarrow \prec_m$)**

(Def$\downarrow_{\beta'}$) : Particular case of (Def$\downarrow_{\mathcal{T}'}$), where $\mathcal{T}' = \{\beta'\}$. **(Def$\downarrow_{\beta'}$:$\prec' \rightsquigarrow \prec_m$)**

□

**Property 4.12** We consider here again a preference relation $\prec'$ in $\mathbf{M}'$ and the preferential entailment $f' = f_{\prec'}$ in $\mathbf{L}'$. We suppose that $f$ is defined from $f'$ in terms of (Def$\downarrow$), (Def$\downarrow_{\mathcal{T}'}$) and (Def$\downarrow_{\beta'}$) respectively. We suppose also here that $f$ is a (classical) preferential entailment in $\mathbf{L}$, i.e that there exists a preference relation $\prec$ on $\mathbf{M}$ such that $f = f_{\prec}$.

In this case, we may choose the preference relation $\prec$ defined as follows from $\prec'$, for any $\mu, \nu$ in $\mathbf{M}$ (remind definition 3.1 for $\mathbf{M}'(\mu)$).

(Def$\downarrow$) $\mu \prec \nu$ iff, either $\mathbf{M}'_{\prec'}(\nu) = \emptyset$ and $\mu = \nu$, or $\mathbf{M}'_{\prec'}(\nu) \neq \emptyset$ and [5] for any $\nu' \in \mathbf{M}'_{\prec'}(\nu)$, there exists $\mu' \in \mathbf{M}'(\mu)$ such that $\mu' \prec' \nu'$.
**(Def$\downarrow$:$\prec' \rightsquigarrow \prec$)**

(Def$\downarrow_{\mathcal{T}'}$) $\mu \prec \nu$ iff, either $\mu = \nu$ and $\mathbf{M}'_{\prec'}(Th'(\nu) \sqcup' \mathcal{T}') = \emptyset$, or $\mathbf{M}'_{\prec'}(Th'(\nu) \sqcup' \mathcal{T}') \neq \emptyset$ and for any $\nu' \in \mathbf{M}'_{\prec'}(Th'(\nu) \sqcup' \mathcal{T}')$,
**(Def$\downarrow_{\mathcal{T}'}$:$\prec' \rightsquigarrow \prec$)**
there exists $\mu' \in \mathbf{M}'(\mu) \cap \mathbf{M}'(\mathcal{T}')$ such that $\mu' \prec' \nu'$.
(Notice that $Th'(\nu) \sqcup' \mathcal{T}' = Th(\nu) \sqcup' \mathcal{T}'$.)

(Def$\downarrow_{\beta'}$) cf case (Def$\downarrow_{\mathcal{T}'}$), with $\mathcal{T}' = \{\beta'\}$. **(Def$\downarrow_{\beta'}$:$\prec' \rightsquigarrow \prec$)**

□

Proof: A preferential entailment $f = f_{\prec}$ is defined as soon as it is defined for the theories $\mathcal{T}$ with one or two models. For $\mathcal{T} = Th(\mu)$ ($\mu \in \mathbf{M}$), we get $f(\mathcal{T}) = \mathcal{T}$ iff $\mathbf{M}'_{\prec'}(\mu) \neq \emptyset$, and $f(\mathcal{T}) = Th(\perp)$ otherwise. For $\mathcal{T} = Th(\mu, \nu)$ with $\mu \neq \nu$, we get that if $f(Th(\nu)) = Th(\perp)$ then $\nu \notin \mathbf{M}(f(\mathcal{T}))$ thus it is useless to add an eventual link $\mu \prec \nu$. If $f(Th(\nu)) \neq \perp$, we get $\nu \notin \mathbf{M}(f(\mathcal{T}))$ iff $\mu \prec \nu$. □

**Remark 4.13** We could alternatively have chosen the following shorter definition instead of $\prec$ as given in (Def$\downarrow_{\mathcal{T}'}$:$\prec' \rightsquigarrow \prec$):

$\mu \prec \nu$ iff for any $\nu' \in \mathbf{M}'(\nu) \cap \mathbf{M}'(\mathcal{T}')$, there exists $\mu' \in (\mathbf{M}'(\mu) \cup \mathbf{M}'(\nu)) \cap \mathbf{M}'(\mathcal{T}')$ such that $\mu' \prec' \nu'$.

The difference is that there are now many "useless" relations $\mu \prec \nu$: each time we have $\nu \prec \nu$ from the first alternative in the definition (Def$\downarrow_{\mathcal{T}'}$:$\prec' \rightsquigarrow \prec$), we add all the $\mu \prec \nu$, for any $\mu \in \mathbf{M}$: clearly, this does not modify the resulting $f_{\prec}$.

Similarly, we could have taken this alternative definition instead of (Def$\downarrow$:$\prec' \rightsquigarrow \prec$): it suffices to suppress the two occurrences of "$\cap \mathbf{M}'(\mathcal{T}')$ "in the alternative definition given just above. □

See below (preservation result 6.21) the case where we start from a multi preferential entailment $f' = f_{\prec'_m}$: then $f$ is a multi preferential entailment, and we describe a multi preference relation $\prec_m$ in $\mathbf{L}$ such that $f = f_{\prec_m}$.

# 5 The main logical properties of preferential entailments

## 5.1 Some of the main possible properties of a pre-circumscription

**Remark 5.1 About the bibliographical references:** In this section, when a result is given as "known", it comes from one of the following texts [Sho88, KLM90, Sat90, Sch92,

[5] Beware that we cannot replace $\mathbf{M}'_{\prec'}(Th'(\nu) \sqcup' \mathcal{T}')$ by $\mathbf{M}'_{\prec'}(\nu) \cap \mathbf{M}'(\mathcal{T}')$ or even by $TC'(\mathbf{M}'_{\prec'}(\nu)) \cap \mathbf{M}'(\mathcal{T}')$ here, and that these three sets may be different.

LM92, FL93, Mak94, MR94a, MR94b, Sch97, MR98b], which are some of the classical references on the subject. As it is not the main purpose of this text, and as these results are known now, we do not always precise more the references of each of these results. Indeed, this would need long developments as generally various authors should be quoted, and some proofs or connections between the different texts are not immediate. We will instead concentrate our attention to the main purpose of this text which is to examine which properties of pre-circumscriptions are preserved when the language is reduced or extended as it is often done in the literature about non monotonic reasoning. □

**Definitions 5.2** Here are various properties a pre-circumscription may possess. $\varphi$ is a formula in $\mathbf{L}$, $\mathcal{T}$ and $\mathcal{T}''$ are sets of formulas in $\mathbf{L}$, while $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_i$ are in $\mathbf{T}$:

| | | |
|---|---|---|
| *Idempotence* : | $f(f(\mathcal{T})) = f(\mathcal{T})$ | **(Idem)** |
| *Reverse monotony* : | $f(\mathcal{T} \cup \mathcal{T}'') \subseteq f(\mathcal{T}) \sqcup \mathcal{T}''$ | **(RM)** |
| *Case reasoning* : | $f(\mathcal{T}_1) \cap f(\mathcal{T}_2) \subseteq f(\mathcal{T}_1 \cap \mathcal{T}_2)$ | **(CR)** |
| *id. (infinite version)* : | $\bigcap_{i\in I} f(\mathcal{T}_i) \subseteq f(\bigcap_{i\in I} \mathcal{T}_i)$ | **(CR$\infty$)** |
| *Conjunctive coherence* : | $f(\mathcal{T} \sqcup \mathcal{T}'') \subseteq f(\mathcal{T}) \sqcup f(\mathcal{T}'')$ | **(CC)** |
| *id. (infinite version)* : | $f(\bigsqcup_{i\in I} \mathcal{T}_i) \subseteq \bigsqcup_{i\in I} f(\mathcal{T}_i) \quad (I \neq \emptyset)$ | **(CC$\infty$)** |
| *Restricted identity* : | if $f(\mathcal{T}_1) \subseteq \mathcal{T}_2$ then $f(\mathcal{T}_2) = \mathcal{T}_2$ | **(RI)** |
| *Disjunctive coherence* : | $f(\mathcal{T}_1 \cap \mathcal{T}_2) \subseteq f(\mathcal{T}_1) \sqcup f(\mathcal{T}_2)$ | **(DC)** |
| *Disjunctive rationality* : | $f(\mathcal{T}_1 \cap \mathcal{T}_2) \subseteq f(\mathcal{T}_1) \cup f(\mathcal{T}_2)$ | **(DR)** |
| *Monotony* : | $f(\mathcal{T}) \subseteq f(\mathcal{T} \sqcup \mathcal{T}'')$ | **(MON)** |
| *Cumulative monotony* : | if $\mathcal{T}'' \subseteq f(\mathcal{T})$ then $f(\mathcal{T}) \subseteq f(\mathcal{T} \sqcup \mathcal{T}'')$ | **(CM)** |
| *Cumulative transitivity* : | if $\mathcal{T}'' \subseteq f(\mathcal{T})$ then $f(\mathcal{T} \sqcup \mathcal{T}'') \subseteq f(\mathcal{T})$ | **(CT)** |
| *Cumulativity* : | if $\mathcal{T}'' \subseteq f(\mathcal{T})$ then $f(\mathcal{T}) = f(\mathcal{T} \sqcup \mathcal{T}'')$ | **(CUMU)** |
| *Preservation of consistency* : | if $f(\mathcal{T}_1) = Th(\bot) = \mathbf{L}$ then $\mathcal{T}_1 = \mathbf{L}$ | **(PC)** |
| *Coherent non monotony* : | if $\bot \notin f(\mathcal{T}) \sqcup \mathcal{T}''$ then $\bot \notin f(\mathcal{T}) \sqcup f(\mathcal{T} \sqcup \mathcal{T}'')$ | **(CNM)** |
| *Rational monotony* : | if $\bot \notin f(\mathcal{T}) \sqcup \mathcal{T}''$ then $f(\mathcal{T}) \subseteq f(\mathcal{T} \sqcup \mathcal{T}'')$ | **(RatM)** |
| *Disjunctive coherence for complete theories* | $f(\mathcal{T}) \subseteq \bigcap_{\mathcal{C}\in\mathbf{C}(\mathcal{T})} \bigsqcup_{\mathcal{C}'\in\mathbf{C}(\mathcal{T})} f(\mathcal{C} \cap \mathcal{C}')$ | **(DCC)** |

□

These properties come from various texts and are well-know except (CC), (CC$\infty$), (CR$\infty$), (RI), and (DCC) which are ours (see [MR98b, MR00] for precise references and more details). Our names come from the literature, except when some conflict existed between the notations used in various texts, or when no specific name has been given before, to our knowledge.

About the "history" of such properties, notice that some of these properties come from the literature about conditional logics, see e.g. [Ben84] for some properties of this kind in this context and some bibliographical references.

(CT) and (CM) appear for the first time in the context of non monotonic reasoning in the pioneer [Gab85] (which has initiated the systematic study of such properties in this context) under the respective names of cut and restricted monotonicity, the names we have given in this text being rather common also.

(PC) appears in a context very close to circumscription (thus in a context close to preferential entailment) in [BS85].

(CR), often called also OR, as in [KLM90], or distributivity as in [Sch92], is also one of the "oldest" properties of this kind which has been considered.

(RM) is called by various names: its formula-only version (called (RM0) below) is called deduction principle in [Sho88] where it makes it first apparition in a context of preferential entailment and circumscription; among the various names given to the full version (RM) we may cite [infinite] conditionalization in [Sch92, Mak94] and, in the lines of Shoham, deductivity in [FL93].

(DC) has no name in [Sat90] where it makes its first apparition to our knowledge (formula-only version, called (DC0) below, which, as remarked by Satoh who was mainly concerned by circumscription, "corresponds [in a context of revision] to the property called (R8) in [KM91]").

The formula-only versions of (DR) and (RatM) (called (DR0) and (RatM0) below) appear in [KLM90].

The formula-only version of (CNM) appears without other name than (13) in [LM92].

Beware that a few texts denote rational monotony by (RM), but the name "rational monotony" is misleading: see in [MR98b] why (among other authors, such as [Gef92, Boc99a]) we consider (DR) and (RatM) as not "rational" at all[6]. Fortunately, as shown in [Sat90], (RatM) is not a property of circumscriptions (except very trivial ones). On the contrary, the variant of (RM) in which $\mathcal{T}''$ must be finitely axiomatizable (see (RM1) below) is a fundamental property of (multi) preferential entailments. As shown by [LM92], (RatM) is satisfied only by very particular multi preferential entailments, which are expressible in terms of the possibility theory (see e.g. [BDP96]). Thus, we consider that (RatM) is exotic in a context of multi preferential entailments and it is much better in texts about general non monotonic reasoning and (multi) preferential entailments to use a short name for (RM) and a longer name for (RatM). (RM) implies (CNM) (property 5.4-2 below) and, from a common sense reasoning point of view, (CNM), which is a kind of "weak (RatM)" (see e.g. property 5.4- 4 below) is much more desirable than (RatM).

Each of the properties listed above has a translation in terms of common sense reasoning. For instance, (CR) means that if we conclude that *Tweety flies* in the two situations where

[6]Please notice that this comment applies in a context of pre-circumscriptions, that is when *skeptical* reasoning is considered. Some results in [Boc99b] show that (RatM) is more desirable in a context of *credulous reasoning* corresponding to *default inferences* as defined since [Rei80]: in this kind of reasoning, $f(\mathcal{T})$ is a *set of theories* instead of a single theory. With credulous reasoning, we no longer have necessarily the equivalence (E): $f(\mathcal{T}) \subseteq f(\mathcal{T}_1) \cup f(\mathcal{T}_2)$ iff $f(\mathcal{T}) \subseteq f(\mathcal{T}_1)$ or $f(\mathcal{T}) \subseteq f(\mathcal{T}_2)$. The failure of (E) makes (DR) and (RatM) much weaker (and from Bochman's results, at least in some cases, more desirable) than in a context of skeptical reasoning, where (E) holds.

all we know about *Tweety* is that *Tweety is a bird* and *Tweety is a bat* respectively, then we get the same conclusion in the situation where all we know about *Tweety* is that *Tweety is a bird or a bat.* For an intuitive meaning of most of these properties, see e.g. [MR98b], which contains also further justifications for some of the names given here.

We need also some weaker versions of the properties given above:

**Definitions 5.3** Formula versions / Formula-only versions

| | Formula versions | | Formula-only versions |
|---|---|---|---|
| **(RM1)** | $f(\mathcal{T} \sqcup \varphi) \subseteq f(\mathcal{T}) \sqcup \varphi$ | **(RM0)** | $f(\psi \wedge \varphi) \subseteq f(\psi) \sqcup \varphi$ |
| **(CR1)** | $f(\mathcal{T} \sqcup \varphi) \cap f(\mathcal{T} \sqcup \psi) \subseteq f(\mathcal{T} \sqcup \varphi \vee \psi)$ | **(CR0)** | $f(\varphi) \cap f(\psi) \subseteq f(\varphi \vee \psi)$ |
| **(DC1)** | $f(\mathcal{T} \sqcup \varphi \vee \psi) \subseteq f(\mathcal{T} \sqcup \varphi) \sqcup f(\mathcal{T} \sqcup \psi)$ | **(DC0)** | $f(\varphi \vee \psi) \subseteq f(\varphi) \sqcup f(\psi)$ |
| **(DR1)** | $f(\mathcal{T} \sqcup \varphi \vee \psi) \subseteq f(\mathcal{T} \sqcup \varphi) \cup f(\mathcal{T} \sqcup \psi)$ | **(DR0)** | $f(\varphi \vee \psi) \subseteq f(\varphi) \cup f(\psi)$ |
| **(CC1)** | $f(\mathcal{T} \sqcup \varphi \wedge \psi) \subseteq f(\mathcal{T} \sqcup \varphi) \sqcup f(\mathcal{T} \sqcup \psi)$ | **(CC0)** | $f(\varphi \wedge \psi) \subseteq f(\varphi) \sqcup f(\psi)$ |
| **(CT1)** | if $\varphi \in f(\mathcal{T})$ then $f(\mathcal{T} \sqcup \varphi) \subseteq f(\mathcal{T})$ | **(CT0)** | if $\varphi \in f(\psi)$ then $f(\varphi \wedge \psi) \subseteq f(\psi)$ |
| **(CM1)** | if $\varphi \in f(\mathcal{T})$ then $f(\mathcal{T}) \subseteq f(\mathcal{T} \sqcup \varphi)$ | **(CM0)** | if $\varphi \in f(\psi)$ then $f(\psi) \subseteq f(\varphi \wedge \psi)$ |
| **(CNM1)** | if $\neg\varphi \notin f(\mathcal{T})$ and $\psi \in f(\mathcal{T})$ then $\neg\psi \notin f(\mathcal{T} \sqcup \varphi)$ | **(CNM0)** | if $\neg\varphi \notin f(\varphi')$, $\psi \in f(\varphi')$ then $\neg\psi \notin f(\varphi \wedge \varphi')$ |
| **(RatM1)** | if $\neg\varphi \notin f(\mathcal{T})$ then $f(\mathcal{T}) \subseteq f(\mathcal{T} \sqcup \varphi)$ | **(RatM0)** | if $\neg\varphi \notin f(\varphi')$ then $f(\varphi') \subseteq f(\varphi \wedge \varphi')$ |
| **(RI1)** | if $f(\mathcal{T}) \subseteq \mathcal{T} \sqcup \varphi$ then $f(\mathcal{T} \sqcup \varphi) = \mathcal{T} \sqcup \varphi$. | **(RI0)** | if $f(\psi) \subseteq Th(\varphi \wedge \psi)$ then $f(\varphi \wedge \psi) = Th(\varphi \wedge \psi)$ |
| **(MON1)** | $f(\mathcal{T}) \subseteq f(\mathcal{T} \sqcup \varphi)$ | **(MON0)** | $f(\varphi) \subseteq f(\varphi \wedge \psi)$. □ |

**Property 5.4** (known or obvious) For any pre-circumscription:

1. (RM1) and (CR1) are equivalent, as are (RM0) and (CR0).
   (RM1) implies (CC1), (CNM1) and (RI1).
2. (RM) implies (CT), (CR), (CNM), (RI), and (CC$\infty$), (CT) implies (Idem), and (RI) implies (Idem).
3. (RM) and (DCC) imply (DC).
4. (DR) implies (DC), ((RatM)+(PC)) implies (DR), (CNM) and (CM).
   (DR1) implies (DC1), ((RatM1) + (PC) implies (DR1), (CNM1) and (CM1).

5. Any full version implies its corresponding formula version, any formula version implies its corresponding formula-only version: (CR) implies (CR1) which in turn implies (CR0). Also, an infinite version implies its corresponding standard full version: (CR$\infty$) implies (CR) (notice that formula versions of (CR$\infty$) could be defined, but we think that our text gives already enough properties...) and (CC$\infty$) implies (CC). $\square$

We examine now some important particular kinds of pre-circumscriptions.

## 5.2 The main properties of (multi) preferential entailments and X-mappings

Here are some properties of a (multi) preference relation which are sometimes useful:

**Definitions 5.5** In points 1, 3, 5, 6 and 7, $\prec$ may denote either a preference relation (denoted by $\prec$ in definition 4.2) or a multi preference relation (denoted by $\prec_m$ in definition 4.5).

1. A (multi) preference relation $\prec$ satisfies the *closure property*, **(cl)** for short, if for any $\mathcal{T} \in \mathbf{T}$ the set $\mathbf{M}_{\prec}(\mathcal{T})$ is a closed set (i.e. $\mathbf{M}_{\prec}(\mathcal{T}) = \mathbf{M}(f_{\prec}(\mathcal{T}))$).
2. A multi preference relation $\prec_m$ is *safely founded*, **(sf)** for short, if for any $\mu_i \in \mathbf{S}(\mathcal{T}) - \mathbf{S}_{\prec_m}(\mathcal{T})$, there exists $\nu_j \in \mathbf{S}_{\prec_m}(\mathcal{T})$ such that $\nu_j \prec_m \mu_i$.

   Thus, a preference relation $\prec$ is safely founded iff, for any $\mu \in \mathbf{M}(\mathcal{T}) - \mathbf{M}_{\prec}(\mathcal{T})$, there exists $\nu \in \mathbf{M}_{\prec}(\mathcal{T})$ such that $\nu \prec \mu$.
3. A (multi) preference relation $\prec$ is *well founded*, **(wf)** for short, if it is transitive, irreflexive, and there exists no infinitely decreasing chain.
4. A multi preference relation $\prec_m$ is *regular*, **(reg)** for short, if for any copies $\mu_1, \mu_2$ and $\nu_1, \nu_2$ of the interpretations $\mu$ and $\nu$ respectively, we have $\mu_1 \prec_m \nu_1$ iff $\mu_2 \prec_m \nu_2$.
5. A (multi) preference relation $\prec$ is *ranked*, **(rk)** for short, if it is a strict order such that we have:

   if $\mu_3 \prec \mu_1$ then ($\mu_3 \prec \mu_2$ or $\mu_2 \prec \mu_1$).
6. A strict order preference relation $\prec$ is *ranked by blocks*, **(rkb)** for short, if we have:

   if $\quad \mu_3 \prec \mu_1$ and $\left( \begin{array}{llll} \text{(a)} & \mu \prec \mu_1 & \text{and} & \mu \prec \mu_2, \quad \text{or} \\ \text{(b)} & \mu_3 \prec \mu & \text{and} & \mu_2 \prec \mu \end{array} \right)$, then $(\mu_2 \prec \mu_1$ or $\mu_3 \prec \mu_2)$.
7. A (multi) preference relation has *isolated loops* if for any distinct (copies of) interpretations $\mu$ and $\nu$, we have: if $\mu \prec \mu$, then $\mu \not\prec \nu$ and $\nu \not\prec \mu$.
8. Let (prop) be some property of a preference relation $\prec$ defined in $\mathbf{M}$. We say that $\prec$ satisfies the property *quasi-(prop)* – q-(prop) for short – if there exists a subset $\mathbf{M}_1$ of $\mathbf{M}$ such that 1) the restriction of $\prec$ to $\mathbf{M}_1$ satisfies (prop), 2) the restriction of $\prec$

to $\mathbf{M} - \mathbf{M}_1$ is equality ($\mu \prec \mu$ and nothing else) and 3) nothing else, in particular if $\mu \in \mathbf{M}_1$ and $\nu \in \mathbf{M} - \mathbf{M}_1$ then $\mu \not\prec \nu$ and $\nu \not\prec \mu$.

By "the restriction of $\prec$ to $\mathbf{M}_1$ satisfies (prop)" we mean precisely what follows: If (prop) is some property of a binary relation (such as transitivity), $\prec$ restricted to the elements of $\mathbf{M}_1$ must satisfy (prop). If (prop) is a property such as (sf) which involves the sets $\mathbf{M}(\mathcal{T})$ and $\mathbf{M}_\prec(\mathcal{T})$ for some theories $\mathcal{T}$, we replace in the definition of (prop) each occurrence of $\mathbf{M}(\mathcal{T})$ by $\mathbf{M}_1(\mathcal{T})$, where $\mathbf{M}_1(\mathcal{T}) = \mathbf{M}(\mathcal{T}) \cap \mathbf{M}_1$. Notice that we have $\mathbf{M}_\prec(\mathcal{T}) = \{\mu \in \mathbf{M}_1(\mathcal{T})\ /\ \text{ for no } \nu \in \mathbf{M}_1(\mathcal{T}) \text{ we have } \nu \prec \mu\}$.

We define in this way the properties *quasi safely founded* – q-(sf) for short –, *quasi ranked* – q-(rk) for short–, *quasi well founded* – q-(wf) for short–, *quasi-(ranked and safely founded)* – q-(rk+sf) for short. Notice that this is not pertinent for any property, e.g. q-(cl) is (cl) and q-(rkb) is (rkb)

If $\prec$ satisfies (prop), then it satisfies q-(prop) and an irreflexive relation $\prec$ satisfies (prop) iff it satisfies q-(prop). □

(cl) is called definability preserving in [Sch92, Sch97] and faithful in [Mak94]. (Multi) preferential entailments in which the relation satisfies (cl) are much easier to deal with, precisely because we get $\mathbf{M}(f_\prec(\mathcal{T})) = \mathbf{M}_\prec(\mathcal{T})$. We will see below (property 8.14) an important consequence of this property in our context.

If $V(\mathbf{L})$ is finite, (cl) is trivially satisfied.

(sf) appears in a context of non monotonic reasoning and preferential entailment as with minimally modelable theories in [BS85], then it was confusingly called well-founded in e.g. [EMR85]. The notion of (sf) for propositional multi preference relations appears as smooth in [KLM90] and as stoppered in [Mak94]. In the "history" of non monotonic reasoning, the first motivation for introducing such a property was to get preservation of consistency (see [BS85, EMR85, Sho88]). It is only later, in [KLM90] and related texts, that it was realized that this property also implied cumulativity. In terms of propositional preference relations, the closest analog of (sf) for multi preference relations as considered in [KLM90] – which does not imply preservation of consistency –, is q-(sf) and not (sf). We make this point precise here because it is very often overlooked.

(wf) is a well known mathematical notion, which implies clearly (sf) (even for preference relations) while the converse is false.

(reg), when satisfied by a multi preference relation $\prec_m$, means that the multi preferential entailment $f_{\prec_m}$ is equal to the preferential entailment $f_\prec$ associated to the preference relation defined as follows: $\mu \prec \mu$ if $\mu \notin l(\mathbf{S})$, $\mu \prec \nu$ if there exists $\mu_i \in l^{-1}(\mu)$ and $\nu_j \in l^{-1}(\nu)$ such that $\mu_i \prec \nu_j$, and nothing else. This is to be compared with (Def$\downarrow$:$\prec' \rightsquigarrow \prec$). Indeed, when $\prec_m$, as defined by (Def$\downarrow$:$\prec' \rightsquigarrow \prec_m$), satisfies (reg), $f$ is a preferential entailment, thus we are in the case in which (Def$\downarrow$:$\prec' \rightsquigarrow \prec$) exists (cf property 4.12). Notice that for any $\mu \in \mathbf{M}$, either $\mathbf{M}'_{\prec'}(\mu) = \emptyset$ or $\mathbf{M}'_{\prec'}(\mu) = \mathbf{M}'(\mu)$. Thus (Def$\downarrow$:$\prec' \rightsquigarrow \prec$) may be modified as follows, without modifying the resulting preferential entailment $f_{\prec}$: $\mu \prec \nu$ iff for any $\nu' \in \mathbf{M}'(\nu)$, there exists $\mu' \in \mathbf{M}'(\mu)$ such that $\mu' \prec' \nu'$. (Def$\downarrow_{\mathcal{T}'}$:$\prec' \rightsquigarrow \prec$) may be simplified similarly.

(rk) appears in [LM92] in order to deal with multi preferential entailments satisfying (RatM0). As noted in [ACS92], (rk) means that $\prec$ is a strict order and that its negation $\not\prec$ is transitive.

q-(prop): This notion is introduced for two reasons: 1) As already explained about (sf) above, it allows to make precise some connections with the literature of [KLM90] and followers. 2) As we will see below, it is convenient when we allow the addition of a fixed theory $\mathcal{T}'$, as done in (Def$\downarrow_{\mathcal{T}'}$).

We state now a few results coming from the literature, which are useful for our purpose (remind remark 5.1).

**Property 5.6** (known) $\prec$ and $\prec^1$ are preference relations, $\prec_m$ and $\prec^1_m$ are multi preference relations.

1. If $\prec$ and $\prec^1$ are two preference relations, $\prec$ being irreflexive, and if $f_{\prec} = f_{\prec^1}$, then $\prec = \prec^1$.
2. If a preference relation $\prec$ satisfies (sf), then $\prec$ is transitive and irreflexive.
3. If a multi preference relation $\prec_m$ satisfies (sf), then there exists a transitive and irreflexive multi preference relation $\prec^1_m$ satisfying (sf) such that $f_{\prec_m} = f_{\prec^1_m}$. Moreover, if $\prec_m$ satisfies also (cl), we may choose a $\prec^1_m$ satisfying also (cl).

   If $V(\mathbf{L})$ is finite, any transitive and irreflexive multi preference relation $\prec_m$ satisfies (sf).
4. If $V(\mathbf{L})$ is finite, then a preference relation satisfies (sf) iff it is transitive and irreflexive.
5. If $V(\mathbf{L})$ is finite, then a multi preference relation $\prec_m$ satisfies (sf) iff there exists a transitive and irreflexive multi preference relation $\prec^1_m$ such that $f_{\prec_m} = f_{\prec^1_m}$.
6. If $V(\mathbf{L})$ is finite, then a preference relation satisfies q-(sf) iff there exists a preference relation $\prec_r$ which is transitive and with isolated loops, such that $f_{\prec} = f_{\prec_r}$. We may choose (and in fact we must: it is the only possibility) $\prec_r$ defined by: $\mu \prec_r \nu$ if $\mu \prec \nu$ and ($\mu = \nu$ or $\nu \not\prec \nu$). $\Box$

**Property 5.7** (known, the results involving (DCC), (CC), (CC$\infty$) and (RI) are from [MR98b, MR00])

1. Any multi preferential entailment satisfies (CT) and (CR), thus (RM1), (CC1), (CN-M1) and (RI1).

A multi preferential entailment may falsify (RI), (CNM), (RM) or (CR$\infty$).

For multi preferential entailments, (RM) and (CC$\infty$) (and (CC) when $V(\mathbf{L})$ is enumerable) are equivalent.

2. Any preferential entailment satisfies (DCC). Thus we get:

   Any preferential entailment satisfying (RM) satisfies also (DC) (see property 5.4); and

   any pre-circumscription satisfying (RM) and (DCC) is a preferential entailment (remind this result in property 5.9, points 5 and 6, given below).

3. Thus, if $V(\mathbf{L})$ is finite, a pre-circumscription is a preferential entailment iff it satisfies (CR) and (DCC), iff it satisfies (CR) and (DC). Remind that in this case (RM) is (RM0), thus also (CR0) or (CR), and similarly (DC) is (DC0). □

**Property 5.8** (known)

1. If a preference relation $\prec$ satisfies (sf), $f_\prec$ satisfies (PC) and (CUMU).

   If a multi preference relation $\prec_m$ satisfies (sf), then $f_{\prec_m}$ satisfies (CUMU).

2. If a preference relation $\prec$ satisfies q-(sf), then $f_\prec$ satisfies (CUMU).

3. If $f_\prec$ is a preferential entailment satisfying (CUMU) and (PC), then $\prec$ is transitive and irreflexive (thus there is only one relation $\prec$ possible). □

There exist various characterization results from which we extract the following ones, useful for circumscriptions, and which are easy to state if not to prove (see [MR98b, MR00] for the proofs and also for more results of this kind).

**Property 5.9** 1. [MR98b] A preferential entailment $f_\prec$ satisfies (RM), (CUMU) and (PC) iff $\prec$ satisfies (sf) and (cl).

2. [MR98b] If $V(\mathbf{L})$ is enumerable, then a preferential entailment $f_\prec$ satisfies (CUMU) and (PC) iff $\prec$ satisfies (sf).

3. [Mak94, Observation 3.4.8] A pre-circumscription $f$ satisfies (RM) and (CM) iff it is a multi preferential entailment defined by a multi preference relation $\prec_m$ satisfying (sf) and (cl). Thus, we get:

4. [KLM90] If $V(\mathbf{L})$ is finite, a pre-circumscription $f$ satisfies (CM0) and (CR0) (i.e. (CR) and (CUMU)) iff it is a multi preferential entailment defined by a multi preference relation $\prec_m$ satisfying (sf) iff (see property 5.6-3) it is a multi preferential entailment defined by a multi preference relation $\prec_m$ which is transitive and irreflexive.

5. [Sch92, theorem 3.1] (or [Sch97, Theorem 2.81]) A pre-circumscription $f$ satisfies (R-M) iff it is a multi preferential entailment defined by a multi preference relation $\prec_m$ satisfying (cl).

   Thus, if $V(\mathbf{L})$ is finite, a pre-circumscription satisfies (RM) (i.e. in this case (RM0) or (CR0) or (CR)) iff it is a multi preferential entailment.

6. [MR98b] A pre-circumscription $f$ satisfies (RM) and (DCC) iff it is a preferential entailment defined by a preference relation $\prec$ satisfying (cl). □

Remind that from proposition 5.6 there is only one preference relation $\prec$ possible in the cases of points 1 and 2. Remind also the two characterization results already given in property 5.7.

Here are a few results about rational monotony:

**Property 5.10** 1. [LM92, MR98b] If $\prec_m$ satisfies q-(rk), $f_{\prec_m}$ satisfies (RatM1) and (DR).

2. [MR98b] If $\prec_m$ satisfies q-(rk) and (cl), or is a strict linear order, $f_{\prec_m}$ satisfies (RatM).

   Notice however that a multi-preferential entailment $f$ may satisfy (RM) and (RatM) even if it does not exist any multi preference relation $\prec_m$ satisfying q-((cl)+(rk)), i.e. (cl) + q-(rk), such that $f = f_{\prec_m}$.

3. [MR98b] If $\prec$ is an irreflexive preference relation, then $f_\prec$ satisfies (RM)+(RatM) iff $\prec$ satisfies (cl)+(rk). □

We end this section by a few results about X-mappings (cf definitions 4.7).

**Property 5.11** [MR98a] Modifying the set $X$:

1. A pre-circumscription $f$ is an X-mapping iff we have $f = f_{I_f}$.

2. For any set $X$[7], $I_{f_X}$ is the $\wedge$-*closure* of $X$: $I_{f_X} = \{\bigwedge_{\varphi \in Y} \varphi \mathbin{/} Y \subseteq X,\ Y \text{ finite}\}$. □

**Property 5.12** [MR98a] Logical properties of X-mappings:

1. Any pre-circumscription $f$ which is an X-mapping satisfies (CUMU) and (CR$\infty$).

2. If $V(\mathbf{L})$ is finite, a pre-circumscription $f$ is an X-mapping iff it satisfies (CM0) and (CR0) (to be compared with property 5.9-4). □

[7] Rigorously, with the definitions given in the present text, we should write here "any set $X$ such that $f_X$ as defined in definition 4.7 exists", however, our formulation is justified by the fact that this result is true for any set $X$, even if $f_X$ is not a pre-circumscription [MR98a, Theorem 3.2].

# 6 The properties preserved by the reductions of the vocabulary (Def$\downarrow$), (Def$\downarrow_{\mathcal{T}'}$) and (Def$\downarrow_{\beta'}$)

**We suppose $\mathbf{L} \subseteq \mathbf{L}'$.**

## 6.1 Introducing two important particular cases

$f'$ is a pre-circumscription in $\mathbf{L}'$, and we define the pre-circumscription $f$ in $\mathbf{L}$ by (Def$\downarrow$), (Def$\downarrow_{\mathcal{T}'}$) or (Def$\downarrow_{\beta'}$) as in definition 4.10. All along this section 6, we will examine which properties of $f'$ are satisfied by $f$. Firstly, let us give again the original definitions, in order to introduce two useful particular cases.

**Definition 6.1** 1. $f(\mathcal{T}) = tr(f'(\mathcal{T})) = f'(\mathcal{T}) \cap \mathbf{L}$. **(Def$\downarrow$)**

2. $f(\mathcal{T}) = tr(f'(\mathcal{T} \cup \mathcal{T}')) = f'(\mathcal{T} \cup \mathcal{T}') \cap \mathbf{L}$ **(Def$\downarrow_{\mathcal{T}'}$)**

   We suppose that we are in this case (Def$\downarrow_{\mathcal{T}'}$).

   (a) If for any $\mu \in \mathbf{M}$, the set $\mathbf{M}'(\mathcal{T}') \cap \mathbf{M}'(\mu) = \mathbf{M}'(\mathcal{T}' \sqcup' Th(\mu))$ is a singleton, we say that we have the *singleton property.* **(singleton property)**

   (b) If for any $\mu \in \mathbf{M}$, the set $\mathbf{M}'(\mathcal{T}') \cap \mathbf{M}'(\mu) = \mathbf{M}'(\mathcal{T}' \sqcup' Th(\mu))$ is not empty, we say that we have the *non empty property.* **(non empty property)**

3. $f(\mathcal{T}) = tr(f'(\mathcal{T} \cup \{\beta'\})) = f'(\mathcal{T} \cup \{\beta'\}) \cap \mathbf{L}$ **(Def$\downarrow_{\beta'}$)**

   (Def$\downarrow_{\beta'}$) may also have the singleton property or the non empty property.

   $\square$

Notice that we have obviously that (Def$\downarrow$) is (Def$\downarrow_{\top}$), and that (Def$\downarrow_{\beta'}$) is (Def$\downarrow_{\{\beta'\}}$). Thus (Def$\downarrow$) implies (Def$\downarrow_{\beta'}$), which in turns implies (Def$\downarrow_{\mathcal{T}'}$). By this formulation, we mean that if $f$ is defined from $f'$ by (Def$\downarrow$), then $f$ is defined from $f'$ by (Def$\downarrow_{\beta'}$) (choose $\beta' = \top$), and if $f$ is defined from $f'$ by (Def$\downarrow_{\beta'}$), then $f$ is defined from $f'$ by (Def$\downarrow_{\mathcal{T}'}$) (choose $\mathcal{T}' = \{\beta'\}$). Thus, if a property is preserved by (Def$\downarrow_{\mathcal{T}'}$), then it is preserved by (Def$\downarrow_{\beta'}$), and if a property is preserved by (Def$\downarrow_{\beta'}$), then it is preserved by (Def$\downarrow$). This amounts to say that, if a property is not preserved by (Def$\downarrow$), then it cannot be preserved by (Def$\downarrow_{\beta'}$). Similarly, if a property is not preserved by (Def$\downarrow_{\beta'}$), then it cannot be preserved by (Def$\downarrow_{\mathcal{T}'}$).

We have also that (Def$\downarrow$) implies (Def$\downarrow_{\beta'}$) with the non empty property. Indeed, $\mathbf{M}(\mu) \cap \mathbf{M}'(\top) = \mathbf{M}'(\mu)$ for any $\mu \in \mathbf{M}$, and $\mathbf{M}'(\mu) \neq \emptyset$. And clearly, (Def$\downarrow_{\beta'}$) with the non empty property implies (Def$\downarrow_{\mathcal{T}'}$) with the non empty property.

Finally we get that (Def$\downarrow_{\beta'}$) with the singleton property implies (Def$\downarrow_{\beta'}$) with the non empty property, and similarly for (Def$\downarrow_{\mathcal{T}'}$) instead of (Def$\downarrow_{\beta'}$).

However, (Def$\downarrow$) implies neither (Def$\downarrow_{\beta'}$) with the singleton property nor (Def$\downarrow_{\mathcal{T}'}$) with the singleton property.

**Remarks 6.2** 1. The singleton property is of particular interest. Indeed, in this case there exists a one-to-one mapping $u'$ from $\mathbf{M}$ onto $\mathbf{M}'(\mathcal{T}')$, defined by: $u'(\mu)$ is the only element in $\mathbf{M}'(\mu) \cap \mathbf{M}'(\mathcal{T}')$. It is immediate to see that $u'^{-1} = tr$: more precisely, $u'^{-1}$ is the restriction of $tr$ to the subset $\mathbf{M}'(\mathcal{T}')$ of $\mathbf{M}'$ (remind that $tr$, considered as a mapping from $\mathbf{M}'$ to $\mathbf{M}$, is not injective, unless $\mathbf{M}' = \mathbf{M}$, i.e. $\mathbf{L}' = \mathbf{L}$). As a consequence of this observation we get that $u'^{-1}$ is continuous, thus, as we deal with compact spaces, $u'$ is also continuous. This means that $u'$ is an homeomorphism between $\mathbf{M}$ and $\mathbf{M}'(\mathcal{T}')$. In particular, for any $\mathcal{T} \in \mathbf{T}$, the restriction of $u'$ to the set $\mathbf{M}(\mathcal{T})$ is a one-to-one mapping from $\mathbf{M}(\mathcal{T})$ onto $\mathbf{M}'(\mathcal{T}' \sqcup' \mathcal{T})$.

Here is another feature of this case: any theory $\mathcal{T}'' \in \mathbf{T}'$ which contains $\mathcal{T}'$ may be written as $\mathcal{T} \sqcup' \mathcal{T}'$ where $\mathcal{T} \in \mathbf{T}$, and we get in this way a one-to-one mapping from $\mathbf{T}$ onto the set $\{\mathcal{T}'' \in \mathbf{T}' \ / \ \mathcal{T}' \subseteq \mathcal{T}''\}$.

If $f'$ is a preferential entailment $f_{\prec'}$, $f$ is necessarily a preferential entailment $f_{\prec}$. For any $\mu \in \mathbf{M}$, $\mathbf{M}'_{\prec'}(Th'(\mu) \sqcup' \mathcal{T}')$ is either the singleton $\mathbf{M}'(Th'(\mu) \sqcup' \mathcal{T}') = \mathbf{M}'(\mu) \cap \mathbf{M}'(\mathcal{T}')$ or the empty set and we get the following simplification of (Def$\downarrow_{\mathcal{T}'}$:$\prec' \rightsquigarrow \prec$):

$$\mu \prec \nu \quad \text{iff} \quad u'(\mu) \prec' u'(\nu) \qquad \textbf{(Def}\downarrow_{\mathcal{T}'}\textbf{:}\prec'\rightsquigarrow\prec\textbf{) (singleton case)}$$

2. The non empty property is far from being so interesting. Let us just notice here that it means that there exists a surjective mapping from $\mathbf{M}'(\mathcal{T}')$ onto $\mathbf{M}$ such that, for any $\mathcal{T} \in \mathbf{T}$, its restriction to $\mathbf{M}'(\mathcal{T} \sqcup' \mathcal{T}')$ is surjective onto $\mathbf{M}(\mathcal{T})$. □

We omit the proofs of point 1 here, as they are without difficulty.

In the following subsections, we examine which properties, among the properties introduced above, are preserved by (Def$\downarrow$), (Def$\downarrow_{\mathcal{T}'}$) or (Def$\downarrow_{\beta'}$). In the following proofs, $\mathcal{T}, \mathcal{T}_i$ are subsets of $\mathbf{L}$ (when intersections are necessary, we require that $\mathcal{T}, \mathcal{T}_i$ are in $\mathbf{T}$), while $\mathcal{T}', \mathcal{T}'_i$ are subsets of $\mathbf{L}'$.

## 6.2 Logical properties of pre-circumscriptions

We examine here the properties in the order of definition 5.2.

**Non preservation result 6.3** (RI) and (Idem) are not preserved by (Def$\downarrow$), even in the finite case. □

**Example 6.4** $V(\mathbf{L}) = \{A\}, V(\mathbf{L}') = \{A, B\}$. $f'$ is a pre-circumscription such that $f'(\top) = f'(A \wedge B) = Th'(A \wedge B)$ and $f'(\varphi) = Th'(\bot)$ for any $\varphi' \in \mathbf{L}' - \{\top, A \wedge B\}$.

If we have $\psi' \models' f(\varphi')$ for some formulas $\varphi', \psi'$ in $\mathbf{L}'$, we must have $\psi' = \bot$ or $\psi' = A \wedge B$. As we have $f'(\bot) = Th'(\bot)$ and $f'(A \wedge B) = Th'(A \wedge B)$, we get in any case $f'(\psi') = Th'(\psi')$: $f'$ satisfies (RI), thus (Idem).

$f$ is the pre-circumscription defined by $f(\mathcal{T}) = f'(\mathcal{T}) \cap \mathbf{L}$ for any $\mathcal{T} \subseteq \mathbf{L}$.

Then, $f(\top) = Th(A)$, $f(\varphi) = Th(\bot)$ for any $\varphi \in \mathbf{L} - \{\top\}$.

Thus we get $A \models f(\top)$ and $f(A) \neq Th(A)$: $f$ falsifies (RI).
We get also $f(\top) = Th(A)$ and $f(A) = Th(\bot)$: $f$ falsifies (Idem). □

**Preservation result 6.5** (RM) and (RM1) are preserved by (Def$\downarrow_{\mathcal{T}'}$). (RM0) is preserved by (Def$\downarrow_{\beta'}$). □

Proof: $f'$ satisfies (RM). $f(\mathcal{T}_1 \cup \mathcal{T}_2) = f'(\mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}') \cap \mathbf{L} \subseteq (f'(\mathcal{T}_1 \cup \mathcal{T}') \sqcup' \mathcal{T}_2) \cap \mathbf{L}$ from (RM) for $f'$. Now, as $\mathcal{T}_2 \subseteq \mathbf{L}$, we know that we have $(f'(\mathcal{T}_1 \cup \mathcal{T}') \sqcup' \mathcal{T}_2) \cap \mathbf{L} = (f'(\mathcal{T}_1 \cup \mathcal{T}') \cap \mathbf{L}) \sqcup \mathcal{T}_2$. Thus $f(\mathcal{T}_1 \cup \mathcal{T}_2) \subseteq f(\mathcal{T}_1) \sqcup \mathcal{T}_2$: $f$ satisfies (RM).
The proofs for (RM1) and (RM0) are similar. □

**Preservation result 6.6** (CR) and (CR1) are preserved by (Def$\downarrow_{\mathcal{T}'}$). (CR$\infty$) and (CR0) are preserved by (Def$\downarrow_{\beta'}$). □

Proof: $f'$ satisfies (CR). $f(\mathcal{T}_1) \cap f(\mathcal{T}_2) = f'(\mathcal{T}_1 \sqcup' \mathcal{T}') \cap f'(\mathcal{T}_2 \sqcup' \mathcal{T}') \cap \mathbf{L} \subseteq f'((\mathcal{T}_1 \sqcup' \mathcal{T}') \cap (\mathcal{T}_2 \sqcup' \mathcal{T}')) \cap \mathbf{L} = f'((\mathcal{T}_1 \cap \mathcal{T}_2) \sqcup' \mathcal{T}') \cap \mathbf{L} = f(\mathcal{T}_1 \cap \mathcal{T}_2)$: $f$ satisfies (CR). The proofs for (CR1) and (CR0) are similar.
$f'$ satisfies (CR$\infty$): We know from classical logic that we have $(\bigcap_{i\in I} \mathcal{T}'_i) \sqcup' \beta' = \bigcap_{i\in I}(\mathcal{T}'_i \sqcup' \beta')$, but that we are only guaranteed of $(\bigcap_{i\in I} \mathcal{T}'_i) \sqcup' \mathcal{T}' \subseteq \bigcap_{i\in I}(\mathcal{T}'_i \sqcup' \mathcal{T}')$ when $I$ is infinite and $\mathcal{T}'$ is not equivalent to a formula. Thus, we may consider (Def$\downarrow_{\beta'}$), but not the general case of (Def$\downarrow_{\mathcal{T}'}$). We suppose $f'$ satisfies (CR$\infty$). $\bigcap_{i\in I} f(\mathcal{T}_i) = \bigcap_{i\in I}(f'(\mathcal{T}_i \sqcup' \beta') \cap \mathbf{L}) = (\bigcap_{i\in I} f'(\mathcal{T}_i \sqcup' \beta')) \cap \mathbf{L} \subseteq f'(\bigcap_{i\in I}(\mathcal{T}_i \sqcup' \beta')) \cap \mathbf{L} = f'((\bigcap_{i\in I} \mathcal{T}_i) \sqcup' \beta') \cap \mathbf{L} = f(\bigcap_{i\in I} \mathcal{T}_i)$: $f$ satisfies (CR$\infty$).□

Notice that (CR$\infty$) is not preserved by (Def$\downarrow_{\mathcal{T}'}$), even when $\mathbf{L}' = \mathbf{L}$, as shown by the following example.

**Example 6.7** $V(\mathbf{L}) = V(\mathbf{L}') = \{P_i \ / \ i \in \mathbb{N}\}$.
We define the following interpretations: $\mu_n = \{P_i \ / \ i \leq n\}$, $\mu_\omega = V(\mathbf{L})$. We take the identity on $\mathbf{T}$ as our pre-circumscription $f'$: $f'(\mathcal{T}) = Th(\mathcal{T})$ for any $\mathcal{T} \subseteq \mathbf{L}$. We define the pre-circumscription $f$ by $f(\mathcal{T}) = f'(\mathcal{T} \sqcup \mathcal{T}')$ for any $\mathcal{T} \subseteq \mathbf{L}$.
$f'$ satisfies obviously (CR$\infty$).
We define the following theories: $\mathcal{T}' = \mathcal{T}'_\omega = Th(\mu_\omega)$, $\mathcal{T}'_n = Th(\mu_n)$.
We get then $f(\mathcal{T}'_n) = f'(\mathcal{T}'_n \sqcup' \mathcal{T}'_\omega) = Th(\bot)$ thus $\bigcap_{n\in \mathbb{N}} f(\mathcal{T}'_n) = Th(\bot)$.
$f(\bigcap_{n\in \mathbb{N}} \mathcal{T}'_n) = f'((\bigcap_{n\in \mathbb{N}} \mathcal{T}'_n) \sqcup \mathcal{T}'_\omega) = f'(\mathcal{T}'_\omega) = \mathcal{T}'_\omega$.
Thus $\bigcap_{n\in \mathbb{N}} f(\mathcal{T}'_n) \not\subseteq f(\bigcap_{n\in \mathbb{N}} \mathcal{T}'_n)$: $f$ falsifies (CR$\infty$). □

**Non preservation result 6.8** (CC) is not preserved by (Def$\downarrow$), even when $V(\mathbf{L}')$ if finite. □

**Example 6.9** $V(\mathbf{L}) = \{A, B\}, V(\mathbf{L}') = \{A, B, C\}$.
We define $f'$ as follows: $f'(\varphi') = \begin{cases} Th'(A \wedge (\neg B \vee C)) & \text{if } \varphi' = A \\ Th'(B \wedge (\neg A \vee \neg C)) & \text{if } \varphi' = B \\ Th'(\bot) & \text{otherwise} \end{cases}$

$f'$ satisfies (CC0), i.e. (CC) or even (CC$\infty$) as we are in the finite case. Indeed, the only non trivial case to check is $f'(A \wedge B) \subseteq f'(A) \sqcup' f'(B)$, and we have $f'(A \wedge B) = f'(A) \sqcup' f'(B) = Th'(\bot)$.

Now, with $f(\varphi) = f'(\varphi) \cap \mathbf{L}$ we get $f(A \wedge B) = Th'(\bot) \cap \mathbf{L} = Th(\bot)$, $f(A) = Th'(A \wedge (\neg B \vee C)) \cap \mathbf{L} = Th(A)$ and $f(B) = Th'(B \wedge (\neg A \vee \neg C)) \cap \mathbf{L} = Th(B)$. Thus, $f((A \wedge B)) \not\subseteq f(A) \sqcup f(B) = Th(A \wedge B)$: $f$ falsifies (CC0). □

**Non preservation result 6.10** (DC) is not preserved by (Def$\downarrow$), even when $V(\mathbf{L}')$ if finite. □

**Example 6.11** $V(\mathbf{L}) = \{A, B\}, V(\mathbf{L}') = \{A, B, C\}$.

We define the following preference relation $\prec'$ on $\mathbf{M}'$: $\{B, C\} \prec' \{A, B, C\}$, $\{A, C\} \prec' \{A, B\}$, and nothing else. Notice that $\prec'$ satisfies (sf).

We take $f' = f_{\prec'}$ as our pre-circumscription on $\mathbf{L}'$. $f'$ being a preferential entailment, we know that it satisfies (DC0), i.e. (DC) or even (DC$\infty$) as we are in the finite case.

We define the pre-circumscrip[tion $f$ in $\mathbf{L}$ by $f(\varphi) = f'(\varphi) \cap \mathbf{L}$ for any $\varphi \in \mathbf{L}$.

We get $f(\varphi) = \begin{cases} Th(\neg A \vee \neg B) & \text{if } \varphi = \top \\ Th(A \Leftrightarrow \neg B) & \text{if } \varphi = A \vee B \\ Th(\varphi) & \text{otherwise} \end{cases}$

We get $f(A) = Th(A)$, $f(B) = Th(B)$ and $f(A \vee B) = Th(A \Leftrightarrow \neg B)$, thus $f(A \vee B) \not\subseteq f(A) \sqcup f(B) = Th(A \wedge B)$: $f$ falsifies (DC0). Thus, $f$ is not a preferential entailment. □

**Preservation result 6.12** (DR) and (DR1) are preserved by (Def$\downarrow_{\mathcal{T}'}$). (DR0) is preserved by (Def$\downarrow_{\beta'}$).
□

<u>Proof:</u> $f'$ satisfies (DR) and $f(\mathcal{T}) = f'(\mathcal{T} \sqcup' \mathcal{T}') \cap \mathbf{L}$. $f(\mathcal{T}_1 \cap \mathcal{T}_2) = f'((\mathcal{T}_1 \cap \mathcal{T}_2) \sqcup' \mathcal{T}') \cap \mathbf{L} = f'((\mathcal{T}_1 \sqcup' \mathcal{T}') \cap (\mathcal{T}_2 \sqcup' \mathcal{T}')) \cap \mathbf{L} \subseteq (f'(\mathcal{T}_1 \sqcup' \mathcal{T}') \cup f'(\mathcal{T}_2 \sqcup' \mathcal{T}')) \cap \mathbf{L} = (f'(\mathcal{T}_1 \sqcup' \mathcal{T}') \cap \mathbf{L}) \cup (f'(\mathcal{T}_2 \sqcup' \mathcal{T}') \cap \mathbf{L}) = f(\mathcal{T}_1) \cup f(\mathcal{T}_2)$: $f$ satisfies (DR).

Notice that the same proof works for the infinite version (DR$\infty$) (definition (DR) as given in definition 5.2 with an infinite number of $\mathcal{T}_i$'s) for (Def$\downarrow_{\beta'}$) but not for (Def$\downarrow_{\mathcal{T}'}$) (cf the proof for (CR$\infty$) above).

The proofs for (DR1) and (DR0) are similar to the proof given above for (DR) with (Def$\downarrow_{\mathcal{T}'}$). □

**Preservation result 6.13** (MON) and (MON1) are preserved by (Def$\downarrow_{\mathcal{T}'}$). (MON0) is preserved by (Def$\downarrow_{\beta'}$). □

<u>Proof:</u> $f'$ satisfies (MON) and $f(\mathcal{T}) = f'(\mathcal{T} \sqcup' \mathcal{T}') \cap \mathbf{L}$. $f(\mathcal{T}_1) = f'(\mathcal{T}_1 \sqcup' \mathcal{T}') \cap \mathbf{L} \subseteq f'((\mathcal{T}_1 \sqcup' \mathcal{T}') \sqcup' \mathcal{T}_2) \cap \mathbf{L} = f'((\mathcal{T}_1 \sqcup \mathcal{T}_2) \sqcup' \mathcal{T}') \cap \mathbf{L} = f(\mathcal{T}_1 \sqcup \mathcal{T}_2)$: $f$ satisfies (MON).

The proofs for (MON1) and (MON0) are similar. □

**Preservation result 6.14** (CM) and (CM1) are preserved by (Def$\downarrow_{\mathcal{T}'}$). (CM0) is preserved by (Def$\downarrow_{\beta'}$). $\square$

Proof: $f'$ satisfies (CM) and $f(\mathcal{T}) = f'(\mathcal{T} \sqcup' \mathcal{T}') \cap \mathbf{L}$. We suppose $\mathcal{T}_1 \subseteq f(\mathcal{T}_2)$, i.e. $\mathcal{T}_1 \subseteq f'(\mathcal{T}_2 \sqcup' \mathcal{T}') \cap \mathbf{L}$. Thus $\mathcal{T}_1 \subseteq f'(\mathcal{T}_2 \sqcup' \mathcal{T}')$. As $f'$ satisfies (CM) we get $f'(\mathcal{T}_2 \sqcup' \mathcal{T}') \subseteq f'(\mathcal{T}_1 \sqcup' \mathcal{T}_2 \sqcup' \mathcal{T}')$. Thus we get $f(\mathcal{T}_2) = f'(\mathcal{T}_2 \sqcup' \mathcal{T}') \cap \mathbf{L} \subseteq f'(\mathcal{T}_1 \sqcup' \mathcal{T}_2 \sqcup' \mathcal{T}') \cap \mathbf{L} = f'((\mathcal{T}_1 \sqcup \mathcal{T}_2) \sqcup' \mathcal{T}') \cap \mathbf{L} = f(\mathcal{T}_1 \sqcup \mathcal{T}_2)$: $f$ satisfies (CM).

The proofs for (CM1) and (CM0) are similar. $\square$

**Preservation result 6.15** (CT) and (CT1) are preserved by (Def$\downarrow_{\mathcal{T}'}$). (CT0) is preserved by (Def$\downarrow_{\beta'}$). $\square$

Proof: $f'$ satisfies (CT) and $f(\mathcal{T}) = f'(\mathcal{T} \sqcup' \mathcal{T}') \cap \mathbf{L}$. We suppose $\mathcal{T}_1 \subseteq f(\mathcal{T}_2)$, i.e. $\mathcal{T}_1 \subseteq f'(\mathcal{T}_2 \sqcup' \mathcal{T}') \cap \mathbf{L}$. Thus $\mathcal{T}_1 \subseteq f'(\mathcal{T}_2 \sqcup' \mathcal{T}')$. As $f'$ satisfies (CT) we get $f'(\mathcal{T}_1 \sqcup' \mathcal{T}_2 \sqcup' \mathcal{T}') \subseteq f'(\mathcal{T}_2 \sqcup' \mathcal{T}')$. Thus we get $f(\mathcal{T}_1 \sqcup \mathcal{T}_2) = f'((\mathcal{T}_1 \sqcup \mathcal{T}_2) \sqcup' \mathcal{T}') \cap \mathbf{L} = f'(\mathcal{T}_1 \sqcup' \mathcal{T}_2 \sqcup' \mathcal{T}') \cap \mathbf{L} \subseteq f'(\mathcal{T}_2 \sqcup' \mathcal{T}') \cap \mathbf{L} = f(\mathcal{T}_2)$: $f$ satisfies (CT).

The proofs for (CT1) and (CT0) are similar. $\square$

**(Non) preservation result 6.16** (PC) is preserved by (Def$\downarrow$), and not preserved by (Def$\downarrow_{\beta'}$) or (Def$\downarrow_{\mathcal{T}'}$). $\square$

Proof: (Def$\downarrow$): $f'$ satisfies (PC). If $\perp \in f(\mathcal{T}) = f'(\mathcal{T}) \cap \mathbf{L}$ then $\perp \in f'(\mathcal{T})$ thus, by (PC), $\perp \in Th'(\mathcal{T})$, i.e., as $\mathcal{T} \subseteq \mathbf{L}$, $\perp \in Th(\mathcal{T})$: $f$ satisfies (PC).

(Def$\downarrow_{\beta'}$) or (Def$\downarrow_{\mathcal{T}'}$): Now, it is obvious that if $\perp \in f'(\mathcal{T} \sqcup' \mathcal{T}') \cap \mathbf{L}$, we may have e.g. $\perp \in \mathcal{T} \sqcup' \mathcal{T}'$ without having $\perp \in Th(\mathcal{T})$. $\square$

**Preservation result 6.17** (CNM) and (CNM1) are preserved by (Def$\downarrow_{\mathcal{T}'}$). (CNM0) is preserved by (Def$\downarrow_{\beta'}$). $\square$

Proof: $f'$ satisfies (CNM). We suppose $\perp \in f(\mathcal{T}_1) \sqcup f(\mathcal{T}_1 \sqcup \mathcal{T}_2)$. We have $f(\mathcal{T}_1) \sqcup f(\mathcal{T}_1 \sqcup \mathcal{T}_2) = (f'(\mathcal{T}_1 \sqcup' \mathcal{T}') \cap \mathbf{L}) \sqcup (f'((\mathcal{T}_1 \sqcup \mathcal{T}_2) \sqcup' \mathcal{T}') \cap \mathbf{L}) \subseteq f'(\mathcal{T}_1 \sqcup' \mathcal{T}') \sqcup' f'((\mathcal{T}_1 \sqcup \mathcal{T}_2) \sqcup' \mathcal{T}') = f'(\mathcal{T}_1 \sqcup' \mathcal{T}') \sqcup' f'(\mathcal{T}_1 \sqcup' \mathcal{T}_2 \sqcup' \mathcal{T}')$. Thus $\perp \in f'(\mathcal{T}_1 \sqcup' \mathcal{T}') \sqcup' \mathcal{T}_2$ by (CNM). Thus, as $\mathcal{T}_2 \subseteq \mathbf{L}$, $\perp \in (f'(\mathcal{T}_1 \sqcup' \mathcal{T}') \cap \mathbf{L}) \sqcup \mathcal{T}_2$, i.e. $\perp \in f(\mathcal{T}_1) \sqcup \mathcal{T}_2$: $f$ satisfies (CNM).

The proofs for (CNM1) and (CNM0) are similar. $\square$

**Preservation result 6.18** (RatM) and (RatM1) are preserved by (Def$\downarrow_{\mathcal{T}'}$). (RatM0) is preserved by (Def$\downarrow_{\beta'}$). $\square$

Proof: $f'$ satisfies (RatM). We suppose $\perp \notin f(\mathcal{T}_1) \sqcup \mathcal{T}_2$, i.e. $\perp \notin (f'(\mathcal{T}_1 \sqcup' \mathcal{T}') \cap \mathbf{L}) \sqcup \mathcal{T}_2$. As $\mathcal{T}_2 \subseteq \mathbf{L}$, this is equivalent to $\perp \notin f'(\mathcal{T}_1 \sqcup' \mathcal{T}') \sqcup' \mathcal{T}_2$. By (RatM), we get $f'(\mathcal{T}_1 \sqcup' \mathcal{T}') \subseteq f'(\mathcal{T}_1 \sqcup' \mathcal{T}_2 \sqcup' \mathcal{T}') = f'((\mathcal{T}_1 \sqcup \mathcal{T}_2) \sqcup' \mathcal{T}')$, thus $f'(\mathcal{T}_1 \sqcup' \mathcal{T}') \cap \mathbf{L} \subseteq f'((\mathcal{T}_1 \sqcup \mathcal{T}_2) \sqcup' \mathcal{T}') \cap \mathbf{L}$. i.e. $f(\mathcal{T}_1) \subseteq f(\mathcal{T}_1 \sqcup \mathcal{T}_2)$: $f$ satisfies (RatM).

The proofs for (RatM1) and (RatM0) are similar. $\square$

Thus, we get two kinds of properties: (Idem), (RI), (CC) and (DC) are *brittle*: they are not preserved by (Def$\downarrow$). The other properties may have various degrees of *robustness*: (RM), (CR), (DR), (MON), (CM), (CT) – thus (CUMU) –, (CNM) and (RatM) are very robust, being preserved by (Def$\downarrow_{\mathcal{T}'}$), while (CR$\infty$) is moderately robust, being preserved by (Def$\downarrow_{\beta'}$), and (PC) is weakly robust, being preserved only by (Def$\downarrow$).

It is interesting to notice that in the literature, the robust properties have been much more studied than the brittle ones. This is particularly clear for (RI), and also for (DR) with respect to the related property (DC). Indeed, (DR) and (DC) have been introduced roughly at the same time ([KLM90] for (DR0) and [Sat90] for (DC0)), and (DC) appears very rarely in the literature, at least when compared to (DR). However we think that (DC) deserves more study than (DR), being much more natural in a context of skeptical non monotonic reasoning. It is possible that the brittleness of (DC) shown here is one of the reasons why it has not deserved more attention. Another reason could be that this kind of properties has been studied in conditional logic before being introduced in non monotonic reasoning, and in conditional logic it is easier to formulate (DR) than (DC). But we do not believe that this reason really matters here. Indeed, it is more tricky to apply this reason to (RM): the formula version (RM1) is easy to formulate in conditional logic, but (RM1) is in fact (CR1), and the full version (RM) is not so immediate to formulate in conditional logic. And (RM), which is robust, has deserved some attention in the literature, since [Sch92]. By the way, it is always possible to formulate all these properties in terms of conditional logic, even it is is slightly more complicated.

## 6.3 Kinds of pre-circumscriptions

We examine now the situation when we start from an X-mapping or from a multi preferential entailment.

**Preservation result 6.19** The notion of X-mapping is preserved by (Def$\downarrow_{\beta'}$). $\square$

Proof: $f' = f_{X'}$ is an X-mapping in $\mathbf{L}'$ ($X' \subseteq \mathbf{L}'$), and $f(\mathcal{T}) = f'(\mathcal{T} \sqcup' \beta') \cap \mathbf{L}$. We define the following subset of $\mathbf{L}$: $X = \{tr^*(\neg\beta' \vee x') \ / \ x' \in X'\}$. We prove $f = f_X$.

Let $\varphi$ be some element of $f(\mathcal{T})$. Then $\varphi \in f'(\mathcal{T} \sqcup' \beta')$ and, from definition 4.7, for any $x' \in X'$, if $\mathcal{T} \cup \{\beta', \varphi\} \models' x'$, we get $\mathcal{T} \cup \{\beta'\} \models' x'$. For any $x \in X$ such that $\mathcal{T} \cup \{\varphi\} \models x$, there exists $x' \in X'$ such that $x = tr^*(\neg\beta' \vee x') = x$ (from $x \in X$), thus we get $\mathcal{T} \cup \{\varphi\} \models' \neg\beta' \vee x'$ from property 3.11-5, i.e. $\mathcal{T} \cup \{\beta', \varphi\} \models' x'$. From $\varphi \in f_{X'}(\mathcal{T} \sqcup' \beta')$, we get then $\mathcal{T} \sqcup' \beta' \models' x'$. Thus, $\mathcal{T} \models' \neg\beta' \vee x'$ and, from property 3.11-5 again, $\mathcal{T} \models x$. This establishes $\varphi \in f_X(\mathcal{T})$.

Let $\varphi$ be some element of $f_X(\mathcal{T})$. Thus $\varphi \in \mathbf{L}$. For any $x \in X$, if $\mathcal{T} \cup \{\varphi\} \models x$, then $\mathcal{T} \models x$ from definition 4.7. Let $x' \in X$ be such that $\mathcal{T} \cup \{\beta', \varphi\} \models' x'$, i.e $\mathcal{T} \cup \{\varphi\} \models' \neg\beta' \vee x'$. Then we get, from property 3.11-5, $\mathcal{T} \cup \{\varphi\} \models tr^*(\neg\beta' \vee x')$, thus, as $\varphi \in f_X(\mathcal{T})$, $\mathcal{T} \models tr^*(\neg\beta' \vee x')$, thus, from property 3.11-5 again, $\mathcal{T} \models' \neg\beta' \vee x'$, i.e. $\mathcal{T} \cup \{\beta'\} \models' x'$: this establishes $\varphi \in f_{X'}(\mathcal{T} \sqcup' \beta')$.

Notice that if we start from the set $X' = I_{f'}$, i.e. from a set $X'$ which is closed for $\wedge$ (cf property 5.11-2), we get a set $X$ which is closed for $\wedge$ (proof immediate, using property 3.11-7).

Thus we get $I_f = \{tr^*(\neg\beta' \vee x') \ / \ x' \in I_{f'}\}$.

It is clear that the use of $\neg\beta'$ is crucial in the proof, thus this proof does not extend to (Def$\downarrow_{\mathcal{T}'}$), and in fact the result is false for (Def$\downarrow_{\mathcal{T}'}$) (even in the case where $\mathbf{L}' = \mathbf{L}$: see our comment about (CR$\infty$) at the end of the proof of preservation result 6.6) as shown by the following example.

**Example 6.20** We take example 6.7 again. $V(\mathbf{L}) = V(\mathbf{L}') = \{P_i \ / \ i \in \mathbb{N}\}$.

As $f'$ is the identity on $\mathbf{T}$, $f'$ is an X-mapping (we may choose $X = \mathbf{L}$). As $f$ defined by $f(\mathcal{T}) = f'(\mathcal{T} \sqcup \mathcal{T}')$ falsifies (CR$\infty$), $f$ is not an X-mapping. □

The proof of preservation result 6.19 extends to X-mappings which are not pre-circumscriptions (definition 4.7 without requiring that $f$ is a pre-circumscription, see note 7 and [SF96, MR98a]). □

**Preservation result 6.21** The property of being a multi preferential entailment is preserved by (Def$\downarrow_{\mathcal{T}'}$). □

Proof: We start from a multi preferential entailment $f' = f_{\prec'_m}$. and we define $f$ from $f'$ by (Def$\downarrow_{\mathcal{T}'}$): $f(\mathcal{T}) = f'(\mathcal{T} \sqcup' \mathcal{T}') \cap \mathbf{L}$ for any $\mathcal{T} \subseteq \mathbf{L}$.

Let us say that the multi preferential entailment $f'$ is defined by the set $\mathbf{S}'$, the mapping $l'$ from $\mathbf{S}'$ to $\mathbf{M}'$ and the multi preference relation $\prec'_m$ defined on $\mathbf{S}'$ (see definition 4.5).

As our set $\mathbf{S}$, we take the set $\mathbf{S} = l'^{-1}(\mathbf{M}'(\mathcal{T}'))$
of all the elements $\mu'_i$ of $\mathbf{S}'$ such that $l'(\mu'_i) \in \mathbf{M}'(\mathcal{T}')$,
and we define the mapping $l$ from $\mathbf{S}$ to $\mathbf{M}$ as follows: **(Def$\downarrow_{\mathcal{T}'}$:$\prec'_m \rightsquigarrow \prec_m$)**
for any $\mu_i \in \mathbf{S}$, $l(\mu_i) = l'(\mu_i) \cap V(\mathbf{L}) = tr(l'(\mu_i))$.
We define the relation $\prec_m$ on $\mathbf{S}$ as the restriction of $\prec'_m$ to $\mathbf{S}$.

We get the following identities, for any $\mathcal{T} \subseteq \mathbf{L}$:

1. $\mathbf{S}(\mathcal{T}) = l'^{-1}(\mathbf{M}'(\mathcal{T} \sqcup' \mathcal{T}')) = \mathbf{S}'(\mathcal{T} \sqcup' \mathcal{T}')$,
2. $\mathbf{S}_{\prec_m}(\mathcal{T}) = l'^{-1}(\mathbf{M}'_{\prec'_m}(\mathcal{T} \sqcup' \mathcal{T}')) = \mathbf{S}'_{\prec'_m}(\mathcal{T} \sqcup' \mathcal{T}')$,
3. $\mathbf{M}_{\prec_m}(\mathcal{T}) = tr(\mathbf{M}'_{\prec'_m}(\mathcal{T} \sqcup' \mathcal{T}'))$.

Proof of identity (1): From definition 4.5 we get $\mathbf{S}(\mathcal{T}) = l^{-1}(\mathbf{M}(\mathcal{T}))$. From the definition of $l$ given in (Def$\downarrow_{\mathcal{T}'}$:$\prec'_m \rightsquigarrow \prec_m$) we get $l = tr \circ l'$. From the definition of $\mathbf{S}$ given in (Def$\downarrow_{\mathcal{T}'}$:$\prec'_m \rightsquigarrow \prec_m$) (remind that $l$ is defined in $\mathbf{S}$ only) we get then $\mathbf{S}(\mathcal{T}) = l^{-1}(\mathbf{M}(\mathcal{T})) = l'^{-1}(tr^{-1}(\mathbf{M}(\mathcal{T}))) \cap l'^{-1}(\mathbf{M}'(\mathcal{T}')) = l'^{-1}(tr^{-1}(\mathbf{M}(\mathcal{T})) \cap \mathbf{M}'(\mathcal{T}')) = l'^{-1}(\mathbf{M}'(\mathcal{T}) \cap \mathbf{M}'(\mathcal{T}'))) = l'^{-1}(\mathbf{M}'(\mathcal{T} \sqcup' \mathcal{T}')) = \mathbf{S}'(\mathcal{T} \sqcup' \mathcal{T}')$.

Proof of identity (2): From definition 4.5 we get $\mathbf{S}_{\prec_m}(\mathcal{T}) = \{\mu_i \in \mathbf{S}(\mathcal{T})$ / for any $\nu_i \in \mathbf{S}(\mathcal{T})$ we have $\nu_i \not\prec_m \mu_i\}$. We know that $\prec_m$ is the restriction to $\mathbf{S}$ of $\prec'_m$. Thus, using also (1), we get: $\mathbf{S}_{\prec_m}(\mathcal{T}) = \{\mu_i \in l'^{-1}(\mathbf{M}'(\mathcal{T} \sqcup' \mathcal{T}'))$ / for any $\nu_i \in l'^{-1}(\mathbf{M}'(\mathcal{T} \sqcup' \mathcal{T}')$ we have $\nu_i \not\prec'_m \mu_i\} = \{\mu_i \in \mathbf{S}$ / $l'(\mu_i) \in \mathbf{M}'(\mathcal{T} \sqcup' \mathcal{T}'))$ and for any $\nu_i \in \mathbf{S}$ such that $l'(\nu_i) \in \mathbf{M}'(\mathcal{T} \sqcup' \mathcal{T}'))$ we have $\nu_i \not\prec'_m \mu_i\} = l'^{-1}(\mathbf{M}'_{\prec'_m}(\mathcal{T} \sqcup' \mathcal{T}'))$.

From definition 4.5 we get also $l'^{-1}(\mathbf{M}'_{\prec'_m}(\mathcal{T} \sqcup' \mathcal{T}')) = \mathbf{S}'_{\prec'_m}(\mathcal{T} \sqcup' \mathcal{T}')$.

Proof of identity (3): From definition 4.5 we get $\mathbf{M}_{\prec_m}(\mathcal{T}) = l(\mathbf{S}_{\prec_m}(\mathcal{T}))$. From the definition of $l$, and from (2), we get then $\mathbf{M}_{\prec_m}(\mathcal{T}) = tr(l'(l'^{-1}(\mathbf{M}'_{\prec'_m}(\mathcal{T} \sqcup' \mathcal{T}')))) = tr(\mathbf{M}'_{\prec'_m}(\mathcal{T} \sqcup' \mathcal{T}'))$.

From remark 4.6 we get $\mathbf{M}(f_{\prec_m}(\mathcal{T})) = TC(\mathbf{M}_{\prec_m}(\mathcal{T}))$. From (3) we get then $\mathbf{M}(f_{\prec_m}(\mathcal{T})) = TC(tr(\mathbf{M}'_{\prec'_m}(\mathcal{T} \sqcup' \mathcal{T}')))$. From remark 3.4-3 we get then $\mathbf{M}(f_{\prec_m}(\mathcal{T})) = tr(TC'(\mathbf{M}'_{\prec'_m}(\mathcal{T} \sqcup' \mathcal{T}')))$. From remark 4.6 again we get $\mathbf{M}(f_{\prec_m}(\mathcal{T})) = tr(\mathbf{M}(f_{\prec'_m}(\mathcal{T} \sqcup' \mathcal{T}'))$. From property 3.5 we get then $\mathbf{M}(f_{\prec_m}(\mathcal{T})) = \mathbf{M}(tr(f_{\prec'_m}(\mathcal{T} \sqcup' \mathcal{T}'))$.

From the definition of $f$, this establishes $\mathbf{M}(f_{\prec_m}(\mathcal{T})) = \mathbf{M}(f(\mathcal{T}))$, i.e. $f = f_{\prec_m}$:

the multi preferential entailment $f_{\prec_m}$ is equal to $f$.

We may define **(Def$\downarrow_{\beta'}$:$\prec'_m \rightsquigarrow \prec_m$)** and **(Def$\downarrow$:$\prec'_m \rightsquigarrow \prec_m$)** by taking respectively $\mathcal{T}' = \{\beta'\}$ and $\mathcal{T}' = \{\emptyset\}$ in definition (Def$\downarrow_{\mathcal{T}'}$:$\prec'_m \rightsquigarrow \prec_m$). □

Notice that, if $\prec'_m = \prec'$ is a preference relation, we get property 4.11 as a particular case of these results: (Def$\downarrow_{\mathcal{T}'}$:$\prec'_m \rightsquigarrow \prec_m$) is simplified into (Def$\downarrow_{\mathcal{T}'}$:$\prec' \rightsquigarrow \prec_m$) as given there, and similarly for (Def$\downarrow_{\beta'}$:$\prec'_m \rightsquigarrow \prec_m$) and (Def$\downarrow$:$\prec'_m \rightsquigarrow \prec_m$).

**Non preservation result 6.22** The property of being a preferential entailment is not preserved by (Def$\downarrow$), even if $V(\mathbf{L}')$ is finite. □

From preservation result 6.21, we already know that if $f'$ is a preferential entailment, $f$ defined from $f'$ by (Def$\downarrow_{\mathcal{T}'}$) is a multi preferential entailment. However, $f$ is not necessarily a preferential entailment, as shown by example 6.11. In fact, any multi preferential entailment $f$ which is not a preferential entailment gives rise to a counter-example (in the finite case, this means any $f$ satisfying (CR)=(CR0) and falsifying (DC)=(DC0)). Indeed, we know from theorem 4.8 that we can express $f$ from some preferential entailment $f'$ by (Def$\downarrow$). □

## 6.4 Properties of a preference relation

Now, we suppose that we start from a preferential entailment $f' = f_{\prec'}$ and that we are in the case of property 4.12: $f$ defined from $f'$ by (Def$\downarrow$), (Def$\downarrow_{\mathcal{T}'}$) or (Def$\downarrow_{\beta'}$) is a preferential entailment. Thus, $f = f_{\prec}$ where $\prec$ is the preference relation defined from $\prec'$ by (Def$\downarrow$:$\prec' \rightsquigarrow \prec$), (Def$\downarrow_{\mathcal{T}'}$:$\prec' \rightsquigarrow \prec$) or (Def$\downarrow_{\mathcal{T}'}$:$\prec' \rightsquigarrow \prec$) in property 4.12.

Then, we examine which properties of $\prec'$ are preserved, i.e. must be satisfied by $\prec$ provided they are satisfied by $\prec'$.

**Preservation result 6.23** (cl) is preserved by (Def$\downarrow_{\mathcal{T}'}$:$\prec'\rightsquigarrow\prec$). □

Proof: We suppose that the preference relation $\prec'$ satisfies (cl) and that $f$, the pre-circumscription defined from $f' = f_{\prec'}$ by (Def$\downarrow_{\mathcal{T}'}$), is a preferential entailment in $\mathbf{L}$. We know from property 4.12 that we have $f = f_{\prec}$ where $\prec$ is a preference relation defined from $\prec'$ by (Def$\downarrow_{\mathcal{T}'}$:$\prec'\rightsquigarrow\prec$). As $\prec'$ satisfies (cl), we get, from (Def$\downarrow_{\mathcal{T}'}$:$\prec'\rightsquigarrow\prec$), $\mathbf{M}(f(\mathcal{T})) = tr(\mathbf{M}'_{\prec'}(\mathcal{T}' \sqcup' \mathcal{T}))$ for any $\mathcal{T} \in \mathbf{T}$ (1). It is easy to show that if $\mu'$ is minimal for $\prec'$ in $\mathbf{M}'(\mathcal{T} \sqcup' \mathcal{T}')$, then $tr(\mu')$ is minimal for $\prec$ in $tr(\mathbf{M}'(\mathcal{T} \sqcup' \mathcal{T}')) = \mathbf{M}(tr(\mathcal{T} \sqcup' \mathcal{T}'))$ (2). Thus, if $\nu \in \mathbf{M}(f(\mathcal{T}))$, from (1) we know that there exists $\nu' \in \mathbf{M}'_{\prec'}(\mathcal{T} \sqcup' \mathcal{T}')$ such that $\nu \in \mathbf{M}_{\prec}(\mathcal{T})$, from which we get $\nu \in \mathbf{M}_{\prec}(\mathcal{T})$ from (2): thus $\prec$ satisfies (cl). □

**Preservation result 6.24** 1. If $\prec'$ satisfies (sf), then $\prec$ defined from $\prec'$ by (Def$\downarrow$:$\prec'\rightsquigarrow\prec$) is transitive and irreflexive.

If moreover $V(\mathbf{L})$ is enumerable, then $\prec$ satisfies (sf).

2. If $\prec'$ satisfies q-(sf), then $\prec$ defined from $\prec'$ by (Def$\downarrow_{\mathcal{T}'}$:$\prec'\rightsquigarrow\prec$) satisfies q-(transitivity + irreflexivity). □

Proof: 1: We suppose that the preference relation $\prec'$ satisfies (sf), and that $\prec$ is a preference relation defined from $\prec'$ by (Def$\downarrow$:$\prec'\rightsquigarrow\prec$) (see property 4.12). We know that $\prec'$ is transitive and irreflexive from property 5.6-2.

We know that $f'$ satisfies (PC) and (CUMU) from property 5.8-1, thus $f$ satisfies (PC) and (CUMU) from preservation results 6.14, 6.15 and 6.16. Using property 5.8-3, we get that $\prec$ is transitive and irreflexive.

If moreover $V(\mathbf{L})$ is enumerable, property 5.9-2 gives that $\prec$ satisfies (sf), but we do not know what is the situation to that respect in the non enumerable case.

2: We suppose now that $\prec'$ is q-(sf) and that $f$ is defined from $f = f_{\prec'}$ by (Def$\downarrow_{\mathcal{T}'}$). We know that $f'$ satisfies (CUMU), thus $f$ satisfies (CUMU) from preservation results 6.14 and 6.15. We may take the set $\mathbf{M}_1 = \{\mu \in \mathbf{M} \ / \ f(Th(\mu)) = Th(\mu)\} = \{\mu \in \mathbf{M} \ / \ \mu \not\prec \mu\}$ as our set $\mathbf{M}_1$ in definition 5.5-8. From (CUMU), we know that we may without real problem "eliminate the interpretations which are outside the set $\mathbf{M}_1$": indeed, for any $\nu \neq \mu$, if $\mu \prec \mu$, we have $\mu \not\prec \nu$ and $\nu \not\prec \mu$ from (Def$\downarrow_{\mathcal{T}'}$:$\prec'\rightsquigarrow\prec$) and (CUMU). The other details of the adaptation of the proof of point 1 are then immediate. □

**Preservation result 6.25** (wf) is preserved by (Def$\downarrow$:$\prec'\rightsquigarrow\prec$), q-(wf) is preserved by (Def$\downarrow_{\mathcal{T}'}$:$\prec'\rightsquigarrow\prec$). □

Proof: We suppose that $\prec'$ satisfies (wf) (thus $\prec'$ satisfies (sf)) and that $\prec$ is defined from $\prec'$ by (Def$\downarrow$:$\prec'\rightsquigarrow\prec$). Then we already know that $\prec$ is transitive and irreflexive from preservation result 6.24. Now, if $\prec$ has an infinitely decreasing chain $\mu_{i+1} \prec \mu_i$, we get that a fortiori $\prec'$ must have such a chain.

Here also, the result for q-(wf) in the more general case of (Def$\downarrow_{\mathcal{T}'}$:$\prec'\rightsquigarrow\prec$) is an immediate consequence of the result for (wf) in the case of (Def$\downarrow$:$\prec'\rightsquigarrow\prec$): again, it suffices to choose the set $\{\mu \ / \ f(Th(\mu)) = Th(\mu)\}$ as our set $\mathbf{M}_1$ in definition 5.5-8. □

**Preservation result 6.26** (rk) is preserved by (Def$\downarrow$:$\prec' \rightsquigarrow \prec$), provided $\prec'$ satisfies (sf): If $\prec'$ satisfies (rk) and (sf), and if $\prec$ is defined from $\prec'$ by (Def$\downarrow$:$\prec' \rightsquigarrow \prec$), then $\prec$ satisfies (rk).

Similarly if $\prec'$ satisfies q-((sf) +(rk)), and if $\prec$ is defined from $\prec'$ by (Def$\downarrow_{\mathcal{T}'}$:$\prec' \rightsquigarrow \prec$), then $\prec$ satisfies q-(rk). □

Proof: We know that $\prec$ is transitive and irreflexive from preservation result 6.24. Let us suppose that we have $\mu_3 \prec \mu_1$. From property 4.12 we get that, for any $\mu'_1 \in \mathbf{M}'_{\prec'}(\mu_1)$, there exists $\mu'_3 \in \mathbf{M}'(\mu_3)$ such that $\mu'_3 \prec' \mu'_1$. Let us suppose that we have also $\mu_3 \not\prec \mu_2$, thus there exists $\mu'_2 \in \mathbf{M}'(\mu_2)$ such that for any $\mu'_3 \in \mathbf{M}'(\mu_3)$ we have $\mu'_3 \not\prec' \mu'_2$. Thus, for any $\mu'_1 \in \mathbf{M}'_{\prec'}(\mu_1)$, there exists $\mu'_3 \in \mathbf{M}'(\mu_3)$ such that we have $\mu'_3 \prec' \mu'_1$ and $\mu'_3 \not\prec' \mu'_2$, thus, from (rk), we get $\mu'_2 \prec' \mu'_1$. This establishes $\mu_2 \prec \mu_1$ from property 4.12: $\prec$ satisfies (rk).

The result with the "quasi" versions is an automatic rewriting of this result, again with $\mathbf{M}_1 = \{\mu \in \mathbf{M} \ / \ f(Th(\mu)) = Th(\mu)\} = \{\mu \in \mathbf{M} \ / \ \mu \not\prec \mu\}$. □

Beware that if $\prec'$ falsifies (sf), we may have that $\prec'$ satisfies (rk) while $\prec$ defined from $\prec'$ by (Def$\downarrow$:$\prec' \rightsquigarrow \prec$) falsifies (rk).

We replace now (rk) by (rkb): even in the presence of (sf), (rkb) is not preserved by (Def$\downarrow$:$\prec' \rightsquigarrow \prec$):

**Non preservation result 6.27** If $\prec'$ satisfies (rkb) then $\prec$ as defined by (Def$\downarrow$:$\prec' \rightsquigarrow \prec$) may falsify (rkb), even if $V(\mathbf{L}')$ is finite, thus even if $\prec'$ satisfies also (sf). □

**Example 6.28** $V(\mathbf{L}) = \{P_1, P_2\}, \mathbf{P}' = \{P'\}, V(\mathbf{L}') = V(\mathbf{L}) \cup \mathbf{P}'$.

$\mu_0 = \emptyset, \mu_1 = \{P_1\}, \mu_2 = \{P_2\}, \mu_3 = \{P_1, P_2\}$ are the four interpretations for $\mathbf{L}$.

We define the following interpretations for $\mathbf{L}'$: $\mu'_i = \mu_i$, $\nu'_i = \mu_i \cup \{P'\}$, for $i \in \{0, 1, 2, 3\}$. We get then all the eight interpretations for $\mathbf{L}'$.

$\prec'$ is the preference relation of $\mathbf{L}'$ defined as follows:

$\mu'_3 \prec' \mu'_1$, $\mu'_3 \prec' \nu'_1$, $\mu'_0 \prec' \mu'_1$, $\mu'_0 \prec' \nu'_1$, $\nu'_0 \prec' \mu'_2$, $\nu'_0 \prec' \nu'_2$ and nothing else.

$\prec'$ satisfies (sf) and (rkb): 1) $\prec'$ is transitive and irreflexive, thus we get (sf) from property 5.6-4. 2) There are two "classes" $E_1 = \{\mu'_0, \mu'_1, \nu'_1, \mu'_3, \nu'_3\}$ and $E_2 = \{\nu'_0, \mu'_2, \nu'_2\}$, such that no element of $E_1$ is connected to an element of $E_2$ and $\prec'$ satisfies (rk) on $E_1$ and on $E_2$. It is easy to see that 1) and 2) imply that $\prec$ satisfies (rkb)[8].

$\prec$ is defined as follows (cf (Def$\downarrow$:$\prec' \rightsquigarrow \prec$) in property 4.12): $\mu_3 \prec \mu_1, \mu_0 \prec \mu_1, \mu_0 \prec \mu_2$ and nothing else.

Then, $\prec$ falsifies (rkb): Indeed we have $\mu_3 \prec \mu_1$, $\mu_0 \prec \mu_1$, $\mu_0 \prec \mu_2$ while $\mu_2 \not\prec \mu_1$ and $\mu_3 \not\prec \mu_2$.

It remains to verify that $f$ is a preferential entailment, i.e. that we have $f = f_\prec$: All we have to do is to check that we get the result of the preferential entailment $f_\prec$ for the theories with more than two models. We get: $f(Th(\{\mu_0, \mu_1, \mu_2\})) =$

[8] This explains the name given to this property: satisfying (rkb) is satisfying (rk) "by blocks" [Moi99].

$Th(\{\mu_0\})$, $f(Th(\{\mu_0,\mu_1,\mu_3\})) = Th(\{\mu_0,\mu_3\})$, $f(Th(\{\mu_0,\mu_2,\mu_3\})) = Th(\{\mu_0,\mu_3\})$, $f(Th(\{\mu_1,\mu_2,\mu_3\})) = Th(\{\mu_2,\mu_3\})$, $f(Th(\{\mu_0,\mu_1,\mu_2,\mu_3\})) = Th(\{\mu_0,\mu_3\})$, in each case we get $f(\mathcal{T}) = f_{\prec}(\mathcal{T})$. □

We give now the similar results for the case where we start from a multi preferential entailment $f'_{\prec'_m}$, getting thus a preferential entailment $f_{\prec_m}$. This means that we examine the preservation results for multi preference relations with (Def$\downarrow_{\mathcal{T}'}$:$\prec'_m \rightsquigarrow \prec_m$), as defined in the proof of preservation result 6.21.

**Preservation result 6.29** (cl), (wf), transitivity, irreflexivity, (sf), (rk) and (rkb) are p-reserved by (Def$\downarrow_{\mathcal{T}'}$:$\prec'_m \rightsquigarrow \prec_m$). □

Proof: The proofs for the properties (wf), (tr), irreflexivity, (rk) and (rkb) are immediate: these properties are always preserved when we restrict the relation to a subset of the original set. This is no longer the case for the properties specific to a preference relation (and not of any binary relation) (cl) and (sf).

In the following proofs, the identity numbers refer to the proof of preservation result 6.21.

Proof for (cl): From identity (3), we get $\mathbf{M}_{\prec_m}(\mathcal{T}) = tr(\mathbf{M}'_{\prec'_m}(\mathcal{T} \sqcup' \mathcal{T}'))$. Thus, we get $TC(\mathbf{M}_{\prec_m}(\mathcal{T})) = TC(tr(\mathbf{M}'_{\prec'_m}(\mathcal{T} \sqcup' \mathcal{T}'))) = tr(TC'(\mathbf{M}'_{\prec'_m}(\mathcal{T} \sqcup' \mathcal{T}')))$ from remark 3.4-3. As $f_{\prec'_m}$ satisfies (cl), we get $TC'(\mathbf{M}'_{\prec'_m}(\mathcal{T} \sqcup' \mathcal{T}')) = \mathbf{M}'(f_{\prec'_m}(\mathcal{T} \sqcup' \mathcal{T}'))$. Thus, we get $\mathbf{M}_{\prec_m}(\mathcal{T}) = tr(\mathbf{M}'(f_{\prec'_m}(\mathcal{T} \sqcup' \mathcal{T}'))) = \mathbf{M}'(tr(f_{\prec'_m}(\mathcal{T} \sqcup' \mathcal{T}')))$ from property 3.5. Using the definition of $f$ we get: $\mathbf{M}_{\prec_m}(\mathcal{T}) = \mathbf{M}'(f(\mathcal{T}))$: $\prec_m$ satisfies (cl).

Proof for (sf): Let $\mu_i$ be some element in $\mathbf{S}(\mathcal{T}) - \mathbf{S}_{\prec_m}(\mathcal{T})$. From identities (1) and (2), we get $\mu_i \in \mathbf{S}'(\mathcal{T} \sqcup' \mathcal{T}')) - \mathbf{S}'_{\prec'_m}(\mathcal{T} \sqcup' \mathcal{T}')$. As $\prec'_m$ satisfies (sf), we get that there exists $\nu_i \in \mathbf{S}'_{\prec'_m}(\mathcal{T} \sqcup' \mathcal{T}') = \mathbf{S}_{\prec_m}(\mathcal{T})$ such that $\nu_i \prec'_m \mu_i$. As $\mu_i$ and $\nu_i$ are in $\mathbf{S}$, we get $\nu_i \prec_m \mu_i$, and $\prec_m$ satisfies (sf).

The present result applies if $\prec'_m = \prec'$ is a preference relation. However, in the yet more particular case where $f = f_{\prec}$ is a preferential entailment, all we show by the present result is that the multi preference relation $\prec_m$ defined from $\prec' = \prec'_m$ in (Def$\downarrow_{\mathcal{T}'}$:$\prec'_m \rightsquigarrow \prec_m$) satisfies indeed (sf). Generally, $\prec_m$ is not a preference relation, even if $f$ is a preferential entailment in $\mathbf{L}$. Thus, the present result does not solve the problem left open in preservation result 6.24: generally, $\prec_m$ is not equal to $\prec$ as defined in (Def$\downarrow_{\mathcal{T}'}$:$\prec' \rightsquigarrow \prec$) because generally $\prec_m$ is not a preference relation. □

Notice that all these results do not have the same importance. For instance the preservation of (cl) has some importance, corresponding to the preservation of (RM) by (Def$\downarrow_{\mathcal{T}'}$). Similarly, the preservation of (sf) corresponds closely to the preservation of (CUMU) by (Def$\downarrow_{\mathcal{T}'}$) and the preservation of (rk) corresponds closely to the preservation of (RatM) by (Def$\downarrow_{\mathcal{T}'}$). However, the preservation of transitivity alone does not seem to have a real significance: it is known that for any multi preference relation $\prec_m$ we may always find some transitive multi preference relation $\prec^1_m$ such that $f_{\prec^1_m} = f_{\prec_m}$: it suffices to introduce enough "useless states" and to complicate accordingly the relation. Similarly, it does not seem that the preservation of (rkb) has another significance than the fact that this property alone for a multi preference relation has no real consequence for the associated multi preferential entailment (a point to be confirmed by a direct investigation, but which seems very likely).

Notice that (reg) is obviously not preserved by $(\mathrm{Def}{\downarrow_{\mathcal{T}'}}{:}{\prec'_m}{\rightsquigarrow}{\prec_m})$, because otherwise this would imply that the notion of preferential entailment is preserved by $(\mathrm{Def}{\downarrow_{\mathcal{T}'}})$.

## 6.5 The case of $(\mathrm{Def}{\downarrow_{\mathcal{T}'}})$ with the singleton property

As already noticed, when the singleton property is satisfied by some particular example of $(\mathrm{Def}{\downarrow_{\mathcal{T}'}})$, we get much more preservation results. We will give only a few indications on this point here, omitting the proofs, which are straightforward, and which would unreasonably augment the size of the present paper.

Thus (cf remark 6.2-1), we suppose here that $f'$ is a pre-circumscription in $\mathbf{L}'$ and that $f$ is defined from $f'$ by $(\mathrm{Def}{\downarrow_{\mathcal{T}'}})$, where $\mathcal{T}'$ is such that the singleton property is satisfied:

For any $\mathcal{T} \subseteq \mathbf{L}$, we have $f(\mathcal{T}) = tr(f'(\mathcal{T} \sqcup' \mathcal{T}')) = f'(\mathcal{T} \sqcup' \mathcal{T}') \cap \mathbf{L}$ and,
for any $\mu \in \mathbf{M}$, $\mathbf{M}'(Th(\mu) \sqcup' \mathcal{T}')$ is a singleton denoted by $\{u'(\mu)\}$.
The one-to one-mapping $u'$ is an homeomorphism between $\mathbf{M}$ and $\mathbf{M}'(\mathcal{T}')$:
$u'$ and $u'^{-1}$ are continuous with respect to the topologies of $\mathbf{M}$ and $\mathbf{M}'(\mathcal{T}')$.

The existence of such a mapping has important consequences on the preservation results. It can be shown that all the properties introduced in definition 5.2 are preserved. Also, all the "unary versions" given in definitions 5.3, such as (RM1), (DC1), are preserved.

The notions of preference relation and of preferential entailment are also preserved, as already seen in remark 6.2, and we have,

for any $\mu, \nu$ in $\mathbf{M}$: $\mu \prec \nu$ iff $u'(\mu) \prec' u'(\nu)$ $\quad(\mathrm{Def}{\downarrow_{\mathcal{T}'}}{:}{\prec'}{\rightsquigarrow}{\prec})$ (singleton case).

For what concerns the properties of a preference relation, using $(\mathrm{Def}{\downarrow_{\mathcal{T}'}}{:}{\prec'}{\rightsquigarrow}{\prec})$ (singleton case), it can be proved that all the properties of a preference relation defined in definition 5.5 are preserved. Similarly, all the properties of a multi preference relation are preserved. Indeed, in the list given in preservation result 6.29, only (reg) was missing, and it is easy to check that (reg) is preserved when we have the singleton property.

Notice however that generally the "zero versions" of the properties given in definitions 5.3 are not preserved. A typical example of this situation is (DC0) with respect to circumscriptions. We will see in remark 7.3 below that any formula circumscription may be defined from an ordinary propositional circumscription in terms of $(\mathrm{Def}{\downarrow_{\mathcal{T}'}})$ with the singleton property. We know that any ordinary propositional circumscription $CIRC(\mathbf{P}, \mathbf{Q}, \mathbf{Z})$ satisfies (DC0) [MR98b, Corollary 4.12], and that a formula circumscription $CIRCF(\Phi, \mathbf{Q}, \mathbf{Z})$ may falsify (DC0) [MR98b, Example 5.18]. Here is an informal description of the problem with the "zero versions". The mapping $u'$ from $\mathbf{M}$ onto $\mathbf{M}'(\mathcal{T}')$ is such that any closed set $\mathbf{M}_1 \subseteq \mathbf{M}$ has an image $u'(\mathbf{M}_1)$ which is closed in $\mathbf{M}'(\mathcal{T}')$, thus, as $\mathbf{M}'(\mathcal{T}')$ is closed in $\mathbf{M}'$, $u'(\mathbf{M}_1)$ is also closed in $\mathbf{M}'$. If we have now an open subset $\mathbf{M}_1 \subseteq \mathbf{M}$, we know that the set $u'(\mathbf{M}_1)$ is open in $\mathbf{M}'(\mathcal{T}')$, but we do not know whether $u'(\mathbf{M}_1)$ is open in $\mathbf{M}'$ or not. Thus, it may exist some formulas $\varphi \in \mathbf{L}$ such that $u'(\mathbf{M}(\varphi))$ is not open and closed in $\mathbf{L}'$, i.e. such that $\varphi$ does not correspond to a formula $\varphi' \in \mathbf{L}'$. This explains why the fact that $f'$ satisfies some property for all the formulas is not enough to ensure that $f$ satisfies this property: we may need to examine some non finitely axiomatizable theory $\mathcal{T}'_1 \in \mathbf{T}'$ in order to recover all the formulas $\varphi \in \mathbf{L}$.

If $\mathcal{T}'$ is finitely axiomatizable, i.e if we are in fact in the case of (Def$\downarrow_{\beta'}$), still with the singleton property, then all the zero versions given in definitions 5.3 are preserved. Indeed, we do not have the problem described above, because $\mathbf{M}'(\beta')$ is open and closed in $\mathbf{M}'$. In particular, this gives another proof of the already known result that, if $\Phi$ is finite, then $CIRCF(\Phi, \mathbf{Q}, \mathbf{Z})$ satisfies (DC0) [MR98b, Proposition 5.17].

Here are two other important **preservation results**, in this case:

The notions of X-mapping and of formula circumscriptions are preserved by (Def$\downarrow_{\mathcal{T}'}$), if the singleton property is satisfied.

We still omit the complete proofs, which are straightforward, but we provide a few indications, by describing how to construct the sets $X$ (from $X'$) and $\Phi$ (from $\Phi'$) in these two cases.

For any formula $\varphi' \in \mathbf{L}'$, let us define the formula $tr_{u'}(\varphi') \in \mathbf{L}$ by $\mathbf{M}(tr_{u'}(\varphi')) = u'^{-1}(\mathbf{M}'(\varphi' \sqcup' \mathcal{T}')) = \{\mu \in \mathbf{M} \;/\; u'(\mu) \in \mathbf{M}'(\mathcal{T}') \cap \mathbf{M}'(\varphi')\}$. As $\mathbf{M}'(\mathcal{T}') \cap \mathbf{M}'(\varphi')$ is open and closed in $\mathbf{M}'(\mathcal{T}')$, and as $u'$ is an homeomorphism, we know that this set is open and closed in $\mathbf{M}$, thus we define a formula in $\mathbf{L}$ in this way. This new definition lies "between the two traces" of $\varphi$ defined in definitions 3.3 and 3.9: we have $tr^*(\varphi) \models tr_{u'}(\varphi) \models tr(\varphi)$, where the two $\models$ may be strict, as shown by the following example.

**Example 6.30** $V(\mathbf{L}) = \{A\}$, $V(\mathbf{L}') = \{A, B\}$ and $\mathcal{T}' = \{\beta'\} = \{\neg B\}$. For any $\mu \in \mathbf{M}$, let us define $u'(\mu) = \mu$. Then, for any $\mu \in \mathbf{M}$, we have $\mathbf{M}'(Th(\mu) \sqcup' \neg B) = \{u'(\mu)\}$: the singleton property holds.
We consider $\varphi' = A \Leftrightarrow B \in \mathbf{L}'$.
Then we have $tr(\varphi') = \top$, $tr^*(\varphi') = \bot$, $tr_{u'}(\varphi') = \neg A$. □

We may now give the following two **constructive preservation results**:
Let $f$ be defined from $f'$ by (Def$\downarrow_{\mathcal{T}'}$), with the singleton property.

1. (a) If $X' \subseteq \mathbf{L}$, if $f'$ is the X-mapping $f_{X'}$, then $f$ is the X-mapping $f_X$, where $X$ is the set defined by $X = tr_{u'}(X') = \{tr_{u'}(\varphi') \;/\; \varphi' \in X'\}$.
   Moreover, if $X' = I_{f'}$ (see in definition 4.7, from property 5.11-1, we know that this means that $X'$ is as above and is its own $\wedge$-closure), then the set $X = tr_{u'}(X')$ is the set $I_f$ (i.e. $X$ is its own $\wedge$-closure).

(b) These two results hold even if we consider the more general notion of X-mapping as introduced in [SF96, MR98a] (cf note 7), which is not necessarily a pre-circumscription. Then $f$ defined from $f'$ by the extension to any mapping from $\mathbf{T}$ to $\mathcal{P}(\mathbf{L})$ of (Def$\downarrow_{\mathcal{T}'}$) (still with the singleton property) is not necessarily a pre-circumscription, but it is still an X-mapping in this full acception, and the sets $X$ and $I_f$ may be obtained from $X'$ or $I_{f'}$ as in (a).

2. If $\Phi' \subseteq \mathbf{L}'$ and if $f' = CIRCF(\Phi', \emptyset, V(\mathbf{L}'))$, then $f$ is a formula circumscription in $\mathbf{L}$. We have $f = CIRCF(\Phi, \emptyset, V(\mathbf{L}))$ where $\Phi = tr_{u'}(\Phi') = \{tr_{u'}(\varphi') \ / \ \varphi' \in \Phi'\}$.

# 7 A few applications of these results

Let us give a few useful applications of some of these results. As these examples deal with (propositional) circumscriptions, let us first remind the definitions involved.

## 7.1 The example of formula circumscription

**Definition 7.1** $V(\mathbf{L}) = \mathbf{P} \cup \mathbf{Q} \cup \mathbf{Z}$. Here and in the following, $\mathbf{P}$, $\mathbf{Q}$ and $\mathbf{Z}$ are disjoint sets[9]. $\mathbf{P}$ is the set of the *circumscribed propositional symbols*, $\mathbf{Z}$ is the set of the *varying* ones, the remaining ones in $\mathbf{Q}$ being *fixed*. We define the preference relation $\prec_{(\mathbf{P},\ \mathbf{Q},\ \mathbf{Z})}$ by:
$\mu \prec_{(\mathbf{P},\ \mathbf{Q},\ \mathbf{Z})} \nu$ if $\mathbf{P} \cap \mu \subset \mathbf{P} \cap \nu$ and $\mathbf{Q} \cap \mu = \mathbf{Q} \cap \nu$ (strict $\subset$, no condition for $\mathbf{Z}$).

The *circumscription* $CIRC(\mathbf{P}, \mathbf{Q}, \mathbf{Z})$, is the preferential entailment $f_{\prec_{(\mathbf{P},\ \mathbf{Q},\ \mathbf{Z})}}$. □

This is the classical (semantical) definition of propositional circumscription, which has been studied in depth in [Sat90], and used by various authors (the present definition may be found e.g. as [LS97, Definitions 3, 4]).

A more general circumscription is often used also [McC86, PM86, Cos98], for which we give the propositional version:

**Definition 7.2** $\Phi$ is a set of formulas in $\mathbf{L}$, $V(\mathbf{L}) = \mathbf{Q} \cup \mathbf{Z}$ (disjoint union), $\mathbf{P}' = \{P'_\varphi\}_{\varphi \in \Phi}$ is a set of distinct propositional symbols not in $\mathbf{L}$. The *formula circumscription* of the formulas $\Phi$, with $\mathbf{Q}$ fixed and $\mathbf{Z}$ varying, is defined as follows, for any $\mathcal{T} \subseteq \mathbf{L}$:

$$CIRCF(\Phi, \mathbf{Q}, \mathbf{Z})(\mathcal{T}) = CIRC(\mathbf{P}', \mathbf{Q}, \mathbf{Z})(\mathcal{T} \cup \{\varphi \Leftrightarrow P'_\varphi\}_{\varphi \in \Phi}) \cap \mathbf{L}.$$

$CIRC$ is defined in the language $\mathbf{L}'$ which is $\mathbf{L}$ augmented by $\mathbf{P}'$: $V(\mathbf{L}') = V(\mathbf{L}) \cup \mathbf{P}'$. □

**Remark 7.3** This provides a widely used example of (Def$\downarrow_{\mathcal{T}'}$). Moreover, this example is particularly interesting because we have here the singleton property introduced in definition 6.1. □

[9] This condition is not absolutely necessary for ordinary or formula circumscription, but it is needed in a few proofs or counter-examples. As this is not a real restriction anyway, it is safer to impose this condition throughout.

Proof: We have here $\mathcal{T}' = \{\varphi \Leftrightarrow P'_\varphi\}_{\varphi \in \Phi}$, thus we get that, for any $\mu \in \mathbf{M}$, there is exactly one $\mu' \in \mathbf{M}'(\mu) \cap \mathbf{M}'(\mathcal{T}')$, namely $\mu' = u'(\mu) = \mu \cup \{P'_\varphi\}_{\mu \models \varphi}$. □

Thus, we may use the results given in subsection 6.5.
Notice that any ordinary circumscription is a formula circumscription:
$CIRC(\mathbf{P}, \mathbf{Q}, \mathbf{Z}) = CIRCF(\mathbf{P}, \mathbf{Q}, \mathbf{Z} \cup \mathbf{P})$.

**Remark 7.4** [folklore] $CIRCF(\Phi, \mathbf{Q}, \mathbf{Z})$ is the preferential entailment $f_\prec$ in $\mathbf{L}$ associated to the preference relation $\prec_{(\Phi;\mathbf{Q},\mathbf{Z})}$ defined by:

$\mu \prec_{(\Phi;\mathbf{Q},\mathbf{Z})} \nu$ if $Th(\mu) \cap \Phi \subset Th(\nu) \cap \Phi$ and $\mathbf{Q} \cap \mu = \mathbf{Q} \cap \nu$. □

**Remarks 7.5** Let us call $\prec'$ the preference relation in $\mathbf{L}'$ associated to $CIRC(\mathbf{P}', \mathbf{Q}, \mathbf{Z})$, i.e. $\prec' = \prec_{(\mathbf{P}',\mathbf{Q},\mathbf{Z})}$ (definition 7.1), and $\prec$ the preference relation in $\mathbf{L}$ associated to $CIRCF(\Phi, \mathbf{Q}, \mathbf{Z})$, i.e. $\prec = \prec_{(\Phi;\mathbf{Q},\mathbf{Z})}$ (remark 7.4).

If we use the preservation results for (Def$\downarrow_{\mathcal{T}'}$) and for (Def$\downarrow_{\mathcal{T}'}$:$\prec' \rightsquigarrow \prec$) of the preceding section, we get the following results about formula circumscription and its associated relation. Notice that most of these results were already known: see e.g. [Poo94, Mak94], or [MR98b] to which we refer the reader for more details and precise references (which may need to combine various texts). However, the method given here is much simpler than previous methods, as we need only to consider the easier ordinary propositional circumscription.

1. (known) $\prec$ satisfies (cl) if $\Phi \cup \mathbf{Q}$ is finite: As we know that $\prec'$ satisfies (cl) iff $\mathbf{P}' \cup \mathbf{Q}$ is finite or $\mathbf{P}' = \emptyset$ [MR98b, Proposition 4.13-2], it suffices to apply preservation result 6.23.

2. (known) A formula circumscription $CIRCF(\Phi, \mathbf{Q}, \mathbf{Z})$ satisfies (RM) as soon as $\Phi \cup \mathbf{Q}$ is finite: either use point 1 together with property 5.9-5, or use preservation result 6.5, and the result stating that an ordinary circumscription $CIRC(\mathbf{P}', \mathbf{Q}', \mathbf{Z}')$ satisfies (RM) iff $\mathbf{P}' \cup \mathbf{Q}'$ is finite or $\mathbf{P}' = \emptyset$ [MR98b, Proposition 4.13-1].

3. (new to our knowledge) Notice that there is no condition on the sets $\mathbf{Q}$ and $\mathbf{Z}$ here.

   If $\Phi$ is finite, then the formula circumscription $CIRCF(\Phi, \mathbf{Q}, \mathbf{Z})$ satisfies (CR$\infty$) (use [MR98b, Proposition 4.19] for ordinary circumscription and preservation result 6.6).

   If $\Phi$ is finite, then the formula circumscription $CIRCF(\Phi, \mathbf{Q}, \mathbf{Z})$ is an X-mapping (use [MR98b, Theorem 6.40] for ordinary circumscription and preservation result 6.19).

4. (known) Any formula circumscription satisfies (CUMU): use the result stating that any ordinary circumscription $CIRC(\mathbf{P}', \mathbf{Q}', \mathbf{Z}')$ satisfies (CUMU) [MR98b, Proposition 4.7], and preservation results 6.14 and 6.15. □

However, we cannot get directly (PC) or (DC) (in some cases) for $CIRCF$ by this general method, and we do not know whether we can prove (sf) for $\prec$ in the non enumerable case by this method or not. Now, we may use the fact that we have the singleton property. This shows that most of the properties of ordinary circumscription are also true for formula circumscription (see subsection 6.5). Thus, we get the following results (which were already known, using other methods which need a specific proof in each case).

**Remarks 7.6** Using the preservation results for (Def$\downarrow_{\mathcal{T}'}$) with the singleton property given in subsection 6.5, and the known results for ordinary circumscription $CIRC(\mathbf{P},\mathbf{Q},\mathbf{Z})$, we get the following results for formula circumscription:

1. A formula circumscription satisfies (PC).
2. The preference relation $\prec_{(\Phi;\mathbf{Q},\mathbf{Z})}$ associated to $CIRCF(\Phi;\mathbf{Q},\mathbf{Z})$ satisfies (sf).
3. If $\Phi \cup \mathbf{Q}$ is finite, then $CIRCF(\Phi;\mathbf{Q},\mathbf{Z})$ satisfies (DC) (using [MR98b, Proposition 4.24]).
4. If $\Phi$ is finite, then $CIRCF(\Phi;\mathbf{Q},\mathbf{Z})$ satisfies (DC0) (already proved as [MR98b, Proposition 5.17]). Here we need only to use the immediate result that any ordinary circumscription satisfies (DC0) [MR98b, Corollary 4.12], and to notice that, when $\Phi$ is finite, we are in the case of (Def$\downarrow_{\beta'}$)). $\square$

As another example of (Def$\downarrow_{\mathcal{T}'}$), let us remind a useful well known result, applied here to propositional circumscription:

**Property 7.7** [dKK89] Any circumscription can be expressed in terms of (Def$\downarrow_{\mathcal{T}'}$) from a circumscription without fixed proposition:

Let us suppose that we have $V(\mathbf{L}) = \mathbf{P}\cup\mathbf{Q}\cup\mathbf{Z}$, with $\mathbf{Q} = \{Q_j\}_{j\in J}$ and that we define a set of new (not in $\mathbf{L}$ and all distinct) propositional symbols $\mathbf{Q}' = \{Q'_j\}_{j\in J}$. For any $\mathcal{T} \subseteq \mathbf{L}$ we get:

$$CIRC(\mathbf{P},\, \mathbf{Q},\, \mathbf{Z})(\mathcal{T}) = CIRC(\mathbf{P}\cup\mathbf{Q}\cup\mathbf{Q}',\, \emptyset,\, \mathbf{Z})(\mathcal{T}\cup\{Q_j \Leftrightarrow \neg Q'_j\}_{j\in J}) \cap \mathbf{L}.$$

The second $CIRC$ is defined in the language $\mathbf{L}'$ such that $V(\mathbf{L}') = V(\mathbf{L})\cup\mathbf{Q}'$. $\square$

Moreover, here again we have the singleton property. Indeed, as $\mathcal{T}' = \{Q_j \Leftrightarrow \neg Q'_j\}_{j\in J}$ here, for any $\mu \in \mathbf{M}$, there is exactly one $\mu' \in \mathbf{M}'(\mu)\cap\mathbf{M}'(\mathcal{T}')$, namely $\mu' = u'(\mu) = \mu\cup\{Q'_j \;/\; \mu \not\models Q_j\}$.

**Remark 7.8** A consequence of this result is that we could have defined the notion of formula circumscription without any fixed proposition. Indeed we get:

$$CIRCF(\Phi,\mathbf{Q},\mathbf{Z}) = CIRCF(\Phi\cup\{Q,\neg Q\}_{Q\in\mathbf{Q}},\emptyset,\mathbf{Q}\cup\mathbf{Z}).$$

Notice however that it is not always a good thing to omit $\mathbf{Q}$ in the notation of a formula circumscription, for instance, this would not allow to distinguish between the two cases given in points 3 and 4 of remarks 7.6. $\square$

As a second example of application of such a study, let us give now a characterization result for finite cumulative multi preferential entailments. Before giving this result, we need to complete an already known result about the characterization of finite formula circumscriptions.

## 7.2 About the characterization of finite formula circumscriptions

**Theorem 7.9** [Fre98, Cos98, MR98b] If $V(\mathbf{L})$ is finite, then a pre-circumscription $f$ is a formula circumscription iff it is a preferential entailment associated to a preference relation $\prec$ which is transitive and irreflexive. □

**Remark 7.10** From properties 5.7-3 and 5.9-1 (in the finite case, (cl) is trivially satisfied), we know that a pre-circumscription $f$ is a preferential entailment $f = f_\prec$ associated to a transitive and irreflexive relation iff it satisfies (CR), (CUMU), (DC) and (PC). □

As theorem 7.9 has independently appeared at least three times as [Fre98, Theorems 13 and 14], [Cos98, Theorem 7] and [MR98b, Proposition 5.24-1], we could omit the proof. However, this result is an interesting application of an "old" and immediate mathematical result, thus we think that it is important to make this clear, which was not done in the referenced texts. [Cos98], in which no constructive definition is given, makes allusion to unprecised mathematical results. We think it is important to make this connection precise, not only for "historical" reasons, but mainly because, as we will see, this will help to provide an interesting constructive method for the characterization result given in theorem 7.16, as explained in examples 7.19 and 7.20 below.

Proof of theorem 7.9: Our description of the proof is longer that it should be, this is to put the similarity of the last two steps in perspective. We hope to convince the reader that this proof is elementary and that it is strange that this result has not appeared before 1998 in the literature about circumscription.

The side "only if" of the proof has already been given as remark 7.4, so we suppose here that we are given a transitive an irreflexive preference relation $\prec$ on $\mathbf{M}$.

The core of this proof is a double application of a result given in [Mac37][10]: any pre-order (i.e. transitive and reflexive) relation $\preceq$ on a set $E$ may be put in correspondence with the relation $\subseteq$ (or by duality $\supseteq$, which is the result that we will use) on a subset of the set of all the sets of $E$. The subsets concerned are very natural, being (when as here we want to get $\supseteq$) the sets of all the elements "following" a given element $e \in E$ for the relation $\preceq$. In the theory of pre-order, any union of such set is sometimes called a filter, the unions of the dual sets, with "preceding elements", being called ideals. To follow closely MacNeille's exposition, it is convenient to introduce a transitive and reflexive relation $\preceq$ such that we have $\mu \prec \nu$ iff $\mu \preceq \nu$ and $\nu \not\preceq \mu$. This is done in "step 0", and this step is necessary if we want to obtain latter one of the smallest possible sets of formulas to circumscribe. The exact formulation of the result used here is as follows: $\preceq$ being a transitive and reflexive relation on a set $E$, we define, for any $e_0 \in E$, the set $s(e_0) = \{e \in E \ / \ e_0 \preceq e\}$. Then we have, for any elements $e_1, e_2$ of $E$,

$$e_1 \preceq e_2 \text{ iff } s(e_2) \subseteq s(e_1) \qquad (MN)$$

[10] This is the oldest reference we know. As we need only the easy finite case, it is possible that older references exist ([Mac37] studies mainly the infinite case, and gives a great number of much deeper results).

This result is given (among other results) in [Mac37, Lemma 11.8 and Theorems 11.6 and 11.9][11].

Step 0, preparation for applying (MN): In order to minimize the size of the set of formulas $\Phi$ obtained directly from $\preceq$ as explained in step 1 below, we must choose a relation $\preceq$ (associated to $\prec$) having a graph as great as possible. The worst possibility to that respect is to choose the (order) relation $\preceq = \preceq_0$ defined by $\mu \preceq_0 \nu$ if $\mu \prec \nu$ or $\mu = \nu$. The best possibility (see remarks 7.12 below) is to choose $\preceq = \preceq_1$ defined by: $\mu \preceq_1 \nu$ if $\mu \prec \nu$ or $(\{\mu' \in \mathbf{M} / \mu \prec \mu'\} = \{\mu' \in \mathbf{M} / \nu \prec \mu'\}$ and $\{\mu' \in \mathbf{M} / \mu' \prec \mu\} = \{\mu' \in \mathbf{M} / \mu' \prec \nu\})$.

Step 1, first application of (MN): We associate to each $\mu \in \mathbf{M}$ the set $s(\mu) = \{\nu \in \mathbf{M} / \mu \preceq \nu\}$. From (MN) we get, for any $\mu, \nu$ in $\mathbf{M}$: $\mu \preceq \nu$ iff $s(\nu) \subseteq s(\mu)$. We get then a set $\mathcal{M}_{\preceq} = \{s(\mu)\}_{\mu \in \mathbf{M}}$ of subsets of $\mathbf{M}$. Now, what we need is a set of formulas instead of a set of subsets of $\mathbf{M}$: For any $\mu \in \mathbf{M}$ we define the formula $\varphi(\mu)$ by $\mathbf{M}(\varphi(\mu)) = s(\mu)$.

Thus we get a set of formulas $\Phi_{\preceq} = \{\varphi(\mu)\}_{\mu \in \mathbf{M}}$, with $\mu \preceq \nu$ iff $\varphi(\nu) \models \varphi(\mu)$. (S1)

Step 2, second application of (MN): We apply (MN) again, starting from the set $\Phi_{\preceq}$ with the transitive and reflexive relation $\models$. For any $\psi \in \Phi_{\preceq}$, we define the set of formulas $\widehat{\varphi}(\psi) = \{\psi' \in \Phi_{\preceq} \ / \ \psi \models \psi'\}$. From (MN), and from the definition of $\Phi_{\preceq}$, we get, for any $\mu, \nu$ in $\mathbf{M}$:

$\varphi(\mu) \models \varphi(\nu)$ iff $\widehat{\varphi}(\varphi(\nu)) \subseteq \widehat{\varphi}(\varphi(\mu))$ (S2)

We get $\varphi(\nu) \in \widehat{\varphi}(\varphi(\mu))$ iff $\varphi(\mu) \models \varphi(\nu)$ from the definition of $\widehat{\varphi}(\varphi(\mu))$, $\varphi(\mu) \models \varphi(\nu)$ iff $\nu \preceq \mu$ from (S1), and $\nu \preceq \mu$ iff $\mu \in s(\nu)$ iff $\mu \models \varphi(\nu)$ from the definitions of $s(\mu)$ and of $\varphi(\mu)$. As we have $\mu \models \varphi(\nu)$ iff $\varphi(\nu) \in Th(\mu)$ from the definition of $Th(\mu)$, we get $\varphi(\nu) \in \widehat{\varphi}(\varphi(\mu))$ iff $\varphi(\nu) \in Th(\mu)$. From the definition of the set $\Phi_{\preceq}$, this means, for any $\mu \in \mathbf{M}$:

$\widehat{\varphi}(\varphi(\mu)) = Th(\mu) \cap \Phi_{\preceq}$. (S3)

From (S1), (S2) and (S3) we get then $\mu \preceq \nu$ iff $Th(\mu) \cap \Phi_{\preceq} \subseteq Th(\nu) \cap \Phi_{\preceq}$.

A fortiori we get: $\mu \prec \nu$ iff $Th(\mu) \cap \Phi_{\preceq} \subset Th(\nu) \cap \Phi_{\preceq}$. From remark 7.4, we know that this proves that we have $f = f_{\prec} = CIRCF(\Phi_{\preceq}, \emptyset, V(\mathbf{L}))$. $\square$

This proof is close to the proofs given in [Fre98, MR98b], but these texts do not consider step 0 and they ignore the similarity between the next two steps.

Step 0 reduces the size of the set $\Phi_{\preceq}$ of formulas to circumscribe, obtained in step 1. Step 1 explains how, starting from a reflexive and transitive preference relation $\preceq$, we may define a set of formulas $\Phi_{\preceq}$ associated to it. Step 2 explains how, starting from any set $\Phi$ of formulas, we get a transitive and reflexive relation $\preceq$ naturally associated to $\Phi$. Let us call $\preceq_{\Phi}$ this relation, defined by $\mu \preceq_{\Phi} \nu$ if $Th(\mu) \cap \Phi \subseteq Th(\nu) \cap \Phi$. Step 2 establishes $\preceq_{\Phi_{\preceq}} = \preceq$.

**Remark 7.11** Remind that $\Phi$ and $\Psi$ have the same closure for $\wedge$ and $\vee$ iff $\preceq_{\Phi} = \preceq_{\Psi}$ [MR98b, Proposition 5.33]. This allows to reduce again the size of the set of formulas to circumscribe.

[11] Notice that [Mac37] examines simultaneously equivalence (MN) and its "dual" form defined as follows: introducing the set $p(e_0) = \{e \in E \ / \ e \preceq e_0\}$, we have also $e_1 \preceq e_2$ iff $p(e_1) \subseteq p(e_2)$.

We also know that we have $\preceq_\Phi = \preceq_\Psi$ iff, for any set of formulas $\Phi_1 \subseteq \mathbf{L}$, we have $CIRCF(\Phi \cup \Phi_1, \emptyset, V(\mathbf{L})) = CIRCF(\Psi \cup \Phi_1, \emptyset, V(\mathbf{L}))$ [MR98b, Proposition 5.33]. □

Let us give here a few additional comments about the size of the set of formulas $\Phi_{\preceq}$ obtained after steps 0 and 1. This is an important matter, as it is a good thing to get a set of formulas $\Phi$ as small as possible. The two relations $\preceq_0$ and $\preceq_1$ introduced in step 0 are respectively the smallest one and the greatest one (see examples 7.13 and 7.14 below). The next remark formalizes this affirmation. Notice also that if we choose $\preceq = \preceq_0$ in step 0, the set $\Phi_{\preceq} = \{\varphi(\mu)\}_{\mu \in \mathbf{M}}$ that we get in step 1 is always made of $card(\mathbf{M})$ different formulas. Thus, any other choice for $\preceq$ in step 0 will give a set $\Phi_{\preceq}$ with at most $card(\mathbf{M})$ formulas.

**Remarks 7.12** Let $\prec$ be some transitive and irreflexive relation on a set $E$, and let us call *associated to* $\prec$ any reflexive and transitive relation $\preceq$ on $E$ such that we have, for any $e_1, e_2$ in $E$:

$$e_1 \prec e_2 \text{ iff } e_1 \preceq e_2 \text{ and } e_2 \not\preceq e_1 \qquad \textbf{(Assoc)}.$$

We denote by $s_r(e)$ and $p_r(e)$ respectively the two sets $s_r(e) = \{e' \in E / e \prec e'\}$ and $p_r(e) = \{e' \in E / e' \prec e\}$[12]. Let us define the following two relations $\preceq_0$ and $\preceq_1$ on $E$:

$e_1 \preceq_0 e_2$ if $e_1 \prec e_2$ or $e_1 = e_2$, and
$e_1 \preceq_1 e_2$ if $e_1 \prec e_2$ or $(s_r(e_1) = s_r(e_2)$ and $p_r(e_1) = p_r(e_2))$.

Then we have:

1. $\preceq_0$ and $\preceq_1$ are associated to $\prec$ and, if $\preceq$ is any relation associated to $\prec$ we have, for any $e_1, e_2$ in $E$: (1) if $e_1 \preceq_0 e_2$ then $e_1 \preceq e_2$ and, (2) if $e_1 \preceq e_2$ then $e_1 \preceq_1 e_2$.

   Thus, any relation $\preceq$ associated to $\prec$ lies somewhere "between $\preceq_0$ and $\preceq_1$".

2. $\preceq_0$ is the only order relation $\preceq$ satisfying (Assoc).

3. Beware that it does not suffice for a transitive and reflexive relation $\preceq$ to be "between $\preceq_0$ and $\preceq_1$" in order to be be associated to $\prec$.

   More precisely, a preference relation $\preceq$ is associated to $\prec$ iff its graph is the reunion of the graphs of $\preceq_0$ and of some universal relations restricted to subsets of $\mathbf{M}$ over which $\preceq_1$ is the universal relation.

4. If $\Phi$ is one of the smallest (in terms of cardinality) sets of formulas having the same closure for $\wedge$ and $\vee$ as $\Phi_{\preceq_1}$, then, for any relation $\preceq$ associated to $\prec$ and for any set $\Psi$ of formulas having the same closure for $\wedge$ and $\vee$ as $\Phi_{\preceq}$, we have $card(\Phi) \leq card(\Psi)$.

5. If $\prec$ is (rk), we get $e_1 \preceq_1 e_2$ iff $e_2 \not\prec e_1$.

   In this case, $\preceq_1$ is also the greatest relation $\preceq$ satisfying (Assoc). □

[12] The index $r$ is here to mean "*r*educed" or "st*r*ict", in order to make a clear difference with the sets $s(e)$ and $p(e)$ introduced in the proof of theorem 7.9, and which are defined from $\preceq$ instead of $\prec$.

Proof: Point 1: It is obvious that $\preceq_0$ is associated to $\prec$ and that any relation $\preceq$ associated to $\prec$ will verify implication (1): Indeed, any such relation $\preceq$ different from $\preceq_0$ will make at least two distinct elements "equivalent" (meaning $\mu \preceq \nu$ and $\nu \preceq \mu$), thus equalizing the formulas $\varphi(\mu)$ and $\varphi(\nu)$, while no distinct "equivalent" elements exist for the relation $\preceq_0$.

It is also obvious that $\preceq_1$ is reflexive. We prove now that $\preceq_1$ is transitive: We suppose $e_1 \preceq_1 e_2$ and $e_2 \preceq_1 e_3$. If we have $e_1 \prec e_2$ and $e_2 \prec e_3$, we get $e_1 \prec e_3$ by the transitivity of $\prec$. If we have $e_1 \prec e_2$ and $s_r(e_2) = s_r(e_3)$ and $p_r(e_2) = p_r(e_3)$, we get $e_1 \prec e_3$ from $p_r(e_2) = p_r(e_3)$. If we have $s_r(e_1) = s_r(e_2)$, $p_r(e_1) = p_r(e_2)$ and $e_2 \prec e_3$, we get $e_1 \prec e_3$ from $s_r(e_1) = s_r(e_2)$. If we have $s_r(e_1) = s_r(e_2)$, $p_r(e_1) = p_r(e_2)$, $s_r(e_2) = s_r(e_3)$ and $p_r(e_2) = p_r(e_3)$, we get $s_r(e_1) = s_r(e_3)$ and $p_r(e_1) = p_r(e_3)$. In any case we get $e_1 \preceq_1 e_3$.

We prove now that $\preceq_1$ is associated to $\prec$. We have $e_1 \preceq_1 e_2$ and $e_2 \not\preceq_1 e_1$ iff we have $[e_1 \prec e_2$ or $(s_r(e_1) = s_r(e_2)$ and $p_r(e_1) = p_r(e_2))]$ and $[e_2 \not\prec e_1$ and not $(s_r(e_1) = s_r(e_2)$ and $p_r(e_1) = p_r(e_2))]$. As $\prec$ is irreflexive and transitive, we have that if $e_1 \prec e_2$ then $e_2 \not\prec e_1$, thus we get: $e_1 \preceq_1 e_2$ and $e_2 \not\preceq_1 e_1$ iff $[e_1 \prec e_2]$ and $[s_r(e_1) \neq s_r(e_2)$ or $p_r(e_1) \neq p_r(e_2)]$. Now, if $e_1 \prec e_2$ we have $e_1 \in p_r(e_2)$, $e_2 \in s_r(e_1)$ and, as $e_1 \notin p_r(e_1)$ and $e_2 \notin s_r(e_2)$, we get $p_r(e_1) \neq p_r(e_2)$ and $s_r(e_1) \neq s_r(e_2)$. This gives $e_1 \preceq_1 e_2$ and $e_2 \not\preceq_1 e_1$ iff $e_1 \prec e_2$, as wanted.

We prove now that implication (2) holds[13]. Let us suppose that $\preceq$ is some relation associated to $\prec$ and that we have $e_1 \preceq e_2$. Then, we have $e_1 \prec e_2$ or $e_2 \preceq e_1$ from the "if" side of (Assoc). If $e_1 \prec e_2$, we get $e_1 \preceq_1 e_2$ from the definition of $\preceq_1$, so, we suppose now that we have $e_1 \not\prec e_2$.

Thus we have $e_2 \preceq e_1$, and still $e_1 \preceq e_2$.

• Let us suppose that we have $e \in p_r(e_1)$, i.e. $e \prec e_1$. Then we get $e \preceq e_1$ from the "only if" side of (Assoc), thus $e \preceq e_2$ by transitivity. Thus we get $e \prec e_2$ or $e_2 \preceq e$ from the "if" side of (Assoc). If we have $e_2 \preceq e$, we get $e_1 \preceq e$ by transitivity, thus $e \not\prec e_1$ from the "only if" side of (Assoc), a contradiction. Thus we must have $e \prec e_2$, i.e. $e \in p_r(e_2)$. Thus, we get $p_r(e_1) \subseteq p_r(e_2)$.

• Let us suppose that we have $e \in s_r(e_2)$, i.e. $e_2 \prec e$. By similar arguments, we get $e_1 \prec e$, i.e. $e \in s_r(e_1)$: $s_r(e_2) \subseteq s_r(e_1)$.

• As our hypothesis is symmetrical, we get, exchanging the roles of $e_1$ and $e_2$ in above proof: $p_r(e_2) \subseteq p_r(e_1)$ and $s_r(e_1) \subseteq s_r(e_2)$.

We get then $p_r(e_1) = p_r(e_2)$ and $s_r(e_1) = s_r(e_2)$, thus $e_1 \preceq_1 e_2$.

Point 2: If $\preceq$ is an order relation satisfying (Assoc), we get that $\mu \preceq \nu$ and $\nu \preceq \mu$ implies $\mu = \nu$. As moreover we know from point 1 that $\mu \preceq_0 \nu$ implies $\mu \preceq \nu$, we get $\preceq = \preceq_0$.

Point 3: Let $\mathbf{M}$ be $\{\mu_1, \mu_2\}$ and $\prec$ be the relation with an empty graph. Then $\preceq_0$ is equality and $\preceq_1$ is the universal relation in $\mathbf{M}$. We define $\preceq$ by $\mu \preceq \nu$ iff $\mu \preceq_0 \nu$ or $\mu = \mu_1$ and $\nu = \mu_2$. Then, $\preceq$ is transitive and reflexive, and (as any reflexive relation in this example) it lies "between $\preceq_0$ and $\preceq_1$. However $\preceq$ falsifies (Assoc): $\mu_1 \preceq \mu_2$ and $\mu_2 \not\preceq \mu_1$ while $\mu_1 \not\prec \mu_2$.

[13] This part of the proof is due to Éric Badouel.

Let now $\preceq$ be some transitive and reflexive relation satisfying (Assoc). Let us suppose that we have $\mu \preceq \nu$ and $\mu \not\preceq_0 \nu$. Then, we must have $\nu \preceq \mu$ from (Assoc). Also, we must have $\mu \preceq_1 \nu$, thus (from (Assoc) again, for $\preceq_1$) $\nu \preceq_1 \mu$. Let us suppose that we have $\mu_1 \preceq \mu_2$, $\mu_2 \preceq \mu_3$, $\mu_1 \not\preceq_0 \mu_2$ and $\mu_2 \not\preceq_0 \mu_3$. Then we get $\mu_1 \preceq \mu_3$ by transitivity, and as above: $\mu_1 \preceq_1 \mu_2$ and $\mu_2 \preceq_1 \mu_3$, $\mu_2 \preceq_1 \mu_1$ and $\mu_3 \preceq_1 \mu_2$: $\preceq_1$ restricted to $\{\mu_1, \mu_2, \mu_3\}$ is the universal relation (again $\preceq_1$ is transitive and reflexive). The same reasoning applies for any subset (even infinite) $\{\mu_i\}_{i \in I}$ of $\mathbf{M}$ which is connected for $\preceq$ and such that $\mu_i \not\preceq_0 \mu_j$ as soon as $i \neq j$ are in $I$.

Thus, we have established that the graph of $\preceq$ is equal to the union of the graph of $\preceq_0$ and of the graphs of the universal relation on some subsets of $\mathbf{M}$ over which the restriction of $\preceq_1$ is universal. Notice that the fact that we know that $\mu \preceq \nu$ implies $\mu \preceq_1 \nu$ is crucial in this proof.

Conversely now, let us suppose that $\prec$ is some irreflexive and transitive relation and that $\preceq$ is some relation which has for graph the union of the graph of $\preceq_0$ and of the graphs of the universal relation on some subsets of $\mathbf{M}$ over which the restriction of $\preceq_1$ is universal.

$\preceq$ is clearly reflexive.

Let us suppose now that we have $\mu_1 \preceq \mu_2$ and $\mu_2 \preceq \mu_3$, and that these three interpretations are all distinct, thus $\mu_i \preceq_0 \mu_j$ iff $\mu_i \prec \mu_j$ for any distinct $i, j$ in $\{1, 2, 3\}$. If $\mu_1 \preceq_0 \mu_2$ and $\mu_2 \preceq_0 \mu_3$, then $\mu_1 \preceq_0 \mu_3$ thus $\mu_1 \preceq \mu_3$. If $\mu_1 \preceq_0 \mu_2$ and $\mu_2 \not\preceq_0 \mu_3$ then $\mu_2 \preceq_1 \mu_3$ and $\mu_3 \preceq_1 \mu_2$ thus $p_r(\mu_2) = p_r(\mu_3)$ and $\mu_1 \prec \mu_3$, thus $\mu_1 \preceq \mu_3$. Similarly, if $\mu_1 \not\preceq_0 \mu_2$ and $\mu_2 \preceq_0 \mu_3$ then $s_r(\mu_1) = s_r(\mu_2)$ and $\mu_1 \prec \mu_3$, thus $\mu_1 \preceq \mu_3$. In any case $\mu_1 \preceq \mu_3$, which establishes the transitivity of $\preceq$.

Let us suppose now that we have $\mu \preceq \nu$ and $\nu \not\preceq \mu$. Then we know that $\preceq$ is not universal on the set $\{\mu, \nu\}$, thus we get $\mu \preceq_0 \nu$, and also $\mu \neq \nu$, thus $\mu \prec \nu$. Let us suppose now that we have $\mu \prec \nu$. Then we get clearly $\mu \preceq \nu$. Let us suppose that we have also $\nu \preceq \mu$, then $\preceq$ is universal on the set $\{\mu, \nu\}$, thus, $\preceq_1$ is also universal on the set $\{\mu, \nu\}$. As $\prec$ is a strict order, we have $\nu \not\prec \mu$ thus, from $\nu \preceq_1 \mu$ we get $s_r(\mu) = s_r(\nu)$, a contradiction with $\nu \in s_r(\mu)$ and $\nu \notin s_r(\nu)$. Thus we get $\mu \preceq \nu$ and $\nu \not\preceq \mu$. We have established that $\preceq$ satisfies (Assoc).

<u>Point 4</u>: The proof of this part is postponed to a forthcoming paper. Indeed, it is rather technical and it lies outside the scope of this text. Notice that this is the only exception to the last sentence of the introduction (section 1). We give this result here because it has an important consequence explained just after the end of the proof of the present remarks.

<u>Point 5</u>: We suppose here that $\prec$ is (rk), i.e. that $\prec$ is transitive and irreflexive, and that $\not\prec$ is transitive. Then, the relation $\preceq_2$ defined on $E$ by $e_1 \preceq_2 e_2$ if $e_2 \not\prec e_1$ is transitive (notice that $\preceq_2$ is always reflexive from the irreflexivity of $\prec$). We have also $e_1 \preceq_2 e_2$ and $e_2 \not\preceq e_1$ iff $e_2 \not\prec e_1$ and $e_1 \prec e_2$, which is equivalent to $e_1 \prec e_2$ because $\prec$ is transitive and irreflexive. This shows that $\preceq_2$ is a relation associated to $\prec$. Now, $\preceq_2$ is clearly the greatest relation associated to $\prec$ possible. Indeed, let us say that a relation $\preceq$ is ***associated to*** $\prec$ ***in the weak meaning*** if we drop the condition that $\preceq$ must be transitive (and reflexive), keeping only condition (Assoc) in the definition of "relation associated to $\prec$" given in remark 7.12. In

our proof that $\preceq_2$ satisfies condition (Assoc) given just above, we did not use the transitivity of $\not\prec$. Moreover, if $\preceq$ is some relation associated to $\prec$ in the weak meaning, we have that $e_2 \preceq e_1$ implies $e_1 \not\prec e_2$. Thus, $\preceq_2$ is always the greatest possible relation associated to $\prec$ in the weak meaning (thus it is reflexive). A fortiori, when, as here, $\preceq_2$ is transitive, then it is associated to $\prec$ and it is the greatest possible relation associated to $\prec$. Thus we get $\preceq_2 = \preceq_1$.

We have established that, if $\prec$ is (rk), then we have: $e \preceq_1 e_2$ iff $e_2 \not\prec e_1$. Notice that the converse is obviously true.

Remind that, when $\prec$ is (rk), there exists a finite (we are in the finite case here) subset $G = \{1, \cdots g\}$ of $\mathbb{N}$, and a surjective mapping $r$ from $\mathbf{M}$ to $G$ such that we have, for any $\mu, \nu$ in $\mathbf{M}$, $\mu \prec \nu$ iff $r(\mu) < r(\nu)$ [LM92]. Thus, we get $\mu \preceq_1 \nu$ iff $r(\mu) \leq r(\nu)$. In step 1 of the proof of theorem 7.9, we get then $s(\mu) = \{\nu \in \mathbf{M} \ / \ r(\mu) \leq r(\nu)\}$, thus we have exactly $g$ different sets, and the set $\Phi_{\preceq_1}$ contains exactly $g$ distinct formulas $\varphi(i)$ $(i \in G)$, defined by $\mathbf{M}(\varphi(i)) = \{\nu \in \mathbf{M}, i \leq r(\nu)\}$, with $\mathbf{M}(\varphi(1)) = \mathbf{M}$, i.e. $\varphi(1) = \top$. We have clearly, for any $i < j$ in $G$, $\mathbf{M}(\varphi(j)) \subset \mathbf{M}(\varphi(i))$, i.e. $\varphi(j) \models \varphi(i)$ and $\varphi(j) \neq \varphi(i)$. Conversely, the construction made in step 2 of the proof of theorem 7.9 (built on (MN)) shows that if we start from a set $\Phi$ of formulas having this property of being linearly ordered by $\models$, then the relation $\prec_{(\Phi;\emptyset,V(\mathbf{L}))}$ is (rk). Notice that [Fre98, Theorems 19 and 20] give a similar result. $\square$

Steps 0 and 1 in the proof of theorem 7.9 provide a very natural mapping from the initial preference relation $\prec$ to a set $\Phi_{\preceq}$ of formulas to circumscribe, with the equivalence $\mu \prec \nu$ iff $\varphi(\nu) \models_{\neq} \varphi(\mu)$. The advantage of this method, inspired by old and easy results about pre-ordered sets, over the previous proofs given in [Fre98, MR98b], is that, when $\preceq_1$ is chosen at step 0, it provides the smallest possible set than can be obtained by any method giving as a result a set of formulas $\varphi(\mu)$. Moreover, remarks 7.11 and 7.12 prove that to get one of the smallest possible sets $\Phi$ (in terms of cardinality) such that $f = CIRCF(\Phi, \emptyset, V(\mathbf{L}))$, it suffices to find one of the smallest sets $\Phi$ having the same closure for $\wedge$ et $\vee$ than the set $\Phi_{\preceq_1}$.

Let us provide now three simple examples of application of theorem 7.9 and its constructive proof. We use the notations introduced in this proof. These examples illustrate the fact that, in any case where these sets are different, the set $\Phi_{\preceq_1}$ has less elements than $\Phi_{\preceq_0}$.

**Example 7.13** $V(\mathbf{L}) = \{A, B\}$, $\mathbf{M} = \{\mu_i\}_{i \in \{1,2,3,4\}}$ with $\mu_1 = \emptyset$, $\mu_2 = \{A\}$, $\mu_3 = \{B\}$, $\mu_4 = \{A, B\}$.

We define $\prec$ by: $\mu_1 \prec \mu_2$, $\mu_1 \prec \mu_3$, $\mu_2 \prec \mu_4$, $\mu_3 \prec \mu_4$ $\mu_1 \prec \mu_4$ and nothing else.

$\prec$ is transitive and irreflexive.

From step 0 of the proof, we get:

$\preceq_0$ is defined by: $\mu_i \preceq_0 \mu_j$ if $\mu_i \prec \mu_j$ or $\mu_i = \mu_j$.

$\preceq_1$ is defined by: $\mu_i \preceq_1 \mu_j$ if $\mu_i \preceq_0 \mu_j$ or $\{i, j\} = \{2, 3\}$.

We examine step 1 now.

1. Here is what we get if we have chosen $\preceq = \preceq_0$ in step 0:

   $s(\mu_1) = \{\mu_1, \mu_2, \mu_3, \mu_4\}$, $s(\mu_2) = \{\mu_2, \mu_4\}$, $s(\mu_3) = \{\mu_3, \mu_4\}$, $s(\mu_4) = \{\mu_4\}$.

   This gives $\Phi_0 = \Phi_{\preceq_0} = \{\varphi(\mu_i)\}_{i \in \{1,2,3,4\}}$ with: $\varphi(\mu_1) = \top$, $\varphi(\mu_2) = A$, $\varphi(\mu_3) = B$, $\varphi(\mu_4) = A \wedge B$.

   We may reduce this set to $\Psi_0 = \{\varphi(\mu_i)\}_{i \in \{2,3\}}$ because the two sets have the same "closure for $\wedge$ and $\vee$" (see remark 7.11), namely $\{\top,\ A \vee B,\ A,\ B,\ A \wedge B,\ \perp\}$.

2. Here is what we get if we have chosen $\preceq = \preceq_1$ in step 0:

   $s(\mu_1) = \{\mu_1, \mu_2, \mu_3, \mu_4\}$, $s(\mu_2) = s(\mu_3) = \{\mu_2, \mu_3, \mu_4\}$, $s(\mu_4) = \{\mu_4\}$.

   This gives $\Phi_1 = \Phi_{\preceq_1} = \{\varphi(\mu_i)\}_{i \in \{1,2,4\}}$ with: $\varphi(\mu_1) = \top$, $\varphi(\mu_2) = A \vee B$, $\varphi(\mu_4) = A \wedge B$.

   Again, this set may be reduced, using the property of the closure for $\wedge$ and $\vee$, to $\Psi_1 = \{\varphi(\mu_i)\}_{i \in \{2,4\}}$.

As this example is very small, we are in a particular case where the advantage of choosing $\preceq_1$ instead of $\preceq_0$, thus getting a set $\Phi_1 = \Phi_{\preceq_1}$ smaller than the set $\Phi_0 = \Phi_{\preceq_0}$, disappears when we reduce the set $\Phi_i$ by using the property of the closure for $\wedge$ and $\vee$.

We have, in this example, $f_\prec = CIRCF(\Phi_0, \emptyset, \{A, B\}) = CIRCF(\Phi_1, \emptyset, \{A, B\}) = CIRCF(\Psi_0, \emptyset, \{A, B\}) = CIRCF(\Psi_1, \emptyset, \{A, B\})$. Also (see remark 7.11), we have, for any set $\Psi \subseteq \mathbf{L}$, $CIRCF(\Phi_i \cup \Psi, \emptyset, \{A, B\}) = CIRCF(\Psi_i \cup \Psi, \emptyset, \{A, B\})$ for $i \in \{1, 2\}$, but we have $CIRCF(\Phi_0 \cup \{A\}, \emptyset, \{A, B\}) \neq CIRCF(\Phi_1 \cup \{A\}, \emptyset, \{A, B\})$. $\square$

As this example was too small to be really interesting, let us provide another example, in which even after reducing the sets thanks to the property of the closure for $\wedge$ and $\vee$, we still get a smaller set when starting from $\preceq_1$ instead of $\preceq_0$.

**Example 7.14** $V(\mathbf{L}) = \{A, B, C\}$, $\mathbf{M} = \{\mu_i\}_{i \in \{1, \cdots, 8\}}$ with $\mu_1 = \emptyset$, $\mu_2 = \{A\}$, $\mu_3 = \{B\}$, $\mu_4 = \{A, B\}$, $\mu_5 = \{A, B, C\}$, $\mu_6 = \{C\}$, $\mu_7 = \{A, C\}$, $\mu_8 = \{B, C\}$.

We define $\prec$ by: for any $i \in \{2, 3, 4\}$, $\mu_1 \prec \mu_i$ and $\mu_i \prec \mu_5$, $\mu_1 \prec \mu_5$ and nothing else.
$\prec$ is transitive and irreflexive.
From step 0 of the proof, we get:
$\preceq_0$ is defined by: $\mu_i \preceq_0 \mu_j$ if $\mu_i \prec \mu_j$ or $\mu_i = \mu_j$.
$\preceq_1$ is defined by: $\mu_i \preceq_1 \mu_j$ if $\mu_i \preceq_0 \mu_j$ or $\{i, j\} \subseteq \{2, 3, 4\}$.

We examine step 1 now.

1. Here is what we get if we have chosen $\preceq = \preceq_0$ in step 0:

   $s(\mu_1) = \{\mu_1, \mu_2, \mu_3, \mu_4, \mu_5\}$, $s(\mu_2) = \{\mu_2, \mu_5\}$, $s(\mu_3) = \{\mu_3, \mu_5\}$, $s(\mu_4) = \{\mu_4, \mu_5\}$ and, if $i \in \{5, 6, 7, 8\}$, $s(\mu_i) = \{\mu_i\}$.

This gives $\Phi_0 = \Phi_{\preceq_0} = \{\varphi(\mu_i)\}_{i\in\{1,\cdots 8\}}$ with: $\varphi(\mu_1) = (A\wedge B)\vee\neg C$, $\varphi(\mu_2) = A\wedge(B \Leftrightarrow C)$, $\varphi(\mu_3) = B \wedge (A \Leftrightarrow C)$, $\varphi(\mu_4) = A \wedge B$, $\varphi(\mu_5) = A \wedge B \wedge C$, $\varphi(\mu_6) = \neg A \wedge \neg B \wedge C$, $\varphi(\mu_7) = A \wedge \neg B \wedge C$, $\varphi(\mu_8) = \neg A \wedge B \wedge C$,

We may reduce this set to $\Psi_0 = \{\varphi(\mu_i)\}_{i\in\{1,\cdots,8\},i\neq 5}$ because the two sets have the same closure for $\wedge$ and $\vee$ (indeed $\varphi(\mu_5) = \varphi(\mu_2) \wedge \varphi(\mu_3)$) and we cannot get a smaller set in this way.

2. Here is what we get if we have chosen $\preceq=\preceq_1$ in step 0:

   $s(\mu_1) = \{\mu_1, \mu_2, \mu_3, \mu_4, \mu_5\}$, $s(\mu_2) = s(\mu_3) = s(\mu_4) = \{\mu_2, \mu_3, \mu_4, \mu_5\}$, $s(\mu_5) = \{\mu_5\}$ and, $s(\mu_6) = s(\mu_7) = s(\mu_8) = \{\mu_6, \mu_7, \mu_8\}$.

   This gives $\Phi_1 = \Phi_{\preceq_1} = \{\varphi(\mu_i)\}_{i\in\{1,2,5,6\}}$ with:

   $\varphi(\mu_1) = (A \wedge B) \vee \neg C$, $\varphi(\mu_2) = (A \vee B) \wedge ((A \wedge B) \vee \neg C)$, $\varphi(\mu_5) = A \wedge B \wedge C$, $\varphi(\mu_6) = (\neg A \vee \neg B) \wedge C$.

   Here, this set cannot be reduced by using the property of the closure for $\wedge$ and $\vee$. Thus, we get directly here a set smaller than the best reduced set obtained when we started from $\preceq_0$. From remarks 7.11 and 7.12-4, together with the proof of theorem 7.9, we know that the set $\Phi_1$ has the smallest possible cardinal such that $f_\prec = CIRCF(\Phi_1, \emptyset, V(\mathbf{L}))$, a fact that can also be checked directly here, as the example is small enough. □

The next example shows what happens when $\prec$ is (rk).

**Example 7.15** $V(\mathbf{L})$ and $\mathbf{M} = \{\mu_i\}_{i\in\{1,2,3,4\}}$ as in example 7.13.
We define $\prec$ by: $\mu_3 \prec \mu_1$, $\mu_3 \prec \mu_2$, $\mu_4 \prec \mu_1$, $\mu_4 \prec \mu_2$ $\mu_4 \prec \mu_3$ and nothing else.
$\prec$ is transitive and irreflexive. Moreover $\not\prec$ is also transitive, thus $\prec$ is (rk).
From step 0 of the proof, we get:
$\preceq_0$ is defined by: $\mu_i \preceq_0 \mu_j$ if $\mu_i \prec \mu_j$ or $\mu_i = \mu_j$.
$\preceq_1$ is defined by: $\mu_i \preceq_1 \mu_j$ if $\mu_i \preceq_1 \mu_j$ or $\{i, j\} = \{1, 2\}$, thus we get $\mu_i \preceq_1 \mu_j$ iff $\mu_j \not\prec \mu_i$, for any $i, j$ in $\{1, 2, 3, 4\}$.

We examine step 1 now.

1. Here is what we get if we have chosen $\preceq=\preceq_0$ in step 0:

   $s(\mu_1) = \{\mu_1\}$, $s(\mu_2) = \{\mu_2\}$, $s(\mu_3) = \{\mu_1, \mu_2, \mu_3\}$, $s(\mu_4) = \{\mu_1, \mu_2, \mu_3, \mu_4\}$.

   This gives $\Phi_0 = \Phi_{\preceq_0} = \{\varphi(\mu_i)\}_{i\in\{1,2,3,4\}}$ with: $\varphi(\mu_1) = \neg A \wedge \neg B$, $\varphi(\mu_2) = A \wedge \neg B$, $\varphi(\mu_3) = \neg A \vee \neg B$, $\varphi(\mu_4) = \top$.

   We may reduce this set to $\Psi_0 = \{\varphi(\mu_i)\}_{i\in\{1,2,3\}}$ because the two sets have the same closure for $\wedge$ and $\vee$, namely $\{\top, \neg A \vee \neg B, \neg B, \neg A \wedge \neg B.A \wedge \neg B, \bot\}$.

2. Here is what we get if we have chosen $\preceq = \preceq_1$ in step 0:

   $s(\mu_1) = s(\mu_2) = \{\mu_1, \mu_2\}$, $s(\mu_3) = \{\mu_1, \mu_2, \mu_3\}$, $s(\mu_4) = \{\mu_1, \mu_2, \mu_3, \mu_4\}$.

   This gives $\Phi_1 = \Phi_{\preceq_1} = \{\varphi(\mu_i)\}_{i \in \{1,3,4\}}$ with: $\varphi(\mu_1) = \neg B$, $\varphi(\mu_3) = \neg A \vee \neg B$, $\varphi(\mu_4) = \top$.

   We may reduce this set to $\Psi_1 = \{\varphi(\mu_i)\}_{i \in \{1,3\}}$ because the two sets have the same closure for $\wedge$ and $\vee$, namely $\{\top, \neg A \vee \neg B, \neg B, \bot\}$.

   Remind that in this case, we always get that the set $\Phi_{\preceq_1}$ is totally ordered by $\models$, thus the reduction by the property of the closure for $\wedge$ and $\vee$ can never do better than eliminating the formula $\top$ (from the definition of $\Phi_{\preceq_1}$, it is obvious that we have $\bot \notin \Phi_{\preceq_1}$ in any case, while we have $\top \in \Phi_{\preceq_1}$ as soon as $\prec$ is (rk)).

   Notice that an immediate consequence of remarks 7.11 and 7.12, together with the proof of theorem 7.9, is that each time the relation $\prec$ is (rk), the set $\Psi_1 = \Phi_{\preceq_1} - \{\top\}$ is always one of the sets with the smallest possible cardinal such that $f_\prec = CIRCF(\Psi_1, \emptyset, V(\mathbf{L}))$. This fact can also be checked directly here, as the example is small enough.

We have, in this example $f_\prec = CIRCF(\Phi_0, \emptyset, \{A, B\}) = CIRCF(\Phi_1, \emptyset, \{A, B\}) = CIRCF(\Psi_0, \emptyset, \{A, B\}) = CIRCF(\Psi_1, \emptyset, \{A, B\})$. Also, from remark 7.11, we have, for any set $\Psi \subseteq \mathbf{L}$, $CIRCF(\Phi_i \cup \Psi, \emptyset, \{A, B\}) = CIRCF(\Psi_i \cup \Psi, \emptyset, \{A, B\})$ for $i \in \{1, 2\}$, but we have $CIRCF(\Phi_0 \cup \{A \wedge \neg B\}, \emptyset, \{A, B\}) \neq CIRCF(\Phi_1 \cup \{A \wedge \neg B\}, \emptyset, \{A, B\})$. □

We are now in position to give the announced characterization result.

## 7.3 A characterization of finite cumulative multi preferential entailments

**Theorem 7.16** If $V(\mathbf{L})$ is finite, then a pre-circumscription $f$ in $\mathbf{L}$ satisfies (CR) and (CUMU) iff it can be expressed in terms of (Def$\downarrow_{\beta'}$) from an ordinary circumscription:

There exist a finite language $\mathbf{L}'$ containing $\mathbf{L}$, two disjoint sets $\mathbf{P}', \mathbf{Z}'$ such that $\mathbf{P}' \cup \mathbf{Z}' = V(\mathbf{L}')$, and a formula $\beta'$ in $\mathbf{L}'$ such that, for any theory $\mathcal{T}$ of $\mathbf{L}$, we have,

$$f(\mathcal{T}) = CIRC(\mathbf{P}', \emptyset, \mathbf{Z}')(\mathcal{T} \sqcup' \beta') \cap \mathbf{L}. \quad \square$$

Remind the other characterization result of these pre-circumscriptions already given in property 5.12-2: they correspond to X-mappings as defined in definition 4.7-2.

Proof: As we are in the finite case ($V(\mathbf{L})$ finite), remind that (CR) is (CR0) or (RM) and that (RM) implies (CT). Also we know from property 5.9 (points 4 and 5) that a pre-circumscription $f$ satisfies (CR) and (CUMU) iff it is a finite cumulative multi preferential entailment, i.e. iff it is a finite multi preferential entailment defined by a relation satisfying (sf). From property 5.6-3 we know that this is equivalent to say that $f$ is a finite multi preferential entailment defined by a relation which is transitive and irreflexive.

"only if", first part: This is the "hard part" in this proof, and it comes mainly from [Cos98, Theorem 15]. This first part gives rise to a formula circumscription. We give the proof in our terms.

Thus, we may suppose that $f = f_{\prec_m}$ is a finite multi preferential entailment where the multi preference relation $\prec_m$ is defined thanks to $\mathbf{S}$ and $l$ (see definition 4.5), and is transitive and irreflexive on the finite set $\mathbf{S}$. From theorem 4.8, we know that there exists a sufficiently large (but finite) language $\mathbf{L}'_1$, with $V(\mathbf{L}'_1) = \mathbf{Z}'_1 \supseteq V(\mathbf{L})$ such that we may express $f$ thanks to a preferential entailment $f_{\prec'_1}$ using (Def$\downarrow$:$\prec_m \rightsquigarrow \prec$)[14]. Now, we must slightly refine the method used in the proof of theorem 4.8 because the preference relation $\prec'$ obtained by ($\uparrow$Def:$\prec_m \rightsquigarrow \prec'$) does not corresponds directly to a formula circumscription. Following Costello, we consider the set $m(\mathbf{S}) \subseteq V(\mathbf{L}'_1)$. As $V(\mathbf{L}'_1)$ is finite, $m(\mathbf{S})$ is the set of all the models of some formula $\beta'_1$ in $\mathbf{L}'_1$. Now, as $\prec_m$ is irreflexive and transitive, we know that $\prec'_1$ as defined from $\prec_m$ by (Def$\downarrow$:$\prec_m \rightsquigarrow \prec$) is transitive and irreflexive inside $m(\mathbf{S})$. What we consider as our preference relation $\prec'_1$ on $\mathbf{M}'_1$ is this relation only, i.e.:

For any $\mu'_1, \nu'_1$ in $\mathbf{M}'_1$, $\mu'_1 \prec'_1 \nu'_1$ iff there exists $\mu_1 \in m^{-1}(\mu'_1)$ and $\nu_1 \in m^{-1}(\nu'_1)$ such that $\mu_1 \prec_m \nu_1$.

This is (Def$\downarrow$:$\prec_m \rightsquigarrow \prec$) as it appears in the proof of theorem 4.8. The mapping $m$ is also introduced in this proof, and concrete examples of $m$ are given in examples 7.18, 7.19 and 7.20 below.

Now, we have $f(\mathcal{T}) = f_{\prec'_1}(\mathcal{T} \cup \{\beta'_1\}) \cap \mathbf{L}$, for any $\mathcal{T} \subseteq \mathbf{L}$. $f_{\prec'_1}$ is a preferential entailment associated to a preference relation which is transitive and irreflexive. We know from theorem 7.9 that this means that $f_{\prec'_1}$ is a formula circumscription: there exists a set of formulas $\Phi'$ in $\mathbf{L}'_1$ such that we have $f_{\prec'_1} = CIRCF(\Phi', \emptyset, \mathbf{Z}'_1)$. Thus, from the proof of theorem 4.8 we get that, for any $\mathcal{T} \subseteq \mathbf{L}$, we have $f(\mathcal{T}) = f_{\prec'_1}(\mathcal{T} \cup \{\beta'_1\}) \cap \mathbf{L} = CIRCF(\Phi', \emptyset, \mathbf{Z}'_1)(\mathcal{T} \cup \{\beta'_1\}) \cap \mathbf{L}$.

Second part: Using definition 7.2, we know that there exists a finite language $\mathbf{L}'$ containing $\mathbf{L}'_1$, with $V(\mathbf{L}') = \mathbf{P}' \cup \mathbf{Z}'$, and a finite theory $\mathcal{T}' = \{\varphi'_1 \Leftrightarrow P'_{\varphi'_1}\}_{\varphi'_1 \in \Phi'}$ in $\mathbf{L}'$ such that $f_{\prec'_1}(\mathcal{T}'_1) = CIRCF(\Phi', \emptyset, \mathbf{Z}'_1)(\mathcal{T}'_1) = CIRC(\mathbf{P}', \emptyset, \mathbf{Z}')(\mathcal{T}'_1 \cup \mathcal{T}') \cap \mathbf{L}'_1$ for any $\mathcal{T}'_1 \subseteq \mathbf{L}'_1$. $\mathcal{T}'$ is equivalent to the formula $\varphi' = \bigwedge_{\varphi'_1 \in \Phi'} (\varphi'_1 \Leftrightarrow P'_{\varphi'_1})$ in $\mathbf{L}'$. Thus, we get, for any $\mathcal{T} \subseteq \mathbf{L}$: $f(\mathcal{T}) = f_{\prec_m}(\mathcal{T}) = CIRC(\mathbf{P}', \emptyset, \mathbf{Z}')(\mathcal{T} \cup \{\beta'_1, \varphi'\}) \cap \mathbf{L}'_1 \cap \mathbf{L} = CIRC(\mathbf{P}', \emptyset, \mathbf{Z}')(\mathcal{T} \sqcup' \beta') \cap \mathbf{L}$, where $\beta' = \beta'_1 \wedge \varphi' \in \mathbf{L}'$. $f$ can be expressed in terms of (Def$\downarrow_{\beta'}$) from an ordinary

[14] Notice that we get a smaller $V(\mathbf{L}'_1)$ than in the proof of [Cos98, Theorem 15] because when Costello requires $\lceil log_2(card(\mathbf{S}))\rceil$ new symbols, we need only $\lceil log_2(Max_{\mu \in \mathbf{M}}\ card(l^{-1}(\mu)))\rceil$ new symbols in the proof of theorem 4.8 (see examples 7.19 and 7.20 below). It can be shown that, if $n = card(\mathbf{M}) = 2^v$ where $v = card(V(\mathbf{L}) \in \mathbb{N}$, for any multi preferential entailment in $\mathbf{L}$, we may choose a $\mathbf{S}$ and a $l$ such that $Max_{\mu \in \mathbf{M}}\ card(l^{-1}(\mu))) \leq n-1$. There exist multi preferential entailments for which we may need a $\mathbf{S}$ of the maximal possible size, $n(n-1)$ (i.e. $card(l^{-1}(\mu) = n-1$ for each $\mu \in \mathbf{M}$). Thus, the difference of the size of the extended vocabulary between the two methods may be significative.

propositional circumscription $f'$.

"if" (the two parts at once): This is an immediate consequence of section 6. Indeed, we know that if $f$ can be expressed in terms of (Def$\downarrow_{\beta'}$) from a circumscription $f'$, then, as any circumscription $f'$ satisfies (CR) and (CUMU), $f$ must satisfy (CR) and (CUMU) (preservation results 6.6 and 6.14, as in the finite case (CR) implies (CT), preservation result 6.15 is not needed). □

**Remark 7.17** This proof has also established the following characterization result (the "only if" part of which, called the first part of "only if" in above proof, is [Cos98, Theorem 15]):

If $V(\mathbf{L})$ is finite, then a pre-circumscription $f$ in $\mathbf{L}$ satisfies (CR) and (CUMU) iff it can be expressed in terms of (Def$\downarrow_{\beta'}$) from a formula circumscription:

There exist a finite language $\mathbf{L}'_1$ containing $\mathbf{L}$, with $V(\mathbf{L}'_1) = \mathbf{Z}'_1$ and a set $\Phi'_1 \cup \{\beta'_1\}$ of formulas in $\mathbf{L}'$ such that, for any theory $\mathcal{T}$ of $\mathbf{L}$, we have:

$$f(\mathcal{T}) = CIRCF(\Phi'_1, \emptyset, \mathbf{Z}'_1)(\mathcal{T} \sqcup'_1 \beta'_1) \cap \mathbf{L}. \quad \Box$$

As we have done with theorem 7.9, we give now an explicit way to get the set $\Phi'$ of formulas to circumscribe. We get here an irreflexive and transitive relation $\prec'_1$ on the subset $m(\mathbf{S}) = \mathbf{M}'(\beta'_1)$ of $\mathbf{M}'$. We take this relation in $\mathbf{M}'$ without modifications, notice however that we could take any relation $\prec'$ irreflexive and transitive on $\mathbf{M}'$ and coinciding with $\prec'_1$ on $\mathbf{M}'(\beta'_1)$. As in step 0 of the proof of theorem 7.9, we define the reflexive and transitive relation $\preceq'_1$ on $\mathbf{M}(\beta'_1)$ by: $\mu' \preceq'_1 \nu'$ if $\mu' \prec'_1 \nu'$ or $(\{\mu'' \ / \ \mu'' \prec'_1 \mu'\} = \{\mu'' \ / \ \mu'' \prec'_1 \nu'\}$ and $\{\mu'' \ / \ \mu' \prec'_1 \mu''\} = \{\mu'' \ / \ \nu' \prec'_1 \mu''\})$. Notice that we need only to consider the subset of $\Phi'_{\preceq'_1}$ consisting of the formulas $\varphi'_1(\mu')$ for the $\mu'$ in $\mathbf{M}'(\beta'_1)$.

In a few occasions (see example 7.18 below), it may be better to refine the definition of the reflexive relation in $\mathbf{M}'$. Here is a method for doing this. We define a relation $\preceq'_2$ in $\mathbf{M}'$ as follows: Let $\mu'_0$ be some element in $\mathbf{M}'(\beta'_1)$. We set $\mu' \preceq'_2 \nu'$ if $\{\mu', \nu'\} \subseteq \mathbf{M}'(\beta'_1)$ and $\mu' \preceq'_1 \nu'$ and also $\mu' \preceq'_2 \nu'$ if $\{\mu', \nu'\} \subseteq \mathbf{M}' - \mathbf{M}'(\beta'_1) \cup \{\mu'_0\}$. Then we "close $\preceq'_2$ by transitivity": we put $\mu' \preceq'_2 \nu'$ for each $\mu' \in \mathbf{M}' - \mathbf{M}'(\beta'_1) = \mathbf{M}'(\neg\beta'_1)$, and each $\nu' \in \mathbf{M}'(\beta'_1)$ such that $\mu'_0 \preceq'_1 \mu'$, and also for each $\mu' \in \mathbf{M}'(\beta'_1)$ such that $\mu' \preceq'_1 \mu'_0$, and each $\nu' \in \mathbf{M}'(\neg\beta'_1)$. Notice that now we may consider the whole set $\Phi'_{\preceq'_2} = \{\varphi'_2(\mu')\}_{\mu' \in \mathbf{M}'}$, for each $\mu' \in \mathbf{M}'$, but this set is equal to the set $\{\varphi'_2(\mu')\}_{\mu' \in \mathbf{M}'(\beta'_1)}$, and it has exactly as many elements as the set $\{\varphi'_1(\mu')\}_{\mu' \in \mathbf{M}'(\beta'_1)}$ obtained when using the relation $\preceq'_1$.

In the case of theorem 7.16, and contrarily to the situation in theorem 7.9, it does not exist a unique reflexive and transitive relation $\preceq'$ on $\mathbf{M}'$ such that $\Phi_{\preceq'}$ is the best possible set in terms of number of elements: this is due to the fact that outside $\mathbf{M}'(\beta'_1)$ we have many possible choices. We have made this comment on the flexibility of the choice of a relation $\preceq$ because it is an important feature of the use of (Def$\downarrow_{\beta'}$) in such circumstances.

Now, in order to give some flesh to this text, we will provide three small examples of application of theorem 7.16. In these examples, we use the constructive method described just above. As we are in the finite case here, the pre-circumscriptions will be indifferently considered as mappings from $\mathbf{T}$ or from $\mathbf{L}$.

These examples are significative because each of the first two lacks exactly one of the two important properties which are "lost" by (Def$\downarrow_{\beta'}$) when we start from a circumscription $f'$, namely (PC) and (DC): indeed in the finite case, a propositional circumscription satisfies (CR), (CUMU), (DC) and (PC) [Sat90] (see also remark 7.10). The third example is a small variant of the second example which lacks the two properties (PC) and (DC).

Each example is the simplest of its kind[15]: vocabularies, sets of states, and graphs of the relations are always as small as possible for the cases where $f$ falsifies (PC) (in example 7.18), $f$ falsifies (DC) and satisfies (PC) (in example 7.19), and $f$ falsifies (PC) and (DC) (in example 7.20).

**Example 7.18** $V(\mathbf{L}) = \{P\}$, thus $\mathbf{M} = \{\mu, \nu\}$ where $\mu = \emptyset$ and $\nu = \{P\}$.

We define $\mathbf{S} = \{\nu_1\}$ with $l(\nu_1) = \nu$: $\mu$ has no copy in $\mathbf{S}$, $\nu$ has one copy.

$\prec_m$ is the relation with an empty graph.

$\prec_m$ is transitive and irreflexive, thus $f = f_{\prec_m}$ is a cumulative multi preferential entailment, i.e. a pre-circumscription satisfying (CR) and (CUMU). We get:

$f(\top) = f(P) = Th(P)$, $f(\bot) = f(\neg P) = Th(\bot) = \mathbf{L}$. $f$ satisfies also (DC) here, thus $f$ is a cumulative preferential entailment. However, $f$ falsifies (PC), thus it is not a formula circumscription. We need no additional symbol here and $\beta'$ will do all the job: we have never more than one copy of the elements of $\mathbf{M}$ in $\mathbf{S}$ and $log_2(1) = 0$. Thus we get $m = l$ (see $m$ in the proof of theorem 7.16).

$m(\mathbf{S}) = \{\nu\}$ and $Th(P) = Th(\nu)$, thus $\beta' = \beta'_1 = P$. As our formula circumscription (remark 7.17, i.e first part in the proof of theorem 7.16), we may take the identity (empty set of formulas: $\Phi' = \emptyset$), thus as our ordinary circumscription we take identity on $\mathbf{T}$ also: no circumscribed proposition.

We get $f(\mathcal{T}) = CIRC(\emptyset, \emptyset, P)(\mathcal{T} \cup \{\beta'\})$ for any $\mathcal{T} \subseteq \mathbf{L}$.

Here is how we get the empty set, if we use the constructive method described just above: We get here $\preceq'_1$ defined as follows on $\mathbf{M}(\beta') = \{\nu\}$: As $\prec$ has the empty graph, we get "only" $\nu \preceq'_1 \nu$ (notice that this is also the "universal relation" on the singleton $\mathbf{M}(\beta')$). Thus, $\preceq'_1$ is defined on $\mathbf{M}'$ by $\nu \preceq'_1 \nu$ and $\mu \preceq'_1 \mu$, but we do not need to examine $\mu$.

From "step 1" in the proof of theorem 7.9 we get then $s(\nu) = \{\nu\}$, thus we are interested in the set $\Phi_{\preceq_0} = \{\varphi(\nu)\} = \{P\}$: as mentioned above, we do not consider $\mu$ which is outside $\mathbf{M}(\beta')$[16]. This give rises to $f(\mathcal{T}) = CIRC(P, \emptyset, \emptyset)(\mathcal{T} \cup \{P\})$, which is also optimal for what concerns the size of the useful part of the set $\Phi_{\preceq_0}$. This is clearly equivalent to the result given

[15] An indirect consequence of this feature, for what concerns "step 0" in the proof of theorem 7.9, is that in each of these three examples, we get $\preceq'_0 = \preceq'_1$, on the set $m(\mathbf{S}) = \mathbf{M}'(\beta'_1)$.

[16] If we considered this other element, we would get the complete set $\Phi_{\preceq_1} = \{\varphi(\nu), \varphi(\nu)\} = \{P, \neg P\}$. This set cannot be optimal in size, but it is correct: indeed $CIRCF((P, \neg P), \emptyset, V(\mathbf{L})) = CIRCF(\emptyset, P, P) = CIRCF(\emptyset, \emptyset, P)$. Notice however that we cannot get the empty set from $\{P, \neg P\}$ by the property of the closure for $\wedge$ and $\vee$.

above: as $\mathcal{T} \cup \{P\} \models P$, we get $CIRC(P, \emptyset, \emptyset)(\mathcal{T} \cup \{P\}) = \mathcal{T} \sqcup P = CIRC(\emptyset, \emptyset, P)(\mathcal{T} \cup \{\beta'\})$ (circumscribing $P$ in a theory containing $P$ cannot do anything). However, this result is not optimal in the absolute, because we cannot get the empty set by using remark 7.11.

Thus, it is a case where we may consider $\preceq'_2$ as described just above. We get only one possible choice here, the universal relation on $\mathbf{M} = \mathbf{M}'$: indeed $\mathbf{M}'(\beta')$ has only one element $\nu$, thus any element of $\mathbf{M}' = \{\mu, \nu\}$ is either in $\mathbf{M}'(\neg\beta') = \{\mu\}$ or in $\{\nu\}$. We get now $\Phi_{\preceq'_2} = \{\varphi(\mu), \varphi(\nu)\} = \{\varphi(\nu)\}$ where $s(\mu) = s(\nu) = \{\mu, \nu\} = \mathbf{M}$, thus giving rise to the set of formulas $\Phi_{\preceq'_2} = \{\top\}$. Notice that, as always, this set has as many elements as the set $\{P\}$ obtained from $\preceq'_0$. However, this set may be reduced by using remark 7.11: $\emptyset$ has the same closure for $\wedge$ and $\vee$ than $\{\top\}$.

This shows that the problem of getting a set $\mathbf{P}'$ (in theorem 7.16) or $\Phi'_1$ (in remark 7.17) as small as possible is even more complicated here by this possibility of modifying the relation $\preceq'$ outside the set $m(\mathbf{S}) \subseteq \mathbf{M}'$. $\Box$

**Example 7.19** $V(\mathbf{L}) = \{P, Q\}$, thus $\mathbf{M} = \{\emptyset, \{P\}, \{Q\}, \{P, Q\}\}$.

We define $\mathbf{S} = \{\emptyset_1, \{P\}_1, \{Q\}_1, \{P, Q\}_1, \{P, Q\}_2\}$, with $l(\mu_1) = \mu$ for any $\mu \in \mathbf{M}$ and $l(\{P, Q\}_2) = \{P, Q\}$: each interpretation in $\mathbf{M}$ has one copy in $\mathbf{S}$, except $\{P, Q\}$ which has two copies. $\prec_m$ is defined on $\mathbf{S}$ by $\{P\}_1 \prec_m \{P, Q\}_1$, $\{Q\}_1 \prec_m \{P, Q\}_2$, and nothing else.

$\prec_m$ is irreflexive and transitive, thus $f = f_{\prec_m}$ is a cumulative preferential entailment, i.e. it satisfies (CR) and (CM). $f$ satisfies also (PC), but $f$ falsifies (DC) (as shown at the end of this example) thus it is not a preferential entailment (a fortiori not a formula circumscription).

One new symbol is enough for the first part which gives a formula circumscription $f_{\prec'_1}$ (cf remark 7.17), because we have never more than two copies in $\mathbf{S}$ and $log_2(2) = 1$. Costello's method would require three new symbols here, because $\mathbf{S}$ has 5 elements, and $\lceil log_2(5) \rceil = 3$.

With $V(\mathbf{L}'_1) = V(\mathbf{L}) \cup \{P'\}$, we define the mapping $m$ from $\mathbf{S}$ to $\mathbf{M}'_1 = \mathcal{P}(V(\mathbf{L}'_1))$ as follows:

$m(\mu_1) = \mu$ for any $\mu \in \mathbf{M}$, $m(\{P, Q\}_2) = \{P, Q, P'\}$. $m(\mathbf{S}) = \{\emptyset, \{P\}, \{Q\}, \{P, Q\}, \{P, Q, P'\}\} \subseteq \mathbf{M}'_1$ and $m(\mathbf{S}) = \mathbf{M}'_1(P' \Rightarrow (P \wedge Q))$, thus we get $\beta'_1 = P' \Rightarrow (P \wedge Q)$. The relation $\prec'_1$ is defined in $\mathbf{M}'_1$ as follows (see ($\uparrow$Def:$\prec_m \rightsquigarrow \prec'$) in the proof of theorem 4.8): $\{P\} \prec'_1 \{P, Q\}$, $\{Q\} \prec'_1 \{P, Q, P'\}$, and nothing else. Notice the similarity with example 6.11.

We must find a set $\Phi'$ of formulas to circumscribe in order to get $f_{\prec'_1}$. Using the construction described in "steps 0 and 1" of the proof of theorem 7.9, we need only to consider the subset $m(\mathbf{S})$ of $\mathbf{M}'_1$, with the relation $\preceq'_1$ defined by $\mu \preceq'_1 \nu$ if $\mu \prec'_1 \nu$ or $\mu = \nu$ (cf note 15). Notice that here we do not have any advantage to modify $\preceq'_1$ on the whole set $\mathbf{M}'$ as indicated above. We get the set $\Phi'' = \{\varphi'_i\}_{i \in \{0,1,2\}} \cup \{\varphi''_i\}_{i \in \{3,4\}}$, with:

$$\begin{array}{rclrclcl}
s(\emptyset) & = & \{\emptyset\}, & \varphi'_0 & = & \varphi(\emptyset) & = & \neg P \wedge \neg Q \wedge \neg P'; \\
s(\{P\}) & = & \{\{P\},\{P,Q\}\}, & \varphi'_1 & = & \varphi(\{P\}) & = & P \wedge \neg P'; \\
s(\{Q\}) & = & \{\{Q\},\{P,Q,P'\}\}, & \varphi'_2 & = & \varphi(\{Q\}) & = & Q \wedge (P \Leftrightarrow P'); \\
s(\{P,Q\}) & = & \{\{P,Q\}\}, & \varphi''_3 & = & \varphi(\{P,Q\}) & = & P \wedge Q \wedge \neg P'; \\
s(\{P,Q,P'\}) & = & \{\{P,Q,P'\}\}, & \varphi''_4 & = & \varphi(\{P,Q,P'\}) & = & P \wedge Q \wedge P'.
\end{array}$$

$\Phi''$ is not the smallest possible set. Indeed, we may take the union of $s(\{P,Q\}) \cup s(\{P,Q,P'\})$, instead of the two sets individually, which corresponds to replace the two formulas $\varphi''_3$ and $\varphi''_4$ by the single formula $\varphi'_3 = \varphi''_3 \vee \varphi''_4 = P \wedge Q$. This merging is an immediate consequence of remark 7.11: the original set and the "merged" set have the same closure for $\wedge$ and $\vee$, indeed, $\varphi''_3 = \varphi'_1 \wedge \varphi'_3$ and $\varphi''_4 = \varphi'_2 \wedge \varphi'_3$. It can be shown that four elements are needed here.

As our set $\Phi'$ we get $\Phi' = \{\varphi'_i\}_{i\in\{0,\cdots,3\}}$, and, if we denote the restriction of a preference relation $\prec'_i$ in $\mathbf{M}'_1$ to the subset $m(\mathbf{S}) = \mathbf{M}'_1(\beta'_1)$ of $\mathbf{M}'_1$ by $\prec'_i \,|\, m(\mathbf{S})$, we get $\prec'_1 \,|\, m(\mathbf{S}) = \prec_{(\Phi';\emptyset,\{P,Q,P'\})} |\, m(\mathbf{S})$. Thus, for any $\mathcal{T} \subseteq \mathbf{L}$, we have $f(\mathcal{T}) = CIRCF(\Phi', \emptyset, (P,Q,P'))(\mathcal{T} \cup \{\beta'_1\}) \cap \mathbf{L}$.

For the second part (result of theorem 7.16), it suffices to apply definition 7.2. We need then four new symbols. $V(\mathbf{L}') = \{P, Q, P', P'_0, P'_1, P'_2, P'_3\}$, and with $\varphi' = (\varphi'_0 \Leftrightarrow P'_0) \wedge (\varphi'_1 \Leftrightarrow P'_1) \wedge (\varphi'_2 \Leftrightarrow P'_2) \wedge (\varphi'_3 \Leftrightarrow P'_3)$, we get, for any $\mathcal{T}'_1 \subseteq \mathbf{L}'_1$:
$CIRCF(\Phi', \emptyset, (P,Q,P'))(\mathcal{T}'_1) = CIRC((P'_0, P'_1, P'_2, P'_3), \emptyset, (P,Q,P'))(\mathcal{T}'_1 \cup \{\varphi'\}) \cap \mathbf{L}'_1$.

Thus, with $\beta' = \beta'_1 \wedge \varphi'$ we get, for any $\mathcal{T} \subseteq \mathbf{L}$:

$$\begin{aligned}
f(\mathcal{T}) = f_{\prec_m}(\mathcal{T}) &= CIRC((P'_0, P'_1, P'_2, P'_3), \emptyset, (P,Q,P'))(\mathcal{T}'_1 \cup \{\beta'\})) \cap \mathbf{L} \\
&= tr(CIRC((P'_0, P'_1, P'_2, P'_3), \emptyset, (P,Q,P'))(\mathcal{T}'_1 \cup \{\beta'\})).
\end{aligned}$$

Finally, let us give an explicit case showing that $f$ falsifies (DC). Remind that the non preservation of (DC) by (Def$\downarrow_{\beta'}$) is one of the facts allowing theorem 7.16 to hold.

$f = f_m$ is defined by: $f_m(P \vee Q) = Th(P \Leftrightarrow \neg Q)$, $f_m(\top) = Th(\neg P \vee \neg Q)$ and $f_m(\varphi) = \varphi$ for the fourteen other elements $\varphi \in \mathbf{L}$. It is easy to check directly that $f_m$ satisfies (CR0) and (CUMU0).

However, as in example 6.11, (DC0) is falsified: $Th(P \Leftrightarrow \neg Q) = f_m(P \vee Q) \not\subseteq f_m(P) \sqcup f_m(Q) = Th(P \wedge Q)$. $\Box$

**Example 7.20** $V(\mathbf{L}) = \{P, Q\}$, thus $\mathbf{M} = \{\emptyset, \{P\}, \{Q\}, \{P,Q\}\}$.

We define $\mathbf{S} = \{\{P\}_1, \{Q\}_1, \{P,Q\}_1, \{P,Q\}_2\}$, with $l(\mu_1) = \mu$ for $\mu \in \mathbf{M} - \{\emptyset\}$ and $l(\{P,Q\}_2) = \{P,Q\}$: each interpretation has one copy, except $\{P,Q\}$ which has two copies and $\emptyset$ which has no copy. $\prec_m$ is defined on $\mathbf{S}$ as in example 7.19: $\{P\}_1 \prec_m \{P,Q\}_1$, $\{Q\}_1 \prec_m \{P,Q\}_2$, and nothing else.

Thus, $f = f_{\prec_m}$ is a cumulative preferential entailment, i.e. it satisfies (CR) and (CM).

Let us call $f_1$ the pre-circumscription called $f$ in example 7.19 (this text uses many subscripts and we try to avoid useless ones, so we have recycled the name $f$). $f$ can be described as follows: for any formula $\varphi \in \mathbf{L}$, we have $f(\varphi) = f_1(\varphi \wedge (P \vee Q))$, or equivalently, for any $\varphi$ such that $\varphi \models P \vee Q$, we have $f(\varphi \vee (\neg P \wedge \neg Q)) = f(\varphi) = f_1(\varphi)$. $f$ falsifies (PC) and (DC): for (PC) we have $f(\neg P \wedge \neg Q) = f_1((\neg P \wedge \neg Q) \wedge (P \vee Q)) = f_1(\perp) = Th(\perp)$, and for (DC) we may keep the example given for $f_1$ because $\varphi \models P \vee Q$ if $\varphi \in \{P \Leftrightarrow \neg Q,\ P,\ Q\}$. Thus $f$ is not a formula circumscription.

One new symbol is enough for the first part, as in example 7.19, while Costello's method would require here two new symbols, because $\mathbf{S}$ has 4 elements, and $\lceil log_2(4) \rceil = 2$.

With $V(\mathbf{L}'_1) = V(\mathbf{L}) \cup \{P'\}$, we define the mapping $m$ from $\mathbf{S}$ to $\mathbf{M}'_1 = \mathcal{P}(V(\mathbf{L}'_1))$ as follows: $m(\mu_1) = \mu$ for any $\mu \in \mathbf{M} - \{\emptyset\}$, $m(\{P,Q\}_2) = \{P,Q,P'\}$. $m(\mathbf{S}) = \{\{P\},\{Q\},\{P,Q\},\{P,Q,P'\}\} \subseteq \mathbf{M}'_1$ and as $m(\mathbf{S}) = \mathbf{M}'_1((P \vee Q) \wedge (P' \Rightarrow (P \wedge Q)))$, we get $\beta'_1 = (P \vee Q) \wedge (P' \Rightarrow (P \wedge Q))$.

The relation $\prec'_1$ is defined by: $\{P\} \prec'_1 \{P,Q\}$, $\{Q\} \prec'_1 \{P,Q,P'\}$, and nothing else. Applying the construction described in "steps 0 and 1" of the proof of theorem 7.9 exactly as in example 7.19, we get the same values than in this example (as $\emptyset$ is no longer in $m(\mathbf{S})$, we suppress $\varphi(\emptyset) = \varphi'_0$). Choosing $\Phi'' = \{\varphi'_i\}_{i \in \{1,2\}} \cup \{\varphi''_i\}_{i \in \{3,4\}}$ with the same $\varphi'_i$'s and $\varphi''_j$ as in example 7.19, we get $\prec'_1 \mid m(\mathbf{S}) = \prec_{(\Phi'';\emptyset,\{P,Q,P'\})} \mid m(\mathbf{S})$. Thus, for any $\mathcal{T} \subseteq \mathbf{L}$, we have $f(\mathcal{T}) = CIRCF(\Phi'', \emptyset, (P,Q,P'))(\mathcal{T} \cup \{\beta'_1\}) \cap \mathbf{L}$. As in example 7.19, the last two formulas may be merged into their disjunction, we get then the set $\Phi' = \{\varphi'_i\}_{i \in \{1,2,3\}}$, and it does not exist a set with fewer elements than in $\Phi'$.

Thus we get, for any $\mathcal{T} \subseteq \mathbf{L}$: $f(\mathcal{T}) = CIRCF(\Phi', \emptyset, (P,Q,P'))(\mathcal{T} \cup \{\beta'_1\}) \cap \mathbf{L}$.

We need three new symbols to get an ordinary circumscription. If $V(\mathbf{L}') = \{P,Q,P',P'_1,P'_2,P'_3\}$, $\varphi' = (\varphi'_1 \Leftrightarrow P'_1) \wedge (\varphi'_2 \Leftrightarrow P'_2) \wedge (\varphi'_3 \Leftrightarrow P'_3)$, and $\beta' = \beta'_1 \wedge \varphi'$ we get, for any $\mathcal{T} \subseteq \mathbf{L}$:

$f(\mathcal{T}) = f_{\prec_m}(\mathcal{T}) = CIRC((P'_1,P'_2,P'_3), \emptyset, (P,Q,P'))(\mathcal{T}'_1 \cup \{\beta'\})) \cap \mathbf{L}$. $\square$

## 7.4 Relating ordinary and cardinality-based circumscriptions

Let us give other examples of application of section 6, concerning the relations between circumscription as defined above and another formalism, cardinality-based circumscription. In the two classical circumscriptions of definitions 7.1 and 7.2 the minimization is made through a $\subset$ (subset) relation. In some applications, such as diagnostic (in the cases where we want to minimize the number of faulty elements) or theory revision, it may be more appropriate to minimize through a $<$ (has less elements than) relation. This is why [LS97] introduces the following definition:

**Definition 7.21** ([LS97, Definitions 5,6])

$\mathbf{P}, \mathbf{Q}, \mathbf{Z}$ being as in definition 7.1, we define the preference relation $<_{(\mathbf{P},\ \mathbf{Q},\ \mathbf{Z})}$ by
$\mu <_{(\mathbf{P},\ \mathbf{Q},\ \mathbf{Z})} \nu$ if $card(\mathbf{P} \cap \mu) < card(\mathbf{P} \cap \nu)$ and $\mathbf{Q} \cap \mu = \mathbf{Q} \cap \nu$.

The *cardinality-based circumscription* $NCIRC(\mathbf{P}, \mathbf{Q}, \mathbf{Z})$ is the preferential entailment $f_{<_{(\mathbf{P},\ \mathbf{Q},\ \mathbf{Z})}}$. $\square$

We will show now that in the finite case we have interdefinability between these two circumscriptions, the two translations being in terms of (Def$\downarrow_{\beta'}$) (remind definition 7.2).

**Theorem 7.22** 1. Any cardinality-based circumscription in which $\mathbf{P}$ is finite is equal to a formula circumscription: $NCIRC(\mathbf{P},\,\mathbf{Q},\,\mathbf{Z}) = CIRCF(\Phi,\mathbf{Q},\mathbf{Z}\cup\mathbf{P})$,

where $\varphi_i = \bigvee_{S\subseteq\mathbf{P},\; card(S)=i}\; \bigwedge_{P\in S} P$ and $\Phi = \{\varphi_i\}_{i\in\{1,\cdots,card(\mathbf{P})\}}$.

2. Any cardinality-based circumscription in which $\mathbf{P}$ is finite can be expressed in terms of (Def$\downarrow_{\beta'}$) from an ordinary propositional circumscription:

if we define a set of new (not in $\mathbf{L}$ and all distinct) propositional symbols $\mathbf{P}' = \{P'_P\}_{P\in\mathbf{P}}$, we have, for any $\mathcal{T}\subseteq\mathbf{L}$,

$$NCIRC(\mathbf{P},\mathbf{Q},\mathbf{Z})(\mathcal{T}) = CIRC(\mathbf{P}',\mathbf{Q},\mathbf{P}\cup\mathbf{Z})(\mathcal{T}\cup\{\varphi_i\Leftrightarrow P'_i\}_{i\in\{1,2,\cdots,card(\mathbf{P})\}})\cap\mathbf{L}.\ \square$$

Proof:

1. With $CIRCF(\Phi,\mathbf{Q},\mathbf{Z}\cup\mathbf{P}) = f_{\prec}$, for any $\mu,\nu$ in $\mathbf{M}$ we have $\mu\prec\nu$ iff $Th(\mu)\cap\Phi\subset Th(\nu)\cap\Phi$ and $\mu\cap\mathbf{Q}=\nu\cap\mathbf{Q}$ from remark 7.4. From the definition of $\varphi_i\in\Phi$ we get $\mu\models\varphi_i$ iff $card(\mu\cap\mathbf{P})\geq i$. Thus we get $Th(\mu)\cap\Phi\subset Th(\nu)\cap\Phi$ iff $card(\mu\cap\mathbf{P}) < card(\nu\cap\mathbf{P})$. Thus we have $\mu\prec\nu$ iff $\mu\cap\mathbf{Q}=\nu\cap\mathbf{Q}$ and $card(\mu\cap\mathbf{P}) < card(\nu\cap\mathbf{P})$. Thus we get $\mu\prec\nu$ iff $\mu <_{\mathbf{P},\mathbf{Q},\mathbf{Z}} \nu$ from definition 7.21, i.e. $CIRCF(\Phi,\mathbf{Q},\mathbf{Z}\cup\mathbf{P}) = NCIRC(\mathbf{P},\,\mathbf{Q},\,\mathbf{Z})$.

As an example, if $\mathbf{P}=\{P_1,P_2,P_3\}$, we get $\Phi=\{\varphi_1,\varphi_2,\varphi_3\}$ with $\varphi_1 = P_1\vee P_2\vee P_3$, $\varphi_2 = (P_1\wedge P_2)\vee(P_1\wedge P_3)\vee(P_2\wedge P_3)$, $\varphi_3 = P_1\wedge P_2\wedge P_3$.

2. Use point 1 above together with definition 7.2. $\square$

**Remark 7.23** 1. (Def$\downarrow_{\beta'}$) used in theorem 7.22-2 has the singleton property. Indeed (see remark 7.3) for any $\mu\in\mathbf{M}$, if we define $u'(\mu) = \mu\cup\{P'_1,\cdots,P'_{card(\mu\cap\mathbf{P})}\}$, we get $\mathbf{M}'(Th(\mu)\sqcup'\beta') = \{u'(\mu)\}$.

2. Thanks to property 7.7 we could even express $NCIRC(\mathbf{P},\mathbf{Q},\mathbf{Z})$ in terms of (Def$\downarrow_{\beta'}$) from an ordinary propositional circumscription without fixed proposition. In this case again, we use a (Def$\downarrow_{\beta'}$) having the singleton property. $\square$

Theorem 7.24 below shows how to extend theorem 7.22-2 to the case where $\mathbf{P}$ is enumerable. However, remark 7.25-3 below shows that theorem 7.22-1 does not extend to the enumerable case: we no longer get a formula circumscription in the same language.

**Theorem 7.24** $\mathbf{P}$ is enumerable here. Any cardinality-based circumscription $NCIRC(\mathbf{P},\mathbf{Q},\mathbf{Z})$ can be expressed in terms of (Def$\downarrow_{\mathcal{T}'}$) from an ordinary circumscription $CIRC(\mathbf{P}',\mathbf{Q},\mathbf{P}\cup\mathbf{Z})$ where $card(\mathbf{P}') = card(\mathbf{P})$:

If we define the set of new (not in $\mathbf{L}$ and all distinct) propositional symbols $\mathbf{P}' = \{P'_i\}_{i\in\,\mathbb{N}-\{0\}}$, we have, for any $\mathcal{T}\subseteq\mathbf{L}$,

$$NCIRC(\mathbf{P},\mathbf{Q},\mathbf{Z})(\mathcal{T}) = CIRC(\mathbf{P}',\mathbf{Q},\mathbf{P}\cup\mathbf{Z})(\mathcal{T}\cup\{(\bigwedge_{P\in S} P)\Rightarrow P'_{card(S)}\}_{\emptyset\subset S\subseteq\mathbf{P},\;S\text{ finite}})\cap\mathbf{L}.\ \square$$

Proof: Contrarily to what happens in theorem 7.22-2, we do not have the singleton property here (all we have is the non empty property), thus the preservation of the notion of preferential entailment is not so immediate. Another complication comes from the fact that the preference relation associated to $CIRC(\mathbf{P}', \mathbf{Q}, \mathbf{P} \cup \mathbf{Z})$ falsifies (cl) here (see e.g. [MR98b, Proposition 4.13]).

We consider the language $\mathbf{L}'$ such that $V(\mathbf{L}') = V(\mathbf{L}) \cup \mathbf{P}'$, with $\mathbf{P}' = \{P'_i\}_{i \in \mathbb{N} - \{0\}}$. For any non empty finite subset $S$ of $\mathbf{P}$, we define the formula $\varphi_S = (\bigwedge_{P \in S} P) \Rightarrow P'_{card(S)}$ of $\mathbf{L}'$. We define then the subset $\mathcal{T}' = \{\varphi_S \ / \ S$ is a non empty finite subset of $\mathbf{P}\}$ of $\mathbf{L}'$. $\prec' = \prec_{(\mathbf{P}', \mathbf{Q}, \mathbf{P} \cup \mathbf{Z})}$ denotes the preference relation in $\mathbf{L}'$ associated to $CIRC(\mathbf{P}', \mathbf{Q}, \mathbf{P} \cup \mathbf{Z})$ and $\prec = <_{(\mathbf{P}, \mathbf{Q}, \mathbf{Z})}$ denotes the preference relation associated to $NCIRC(\mathbf{P}, \mathbf{Q}, \mathbf{Z})$ in $\mathbf{L}$. Thus we have $f_\prec = NCIRC(\mathbf{P}, \mathbf{Q}, \mathbf{Z})$ and $f_{\prec'} = CIRC(\mathbf{P}', \mathbf{Q}, \mathbf{P} \cup \mathbf{Z})$. We define also the pre-circumscription $f$ in $\mathbf{L}$ by $f(\mathcal{T}) = f_{\prec'}(\mathcal{T} \sqcup' \mathcal{T}') \cap \mathbf{L} = tr(f_{\prec'}(\mathcal{T} \sqcup' \mathcal{T}'))$, for any $\mathcal{T} \subseteq \mathbf{L}$.

For any $\mu \in \mathbf{M}$, we define $v'(\mu) \in \mathbf{M}'$ as follows: $v'(\mu) = \mu \cup \{P'_i \ / \ 1 \leq i \leq card(\mu \cap \mathbf{P})\}$.

From the definition of $\mathcal{T}'$ we get $v'(\mu) \in \mathbf{M}'(\mathcal{T}')$, and more precisely $v'(\mu) \in \mathbf{M}'(Th'(\mu) \sqcup' \mathcal{T}')$. Also from the definition of $\mathcal{T}'$, for any $\nu' \in \mathbf{M}'(Th'(\mu) \sqcup' \mathcal{T}')$, we get $\nu' \models' P'_i$ for any $i \in \{1, \cdots, card(\mu \cap \mathbf{P})\}$, thus we have $v'(\mu) = \nu'$ or $v'(\mu) \prec' \nu'$ (R1).

Thus we get $\mathbf{M}_{\prec'}(Th'(\mu) \sqcup' \mathcal{T}') = \{v'(\mu)\}$, for any $\mu \in \mathbf{M}$.

For any $\mu, \nu$ in $\mathbf{M}$, from definition 7.21 we get: $\mu \prec \nu$ iff $card(\mu \cap \mathbf{P}) < card(\nu \cap \mathbf{P})$ and $\mu \cap \mathbf{Q} = \nu \cap \mathbf{Q}$. We have also $card(\mu \cap \mathbf{P}) < card(\nu \cap \mathbf{P})$ iff $\{1, \cdots, card(\mu \cap \mathbf{P})\} \subset \{1, \cdots, card(\nu \cap \mathbf{P})\}$. For any $\mu, \nu$ in $\mathbf{M}$, we get from definition 7.1: $v'(\mu) \prec' v'(\nu)$ iff $v'(\mu) \cap \mathbf{P}' \subset v'(\nu) \cap \mathbf{P}'$ and $v'(\mu) \cap \mathbf{Q} = v'(\nu) \cap \mathbf{Q}$. From the definition of $v'(\mu)$ we have, for any $\mu$ in $\mathbf{M}$: $v'(\mu) \cap \mathbf{Q} = \mu \cap \mathbf{Q}$ and $v'(\mu) \cap \mathbf{P}' = \{P'_i \ / \ i \in \{1, \cdots, card(\mu \cap \mathbf{P})\}\}$. Thus we get $\mu \prec \nu$ iff $v'(\mu) \prec' v'(\nu)$ (R2).

Let us now suppose that, for some $\mathcal{T} \subseteq \mathbf{L}$ and $\nu \in \mathbf{M}$, we have $\nu \in \mathbf{M}_\prec(\mathcal{T})$ and $v'(\nu) \notin \mathbf{M}'_{\prec'}(\mathcal{T} \sqcup' \mathcal{T}')$. As we know that we have $v'(\nu) \in \mathbf{M}'(Th'(\nu) \sqcup' \mathcal{T}') = \mathbf{M}'(Th(\nu) \sqcup' \mathcal{T}') \subseteq \mathbf{M}'(\mathcal{T} \sqcup' \mathcal{T}')$, this means that there exists $\mu' \in \mathbf{M}'(\mathcal{T} \sqcup' \mathcal{T}')$ such that $\mu' \prec' v'(\nu)$. We know from (R1) that, for $\mu = tr(\mu')$, we have $\mu \in \mathbf{M}(\mathcal{T})$ with $v'(\mu) = \mu'$ or $v'(\mu) \prec' \mu'$ and, as $\prec'$ is transitive we get $v'(\mu) \prec' v'(\nu)$. From (R2) we get $\mu \prec \nu$, a contradiction with $\nu \in \mathbf{M}_\prec(\mathcal{T})$ and $\mu \in \mathbf{M}(\mathcal{T})$.

Let us suppose now that, for some $\mathcal{T} \subseteq \mathbf{L}$ and $\nu \in \mathbf{M}$, we have $v'(\nu) \in \mathbf{M}'_{\prec'}(\mathcal{T} \sqcup' \mathcal{T}')$ and $\nu \notin \mathbf{M}_\prec(\mathcal{T})$. Then we have $v'(\nu) \in \mathbf{M}'(\mathcal{T} \sqcup' \mathcal{T}') \subseteq \mathbf{M}'(\mathcal{T})$, thus $\nu = tr(v'(\nu)) \in tr(\mathbf{M}'(\mathcal{T})) = \mathbf{M}(\mathcal{T})$. Thus, there exists $\mu \in \mathbf{M}(\mathcal{T})$ such that $\mu \prec \nu$. From (R2) we get $v'(\mu) \prec' v'(\nu)$. From $\mu \in \mathbf{M}(\mathcal{T})$ we get $v'(\mu) \in \mathbf{M}'(\mathcal{T} \sqcup' \mathcal{T}')$, a contradiction with $v'(\nu) \in \mathbf{M}'_{\prec'}(\mathcal{T} \sqcup' \mathcal{T}')$.

These two contradictions prove that we have, for any $\nu \in \mathbf{M}$:
$\nu \in \mathbf{M}_\prec(\mathcal{T})$ iff $v'(\nu) \in \mathbf{M}'_{\prec'}(\mathcal{T} \sqcup' \mathcal{T}')$ (R3).

We prove now that we have $\mathbf{M}_\prec(\mathcal{T}) = tr(\mathbf{M}'_{\prec'}(\mathcal{T} \sqcup' \mathcal{T}'))$ (R4):

Let us suppose that we have $\nu \in \mathbf{M}_\prec(\mathcal{T})$, for some $\mathcal{T} \subseteq \mathbf{L}$. Then we have $\nu = tr(v'(\nu))$, and, from (R3), $v'(\nu) \in \mathbf{M}'_{\prec'}(\mathcal{T} \sqcup' \mathcal{T}')$, which gives the side $\subseteq$ of equality (R4).

Let us suppose that we have $\nu \in tr(\mathbf{M}'_{\prec'}(\mathcal{T} \sqcup' \mathcal{T}'))$, i.e. there exists $\nu' \in \mathbf{M}'_{\prec'}(\mathcal{T} \sqcup' \mathcal{T}')$ such that $\nu = tr(\nu')$. Then we have $\nu' \in \mathbf{M}'(Th'(\nu) \sqcup' \mathcal{T}')$, thus, from (R1), $v'(\nu) = \nu'$ or

$v'(\nu) \prec' \nu'$. As $\nu' \in \mathbf{M}'_{\prec'}(\mathcal{T} \sqcup' \mathcal{T}')$, the only possibility is $v'(\nu) = \nu'$ and, from (R3) we get $\nu \in \mathbf{M}_{\prec}(\mathcal{T})$.

We get then $\mathbf{M}(f_{\prec}(\mathcal{T})) = TC(\mathbf{M}_{\prec}(\mathcal{T}))$ from remark 4.3, $TC(\mathbf{M}_{\prec}(\mathcal{T})) = TC(tr(\mathbf{M}'_{\prec'}(\mathcal{T} \sqcup' \mathcal{T}')))$ from (R4), $TC(tr(\mathbf{M}'_{\prec'}(\mathcal{T} \sqcup' \mathcal{T}'))) = tr(TC'(\mathbf{M}'_{\prec'}(\mathcal{T} \sqcup' \mathcal{T}')))$ from remark 3.4-3, $tr(TC'(\mathbf{M}'_{\prec'}(\mathcal{T} \sqcup' \mathcal{T}')))) = tr(\mathbf{M}'(f_{\prec'}(\mathcal{T} \sqcup' \mathcal{T}')))$ from remark 4.3 again, $tr(\mathbf{M}'(f_{\prec'}(\mathcal{T} \sqcup' \mathcal{T}'))) = \mathbf{M}(tr(f_{\prec'}(\mathcal{T} \sqcup' \mathcal{T}')))$ from property 3.5 and $tr(\mathbf{M}'(f_{\prec'}(\mathcal{T} \sqcup' \mathcal{T}'))) = \mathbf{M}(f(\mathcal{T}))$ from the definition of $f$ and property 3.5 again. Thus we have $f_{\prec} = f$. □

**Remarks 7.25** 1. As already noticed, we have here an example of (Def$\downarrow_{\mathcal{T}'}$) having the non empty property, but not the singleton property.

2. We may go further, getting a circumscription $f'$ without fixed propositions:

   If $\mathbf{P}$ is enumerable, and if $V(\mathbf{L}) = \mathbf{P} \cup \mathbf{Q} \cup \mathbf{Z}$, there exist a language $\mathbf{L}' \supseteq \mathbf{L}$, two disjoint sets $\mathbf{P}'$, $\mathbf{Z}'$ with $V(\mathbf{L}') = \mathbf{P}' \cup \mathbf{Z}'$, and a set of formulas $\mathcal{T}'$ in $\mathbf{L}'$ such that we have, for any $\mathcal{T} \subseteq \mathbf{L}$:

   $$NCIRC(\mathbf{P}, \mathbf{Q}, \mathbf{Z})(\mathcal{T}) = CIRC(\mathbf{P}', \emptyset, \mathbf{Z}')(\mathcal{T} \sqcup' \mathcal{T}') \cap \mathbf{L}.$$

   Here again, we have an example of (Def$\downarrow_{\mathcal{T}'}$) having the non empty property, but not the singleton property.

3. If $\mathbf{P}$ is infinite, it cannot exist a formula circumscription in $\mathbf{L}$ such that we have $NCIRC(\mathbf{P}, \mathbf{Q}, \mathbf{Z}) = CIRCF(\Phi, \mathbf{Q}_1, \mathbf{Z}_1)$. □

Proof:

Point 1: Immediate.

Notice that, even if we do not have the singleton property here, we are "close to". Indeed, we have the non empty property (for any $\mu \in \mathbf{M}$, $\mathbf{M}'(\mu) \cap \mathbf{M}'(\mathcal{T}') \neq \emptyset$), and also, for any $\mu \in \mathbf{M}$, the set $\mathbf{M}'_{\prec'}(Th'(\mu) \sqcup' \mathcal{T}')$ is a singleton, namely $\{v'(\mu)\}$. We leave for another work the specific study of the properties which are preserved in such a case.

Point 2: Immediate consequence of point 1 and of property 7.7. Notice that we are still in the particular case "close to the singleton property" described in point 1.

Point 3: This is not a direct consequence of the preservation results of section 6, but it can be proved by similar (and easier) methods. We prove the (apparently) more powerful following restriction:

Once the sets $\mathbf{P}, \mathbf{Q}, \mathbf{Z}$ are given, such that $\mathbf{P} \cup \mathbf{Q} \cup \mathbf{Z} = V(\mathbf{L})$ with $\mathbf{P}$ infinite, it cannot exist a set $\Phi$ of formulas, two sets $\mathbf{Q}'$, $\mathbf{Z}'$ such that $\mathbf{Q}' \cup \mathbf{Z}' = V(\mathbf{L})$ and a theory $\mathcal{T}'$, all in $\mathbf{L}$, such that we have, for any $\mathcal{T} \subseteq \mathbf{L}$: $NCIRC(\mathbf{P}, \mathbf{Q}, \mathbf{Z})(\mathcal{T}) = CIRCF(\Phi, \mathbf{Q}', \mathbf{Z}')(\mathcal{T} \sqcup \mathcal{T}')$.

Let us suppose such a translation is possible, then we are in a special (and easy) case of property 4.12, in which $\mathbf{L} = \mathbf{L}'$. Let us call $\prec' = \prec_{(\Phi, \mathbf{Q}', \mathbf{Z}')}$ and $\prec = <_{(\mathbf{P}, \mathbf{Q}, \mathbf{Z})}$ the relations associated to $CIRCF(\Phi, \mathbf{Q}', \mathbf{Z}')$ and $NCIRC(\mathbf{P}, \mathbf{Q}, \mathbf{Z})$ respectively. We know that these two relations are irreflexive, thus we may use property 5.6-1. From (Def$\downarrow_{\mathcal{T}'}$:$\prec'\rightsquigarrow\prec$) (see property 4.12) we know that we have $\mu \prec \nu$ iff, either $\mu = \nu$ and $\mathbf{M}'_{\prec'}(Th'(\nu) \sqcup' \mathcal{T}') = \emptyset$, or $\mathbf{M}'_{\prec'}(Th'(\nu) \sqcup' \mathcal{T}') \neq \emptyset$ and for any $\nu' \in \mathbf{M}'_{\prec'}(Th'(\nu) \sqcup' \mathcal{T}')$, there exists $\mu' \in \mathbf{M}'(\mu) \cap \mathbf{M}'(\mathcal{T}')$ such that $\mu' \prec' \nu'$. As $\prec$ is irreflexive, we must then have , for any $\nu \in \mathbf{M}$, $\mathbf{M}'_{\prec'}(Th'(\nu) \sqcup' \mathcal{T}') \neq \emptyset$. A fortiori $\mathbf{M}'(Th'(\nu) \sqcup' \mathcal{T}') \neq \emptyset$, where $\mathbf{M}'(Th'(\nu) \sqcup' \mathcal{T}') = \mathbf{M}'(\nu) \cap \mathbf{M}'(\mathcal{T}')$. As here $V(\mathbf{L}) = V(\mathbf{L}')$, we get $\mathbf{M}'(\nu) = \{\nu\}$ for any $\nu \in \mathbf{M}$. Thus, we get $\nu \in \mathbf{M}'(\mathcal{T}')$ for any $\nu \in \mathbf{M}$. As $\mathcal{T}' \in \mathbf{T}' = \mathbf{T}$, this means $\mathcal{T}' = Th(\top)$. Thus, we get $NCIRC(\mathbf{P}, \mathbf{Q}, \mathbf{Z})(\mathcal{T}) = CIRCF(\Phi, \mathbf{Q}', \mathbf{Z}')(\mathcal{T})$, for any $\mathcal{T} \subseteq \mathbf{L}$, and $\prec' = \prec$. This is impossible because (sf) is satisfied by $\prec'$ and also by its converse relation $\succ'$ defined by $\mu \succ' \nu$ iff $\nu \prec' \mu$ [MR98b, Corollary 5.7-4], while the relation $\succ$ (converse of $\prec$) falsifies (sf) if $\mathbf{P}$ is infinite as shown by the following example:

$\mathbf{P} = \{P_i\}_{i \in \mathbb{N}-\{0\}}$, $\mathbf{Q} = \mathbf{Z} = \emptyset$. We define $\mu_i = \{P_j\}_{i \leq j \leq 2i}$ (notice that $\mu_0 = \emptyset$) and $\mathcal{T} = Th(\{\mu_i \ / \ i \in \mathbb{N}\})$. We get $\mathbf{M}(\mathcal{T}) = \{\mu_i \ / \ i \in \mathbb{N}\}$ and $\mathbf{M}_{\succ}(\mathcal{T}) = \emptyset$, thus $\succ$ falsifies (sf).

The argument extends easily to greater (non enumerable) sets $\mathbf{P}$ and to any sets $\mathbf{Q}$ and $\mathbf{Z}$. $\square$

Here is the opposite of the translation given in theorem 7.22, in the finite case.

**Theorem 7.26** Any ordinary circumscription $CIRC(\mathbf{P}, \mathbf{Q}, \mathbf{Z})$ in which $\mathbf{P}$ is finite can be expressed in terms of (Def$\downarrow_{\beta'}$) from a cardinality-based circumscription $NCIRC(\mathbf{P}, \mathbf{Q}', \mathbf{Z})$:

If $\mathbf{P} = \{P_1, \cdots, P_n\}$ is finite, and if $V(\mathbf{L}) = \mathbf{P} \cup \mathbf{Q} \cup \mathbf{Z}$ (disjoint sets), there exist a set $\mathbf{P}' = \{P'_1, \cdots, P'_n\}$ of propositional symbols not in $V(\mathbf{L})$ and a formula $\beta' = \bigwedge_{i \in \{1, \cdots, n\}} (P_i \Rightarrow \neg P'_i)$ such that we have, for any $\mathcal{T} \subseteq \mathbf{L}$:

$$CIRC(\mathbf{P}, \mathbf{Q}, \mathbf{Z})(\mathcal{T}) = NCIRC(\mathbf{P}, \mathbf{Q} \cup \mathbf{P}', \mathbf{Z})(\mathcal{T} \cup \{\beta'\}) \cap \mathbf{L}. \quad \square$$

Proof: The idea of the proof is as follows: Thanks to $\beta'$, we have reversed for $\mathbf{P}'$ in $\mathbf{M}' = \mathcal{P}(V(\mathbf{L}'))$ the relation $\subseteq$ with respect to $\mathbf{P}$ in $\mathbf{M}$. Notice that in $V(\mathbf{L}')$ we have the two sets $\mathbf{P}$ and $\mathbf{P}'$. The problem is that the relation "has a smaller cardinal than", with respect to $\mathbf{P}$, is much greater (for its graph) than the relation "is strictly included in", still with respect to $\mathbf{P}$. Thanks to the elements of $\mathbf{P}'$, we are able to "cut the relations in excess": each time we have a cardinality relation which does not correspond to an inclusion, we suppress it thanks to the "fixed" elements of $\mathbf{P}'$: if $card(\mu \cap \mathbf{P}) < card(\nu \cap \mathbf{P})$ and $\mu \cap \mathbf{P} \not\subseteq \nu \cap \mathbf{P}$, then there exists in $\mathbf{M}'(\beta')$ an element representing $\nu$ which is preceded by no element representing $\mu$ (i.e. there exists an element in $\mathbf{M}'(Th(\nu) \sqcup' \beta')$ which is preceded by no element in $\mathbf{M}'(Th(\mu) \sqcup' \beta')$).

Here is the formal proof: We have $V(\mathbf{L}) = \mathbf{P} \cup \mathbf{Q} \cup \mathbf{Z}$ and $V(\mathbf{L}') = V(\mathbf{L}) \cup \mathbf{P}'$. Let us call $\prec = \prec_{(\mathbf{P}, \mathbf{Q}, \mathbf{Z})}$ the prefence relation in $\mathbf{L}$ associated to $CIRC(\mathbf{P}, \mathbf{Q}, \mathbf{Z})$ and $\prec' = <_{(\mathbf{P}, \mathbf{Q} \cup \mathbf{P}', \mathbf{Z})}$, the preference relation in $\mathbf{L}'$ associated to $NCIRC(\mathbf{P}, \mathbf{Q} \cup \mathbf{P}', \mathbf{Z})$. We define the pre-circumscription $f$ in $\mathbf{L}$ by $f(\mathcal{T}) = tr(f_{\prec'}(\mathcal{T} \sqcup' \beta')) = f_{\prec'}(\mathcal{T} \sqcup' \beta') \cap \mathbf{L}$, for any $\mathcal{T} \subseteq \mathbf{L}$.

For any $\mu \in \mathbf{M}$, we define $e'(\mu) = \{P'_i \ / \ P_i \notin \mu\}$ and $v'(\mu) = \mu \cup e'(\mu)$, thus we have $v'(\mu) \in \mathbf{M}'(\mu) \cap \mathbf{M}'(\beta') = \mathbf{M}'(Th(\mu) \sqcup' \beta')$.

We have, for any $\mu, \nu$ in $\mathbf{M}$: $\mu \prec \nu$ iff $\mu \cap \mathbf{P} \subset \nu \cap \mathbf{P}$ and $\mu \cap \mathbf{Q} = \nu \cap \mathbf{Q}$.

As we have $\mu \cap \mathbf{P} \subseteq \nu \cap \mathbf{P}$ iff $e'(\nu) \subseteq e'(\mu)$ (N0),
we get that, if $\mu \prec \nu$, then $e'(\nu) \subseteq e'(\mu)$ (N1).
From the definition of $\beta'$, for any $\mu \in \mathbf{M}$,
we have $\mathbf{M}'(Th(\mu) \sqcup' \beta') = \{\mu' \in \mathbf{M}'(\mu) \ / \ \mu' \cap \mathbf{P}' \subseteq e'(\mu)\}$ (N2).

Let us suppose $\nu \in \mathbf{M}_{\prec}(\mathcal{T})$ for some $\mathcal{T} \subseteq \mathbf{L}$. We define $\nu' = v'(\nu)$. Then $\nu \in \mathbf{M}(\mathcal{T})$, thus $\nu' \in \mathbf{M}'(\mathcal{T})$ and more precisely $\nu' \in \mathbf{M}'(\mathcal{T} \sqcup' \beta')$. Let us suppose that there exists $\mu' \in \mathbf{M}'(\mathcal{T} \sqcup' \beta')$ such that $\mu' \prec' \nu'$. We define $\mu = tr(\mu') = \mu' \cap V(\mathbf{L})$. Then, as $\mu' \in \mathbf{M}'(\mathcal{T})$, we have $\mu \in tr(\mathbf{M}'(\mathcal{T})) = \mathbf{M}(\mathcal{T})$. Then $card(\mu \cap \mathbf{P}) = card(\mu' \cap \mathbf{P}) < card(\nu' \cap \mathbf{P}) = card(\nu \cap \mathbf{P})$, $\mu \cap \mathbf{Q} = \mu' \cap \mathbf{Q} = \nu' \cap \mathbf{Q} = \nu \cap \mathbf{Q}$ and $\mu' \cap \mathbf{P}' = \nu' \cap \mathbf{P}' = e'(\nu)$. As $\mu' \in \mathbf{M}'(\beta')$, we get $\mu' \cap \mathbf{P}' \subseteq e'(\mu)$ from (N2). Thus we get $e'(\nu) \subseteq e'(\mu)$, and, from (N0), $\mu \cap \mathbf{P} \subseteq \nu \cap \mathbf{P}$. As we have $card(\mu \cap \mathbf{P}) < card(\nu \cap \mathbf{P})$, we get $\mu \cap \mathbf{P} \subset \nu \cap \mathbf{P}$. Thus we get $\mu \cap \mathbf{P} \subset \nu \cap \mathbf{P}$ and $\mu \cap \mathbf{Q} = \nu \cap \mathbf{Q}$, i.e. $\mu \prec \nu$, which contradicts $\nu \in \mathbf{M}_{\prec}(\mathcal{T})$ and $\mu \in \mathbf{M}(\mathcal{T})$. Thus no such $\mu'$ can exist and we have $\nu' \in \mathbf{M}'_{\prec'}(\mathcal{T} \sqcup' \beta')$. As we have clearly $\nu = tr(\nu')$, we get $\nu \in tr(\mathbf{M}'_{\prec'}(\mathcal{T} \sqcup' \beta'))$.

Let us suppose $\nu \in tr(\mathbf{M}'_{\prec'}(\mathcal{T} \sqcup' \beta'))$ for some $\mathcal{T} \subseteq \mathbf{L}$. Then there exists $\nu' \in \mathbf{M}'_{\prec'}(\mathcal{T} \sqcup' \beta')$ such that $\nu = tr(\nu') = \nu' \cap V(\mathbf{L})$. As we have $\nu' \in \mathbf{M}'(\mathcal{T})$, we get $\nu \in tr(\mathbf{M}'(\mathcal{T})) = \mathbf{M}(\mathcal{T})$. Let us suppose that there exists $\mu \in \mathbf{M}(\mathcal{T})$ such that $\mu \prec \nu$. Let us define $\mu' = \mu \cup (\nu' \cap \mathbf{P}')$. As $\mu \prec \nu$, we have $\mu' \cap \mathbf{P} = \mu \cap \mathbf{P} \subset \nu \cap \mathbf{P} = \nu' \cap \mathbf{P}$, and $\mu' \cap \mathbf{Q} = \mu \cap \mathbf{Q} = \nu \cap \mathbf{Q} = \nu' \cap \mathbf{Q}$. As we have $\mu' \cap \mathbf{P}' = \nu' \cap \mathbf{P}'$ from the definition of $\mu'$, and as $\mu' \cap \mathbf{P} \subset \nu' \cap \mathbf{P}$ implies $card(\mu' \cap \mathbf{P}) < card(\nu' \cap \mathbf{P})$ from the finiteness of $\mathbf{P}$, we get $\mu' \prec' \nu'$. For any $P_i \in \mathbf{P}$, if $P_i \in \mu$, we have $P_i \in \nu$, thus $P_i \in \nu'$ thus, as $\nu' \in \mathbf{M}'(\beta')$, $P'_i \notin \nu'$, thus $P'_i \notin \mu'$: this shows that we have $\mu' \in \mathbf{M}'(\beta')$. As we have $\mu' \in \mathbf{M}'(\mathcal{T})$, we get $\mu' \in \mathbf{M}'(\mathcal{T} \sqcup' \beta')$ which contradicts $\nu' \in \mathbf{M}'_{\prec'}(\mathcal{T} \sqcup' \beta')$. No such $\mu$ can exist, and we get $\nu \in \mathbf{M}_{\prec}(\mathcal{T})$.

We have established: $\mathbf{M}_{\prec}(\mathcal{T}) = tr(\mathbf{M}'_{\prec'}(\mathcal{T} \sqcup' \beta'))$.

Thus we get, as in the end of the proof of theorem 7.24, $\mathbf{M}(f_{\prec}(\mathcal{T})) = TC(\mathbf{M}_{\prec}(\mathcal{T})) = TC(tr(\mathbf{M}'_{\prec'}(\mathcal{T} \sqcup' \beta'))) = tr(TC'(\mathbf{M}'_{\prec'}(\mathcal{T} \sqcup' \beta'))) = tr(\mathbf{M}'(f'_{\prec'}(\mathcal{T} \sqcup' \beta'))) = \mathbf{M}(tr(f'_{\prec'}(\mathcal{T} \sqcup' \beta'))) = \mathbf{M}(f(\mathcal{T}))$.

Thus, $f = f_{\prec} = CIRC(\mathbf{P}, \mathbf{Q}, \mathbf{Z})$. $\square$

This translation is even easier than the opposite translation given in theorem 7.22-2: the size $card(\mathbf{P})$ of the extended vocabulary is the same in the two results, but the size of $\beta'$ is linear in $card(\mathbf{P})$ here while the size of $\beta' = \bigwedge_{i=1}^{card(\mathbf{P})}((\bigvee_{S \subseteq \mathbf{P},\ card(S)=i} \ \bigwedge_{P \in S} P) \Leftrightarrow P'_i)$ is exponential in theorem 7.22-2. Notice however that the translation given in theorem 7.22-2 has the singleton property, which is not the case here. Moreover, theorem 7.22-2 extends to any enumerable $\mathbf{P}$, as seen in theorem 7.24 (even if the singleton property is lost), while we will see now why theorem 7.26-1 cannot be extended to an infinite $\mathbf{P}$. The reason is that the relation $<_{(\mathbf{P},\mathbf{Q},\mathbf{Z})}$ associated to cardinality-based circumscription satisfies clearly (wf) while the relation $\prec_{(\mathbf{P},\mathbf{Q},\mathbf{Z})}$ associated to ordinary circumscription is known to falsify (wf) as soon as $\mathbf{P}$ is infinite (see e.g [MR98b, example 4.6]). If we could express any $CIRC$ in terms of $NCIRC$ as we have done in theorem 7.26, i.e. thanks to (Def$\downarrow_{\mathcal{T}'}$), this would force the relation $\prec_{(\mathbf{P},\mathbf{Q},\mathbf{Z})}$ to satisfy (wf) from preservation result 6.25. Indeed, we know that $\prec_{(\mathbf{P},\mathbf{Q},\mathbf{Z})}$ is irreflexive, thus q-(wf) is equivalent to (wf). Thus our preservation result about q-(wf) shows that no translation such as the one given in theorem 7.26 can exist as soon as $\mathbf{P}$ is infinite.

Let us add a few words about the translation given in theorem 7.26. It is easy to show that cardinality-based circumscriptions are preferential entailments with a preference relation satisfying (rkb) [Moi99], while the relation associated to ordinary circumscription generally falsifies (rkb): as soon as $\mathbf{P}$ has at least three elements, the relation $\prec_{(\mathbf{P},\mathbf{Q},\mathbf{Z})}$ falsifies (rkb), because the set inclusion $\subset$ among the subsets of a set of at least three elements falsifies (rkb). Thus, the non preservation of (rkb) by (Def$\downarrow_{\mathcal{T}'}$) is the crucial point which makes this translation possible. Notice that this provides a second counter-example (after example 6.28) of the non preservation of (rkb) by (Def$\downarrow_{\mathcal{T}'}$).

Let us give a last example of the use of the results of section 6 for getting interesting results about preferential entailments.

## 7.5 Varying and fixed propositions in cardinality-based circumscription

It is easy to verify that the preference relation $<_{(\mathbf{P},\emptyset,\mathbf{Z})}$ associated to $NCIRC(\mathbf{P},\emptyset,\mathbf{Z})$ satisfies (rk) (thus $NCIRC(\mathbf{P},\emptyset,\mathbf{Z})$ satisfies (RatM1)). However, as soon as $\mathbf{P}$ and $\mathbf{Q}$ are not empty, the relation $<_{(\mathbf{P},\mathbf{Q},\mathbf{Z})}$ associated to $NCIRC(\mathbf{P},\mathbf{Q},\mathbf{Z})$ falsifies (rk) (and $NCIRC(\mathbf{P},\mathbf{Q},\mathbf{Z})$ falsifies (RatM0)). One way to see this is to use the corresponding result for $CIRC(\mathbf{P},\mathbf{Q},\mathbf{Z})$ given in [Sat90]: as soon as $\mathbf{P}$ and $\mathbf{Q}$ are not empty, $CIRC(\mathbf{P},\mathbf{Q},\mathbf{Z})$ falsifies (RatM0), thus $\prec_{(\mathbf{P},\mathbf{Q},\mathbf{Z})}$ falsifies (rk). Now, if $\mathbf{P}$ as no more than two elements, we get $\prec_{(\mathbf{P},\mathbf{Q},\mathbf{Z})}=<_{(\mathbf{P},\mathbf{Q},\mathbf{Z})}$: indeed, if $card(B) \leq 2$, we have $A \subset B$ iff $card(A) < card(B)$. This provides examples (easily extended to greater sets $\mathbf{P},\mathbf{Q},\mathbf{Z}$) of cardinality-based circumscriptions falsifying (RatM0), thus of relations $<_{(\mathbf{P},\mathbf{Q},\mathbf{Z})}$ falsifying (rk).

From preservation result 6.26, or from the result 6.18, we get then the following result:

**Property 7.27** A non trivial cardinality-based circumscription with fixed proposition(s) cannot be expressed in terms of (Def$\downarrow_{\mathcal{T}'}$) from a cardinality-based circumscription without any fixed proposition:

If $V(\mathbf{L}) = \mathbf{P} \cup \mathbf{Q} \cup \mathbf{Z}$, if $\mathbf{P}$ and $\mathbf{Q}$ are not empty, it cannot exist any language $\mathbf{L}' \supseteq \mathbf{L}$ with a theory $\mathcal{T}' \subseteq \mathbf{L}'$ such that we have, for any $\mathcal{T} \subseteq \mathbf{L}$:
$NCIRC(\mathbf{P},\mathbf{Q},\mathbf{Z})(\mathcal{T}) = NCIRC(\mathbf{P}',\emptyset,\mathbf{Z}')(\mathcal{T} \sqcup \mathcal{T}') \cap \mathbf{L}$. □

This amounts to say that property 7.7 cannot be applied to non trivial cardinality-based circumscriptions (even in the finite case). An immediate consequence is that [LS97, corollary

9] is false. Notice that [Moi99] shows that a result similar to property 7.7 indeed exists for finite cardinality-based circumscriptions, but when the roles of the fixed and varying propositions are exchanged. As this provides another example of using (Def$\downarrow_{\beta'}$), and even with the singleton property, let us remind this result here.

**Property 7.28** [Moi99] Any cardinality-based circumscription with finitely many varying propositions can be expressed in terms of (Def$\downarrow_{\beta'}$) from a cardinality-based circumscription without any varying proposition:

$\mathbf{P} \cup \mathbf{Q} \cup \mathbf{Z} = V(\mathbf{L})$ (disjoint union as usual) with $\mathbf{Z} = \{Z_k\}_{k \in \{1,\cdots n\}}$ for some finite $n$. We define a set of new (not in $\mathbf{L}$ and all distinct) symbols $\mathbf{Z}' = \{Z'_k\}_{k \in \{1,\cdots n\}}$. We get, for any $\mathcal{T} \subseteq \mathbf{L}$:

$$NCIRC(\mathbf{P}, \mathbf{Q}, \mathbf{Z})(\mathcal{T}) = NCIRC(\mathbf{P} \cup \mathbf{Z} \cup \mathbf{Z}', \mathbf{Q}, \emptyset)(\mathcal{T} \sqcup \bigwedge_{k=1}^{n} (Z_k \Leftrightarrow \neg Z'_k)) \cap \mathbf{L}. \quad \square$$

<u>Proof:</u> $V(\mathbf{L}') = V(\mathbf{L}) \cup \mathbf{Z}' = \mathbf{P} \cup \mathbf{Q} \cup \mathbf{Z} \cup \mathbf{Z}'$. As usual, we define $\mathbf{M} = \mathcal{P}(V(\mathbf{L}))$ and $\mathbf{M}' = \mathcal{P}(V(\mathbf{L}'))$. Let us denote by $\mathcal{T}'_0$ the following subset of $\mathbf{L}'$: $\mathcal{T}'_0 = \{Z_k \Leftrightarrow \neg Z'_k\}_{k \in \{1,\cdots n\}}$.
To any $\mu \in \mathbf{M}$, we associate $e'(\mu) = \mu \cup \{Z'_k \ / \ k \in \{1, \cdots n\}, Z_k \notin \mu\}$ (0).
For any $\mathcal{T} \subseteq \mathbf{L}$, the restriction of $e'$ to $\mathbf{M}(\mathcal{T}) \subseteq \mathbf{M}$ defines a one-to-one mapping from $\mathbf{M}(\mathcal{T})$ to $\mathbf{M}'(\mathcal{T} \cup \mathcal{T}'_0)$ (1).
As $\mathbf{P}, \mathbf{Z}$ and $\mathbf{Z}'$ are disjoint, for any $\mu' \in \mathbf{M}'$ we get $card(\mu' \cap (\mathbf{P} \cup \mathbf{Z} \cup \mathbf{Z}')) = card(\mu' \cap \mathbf{P}) + card(\mu' \cap (\mathbf{Z} \cup \mathbf{Z}'))$ (2).
If $\mu' \in \mathbf{M}'(\mathcal{T}'_0)$, we get $card(\mu' \cap (\mathbf{Z} \cup \mathbf{Z}')) = card(\mathbf{Z}) = card(\mathbf{Z}')$ (3).
As $\mathbf{Z}$ is finite, from (0), (2) and (3) we get, for any $\mu, \nu$ in $\mathbf{M}$:
$card(e'(\mu) \cap (\mathbf{P} \cup \mathbf{Z} \cup \mathbf{Z}')) < card(e'(\nu) \cap (\mathbf{P} \cup \mathbf{Z} \cup \mathbf{Z}'))$ iff $card(\mu \cap \mathbf{P}) < card(\nu \cap \mathbf{P})$.
From (1) we get then, for any $\mathcal{T} \subseteq \mathbf{L}$: $\mu \in \mathbf{M}_{\prec_{(\mathbf{P},\mathbf{Q},\mathbf{Z})}}(\mathcal{T})$ iff $e'(\mu) \in \mathbf{M}'_{\prec_{(\mathbf{P}\cup\mathbf{Z}\cup\mathbf{Z}',\mathbf{Q},\emptyset)}}(\mathcal{T} \cup \mathcal{T}'_0)$.
As for any $\varphi \in \mathbf{L}$ we have $\mu \models \varphi$ iff $e'(\mu) \models \varphi$, we get the result. Moreover, we have the singleton property with $e'$ taking the role of $u'$ as described in remarks 6.2. $\square$

Another corollary of property 7.27 and of the satisfaction of (rk) by $<_{(\mathbf{P},\emptyset,\mathbf{Z})}$ – thus of the satisfaction of (RatM) by $NCIRC(\mathbf{P}, \emptyset, \mathbf{Z})$ when $\mathbf{P}$ is finite –, is that in theorem 7.26 we cannot require that we get a cardinality-based circumscription without fixed propositions. Indeed, this is a consequence of the preservation result 6.26, or 6.18. However, as in property 7.28, if $\mathbf{Z}$ is finite, we may require that we get a cardinality-based circumscription without varying propositions:

**Remark 7.29** Any ordinary circumscription in which finitely many propositions are circumscribed or varying can be expressed in terms of (Def$\downarrow_{\beta'}$) from a cardinality-based circumscription without any varying proposition:

If $\mathbf{P}$ and $\mathbf{Z}$ are finite, and if $V(\mathbf{L}) = \mathbf{P} \cup \mathbf{Q} \cup \mathbf{Z}$, there exist a language $\mathbf{L}'$, two sets $\mathbf{P}'$ and $\mathbf{Q}'$ such that $\mathbf{P}' \cup \mathbf{Q}' = V(\mathbf{L}') \supseteq V(\mathbf{L})$, and a formula $\beta' \in \mathbf{L}'$ such that we have, for any $\mathcal{T} \subseteq \mathbf{L}$:

$$CIRC(\mathbf{P}, \mathbf{Q}, \mathbf{Z})(\mathcal{T}) = NCIRC(\mathbf{P}', \mathbf{Q}', \emptyset)(\mathcal{T} \cup \{\beta'\}) \cap \mathbf{L}. \quad \square$$

# 8 Extending the vocabulary

## 8.1 Definitions: introducing (Def↑)

We examine now another modification of the vocabulary which is useful in non monotonic reasoning.

Beware that here, contrarily to the preceding sections, we make the following supposition[17]:

**We suppose** $V(\mathbf{L}') \subseteq V(\mathbf{L})$ **(i.e.** $\mathbf{L}' \subseteq \mathbf{L}$**).**

We will now denote the traces in the smaller language $\mathbf{L}'$ by $tr'$. Thus, for any $\mathcal{T} \in \mathbf{T}$, $tr'(\mathcal{T}) = \mathcal{T} \cap \mathbf{L}'$.

We suppose also that $f'$ is a pre-circumscription defined in the language $\mathbf{L}'$[18].
Then, we may define a pre-circumscription $f$ in $\mathbf{L}$ by:

$$\text{For any } \mathcal{T} \in \mathbf{T},\ f(\mathcal{T}) = f'(\mathcal{T} \cap \mathbf{L}') \sqcup \mathcal{T} = f'(tr'(\mathcal{T})) \sqcup \mathcal{T} \qquad \textbf{(Def}\uparrow\textbf{)}$$

This method is used in [Lif85] in order to express any circumscription in terms of a circumscription without varying predicates (see theorem 8.18 below for the propositional calculus case).

We could introduce more general forms of this method, as we have done with (Def$\downarrow_{\beta'}$) and (Def$\downarrow_{\mathcal{T}'}$) from (Def↓), but we do not know of any general use of such extensions, so we will not study them here. Moreover, we may design a method encompassing both (Def↓) and (Def↑), and also their respective extensions: this is done in [MR99].

We will examine now which properties are preserved or not by this "extension of the vocabulary", again following the order in which they appear in definition 5.2.

## 8.2 Preservation results with (Def↑)

**Preservation result 8.1** (Idem) is preserved by (Def↑): if $f'$ satisfies (Idem), then $f$ defined from $f'$ by (Def↑) satisfies (Idem). □

Proof: We suppose here that $f'$ is a pre-circumscription in $\mathbf{L}'$, which satisfies (Idem), and we define $f$ as follows: for any $\mathcal{T} \subseteq \mathbf{L}$, $f(\mathcal{T}) = f'(tr'(\mathcal{T})) \sqcup \mathcal{T}$.

Then, $f(f(\mathcal{T})) = f'(tr'(f'(tr'(\mathcal{T})) \sqcup \mathcal{T})) \sqcup f'(tr'(\mathcal{T})) \sqcup \mathcal{T}$. As $f'(tr'(\mathcal{T})) \subseteq \mathbf{L}'$, we get $tr'(f'(tr'(\mathcal{T})) \sqcup \mathcal{T}) = f'(tr'(\mathcal{T})) \sqcup' tr'(\mathcal{T})$. As $f'$ is a pre-circumscription, we get $f'(tr'(\mathcal{T})) \sqcup'$

[17] This is to keep, all along the text (at least in sections 6, 7 and 8), the fact that we start from a pre-circumscription $f'$ from which we define a pre-circumscription $f$.

[18] This is to keep things reasonably simple. Notice that we could start from any mapping $f'$ from $\mathbf{T}'$ to $\mathcal{P}(\mathbf{L}')$ (i.e. any inference operation satisfying LLE in terms of [KLM90]), we would get that $f$ is a pre-circumscription in $\mathbf{L}$.

$tr'(\mathcal{T}) = f'(tr'(\mathcal{T}))$. Thus $f(f(\mathcal{T})) = f'(f'(tr'(\mathcal{T}))) \sqcup f'(tr'(\mathcal{T})) \sqcup \mathcal{T} = f'(f'(tr'(\mathcal{T}))) \sqcup \mathcal{T}$. As $f'$ satisfies (Idem), we get $f(f(\mathcal{T})) = f'(tr'(\mathcal{T})) \sqcup \mathcal{T} = f(\mathcal{T})$: $f$ satisfies (Idem). □

**Preservation result 8.2** (RM) is preserved by (Def↑). □

Proof We suppose here that $f'$ is a pre-circumscription in $\mathbf{L}'$ which satisfies (RM), and that we have $f(\mathcal{T}) = f'(tr'(\mathcal{T})) \sqcup \mathcal{T}$ for any $\mathcal{T} \subseteq \mathbf{L}$. $\mathcal{T}_1$ and $\mathcal{T}_2$ are two subsets of $\mathbf{L}$. $f(\mathcal{T}_1 \sqcup \mathcal{T}_2) = f'(tr'(\mathcal{T}_1 \sqcup \mathcal{T}_2)) \sqcup \mathcal{T}_1 \sqcup \mathcal{T}_2 = f'(tr'(\mathcal{T}_1) \sqcup' tr'(\mathcal{T}_1 \sqcup \mathcal{T}_2)) \sqcup \mathcal{T}_1 \sqcup \mathcal{T}_2 \subseteq f'(tr'(\mathcal{T}_1)) \sqcup' tr'(\mathcal{T}_1 \sqcup \mathcal{T}_2) \sqcup \mathcal{T}_1 \sqcup \mathcal{T}_2$ (by (RM) for $f'$). As $tr'(\mathcal{T}_1 \sqcup \mathcal{T}_2) \sqcup \mathcal{T}_1 \sqcup \mathcal{T}_2 = \mathcal{T}_1 \sqcup \mathcal{T}_2$, and $f'(tr'(\mathcal{T}_1)) \sqcup \mathcal{T}_1 \sqcup \mathcal{T}_2 = (f'(tr'(\mathcal{T}_1)) \sqcup \mathcal{T}_1) \sqcup \mathcal{T}_2 = f(\mathcal{T}_1) \sqcup \mathcal{T}_2$ we get $f(\mathcal{T}_1 \sqcup \mathcal{T}_2) \subseteq f(\mathcal{T}_1) \sqcup \mathcal{T}_2$: $f$ satisfies (RM). □

**Non preservation result 8.3** (CR) is not preserved by (Def↑).

More precisely, if $f$ is non trivially (meaning $V(\mathbf{L}') \subset V(\mathbf{L})$) defined from $f'$ by (Def↑), then $f$ satisfies (CR) iff $f'$ satisfies (RM). □

Proof We will prove the stronger result of the second sentence. We already know that if $f'$ satisfies (RM), then $f$ satisfies (RM) thus (property 5.4-2) (CR).

Let us suppose now that $f'$ falsifies (RM). Thus, there exist some theories $\mathcal{T}'_1$ and $\mathcal{T}'_2$ of $\mathbf{L}'$ such that $f'(\mathcal{T}'_1 \sqcup' \mathcal{T}'_2) \not\subseteq f'(\mathcal{T}'_1) \sqcup' \mathcal{T}'_2$. This means that there exists some model $\mu'$ of $f'(\mathcal{T}'_1)$ and of $\mathcal{T}'_2$ which is not a model of $f'(\mathcal{T}'_1 \sqcup' \mathcal{T}'_2)$. We consider some $Z \in V(\mathbf{L}) - V(\mathbf{L}')$ and we define the following theories in $\mathbf{L}$: $\mathcal{T}_1 = \mathcal{T}'_1 \sqcup \neg Z$ and $\mathcal{T}_2 = \mathcal{T}'_1 \sqcup \mathcal{T}'_2 \sqcup Z$.

Then $tr'(\mathcal{T}_1) = \mathcal{T}'_1$, $tr'(\mathcal{T}_2) = \mathcal{T}'_1 \sqcup' \mathcal{T}'_2$, and $tr'(\mathcal{T}_1 \sqcap \mathcal{T}_2) = tr'(\mathcal{T}'_1) \sqcap tr'(\mathcal{T}'_2) = \mathcal{T}'_1$.

Let $\mu$ be the interpretation of $\mathbf{L}$ defined by $\mu = \mu' \cup \{Z\}$. Then, $\mu$ is a model of $f'(\mathcal{T}'_1) \sqcup \mathcal{T}'_2 \sqcup Z$, thus of $\mathcal{T}'_1$ ($f'$ is a pre-circumscription). Thus, $\mu$ is a model of $\mathcal{T}_2$, and a fortiori of $\mathcal{T}_1 \sqcap \mathcal{T}_2$. As $\mu$ is a model of $f'(tr'(\mathcal{T}_1 \sqcap \mathcal{T}_2)) = f'(\mathcal{T}'_1)$ and of $\mathcal{T}_1 \sqcap \mathcal{T}_2$, we get that $\mu$ is a model of $f'(tr'(\mathcal{T}_1 \sqcap \mathcal{T}_2)) \sqcup \mathcal{T}_1 \sqcap \mathcal{T}_2 = f(\mathcal{T}_1 \sqcap \mathcal{T}_2)$. As $\mu \not\models \neg Z$, we get $\mu \not\models \mathcal{T}_1$ and a fortiori $\mu$ is not a model of $f(\mathcal{T}_1)$. As $f'(\mathcal{T}'_1 \sqcup' \mathcal{T}'_2) \subseteq \mathbf{L}'$, we have $\mu \models f'(\mathcal{T}'_1 \sqcup' \mathcal{T}'_2)$ iff $\mu' \models' f'(\mathcal{T}'_1 \sqcup' \mathcal{T}'_2)$. As $\mu'$ is not a model of $f'(\mathcal{T}'_1 \sqcup' \mathcal{T}'_2)$, we get that $\mu$ is not a model of $f'(\mathcal{T}'_1 \sqcup' \mathcal{T}'_2)$ and a fortiori $\mu$ is not a model of $f'(\mathcal{T}'_1 \sqcup' \mathcal{T}'_2) \sqcup \mathcal{T}_2 = f(\mathcal{T}_2)$. This shows that we have $\mathbf{M}(f(\mathcal{T}_1 \sqcap \mathcal{T}_2)) \not\subseteq \mathbf{M}(f(\mathcal{T}_1)) \cup \mathbf{M}(f(\mathcal{T}_2)) = \mathbf{M}(f(\mathcal{T}_1) \sqcap f(\mathcal{T}_2))$, i.e. $f(\mathcal{T}_1) \sqcap f(\mathcal{T}_2) \not\subseteq f(\mathcal{T}_1 \sqcap \mathcal{T}_2)$: $f$ falsifies (CR). □

Notice that we will provide below (example 8.21) an explicit example in which $f'$ satisfies (CR) and falsifies (RM), and that this example will show that this negative result has some impact on one of the best known non monotonic formalism, circumscription. Moreover, in this example, $f'$ satisfies (CUMU) and (CR$\infty$), which shows that even in this case, (CR) is not necessarily preserved by (Def↑).

Notice also that we know from preservation result 8.2 and property 5.4-1 that if $V(\mathbf{L}')$ is finite, (CR) is preserved. We will come back to this importance of the preservation of (RM) and of this non preservation of (CR) when studying the problem of the preservation of multi preferential entailment.

Let us give a last comment about results 8.2 and 8.3. Together with property 5.4-2, these two results show that (RM) may be considered as a rather desirable extension of (CR). This provides in our opinion another argument in favor of the importance of (RM).

**Non preservation result 8.4** (CC) is not preserved by (Def$\uparrow$), even if $V(\mathbf{L})$ is finite. □

**Example 8.5** $V(\mathbf{L}) = \{A, B\}, V(\mathbf{L}') = \{A\}$.
We define $f'$ and $f$ as follows:
$f'(\mathcal{T}') = Th'(\perp)$ for $\mathcal{T}' \in \{Th'(\perp), Th'(A), Th'(\neg A)\}$ and $f'(Th'(\top)) = Th'(\top)$.
$f(\mathcal{T}) = f'(\mathcal{T} \cap \mathbf{L}') \sqcup \mathcal{T}$.

It is obvious that $f'$ satisfies (CC).

We define $\mathcal{T}_1 = Th(A \vee \neg B)$, $\mathcal{T}_2 = Th(B)$. We get $\mathcal{T}_1 \sqcup \mathcal{T}_2 = Th((A \vee \neg B) \wedge B) = Th(A \wedge B)$ and $\mathcal{T}_1 \cap \mathbf{L}' = \mathcal{T}_2 \cap \mathbf{L}' = Th'(\top)$ while $(\mathcal{T}_1 \sqcup \mathcal{T}_2) \cap \mathbf{L}' = Th'(A)$. Thus, $f(\mathcal{T}_1) = f'(\mathcal{T}_1 \cap \mathbf{L}') \sqcup \mathcal{T}_1 = \mathcal{T}_1$, $f(\mathcal{T}_2) = f'(\mathcal{T}_2 \cap \mathbf{L}') \sqcup \mathcal{T}_2 = \mathcal{T}_2$, $f(\mathcal{T}_1 \sqcup \mathcal{T}_2) = f'((\mathcal{T}_1 \sqcup \mathcal{T}_2) \cap \mathbf{L}') \sqcup \mathcal{T}_1 \sqcup \mathcal{T}_2 = Th(\perp)$: $f$ falsifies (CC). □

Remind (property 5.7-1 and preservation result 8.2) that in such an example, neither $f$ nor $f'$ can be a multi preferential entailment.

**Preservation result 8.6** (RI) is preserved by (Def$\uparrow$). □

Proof: $\mathcal{T}_1$ and $T_2$ are two theories of $\mathbf{L}$ (elements of $\mathbf{T}$). We suppose $f(\mathcal{T}_1) \subseteq \mathcal{T}_2$, i.e. $f'(tr'(\mathcal{T}_1)) \sqcup \mathcal{T}_1 \subseteq \mathcal{T}_2$. Then $(f'(tr'(\mathcal{T}_1)) \sqcup \mathcal{T}_1) \cap \mathbf{L}' \subseteq \mathcal{T}_2 \cap \mathbf{L}'$, thus $f'(tr'(\mathcal{T}_1)) \subseteq \mathcal{T}_2 \cap \mathbf{L}' = tr'(\mathcal{T}_2)$. From (RI) we get then $f'(tr'(\mathcal{T}_2)) = tr'(\mathcal{T}_2)$. Thus, $f(\mathcal{T}_2) = f'(tr'(\mathcal{T}_2)) \sqcup \mathcal{T}_2 = tr'(\mathcal{T}_2) \sqcup \mathcal{T}_2 = \mathcal{T}_2$: $f$ satisfies (RI). □

**Preservation result 8.7** (DC) is preserved by (Def$\uparrow$). □

Proof: $\mathcal{T}_1 \in \mathbf{T}, \mathcal{T}_2 \in \mathbf{T}$, $f'$ satisfies (DC). $f(\mathcal{T}_1 \cap \mathcal{T}_2) = f'(\mathcal{T}_1 \cap \mathcal{T}_2 \cap \mathbf{L}') \sqcup (\mathcal{T}_1 \cap \mathcal{T}_2) = f'((\mathcal{T}_1 \cap \mathbf{L}') \cap (\mathcal{T}_2 \cap \mathbf{L}')) \sqcup (\mathcal{T}_1 \cap \mathcal{T}_2)$. As $f'$ satisfies (DC) we get $f(\mathcal{T}_1 \cap \mathcal{T}_2) \subseteq (f'(\mathcal{T}_1 \cap \mathbf{L}') \sqcup' f'(\mathcal{T}_2 \cap \mathbf{L}')) \sqcup (\mathcal{T}_1 \cap \mathcal{T}_2) \subseteq f'(\mathcal{T}_1 \cap \mathbf{L}') \sqcup f'(\mathcal{T}_2 \cap \mathbf{L}') \sqcup \mathcal{T}_1 \sqcup \mathcal{T}_2 = (f'(\mathcal{T}_1 \cap \mathbf{L}') \sqcup \mathcal{T}_1) \sqcup (f'(\mathcal{T}_2 \cap \mathbf{L}')) \sqcup \mathcal{T}_2) = f(\mathcal{T}_1) \sqcup f(\mathcal{T}_2)$: $f$ satisfies (DC). □

**Preservation result 8.8** (CT) is preserved by (Def$\uparrow$). □

Proof: $\mathcal{T}_1 \in \mathbf{T}, \mathcal{T}_2 \in \mathbf{T}$. We suppose that we have: $\mathcal{T}_2 \subseteq f(\mathcal{T}_1)$ and $f'$ satisfies (CT). As we already know that $f$ is a pre-circumscription, we get then $\mathcal{T}_1 \sqcup \mathcal{T}_2 \subseteq f(\mathcal{T}_1)$. Thus, taking the intersections with $\mathbf{L}'$, we get $(\mathcal{T}_1 \sqcup \mathcal{T}_2) \cap \mathbf{L}' \subseteq (f'(\mathcal{T}_1 \cap \mathbf{L}') \sqcup \mathcal{T}_1) \cap \mathbf{L}'$. As $f'(\mathcal{T}_1 \cap \mathbf{L}') \subseteq \mathbf{L}'$, we get $(f'(\mathcal{T}_1 \cap \mathbf{L}') \sqcup \mathcal{T}_1) \cap \mathbf{L}' = f'(\mathcal{T}_1 \cap \mathbf{L}') \sqcup' (\mathcal{T}_1 \cap \mathbf{L}')$ and, as $f'$ is a pre-circumscription, $f'(\mathcal{T}_1 \cap \mathbf{L}') \sqcup' (\mathcal{T}_1 \cap \mathbf{L}') = f'(\mathcal{T}_1 \cap \mathbf{L}')$. Thus we have $(\mathcal{T}_1 \sqcup \mathcal{T}_2) \cap \mathbf{L}' \subseteq f'(\mathcal{T}_1 \cap \mathbf{L}')$. From (CT) for $f'$ we get then $f'((\mathcal{T}_1 \sqcup \mathcal{T}_2) \cap \mathbf{L}') \subseteq f'(\mathcal{T}_1 \cap \mathbf{L}')$. Thus we get $f(\mathcal{T}_1 \sqcup \mathcal{T}_2) = f'((\mathcal{T}_1 \sqcup \mathcal{T}_2) \cap \mathbf{L}') \sqcup \mathcal{T}_1 \sqcup \mathcal{T}_2 \subseteq f'(\mathcal{T}_1 \cap \mathbf{L}') \sqcup \mathcal{T}_1 \sqcup \mathcal{T}_2 \subseteq f'(\mathcal{T}_1 \cap \mathbf{L}') \sqcup f(\mathcal{T}_1) = f(\mathcal{T}_1)$: $f$ satisfies (CT). □

**Preservation result 8.9** (CM) is preserved by (Def$\uparrow$). $\Box$

Proof: $\mathcal{T}_1 \in \mathbf{T}, \mathcal{T}_2 \in \mathbf{T}$. We suppose that we have: $\mathcal{T}_2 \subseteq f(\mathcal{T}_1)$ and $f'$ satisfies (CM). As in the proof of preservation result 8.8, we get $(\mathcal{T}_1 \sqcup \mathcal{T}_2) \cap \mathbf{L}' \subseteq f'(\mathcal{T}_1 \cap \mathbf{L}')$. As $f'$ satisfies (CM), we get then $f'(\mathcal{T}_1 \cap \mathbf{L}') \subseteq f'((\mathcal{T}_1 \sqcup \mathcal{T}_2) \cap \mathbf{L}')$. Thus we get $f(\mathcal{T}_1) = f'(\mathcal{T}_1 \cap \mathbf{L}') \sqcup \mathcal{T}_1 \subseteq f'((\mathcal{T}_1 \sqcup \mathcal{T}_2) \cap \mathbf{L}') \sqcup \mathcal{T}_1 \subseteq f'((\mathcal{T}_1 \sqcup \mathcal{T}_2) \cap \mathbf{L}') \sqcup \mathcal{T}_1 \sqcup \mathcal{T}_2 = f(\mathcal{T}_1 \sqcup \mathcal{T}_2)$: $f$ satisfies (CM). $\Box$

Thus, (CUMU) is also preserved by (Def$\uparrow$).

**Preservation result 8.10** (PC) is preserved by (Def$\uparrow$). $\Box$

Proof: Let us suppose that we have: $f'$ satisfies (PC) and $f(\mathcal{T}) = f'(tr'(\mathcal{T})) \sqcup \mathcal{T} = Th(\perp)$. We have $f'(tr'(\mathcal{T})) \sqcup \mathcal{T} = Th(\perp)$ iff $tr'(f'(tr'(\mathcal{T})) \sqcup \mathcal{T}) = Th(\perp)$ from property 3.6-3. As $f'(tr'(\mathcal{T})) \subseteq \mathbf{L}'$, we get $tr'(f'(tr'(\mathcal{T})) \sqcup \mathcal{T}) = f'(tr'(\mathcal{T})) \sqcup' tr'(\mathcal{T})$. As $f'$ is a pre-circumscription we get $f'(tr'(\mathcal{T})) \sqcup' tr'(\mathcal{T}) = f'(tr'(\mathcal{T}))$ and as $f'$ satisfies (PC) we get $\perp \in tr'(\mathcal{T})$, i.e. $\perp \in Th(\mathcal{T})$ from property 3.6-3 again. $\Box$

**Preservation result 8.11** (DR) is preserved by (Def$\uparrow$). $\Box$

Proof: We suppose that $f'$ satisfies (DR). Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be in $\mathbf{T}$. $f(\mathcal{T}_1 \cap \mathcal{T}_2) = f'(\mathcal{T}_1 \cap \mathcal{T}_2 \cap \mathbf{L}') \sqcup (\mathcal{T}_1 \cap \mathcal{T}_2)$. As $f'$ satisfies (DR), we know that we have $f'(\mathcal{T}_1 \cap \mathcal{T}_2 \cap \mathbf{L}') = f'((\mathcal{T}_1 \cap \mathbf{L}') \cap (\mathcal{T}_2 \cap \mathbf{L}')) \subseteq f'(\mathcal{T}_1 \cap \mathbf{L}') \cup f'(\mathcal{T}_2 \cap \mathbf{L}')$. As we know that a theory is included in a union of two theories iff it is included in at least one of these two theories (this is "equivalence (E)" in note 6), we get that, for some $i \in \{1,2\}$, we have $f'(\mathcal{T}_1 \cap \mathcal{T}_2 \cap \mathbf{L}') \subseteq f'(\mathcal{T}_i \cap \mathbf{L}')$. Thus we get, for some $i \in \{1,2\}$, $f(\mathcal{T}_1 \cap \mathcal{T}_2) = f'(\mathcal{T}_1 \cap \mathcal{T}_2 \cap \mathbf{L}') \sqcup (\mathcal{T}_1 \cap \mathcal{T}_2) \subseteq f'(\mathcal{T}_i \cap \mathbf{L}') \sqcup (\mathcal{T}_1 \cap \mathcal{T}_2) \subseteq f'(\mathcal{T}_i \cap \mathbf{L}') \sqcup \mathcal{T}_i = f(\mathcal{T}_i)$. A fortiori we get $f(\mathcal{T}_1 \cap \mathcal{T}_2) \subseteq f(\mathcal{T}_1) \cup f(\mathcal{T}_2)$: $f$ satisfies (DR). $\Box$

**Preservation result 8.12** (CNM) is preserved by (Def$\uparrow$). $\Box$

Proof: We suppose: $f$ satisfies (CNM), $\mathcal{T}_1 \subseteq \mathbf{L}, \mathcal{T}_2 \subseteq \mathbf{L}$ and $\perp \notin f(\mathcal{T}_1) \sqcup \mathcal{T}_2$. Thus, $\perp \notin f'(tr'(\mathcal{T}_1)) \sqcup \mathcal{T}_1 \sqcup \mathcal{T}_2$, and, as $f'(tr'(\mathcal{T}_1)) \subseteq \mathbf{L}'$, this is equivalent to $\perp \notin f'(tr'(\mathcal{T}_1)) \sqcup tr'(\mathcal{T}_1 \sqcup \mathcal{T}_2)$ (cf properties 3.6-4 and 3.8-2c). From (CNM) for $f'$, we get then $\perp \notin f'(tr'(\mathcal{T}_1)) \sqcup' f'(tr'(\mathcal{T}_1 \sqcup \mathcal{T}_2))$. From properties 3.6-4 and 3.8-2c again, we know that we have $\perp \in f'(tr'(\mathcal{T}_1)) \sqcup' f'(tr'(\mathcal{T}_1 \sqcup \mathcal{T}_2)) \sqcup \mathcal{T}_1 \sqcup \mathcal{T}_2$ iff $\perp \in f'(tr'(\mathcal{T}_1)) \sqcup' f'(tr'(\mathcal{T}_1 \sqcup \mathcal{T}_2)) \sqcup' tr'(\mathcal{T}_1) \sqcup' tr'(\mathcal{T}_1 \sqcup \mathcal{T}_2)$, iff (as $f'$ is a pre-circumscription) $\perp \in f'(tr'(\mathcal{T}_1)) \sqcup' f'(tr'(\mathcal{T}_1 \sqcup \mathcal{T}_2))$. Thus we get $\perp \notin f'(tr'(\mathcal{T}_1)) \sqcup' f'(tr'(\mathcal{T}_1 \sqcup \mathcal{T}_2)) \sqcup \mathcal{T}_1 \sqcup \mathcal{T}_2$, i.e. $\perp \notin f(\mathcal{T}_1) \sqcup f(\mathcal{T}_1 \sqcup \mathcal{T}_2)$: $f$ satisfies (CNM). $\Box$

**Preservation result 8.13** (RatM) is preserved by (Def$\uparrow$). $\Box$

Proof: We suppose: $f$ satisfies (RatM), $\mathcal{T}_1 \subseteq \mathbf{L}, \mathcal{T}_2 \subseteq \mathbf{L}$ and $\perp \notin f(\mathcal{T}_1) \sqcup \mathcal{T}_2$. Thus, $\perp \notin f'(tr'(\mathcal{T}_1)) \sqcup \mathcal{T}_1 \sqcup \mathcal{T}_2$, and, as $f'(tr'(\mathcal{T}_1)) \subseteq \mathbf{L}'$. As in the proof of preservation result 8.12, we get that this is equivalent to $\perp \notin f'(tr'(\mathcal{T}_1)) \sqcup tr'(\mathcal{T}_1 \sqcup \mathcal{T}_2)$. From (RatM) for $f'$, we get then $f'(tr'(\mathcal{T}_1)) \subseteq f'(tr'(\mathcal{T}_1 \sqcup \mathcal{T}_2))$. Thus $f'(tr'(\mathcal{T}_1)) \sqcup \mathcal{T}_1 \subseteq f'(tr'(\mathcal{T}_1 \sqcup \mathcal{T}_2)) \sqcup \mathcal{T}_1 \sqcup \mathcal{T}_2$, i.e. $f(\mathcal{T}_1) \subseteq f(\mathcal{T}_1 \sqcup \mathcal{T}_2)$: $f$ satisfies (RatM). □

Now, we examine the situation with multi preferential entailments and X-mappings.

**Preservation result 8.14** 1. We suppose here that $V(\mathbf{L}')$ is strictly included in $V(\mathbf{L})$.

(a) The notions of multi preferential entailment, and of preferential entailment, are preserved by (Def↑) iff (RM) is satisfied.

(b) The notions of multi preferential entailment, and of preferential entailment are preserved by (Def↑) iff an associated (multi) preference relation satisfies (cl).

2. Thus, if $V(\mathbf{L})$ is finite, the notions of multi preferential entailment, and of preferential entailment, are always preserved by (Def↑). □

Proof: We know that (a) and (b) are equivalent from properties 5.9 (points 5 and 6), and 5.7-2.

Also, point 2 is an immediate consequence of point 1: in the finite case, any (multi) preference relation satisfies trivially (cl), thus any (multi) preference relation satisfies (RM).

Thus, we prove point 1 (a) here:

From non preservation result 8.3, we already know that if $f'$ falsifies (RM), then $f$ cannot be a multi preferential entailment, because it falsifies (CR). Thus we get that (multi) preferential entailments are not preserved if they falsify (RM).

We suppose now that $f'$ is a pre-circumscription in $\mathbf{L}'$ satisfying (RM). We know from property 5.9-5 that $f' = f_{\prec'_m}$ where $\prec'_m$ is a multi preference relation in $\mathbf{L}'$ satisfying (cl)[19]. $\prec'_m$ is defined (definition 4.5) by a set of states $\mathbf{S}'$, a mapping $l'$ from $\mathbf{S}'$ to $\mathbf{M}'$, and the relation $\prec'_m$ on $\mathbf{S}'$ itself.

We define $f$ in $\mathbf{L}$ by (Def↑): $f(\mathcal{T}) = f(tr'(\mathcal{T})) \sqcup \mathcal{T}$ for any $\mathcal{T} \subseteq \mathbf{L}$. We consider the multi preferential entailment $f_{\prec_m}$ in $\mathbf{L}$ defined as follows, with $\mathbf{Z} = V(\mathbf{L}) - V(\mathbf{L}') \neq \emptyset$. $\mathbf{S} = \mathbf{S}' \times \mathcal{P}(\mathbf{Z})$, $l$ is the mapping from $\mathbf{S}$ to $\mathbf{M}$ defined by: if $w = (w', \mathbf{Z}_w)$ where $w' \in \mathbf{S}'$ and $\mathbf{Z}_w \subseteq \mathbf{Z}$, we define $l(w) = l'(w') \cup \mathbf{Z}_w$: we map any state $w$ to the interpretation of $\mathbf{L}$ which corresponds to the interpretation $l'(w')$ for the symbols of $V(\mathbf{L}')$, and which corresponds to $\mathbf{Z}_w$ for the symbols of $V(\mathbf{L}) - V(\mathbf{L}')$.

[19]Notice that the hypothesis that $f'$ is a multi preferential entailment is redundant here. Moreover, the part of the following proof dealing with multi preference relations $\prec_m$ and $\prec'_m$ could have been avoided, the preservation of multi preferential entailments satisfying (RM) being a consequence of preservation result 8.2 and of property 5.9-5. However, an interest of this part of the proof is that it provides an explicit definition of $\prec_m$ from $\prec'_m$, in all the cases where this has a meaning (see also note 20 below).

We define $\prec_m$ on $\mathbf{S}$ by $(w', \mathbf{Z}_w) \prec_m (w'_1, \mathbf{Z}_{w_1})$ iff $w' \prec_m w'_1$. **(Def↑: $\prec'_m \leadsto \prec_m$)**

We prove now $f = f_{\prec_m}$.

$\mu \in \mathbf{M}(f(\mathcal{T}))$ iff $\mu \in \mathbf{M}(\mathcal{T})$ and $\mu' = \mu \cap V(\mathbf{L}') \in \mathbf{M}'(f'(tr'(\mathcal{T})))$. $\mathbf{M}'(f'(tr'(\mathcal{T}))) = \mathbf{M}_{\prec'_m}(tr'(\mathcal{T}))$ from (cl) for $\prec'_m$. $\mu \in \mathbf{M}_{\prec_m}(\mathcal{T})$ iff there exists $w = (w', \mathbf{X}) \in \mathbf{S}_{\prec_m}(\mathcal{T})$ such that $l'(w') = \mu \cap V(\mathbf{L}')$ and $\mathbf{X} = \mu \cap \mathbf{Z}$ (from definition 4.5 and (Def↑: $\prec'_m \leadsto \prec_m$)). $w \in \mathbf{S}_{\prec_m}(\mathcal{T})$ iff $l(w) = \mu \in \mathbf{M}(\mathcal{T})$ and for any $w_1 = (w'_1, \mathbf{X}_1) \in \mathbf{S}$ such that $l(w_1) = l(w'_1) \cup \mathbf{X}_1 \in \mathbf{M}(\mathcal{T})$, we have $w_1 \not\prec_m w$.

$w_1 \prec_m w$ iff $l'(w'_1) \prec'_m l'(w')$ from (Def↑: $\prec'_m \leadsto \prec_m$).

Thus we get $\mu \in \mathbf{M}_{\prec_m}(\mathcal{T})$ iff 1) $\mu \in \mathbf{M}(\mathcal{T})$, 2) there exists $w' \in \mathbf{S}'$ such that $l'(w') \in \mathbf{M}'(tr'(\mathcal{T}))$ and 3) for any $w'_1 \in \mathbf{S}'$ such that $l'(w'_1) \in \mathbf{M}'(tr'(\mathcal{T}))$ we have $l'(w'_1) \not\prec'_m l'(w')$. This means $\mu \in \mathbf{M}_{\prec_m}(\mathcal{T})$ iff $\mu \in \mathbf{M}(f(\mathcal{T}))$. Thus $f = f_{\prec_m}$.

Notice that, as $\mathbf{M}(f(\mathcal{T}))$ is a closed set, we get also the result that $\prec_m$ satisfies (cl).

This establishes the result of the preservation of the notion of closed multi preferential entailment (i.e. of multi preferential entailments satisfying (RM)), and this provides an explicit definition of the multi preference relation $\prec_m$.

Let us suppose now that our multi preference relation $\prec'_m$ is in fact a preference relation $\prec'$ in $\mathbf{L}'$: $\prec'_m = \prec'$. Above proof still works, and $\prec_m$ defined from $\prec'$ by (Def↑: $\prec'_m \leadsto \prec_m$) is a preference relation in $\mathbf{L}$. We get the following definition of $\prec$ from $\prec'$, which is (Def↑: $\prec'_m \leadsto \prec_m$) simplified to the case where $\prec'_m = \prec'$:

We define $f$ in $\mathbf{L}$ by (Def↑): $f(\mathcal{T}) = f(tr'(\mathcal{T})) \sqcup \mathcal{T}$ for any $\mathcal{T} \subseteq \mathbf{L}$. We consider the following preferential entailment $f_\prec$ in $\mathbf{L}$ defined as follows, with $\mathbf{Z} = V(\mathbf{L}) - V(\mathbf{L}')$:

$\mu \prec \nu$ iff $\mu \cap V(\mathbf{L}') \prec' \nu \cap V(\mathbf{L}')$. **(Def↑: $\prec' \leadsto \prec$)**

The proof given above still works, and shows that $f = f_\prec$ is a closed preferential entailment, i.e. a preferential entailment satisfying (RM). Remind that any preferential entailment satisfying (RM) satisfies also (DC)[20] a result to be compared with the preservation of (DC) by (Def↑). □

**Non preservation result 8.15** The notion of X-mapping (which are pre-circumscriptions) is not preserved by (Def↑). □

Proof: It is known (see [MR98a]) that there exist X-mappings which falsify (RM) (see example 8.21 below), while any X-mapping satisfies (CR). Then, use non preservation result

[20] There exist pre-circumscriptions satisfying (RM) and (DC) which falsify (DCC), thus which are not preferential entailments [MR00]. Thus, we cannot simply use preservation result 8.7 for dealing with the case of preferential entailments, and we need a specific proof in this case. Doing so, it was easy to give also the similar proof for multi preferential entailments (cf note 19).

8.3. $\square$

As for multi preferential entailments, in the finite case the notion of X-mapping is preserved by (Def$\uparrow$):

**Preservation result 8.16** If $V(\mathbf{L})$ is finite, the notion of X-mapping (which are pre-circumscriptions) is preserved by (Def$\uparrow$). $\square$

Proof: In a finite language, a pre-circumscription is an X-mapping iff it is a cumulative multi preferential entailment [MR98a], and we have seen in preservation results 8.8, 8.9 and 8.14 that in the finite case the notion of cumulative multi preferential entailment is preserved by (Def$\uparrow$).

We have achieved the proof, but we have not yet given an explicit way of defining a set $X$ such that $f = f_X$ from the set $X'$, contrarily to what we have done with preservation result 6.19. At first sight, it does not seem that such a definition is so obvious. However, using results of [MR98a], we provide now a way of defining a set $X$ from $X'$.

This method has also the advantage of giving a constructive definition of any finite X-mapping in terms of a multi preferential entailment, and the multi preference relation obtained is the smallest possible, in terms of size of the set $\mathbf{S}'$ and of the graph of the transitive and irreflexive relation (step 4 below, in the language $\mathbf{L}'$).

The method provides also the opposite construction: starting from a given transitive and irreflexive multi preference relation $\prec_m$, we show how to construct a set $X$ such that $f_{\prec_m} = f_X$ (step 6, in the language $\mathbf{L}$, in the case occurring there, as we start from a relation $\prec_m$ which has the smallest possible graph, we get a set of formulas $X$ which is the smallest possible).

This is why we detail the method here, even if, for our final goal, which is defining the set $X$ from the set $X'$, a shorter method could certainly be given.

1. Define the set $X'_\wedge$ which is the smallest which has the same $\wedge$-closure than the set $X'$: from property 5.11-2, we know that $X'_\wedge$ is the smallest set such that we have $f_{X'} = f_{X'_\wedge}$.
2. For any formula $x' \in X'_\wedge$, compute the formula $\varphi(x') = \neg x' \wedge (\bigwedge_{y' \in X'_\wedge,\, y' \neq x',\, x' \models' y'} y')$ (cf [MR98a, Definition 5.9]).
3. As we suppose that $f_{X'}$ is a pre-circumscription, we know that each $\varphi(x')$ is equivalent to a complete theory in $\mathbf{L}'$ [MR98a, Property 5.14]. Let us call $l'(x')$ the interpretation of $\mathbf{L}'$ such that $\mathbf{M}'(\varphi(x')) = \{l'(x')\}$, i.e. $l'(x')$ is the model of the complete theory $Th'(\varphi(x'))$ in $\mathbf{L}'$.
4. We define then the multi preference relation $\prec'_m$ as follows: $\mathbf{S}' = X'_\wedge$, $l'$ is as defined above, and we define $\prec'_m$ as follows: for any $x', y'$ in $\mathbf{S}'$, we define $x' \prec'_m y'$ if $y' \models' x'$ and $y' \neq x'$. From [MR98a, Property 5.14], we get $f_{\prec'_m} = f_{X'}$.

5. Define $\prec_m$ from $\prec'_m$ as done in (Def$\uparrow$:$\prec'_m \rightsquigarrow \prec_m$) (proof of preservation result 8.14).

   Thus we get: $\mathbf{S} = \mathbf{S}' \times \mathcal{P}(\mathbf{Z})$ where $\mathbf{Z} = V(\mathbf{L}) - V(\mathbf{L}')$, $l(x', w'') = l'(x') \cup w''$ for any $x' \in X'_\wedge = \mathbf{S}', w'' \subseteq \mathbf{Z}$, and for any $x', y'$ in $\mathbf{S}' = X'_\wedge$ and any $w'', w'''$ in $\mathcal{P}(\mathbf{Z})$, $(x', w'') \prec_m (y', w''')$ iff $x' \prec'_m y'$.

6. Then, in order to get the set $X$ from $\prec_m$, we do the reverse operation of what we have done above in order to get $\prec'_m$ from the set $X'_\wedge$[21]. Thus, for any element $w = (x', w'') \in \mathbf{S}$, where $x' \in \mathbf{S}' = X'_\wedge$ and $w'' \subseteq \mathbf{Z}$, we define the formula $\xi(w)$ as follows [MR98a, Definitions 5.5]: $\mathbf{M}(\neg\xi(w)) = \{l'(x') \cup w''\} \cup \{l'(y') \cup w'''\}_{(y', w''') \prec_m (x', w'')}$. We take the set $\{\xi(w)\}_{w \in \mathbf{S}}$ as our set $X$ and we get $f = f_X$ from [MR98a, Property 5.6] applied to the multi preferential entailment defined by the multi preference relation $\prec_m$.

If $X'_\wedge$ has $n$ elements and $\mathbf{Z}$ has $z$ elements, we get a set $X = X_\wedge$ with $n \times 2^z$ elements, and this is the smallest possible set (from [MR98a, Property 5.14] and property 5.11-2).

We define now directly $X = X_\wedge$ in terms of $X'_\wedge$ as follows:

Firstly, we have the following two equalities, for any $x' \in X'_\wedge$ (see $\varphi'(x')$ in point 2 above):

$$\bigwedge_{x' \models' y',\ y' \in X'_\wedge} \neg\varphi'(y') = \bigwedge_{x' \models' y',\ y' \in X'_\wedge} y' = x' \quad (1)$$

$$\bigwedge_{x' \models' y',\ y' \in X'_\wedge - \{x'\}} \neg\varphi'(y') = \bigwedge_{x' \models' y',\ y' \in X'_\wedge - \{x'\}} y' \quad (2)$$

The proofs have no difficulty (use a recurrence on the length of the greatest chain of strict entailments in the set $X'_\wedge$, starting the recurrence with the elements which are at the end of these chains).

Secondly, from the definition of $\xi(x', y')$ (point 6 above), we get:

$\mathbf{M}(\neg\xi(x', w'')) = \{l'(x') \cup w''\} \cup \{l'(y') \cup w'''\}_{(y', w''') \prec_m (x', w'')}$,

thus, from the definitions of $l'(x')$ (point 3), of $\prec_m$ (point 5) and of $\prec'_m$ (point 4), we get:

$\neg\xi(x', w'') = (\varphi'(x') \wedge \bigwedge_{Z \in w''} Z \wedge \bigwedge_{Z \in \mathbf{Z} - w''} \neg Z) \vee (\bigvee_{x' \models' y',\ y' \in X'_\wedge - \{x'\}} \varphi'(y'))$, i.e.

$\xi(x', w'') = (\neg\varphi'(x') \vee \bigvee_{Z \in w''} \neg Z \vee \bigvee_{Z \in \mathbf{Z} - w''} Z) \wedge (\bigwedge_{x' \models' y',\ y' \in X'_\wedge - \{x'\}} \neg\varphi'(y'))$

$=$

$(\bigwedge_{x' \models' y',\ y' \in X'_\wedge} \neg\varphi'(y')) \vee ((\bigvee_{Z \in w''} \neg Z \vee \bigvee_{Z \in \mathbf{Z} - w''} Z) \wedge (\bigwedge_{x' \models' y',\ y' \in X'_\wedge - \{x'\}} \neg\varphi'(y')))$.

Thirdly, using equalities (1) and (2) given above, we get a direct definition **(Def$\uparrow$:** $X' \rightsquigarrow X$**)**:

[21] Notice that, as $\prec'_m$ obtained by the method given is in a "reduced form" (the set $\mathbf{S}'$ and the graph for $\prec'_m$ have the smallest possible number of elements), so is $\prec_m$. This implies that we will get a set $X$ which is the smallest possible: $X = X_\wedge$.

$$\xi(x', w'') = x' \vee ((\bigvee_{Z \in w''} \neg Z \vee \bigvee_{Z \in \mathbf{Z} - w''} Z) \wedge (\bigwedge_{x' \models' y',\ y' \in X'_\wedge - \{x'\}} y')) \qquad (3)$$

Using the definition of $\varphi'(x')$ again, we get:

$$\xi(x', w'') = x' \vee ((\bigvee_{Z \in w''} \neg Z \vee \bigvee_{Z \in \mathbf{Z} - w''} Z) \wedge \varphi'(x')). \qquad (3')$$

Now, starting from (3) and applying distributivity, we get a third possible writing:

$$\xi(x', w'') = (x' \vee (\bigvee_{Z \in w''} \neg Z \vee \bigvee_{Z \in \mathbf{Z} - w''} Z)) \wedge (\bigwedge_{x' \models' y',\ y' \in X'_\wedge - \{x'\}} y'). \qquad (4)$$

We get then directly (and in three forms) the smallest possible set $X = \{\xi(x', w'')\}_{x' \in X'_\wedge, w'' \subseteq \mathbf{Z}}$. $\square$

Let us give a small example:

**Example 8.17** $V(\mathbf{L}') = \{A, B\}$, $\mathbf{Z} = \{Z\}$, $V(\mathbf{L}) = V(\mathbf{L}') \cup \mathbf{Z}$.
$X' = \{A \vee B,\ A \vee \neg B,\ \neg A \vee \neg B,\ \neg A \wedge B,\ \neg A \wedge \neg B)\}$.

$f' = f_{X'}$ is the X-mapping defined on $\mathbf{L}'$ by the set $X'$ (we have chosen a set $X'$ such that $f'$ is a pre-circumscription, see point 3 below) and $f$ is the pre-circumscription defined in $\mathbf{L}$ by $f(\mathcal{T}) = f'(Th(\mathcal{T}) \cap \mathbf{L}') \sqcup \mathcal{T}$.

From above result, we know that there exists in $\mathbf{L}$ a set $X$ such that $f$ is the X-mapping defined by $X$: $f = f_X$.

Firstly, let us use the indirect method of computation of the set $X$ given in points 1–6 above, which has the advantage of describing $f'$ and $f$ also in terms of multi preferential entailments. Another advantage of the method used here (described in the proof of preservation result 8.16) is that we get the most economical way, in terms of the size of the set of states and of the graph of the multi preference relation, to describe such multi preferential entailments.

1. $X' = X'_\wedge$ here.
2. Calculating all the $\varphi'(x')$ ($x' \in X'_\wedge$), we get: $\varphi'(A \vee B) = \neg A \wedge \neg B$, $\varphi'(A \vee \neg B) = \neg A \wedge B$, $\varphi'(\neg A \vee \neg B) = A \wedge B$, $\varphi'(\neg A \wedge B) = A \wedge \neg B$, $\varphi'(\neg A \wedge \neg B) = A \wedge \neg B$.
3. Each $\varphi'(x')$ corresponds to a theory complete in $\mathbf{L}'$, thus $f_{X'}$ is a pre-circumscription in $\mathbf{L}'$[22].

[22]Notice that the $\varphi'(x')$'s are not all distinct, which means that $f' = f_{X'}$ is not a preferential entailment in $\mathbf{L}'$ [MR98a]. And in fact it is easy to show that $f'$ falsifies (DC): $f'(\neg B) = Th(\neg B), f'(\neg(A \Leftrightarrow B)) = Th(\neg(A \Leftrightarrow B))$, $Th(\neg B) \cap Th(\neg(A \Leftrightarrow B)) = Th(\neg A \vee \neg B)$, $f'(\neg A \vee \neg B) = Th(\neg A) \not\subseteq Th((\neg B) \wedge \neg(A \Leftrightarrow B)) = Th(A \wedge \neg B)$. For these computations, it may be easier (at least manually) to use the multi preferential entailment definition of $f'$, i.e. $f_{\prec'_m}$, described in points 3 and 4.

We get: $l'(A \vee B) = \emptyset$, $l'(A \vee \neg B) = \{B\}$, $l'(\neg A \vee \neg B) = \{A, B\}$, $l'(\neg A \wedge B) = l'(\neg A \wedge \neg B) = \{A\}$.

4. $\prec'_m$ is defined as the reverse relation of "$\models'$ and $\neq$" on $X'_\wedge = \mathbf{S}'$, i.e.: $A \vee B \prec'_m \neg A \wedge B$, $A \vee \neg B \prec'_m \neg A \wedge \neg B$, $\neg A \vee \neg B \prec'_m \neg A \wedge B$, $\neg A \vee \neg B \prec'_m \neg A \wedge \neg B$ and nothing else.

5. $\mathbf{S} = \{w_i\}_{i \in \{1, \cdots, 10\}}$ with $w_1 = (A \vee B, \emptyset)$, $w_2 = (A \vee B, \{Z\})$, $w_3 = (A \vee \neg B, \emptyset)$, $w_4 = (A \vee \neg B, \{Z\})$, $w_5 = (\neg A \vee \neg B, \emptyset)$, $w_6 = (\neg A \vee \neg B, \{Z\})$, $w_7 = (\neg A \wedge B, \emptyset)$, $w_8 = (\neg A \wedge B, \{Z\})$, $w_9 = (\neg A \wedge \neg B, \emptyset)$, $w_{10} = (\neg A \wedge \neg B, \{Z\})$. $\prec_m$ is defined on $\mathbf{S}$ by $(x', x'') \prec_m (y', y'')$ iff $x' \prec'_m y'$, where $x', y'$ are in $\mathbf{S}' = X'_\wedge$ and $x'', y''$ are any subsets of $\mathbf{Z}$.

6. $\xi(w_1) = A \vee B \vee Z$, $\xi(w_2) = A \vee B \vee \neg Z$, $\xi(w_3) = A \vee \neg B \vee Z$, $\xi(w_4) = A \vee \neg B \vee \neg Z$, $\xi(w_5) = \neg A \vee \neg B \vee Z$, $\xi(w_6) = \neg A \vee \neg B \vee \neg Z$, $\xi(w_7) = (\neg A \wedge B) \vee (A \wedge \neg B \wedge Z)$, $\xi(w_8) = (\neg A \wedge B) \vee (A \wedge \neg B \wedge \neg Z)$, $\xi(w_9) = (\neg A \wedge \neg B) \vee (A \wedge \neg B \wedge Z)$, $\xi(w_{10}) = (\neg A \wedge \neg B) \vee (A \wedge \neg B \wedge \neg Z)$.

   Thus, we get our set $X = X_\wedge = \{\xi(w_i)\}_{i \in \{1, \cdots, 10\}}$. It may be checked that we have indeed $f(\mathcal{T}) = f_X(\mathcal{T}) = f_{\prec_m}(\mathcal{T}) = f'(tr'(\mathcal{T})) \sqcup \mathcal{T}$, for any $\mathcal{T} \subseteq \mathbf{L}$, where $tr'(\mathcal{T}) = Th(\mathcal{T}) \cap \mathbf{L}'$ and $f'(\mathcal{T}') = f_{X'}(\mathcal{T}') = f_{\prec'_m}(\mathcal{T}')$ for any $\mathcal{T}' \subseteq \mathbf{L}'$.

We use now the direct method for computing the set $X$ from $X'_\wedge$ given above (Def$\uparrow$: $X' \rightsquigarrow X$), computing directly all the $\xi(w_i)$ for $i \in \{1, \cdots, 10\}$.

We get, applying the direct computation given by formulas (3), (3') and (4) respectively:

| $w_i$ | $(x', w'')$ | $\xi(w_i)$ form (3) | $\xi(w_i)$ form (3′) | $\xi(w_i)$ form (4) |
|---|---|---|---|---|
| $w_1$ | $(A \vee B, \emptyset)$ | $A \vee B \vee Z$ | $A \vee B \vee$ $(\neg A \wedge \neg B \wedge Z)$ | $A \vee B \vee Z$ |
| $w_2$ | $(A \vee B, \{Z\})$ | $A \vee B \vee \neg Z$ | $A \vee B \vee$ $(\neg A \wedge \neg B \wedge \neg Z)$ | $A \vee B \vee \neg Z$ |
| $w_3$ | $(A \vee \neg B, \emptyset)$ | $A \vee \neg B \vee Z$ | $A \vee \neg B \vee$ $(\neg A \wedge B \wedge Z)$ | $A \vee \neg B \vee Z$ |
| $w_4$ | $(A \vee \neg B, \{Z\})$ | $A \vee \neg B \vee \neg Z$ | $A \vee \neg B \vee$ $(\neg A \wedge B \wedge \neg Z)$ | $A \vee \neg B \vee \neg Z$ |
| $w_5$ | $(\neg A \vee B, \emptyset)$ | $\neg A \vee B \vee Z$ | $\neg A \vee B \vee$ $(A \wedge \neg B \wedge Z)$ | $\neg A \vee B \vee Z$ |
| $w_6$ | $(\neg A \vee B, \{Z\})$ | $\neg A \vee B \vee \neg Z$ | $\neg A \vee B \vee$ $(A \wedge \neg B \wedge \neg Z)$ | $\neg A \vee B \vee \neg Z$ |
| $w_7$ | $(\neg A \wedge B, \emptyset)$ | $(\neg A \wedge B) \vee$ $(\neg(A \Leftrightarrow B) \wedge Z)$ | $(\neg A \wedge B) \vee$ $(A \wedge \neg B \wedge Z)$ | $((\neg A \wedge B) \vee Z) \wedge$ $\neg(A \Leftrightarrow B)$ |
| $w_8$ | $(\neg A \wedge B, \{Z\})$ | $(\neg A \wedge B) \vee$ $(\neg(A \Leftrightarrow B) \wedge \neg Z)$ | $(\neg A \wedge B) \vee$ $(A \wedge \neg B \wedge \neg Z)$ | $((\neg A \wedge B) \vee \neg Z) \wedge$ $\neg(A \Leftrightarrow B)$ |
| $w_9$ | $(\neg A \wedge \neg B, \emptyset)$ | $(\neg A \wedge \neg B) \vee$ $(\neg B \wedge Z)$ | $(\neg A \wedge \neg B) \vee$ $(A \wedge \neg B \wedge Z)$ | $((\neg A \wedge \neg B) \vee Z) \wedge$ $\neg B$ |
| $w_{10}$ | $(\neg A \wedge \neg B, \{Z\})$ | $(\neg A \wedge \neg B) \vee$ $(\neg B \wedge \neg Z)$ | $(\neg A \wedge \neg B) \vee$ $(A \wedge \neg B \wedge \neg Z)$ | $((\neg A \wedge \neg B) \vee \neg Z) \wedge$ $\neg B$ |

We find again the set $X$, this time with (at most) three writings for each formula in $X$. □

We give now a few applications of these results to propositional circumscriptions.

## 8.3 A few applications of the results about (Def↑)

We suppose $V(\mathbf{L}') = \mathbf{P} \cup \mathbf{Q}$ and $V(\mathbf{L}) = V(\mathbf{L}') \cup \mathbf{Z}$ (disjoint unions). We consider $f' = f_{\prec'}$ with $\prec' = \prec_{(\mathbf{P},\, \mathbf{Q},\, \emptyset)}$ (definition 7.1). Thus $f' = CIRC(\mathbf{P}, \mathbf{Q}, \emptyset)$ is the circumscription of $\mathbf{P}$ with $\mathbf{Q}$ fixed, and without varying propositions. Then, if we define $\prec$ in $\mathbf{L}$ from $\prec'$ by (Def↑: $\prec' \rightsquigarrow \prec$), we get $\mu \prec \nu$ iff $\mu \cap \mathbf{P} \subset \nu \cap \mathbf{P}$ and $\mu \cap \mathbf{Q} = \nu \cap \mathbf{Q}$, i.e. $\prec = \prec = \prec_{(\mathbf{P},\, \mathbf{Q},\, \mathbf{Z})}$: $f = f_{\prec}$ is the circumscription of $\mathbf{P}$ with $\mathbf{Q}$ fixed and $\mathbf{Z}$ varying.

We get then the following result:

**Theorem 8.18** A propositional circumscription $CIRC(\mathbf{P}, \mathbf{Q}, \mathbf{Z})$ can be expressed in terms of (Def↑) from the circumscription without varying proposition $CIRC(\mathbf{P}, \mathbf{Q}, \emptyset)$ iff $\mathbf{P} \cup \mathbf{Q}$ is finite (or $\mathbf{P} = \emptyset$ or $\mathbf{Z} = \emptyset$). In this case we get, with $V(\mathbf{L}) = \mathbf{P} \cup \mathbf{Q} \cup \mathbf{Z}$ and $V(\mathbf{L}') = \mathbf{P} \cup \mathbf{Q}$, for any $\mathcal{T} \subseteq \mathbf{L}$:

$$CIRC(\mathbf{P}, \mathbf{Q}, \mathbf{Z})(\mathcal{T}) = f(\mathcal{T}) \text{ where } f(\mathcal{T}) = CIRC(\mathbf{P}, \mathbf{Q}, \emptyset)(tr'(\mathcal{T})) \sqcup \mathcal{T}.$$

In the other cases ($\mathbf{P} \cup \mathbf{Q}$ infinite, $\mathbf{P}$ and $\mathbf{Z}$ non empty), the pre-circumscription $f$ defined above is not a multi preferential entailment (thus it is not a circumscription). □

Proof: We have already given the main parts. It suffices to add the known result (cf remark 7.5-2) that a propositional circumscription $CIRC(\mathbf{P}, \mathbf{Q}, \mathbf{Z})$ satisfies (RM) iff $\mathbf{P} \cup \mathbf{Q}$ is finite (or $\mathbf{P} = \emptyset$, as $CIRC(\emptyset, \mathbf{Q}, \mathbf{Z})$ is the identity). Thus, if $\mathbf{P} \cup \mathbf{Q}$ is finite, we get the result from the proof given for proposition 8.14 applied, as shown just above, to this particular case. If $\mathbf{P} \cup \mathbf{Q}$ is infinite, $\mathbf{P} \neq \emptyset$ and $\mathbf{Z} \neq \emptyset$, we know that $f$ as defined here falsifies (CR), thus it is not a multi preferential entailment, and a fortiori not a circumscription. □

Notice that this result is the translation to the propositional case of the result given in [Lif85, Proposition 2], also as [Lif94, proposition 3.2.1] in the case of the "second order" predicate circumscription (in a first order framework), for finitely axiomatizable theories. The simplification provoked by the propositional case instead of the predicate case has allowed us to give the precise range of application of the result, while, to our knowledge, the exact limits of liability of Lifschitz's results are still unknown. The present study (transposed to the predicate calculus case) shows that in Lifschitz result, some restrictions are indeed mandatory. The restriction to finitely axiomatizable theories is probably too restrictive, but we do not know to which extend. In the propositional case, we are not restricted to finitely axiomatizable theories, which is a good thing because this restriction is very severe in the propositional case. However, it is an obvious corollary of our result that for any propositional circumscription we get $CIRC(\mathbf{P}, \mathbf{Q}, \mathbf{Z})(\mathcal{T}) = f(\mathcal{T})$ if $\mathcal{T}$ is finitely axiomatizable: indeed, in this case we may work in a finite vocabulary (the vocabulary $V(\mathcal{T})$ of some writing of $\mathcal{T}$) and add afterwards (meaning once the finite circumscription is made) all the formulas $\neg P$ for any $P \in \mathbf{P} - V(\mathcal{T})$.

Now, we get a result similar to theorem 8.18 for cardinality-based circumscription. Before stating the result, we need a lemma.

**Lemma 8.19** If $\mathbf{P}$ is enumerable and if $\mathbf{Q}$ is finite, then $NCIRC(\mathbf{P}, \mathbf{Q}, \mathbf{Z})$ satisfies (RM).
If $\mathbf{P}$ is not enumerable, or if $\mathbf{Q}$ is infinite (and $\mathbf{P} \neq \emptyset$), then $NCIRC(\mathbf{P}, \mathbf{Q}, \mathbf{Z})$ falsifies (RM). □

Proof: We write $\prec$ for the preference relation $<_{(\mathbf{P},\mathbf{Q},\mathbf{Z})}$ associated to $NCIRC(\mathbf{P}, \mathbf{Q}, \mathbf{Z})$.

1) $\mathbf{P}$ is enumerable and $\mathbf{Q}$ is finite. We prove that $\prec$ satisfies (cl), then property 5.9-5 will give that $NCIRC(\mathbf{P}, \mathbf{Q}, \mathbf{Z})$ satisfies (RM). If $\mathbf{P}$ is finite, we get the conclusion from theorem 7.22, preservation result 6.5 and remark 7.5-2. So we suppose $\mathbf{P}$ denumerable (meaning enumerable and infinite) here (the proof also works with a finite $\mathbf{P}$, even if it becomes rather trivial).

Let us suppose that, for some $\mathcal{T} \subseteq \mathbf{L}$, there exists $\mu \in \mathbf{M}(f_\prec(\mathcal{T})) = TC(\mathbf{M}_\prec(\mathcal{T}))$ such that $\mu \notin \mathbf{M}_\prec(\mathcal{T})$. We define the formula $\varphi = (\bigwedge_{Q \in \mu \cap \mathbf{Q}} Q) \wedge (\bigwedge_{Q \in \mathbf{Q} - \mu} \neg Q)$. We have $\mu \in TC(\mathbf{M}_\prec(\mathcal{T}))$, $\mathbf{M}(\varphi)$ is open and $\mu \in \mathbf{M}(\varphi)$, thus there exists $\nu \in \mathbf{M}(\varphi) \cap \mathbf{M}_\prec(\mathcal{T})$. We have $\nu \cap \mathbf{Q} = \mu \cap \mathbf{Q}$ from the definition of $\varphi$. If $card(\mu \cap \mathbf{P}) \leq card(\nu \cap \mathbf{P})$, as we have also $\nu \in \mathbf{M}_\prec(\mathcal{T})$ and $\mu \in \mathbf{M}(f(\mathcal{T})) \subseteq \mathbf{M}(\mathcal{T})$, we get $card(\mu \cap \mathbf{P}) = card(\nu \cap \mathbf{P})$, but then we get also $\mu \in \mathbf{M}_\prec(\mathcal{T})$, a contradiction. Thus we have $card(\nu \cap \mathbf{P}) < card(\mu \cap \mathbf{P})$ and $\nu \prec \mu$, in particular, $n = card(\nu \cap \mathbf{P})$ is finite. Thus, there exists a subset $\mathbf{P}_1$ of $\mu \cap \mathbf{P}$ with $n+1$ elements. We define the formula $\psi = \varphi \wedge (\bigwedge_{P \in \mathbf{P}_1} P)$. For any $\mu' \in \mathbf{M}(\psi) \cap \mathbf{M}(\mathcal{T})$ we get $card(\mu' \cap \mathbf{P}) > n$, thus $\nu \prec \mu'$ and $\mu' \notin \mathbf{M}_\prec(\mathcal{T})$. Thus $\mathbf{M}(\psi) \cap \mathbf{M}_\prec(\mathcal{T}) = \emptyset$. Now, from the definition of $\psi$, we get $\mu \in \mathbf{M}(\psi)$, thus $\mathbf{M}(\psi)$ is an open set containing $\mu$. As $\mu \in TC(\mathbf{M}_\prec(\mathcal{T}))$, we get $\mathbf{M}(\psi) \cap \mathbf{M}_\prec(\mathcal{T}) \neq \emptyset$, a contradiction. This establishes $TC(\mathbf{M}_\prec(\mathcal{T})) = \mathbf{M}_\prec(\mathcal{T})$: $\prec$ satisfies (cl).

2) Let us suppose now that $\mathbf{P} = \{P_i\}_{i<\omega_1}$, $\mathbf{Q} = \mathbf{Z} = \emptyset$ ($\omega_1$ denotes here the smallest cardinal greater than $\omega = card(\mathbb{N})$). We define $\mathcal{T} = Th(\{P_i\}_{i<\omega})$ and $\mu = \mathbf{P}$. Then $\mathbf{M}_\prec(\mathcal{T}) = \{\nu / card(\nu) = \omega\}$ and $\mu \in TC(\mathbf{M}_\prec(\mathcal{T})) - \mathbf{M}_\prec(\mathcal{T})$: $\prec$ falsifies (cl). From properties 5.9-5 and 5.6-1 we get then that $f_\prec$ falsifies (RM). This counter-example may be easily generalized to any sets $\mathbf{P}, \mathbf{Q}, \mathbf{Z}$ containing the respective sets given here (just choose theories with a vocabulary included in the set $\mathbf{P}$ given here).

3) For the case $\mathbf{P} \neq \emptyset$ and $\mathbf{Q}$ infinite, see example 8.21 below: it gives an example of an ordinary circumscription $CIRC(\{P\}, \mathbf{Q}, \emptyset)$ falsifying (RM). As $\mathbf{P} = \{P\}$ is a singleton there, we get $NCIRC(\{P\}, \mathbf{Q}, \emptyset) = CIRC(\{P\}, \mathbf{Q}, \emptyset)$, thus this is an example of a cardinality-based circumscription falsifying (RM). As $\mathbf{Q}$ is enumerable there, any infinite set has some set equipotent to this $\mathbf{Q}$ as a subset. As in point 2, we may easily generalize this counter-example to any sets $\mathbf{P}, \mathbf{Q}, \mathbf{Z}$ containing the respective sets given here. □

**Theorem 8.20** Any cardinality-based circumscription with countably many circumscribed propositions and finitely many fixed ones can be expressed in terms of (Def↑) from a cardinality-based circumscription without varying proposition:

$V(\mathbf{L}') = \mathbf{P} \cup \mathbf{Q}$ and $V(\mathbf{L}) = V(\mathbf{L}') \cup \mathbf{Z}$ (disjoint unions). If $\mathbf{P}$ is enumerable and $\mathbf{Q}$ finite, we have, for any $\mathcal{T} \in \mathbf{T}$:

$$NCIRC(\mathbf{P}, \mathbf{Q}, \mathbf{Z})(\mathcal{T}) = NCIRC(\mathbf{P}, \mathbf{Q}, \emptyset)(\mathcal{T} \cap \mathbf{L}') \sqcup \mathcal{T}. \quad \square$$

Proof: From lemma 8.19 we know that we are in a case where $NCIRC(\mathbf{P}, \mathbf{Q}, \emptyset)$ satisfies (RM). We get then the result by applying preservation result 8.14 to this particular case. Indeed, it is obvious from the definition of $<_{(\mathbf{P},\mathbf{Q},\mathbf{Z})}$ in definition 7.21 that if $\prec' = <_{(\mathbf{P},\mathbf{Q},\emptyset)}$, and if we define $\prec$ by (Def↑:$\prec' \rightsquigarrow \prec$), then we get $\prec = <_{(\mathbf{P},\mathbf{Q},\mathbf{Z})}$.

Notice that from non preservation result 8.3 and from lemma 8.19 we get also that this equality is false if $\mathbf{Z}$ is not empty and if either $\mathbf{P}$ is not enumerable, or $\mathbf{Q}$ is infinite (and $\mathbf{P} \neq \emptyset$). □

Thus, we get an (exceptional? at least interesting...) case where we have the choice of applying a transformation of the kind (Def$\downarrow_{\beta'}$) (remind property 7.28) or of the kind (Def↑) as done here, when we want to suppress the varying propositions in a cardinality-based circumscriptions. Indeed, if $\mathbf{P}$ is enumerable and $\mathbf{Q}$ and $\mathbf{Z}$ finite (in particular if $V(\mathbf{L})$ is finite), we may apply the two results, and it remains to examine which one is the most useful, for instance for what concerns the simplification of the effective computation.

Let us give now an example, dealing with circumscription, as an illustration of (non) preservation results 8.3, 8.14, 8.15.

**Example 8.21** $\mathbf{P} = \{P\}$, $\mathbf{Z} = \{Z\}$, $\mathbf{Q} = \{Q_i\}_{i\in \mathbb{N}}$.

We consider $V(\mathbf{L}') = \mathbf{P} \cup \mathbf{Q}$ and $V(\mathbf{L}) = V(\mathbf{L}') \cup \mathbf{Z}$.

We define $f' = f_{\prec'} = CIRC(\mathbf{P}, \mathbf{Q}, \emptyset)$ in $\mathbf{L}'$ and $f_1 = f_{\prec_1} = CIRC(\mathbf{P}, \mathbf{Q}, \mathbf{Z})$ in $\mathbf{L}$. Thus $\prec' = \prec_{(\mathbf{P},\ \mathbf{Q},\ \emptyset)}$ and $\prec_1 = \prec_{(\mathbf{P},\ \mathbf{Q},\ \mathbf{Z})}$ (definition 7.1).

We define the following interpretations in $\mathbf{L}'$: $\mu'_n = \{P\} \cup \{Q_i\}_{i\geq n}$, $\mu'_\omega = \{P\}$, $\nu' = \emptyset$.

We define also the following theory of $\mathbf{L}'$: $\mathcal{T}' = \bigcap_{n\in \mathbb{N}} Th'(\mu'_n)\ \cap\ Th'(\nu')$.

We get $\mathbf{M}'(\mathcal{T}') = \{\mu'_n\}_{n\in \mathbb{N}} \ \cup\ \{\mu'_\omega, \nu')\}$ and $\mathbf{M}'_{\prec'}(\mathcal{T}') = \mathbf{M}'(\mathcal{T}') - \{\mu'_\omega\}$ (indeed, $\nu' \prec' \mu'_\omega$).

Thus $\mathbf{M}'_{\prec'}(\mathcal{T}')$ is not a closed set, and $f' = f_{\prec'}$ falsifies (RM).

We will also use the following interpretations for $\mathbf{L}$: $\mu_n = \mu'_n \cup \{Z\}, \mu_\omega = \mu'_\omega \cup \{Z\}$ and $\nu = \nu' \cup \{Z\}$. Also, we will sometimes need to consider the sets $\mu'_n, \mu'_\omega$ and $\nu'$ as interpretations for $\mathbf{L}$ (then, this will be made explicit).

We give here an explicit counter-example to (RM): $\mathcal{T}'_1 = \mathcal{T}'$, $\mathcal{T}'_2 = Th'(\mu'_\omega) \cap Th'(\nu')$ , thus $\mathcal{T}'_1 \subseteq \mathcal{T}'_2$, i.e. $\mathcal{T}'_1 \sqcup' \mathcal{T}'_2 = \mathcal{T}'_2$. $f'(\mathcal{T}'_1) = \mathcal{T}'_1$ and $f'(\mathcal{T}'_1 \sqcup' \mathcal{T}'_2) = f'(\mathcal{T}'_2) = Th'(\nu') \not\subseteq f'(\mathcal{T}'_1) \sqcup \mathcal{T}'_2 = \mathcal{T}'_2$. Notice that we cannot choose a theory $\mathcal{T}'_2$ which is finitely axiomatizable (even "with respect to $\mathcal{T}_1$") as we know that (RM1) is satisfied by any multi preferential entailment, thus by any circumscription.

Now, we define the following theories in $\mathbf{L}$: $\mathcal{T}_1 = \mathcal{T}'_1 \sqcup \neg Z$, $\mathcal{T}_2 = \mathcal{T}'_2 \sqcup Z$.

We define $f(\mathcal{T}) = f(tr'(\mathcal{T})) \sqcup \mathcal{T}$ for any $\mathcal{T} \in \mathbf{T}$, where $tr'(\mathcal{T}) = \mathcal{T} \cap \mathbf{L}'$.

We get $f(\mathcal{T}_1) = f'(\mathcal{T}'_1) \sqcup \mathcal{T}_1 = \mathcal{T}'_1 \sqcup \mathcal{T}_1 = \mathcal{T}_1$, $f(\mathcal{T}_2) = f'(\mathcal{T}'_2) \sqcup \mathcal{T}_2 = Th'(\nu') \sqcup \mathcal{T}_2 = Th(\nu)$. $f\mathcal{T}_1 \cap \mathcal{T}_2) = f'(tr'(\mathcal{T}_1 \cap \mathcal{T}_2)) \sqcup (\mathcal{T}_1 \cap \mathcal{T}_2) = f'(tr'(\mathcal{T}_1) \cap tr'(\mathcal{T}_2)) \sqcup (\mathcal{T}_1 \cap \mathcal{T}_2) = f'(\mathcal{T}'_1 \cap \mathcal{T}'_2) \sqcup (\mathcal{T}_1 \cap \mathcal{T}_2) = f'(\mathcal{T}'_1) \sqcup (\mathcal{T}_1 \cap \mathcal{T}_2) = \mathcal{T}'_1 \sqcup (\mathcal{T}_1 \cap \mathcal{T}_2) = \mathcal{T}_1 \cap \mathcal{T}_2$.

$f(\mathcal{T}_1) \cap f(\mathcal{T}_2) \not\subseteq f(\mathcal{T}_1 \cap \mathcal{T}_2)$: we get $\mu_\omega \in \mathbf{M}(f(\mathcal{T}_1 \cap \mathcal{T}_2)) = \mathbf{M}(\mathcal{T}_1) \cup \mathbf{M}(\mathcal{T}_2)$ (indeed, $\mu_\omega \in \mathbf{M}(\mathcal{T}_2)$). However, $\mu_\omega \notin \mathbf{M}(f(\mathcal{T}_1)) = \mathbf{M}(\mathcal{T}_1)$ and $\mu_\omega \notin \mathbf{M}(f(\mathcal{T}_2)) = \{\nu\}$, thus $\mu_\omega \notin \mathbf{M}(f(\mathcal{T}_1) \cap f(\mathcal{T}_2))$: $f$ falsifies (CR), thus $f \neq f_1$.

Precisely, we get here $\mathbf{M}_{\prec_1}(\mathcal{T}_1 \cap \mathcal{T}_2)) = \{\mu'_n\}_{n\in \mathbb{N}} \cup \{\nu'\} \cup \{\nu\}$ (here, all the elements are considered as interpretations for $\mathbf{L}$): indeed, the only models of $\mathcal{T}_1 \cap \mathcal{T}_2$ which are removed are $\mu'_\omega$ and $\mu_\omega$: $\nu \prec_1 \mu'_\omega$ and $\nu \prec_1 \mu_\omega$ (similarly for $\nu'$ instead of $\nu$ by the way). Now, the topological closure of this set does not contain $\mu_\omega$, it contains only the additional element $\mu'_\omega$ which is the limit of the sequence $(\mu'_n)_{n\in \mathbb{N}}$. Thus, $\mathbf{M}(f_1(\mathcal{T})) = \{\mu'_n\}_{n\in \mathbb{N}} \cup \{\mu'_\omega\} \cup \{\nu'\} \cup \{\nu\}$. As we have seen that we have $\mathbf{M}(f(\mathcal{T}_1 \cap \mathcal{T}_2)) = \mathbf{M}(\mathcal{T}_1) \cup \mathbf{M}(\mathcal{T}_2) = \mathbf{M}(f_1(\mathcal{T})) \cup \{\mu_\omega\}$, this shows that we have $f(\mathcal{T}_1 \cap \mathcal{T}_2) \neq f_1(\mathcal{T}_1 \cap \mathcal{T}_2)$: we have given an explicit example in which $CIRC(\mathbf{P}, \mathbf{Q}, \mathbf{Z})$ and $f$ differ. Also, we see why this result is closely linked to the falsification of (CR) by $f$, which itself comes from the falsification of (RM) by the circumscription $f'$.

Now, we can get more from this example. Indeed, it is known (see [MR98b, Theorem 6.40]) that a formula circumscription $CIRC(\mathbf{P}, \mathbf{Q}, \mathbf{Z})$ is an X-mapping iff $\mathbf{P}$ is finite or $\mathbf{Z}$ is empty. Thus, in above example, $f'$ is an X-mapping. We may choose the set of its inaccessible formulas as our set $X'$ such that $f = f_{X'}$ (see [MR98a, MR98b] for a syntactic description of this set, and also of the smallest set $X'$ possible here). Now, as $f$ falsifies (CR), it cannot be an X-mapping. This provides an explicit example of an X-mapping $f'$ such that $f$ as defined by (Def$\uparrow$) is not an X-mapping (see non preservation result 8.15).

Remind also that $f'$ satisfies (CUMU) (as any propositional circumscription) and (CR$\infty$) (as any X-mapping). $\square$

# 9 Conclusion and future work

We have given a few results about the most widely used methods of extension or reduction of the vocabulary in non monotonic reasoning. We have particularly examined which properties are preserved by these modifications of the vocabulary. The simplification of the effective computation of preferential entailments is one possible application of this kind of result.

In our first study (called here reduction of the vocabulary, but which may be applied in both ways), we have shown how the study of multi preferential entailments originating in the work of Kraus, Lehmann and Magidor [KLM90] may be done from the study of the much simpler notion of ordinary preferential entailment originating in Shoham [Sho88]. The method proposed here is a significative generalization of a recent result given by Costello [Cos98] about propositional circumscription. This study has separated the classical properties in two classes, and surprisingly it appears that the properties preserved are those which have received the more attention in the literature. We think that this stability is one possible hidden reason for this fact. Notice that from a pure knowledge representation perspective, nothing justifies e.g. to study the rather contestable disjunctive rationality (p-reserved, and studied in many texts) and to ignore the more desirable disjunctive coherence (not preserved, and ignored in many texts).

In our second study (called extension of the vocabulary), we have exhibited a case when the property that we call (RM) is very desirable. This (already well known) property is an extension of the property that we call (CR) (called "OR" in [KLM90]) and the desirability of (RM) has sometimes been contested in the literature. For instance, we read a comment in [Mak94] stating, without further justification: "The intuitive status [of this property] is rather debatable". We have exhibited one situation in which this property has important consequences: when satisfied, the notions of preferential entailments and of multi preferential entailments are preserved by a very simple kind of extension of the vocabulary already evoked in "old" texts in the literature, while no preservation occur when this property is falsified. This example exhibits an important case in which (RM) has the behavior of a (CR) much more stable than the basic (CR) alone.

The two cases ("reduction" and "extension") considered here, with their associated preservation results, have many useful consequences. We have given a few examples, concentrating our attention to circumscriptions. We have shown what has made possible the interdefinability between ordinary circumscription and cardinality-based circumscription in the finite case, discovered recently [Moi99]. Also we have shown why only one of these two translations can be extended to any enumerable case. We have also made precise the range of viability of a known theorem of suppression of varying propositions in circumscriptions. To our knowledge, such limitations, which are obvious consequences of our preservation results, had never been studied before. Applying again these results to cardinality-based circumscription, we have exhibited a particular case in which the two transformations examined in the present text (reduction, and extension of the vocabulary) may be applied. We have shown that these two methods allow to suppress the varying propositions in this kind of circumscription, in the finite case. Such results should greatly improve the efficiency of automatic computation methods. It remains to examine which method is the best one, or to determine the cases in which one method is more efficient.

Despite the fact that we have already given various applications of our study, a lot of work still remains.

Firstly, there are still a few "holes" in our study.

One hole is that, in our reduction process called (Def$\downarrow$), we were unable to state about the preservation or not of safe foundedness (sf), in the case where the small language $\mathbf{L}$ is not enumerable. This is of some importance because (sf) is a very widely used and desirable property of (multi) preference relations.

A second hole concerns our extension process, and deals with circumscription with varying predicates: we need a property, not present in the list given in definitions 5.2, which explains why it is apparently so difficult to express an ordinary circumscription in terms of a circumscription without varying objects (i.e. with an empty $\mathbf{Z}$). Indeed, the restriction made in theorem 8.18 is rather severe, and we do not know of any property which explains precisely why. It is known that (CR$\infty$) is a good property which separates the circumscriptions with $\mathbf{Z}$ and those without $\mathbf{Z}$, when the set $\mathbf{P}$ of the circumscribed propositions is infinite. But this does not explain the limits in theorem 8.18: the restriction $\mathbf{P} \cup \mathbf{Q}$ finite is more severe than $\mathbf{P}$ finite, and anyway (CR) itself is not preserved there. The property (RM), shown to block any immediate extension of applicability of theorem 8.18, applies equally to the case when $\mathbf{Z}$ is empty or not in this case. So, the question we ask is the following one: is there a possibility of expressing precisely such impossibility in terms of a property of knowledge representation such as the ones given in definitions 5.2, or have we missed something else here? Various questions of this kind remain unanswered, and we consider that this is however a good point of the present study, as it shows how it may help us in improving our basic understanding of the leading (multi) preferential entailments used in the literature.

A third hole concerns an eventual extension of the characterization of finite cumulative multi preferential entailments given in theorem 7.16 to the infinite case. A characterization of infinite propositional formula circumscription is given in [MR98b]. Could this result, or a similar result, be used in order to help extending this theorem to the case of infinite cumulative multi preferential entailments?

Another hole, or more precisely unachieved study, concerns also theorem 7.16, more precisely now the finite case, compared with the results of remark 7.29. In these two cases, we got a very constrained $f'$ allowing to express any finite cumulative multi preferential entailment $f$ in terms of $f(\mathcal{T}) = f'(\mathcal{T} \cup \{\beta'\}) \cap \mathbf{L}$. A natural question to ask is: how far can we go in our requirements for $f'$. Can we go further yet? Or do we have here two different examples of $f'$ which are, so to say, "maximally constrained". And if so, can we describe all theses maximally constrained $f'$, what do they have as particular properties? Precisely in this example, which are the precise properties which make ordinary circumscriptions (even without fixed proposition) and cardinality-based circumscriptions (even without varying proposition) exactly equally powerful to that respect? What important properties do they have, which apparently is missing in our study? Clearly, this last question could demand a lot of work, and is at the frontier between a simple suppression of small holes, and a really new perspective. This is an important matter because it is very likely that automatic computation becomes easier when $f'$ is more constrained.

We will evoke below three other possible future works of this kind, or at least questions, originating from the present study. Before that, let us add here that we have given only a few applications of our study to circumscription. This was in order to explain on a few concrete examples how this study can be used. But there are many other possible uses of this study, for example about the most general notion of preferential entailment.

A significative part of the present text provides constructive definitions in most of the applications of our study to "concrete" cases. We have shown how to compute a set of formulas in the new vocabulary, when starting from a set of formulas in the original vocabulary. The aim of these constructive definitions is to give a first step towards a real simplification of the automatic computation. But it remains to really apply our results to automatic computation. We think that this is possible, and that we have given serious indications in favor of this opinion, but a lot of work remains to be done.

We have only studied the propositional case, for the sake of clarity, but extensions to the predicate calculus case should be done. As we have studied the infinite propositional case, we have given a few clues towards this much complicated study, but a lot of work remains, because the conditions of viability of the properties studied here are generally rather complex in the predicate calculus case.

We think also that it would be very interesting to extend the present study to the case of "credulous reasoning", where conclusions are not necessarily deductively closed theories. Bochman has made precise some theoretical foundations of such a way of reasoning in terms of (multi) preference relations [Boc99b], and we think that combining a study of this kind with the work presented here could bring interesting results.

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