High Frequency Limit of the Helmholtz Equations

Jean-David Benamou, François Castella, Thodoros Katsaounis, Benoît Perthame

INSTITUT NATIONAL DE RECHERCHE EN INFORMATIQUE ET EN AUTOMATIQUE

High Frequency limit of the Helmholtz Equations

Jean-David Benamou , François Castella , Thodoros Katsaounis , Benoit Perthame

N ° 3785

Octobre 1999

THÈME 4

Rapport de recherche

ISSN 0249-6399 ISRN INRIA/RR--3785--FR+ENG

High Frequency limit of the Helmholtz Equations

Jean-David Benamou , François Castella , Thodoros Katsaounis , Benoit Perthame

Thème 4 — Simulation et optimisation
de systèmes complexes
Projet M3N, Ondes

Rapport de recherche n ° 3785 — Octobre 1999 — pages

Abstract: We derive the high frequency limit of the Helmholtz equations in terms of quadratic observables. We prove that it can be written as a stationary Liouville equation with source terms. Our method is based on the Wigner Transform, which is a classical tool for evolution dispersive equations. We extend its use to the stationary case after an appropriate scaling of the Helmholtz equation. Several specific difficulties arise here; first, the identification of the source term (which does not share the quadratic aspect) in the limit, then, the lack of L2 bounds which can be handled with homogeneous Morrey-Campanato estimates, and finally the problem of uniqueness which, at several stage of the proof, is related to outgoing conditions at infinity.

(Résumé : tsvp)

* INRIA, Ondes.
† CNRS et IRMAR, Campus de Beaulieu.
‡ Institute of Applied and Computational Mathematics, FORTH.
§ INRIA, M3N et ENS, DMI.

Limite haute fréquence de l'équation de Helmholtz

Résumé : Nous obtenons la limite haute fréquence de l'équation de Helmholtz à l'aide d'observations quadratiques de sa solution. Nous démontrons que cette limite peut être écrite comme une équation de Liouville stationnaire avec des termes sources. Notre méthode s'appuie sur un outil classique dans l'étude des équations d'évolution dispersives : la transformée de Wigner. Son utilisation est ici étendue dans un cas stationnaire grâce à un dimensionnnement particulier de l'équation de Helmholtz. Les difficultés spécifique sont dans ce cas : l'identification de la limite du terme source (non quadratique), l'absence de bornes L2 sur la solution qui sont ici remplacées par des estimations homogènes de type Morrey-Campanato et finalement le problème de l'unicité liè aux conditions de radiations à l'infini qui apparaît à plusieur niveau de la démonstration.

1 Introduction

This paper is concerned with the analysis of the high frequency limit of the Helmholtz equation in a three dimensional inhomogeneous medium; some formulas and the scaling depends on the dimension although the method works in any dimensions. The index of refraction $n(x)$ is smooth, positive and normalized with

$$n(0) = 1. \tag{1.1}$$

The Helmholtz equation is then written, after appropriate scaling

$$-i\frac{\alpha_\varepsilon}{\varepsilon}u_\varepsilon + \Delta u_\varepsilon + \Big(\frac{n(x)}{\varepsilon}\Big)^2 u_\varepsilon = -S_\varepsilon(x) := -\frac{1}{\varepsilon^3}S(\frac{x}{\varepsilon}), \quad x \in \mathbb{R}^3. \tag{1.2}$$

Here, the parameter $\varepsilon \in (0,1)$ is related to the wave length $\lambda = 2\pi\varepsilon$ and we are interested in the limit $\varepsilon \to 0$. The source term S is a given function. We assume that,

$$\alpha_\varepsilon > 0 \ , \quad \alpha_\varepsilon \to \alpha \geq 0 \ , \tag{1.3}$$

and thus there is a unique L^2 solution to (1.2). Although our basic estimates and method allow α_ε to vanish, this avoids to write outgoing conditions at infinity, which we cannot do with the weak assumptions on n we will use here (especially n needs not be constant at infinity).

We would like to explain how in the limit $\varepsilon \to 0$, the energy (or more generally quadratic observables) can be globally described by the geometrical optics, written under the form of the Liouville equation

$$+\alpha f + \xi \cdot \nabla_x f(x,\xi) + \frac{1}{2}\nabla_x n^2 \cdot \nabla_\xi f(x,\xi) = +\frac{1}{(4\pi)^2}\delta(x)|\widehat{S}(\xi)|^2\delta(|\xi| = 1) \ , \tag{1.4}$$

completed with the radiation condition, when $\alpha = 0$,

$$f(x,\xi) \to 0, \quad \text{as } |x| \to \infty \quad \text{with} \quad x\cdot\xi \leq 0 \, . \tag{1.5}$$

Our convention is that the total mass of the measure $\delta(|\xi| = 1)$ is 4π and $\widehat{S}$ denotes the Fourier Transform of S. Of course, the existence of a solution f, and thus the derivation of the high frequency limit, requires some assumptions on the function $n(x)$: namely the dispersion of the trajectories of the following differential system (geometrical optics or ray tracing)

$$\begin{aligned} &\dot{X}(t) = \zeta(t), \quad X(0) = x \ , \\ &\dot{\zeta}(t) = \tfrac{1}{2}\nabla n^2\Big(X(t)\Big), \quad \zeta(0) = \xi. \end{aligned} \tag{1.6}$$

Indeed, the particular solution f is given by the representation formula

$$f(x,\xi) = \frac{1}{(4\pi)^2}\int_0^{+\infty} \delta\Big(X(s)\Big)\ |\widehat{S}(\zeta(s))|^2\ \delta\Big(|\zeta(s)| = 1\Big)e^{-\alpha s}ds \, . \tag{1.7}$$

In the sequel we will give a derivation which relies on the Wigner measures introduced by P. Gérard [10], P.-L. Lions and T. Paul [14], L. Tartar [18]. One of the new points here is the treatment of the inhomogeneous term S which does not follow the general method. It can be handled mainly thanks to the particular scaling we have introduced in (1.2) which concentrates the source at the origin, and allows to recover locally the solution with an explicit form. The counterpart is the singular source in the righthand side of (1.4). Several technical difficulties also specifically arise for the Helmholtz equation. Uniform (in ε) L^2 bounds are not available. We replace them by some weighted L^2 estimates, called Morrey-Campanato estimates, derived in B. Perthame and L. Vega [17], (see also S. Agmon and L. Hörmander [1] for $n=$ constant, C. Kenig, G. Ponce and L. Vega [13] for the case of Schroedinger equation, P.L. Lions and B. Perthame [15] or I. Gasser, P. Markowich and B. Perthame [9] for the relations between these estimates and moments lemmas in kinetic theory) which are space homogeneous and thus appropriate for the high frequency limit. Another technical difficulty comes from the interpretation of radiation conditions at infinity, which in turn leads to the condition (1.5).

We would like also to point out that the understanding of high frequencies in PDEs is a very active field, see for instance P. Gérard, P.A. Markowich, N.J. Mauser, F. Poupaud [11] for periodic media for instance, G. Papanicolaou and L. Ryzhik [16], and the references therein, for a survey of the theory and an introduction to the questions related to random media, J.D. Benamou, F. Castella, B. Perthame, and O. Runborg [4] for generalisations of the present results to more general source terms, F. Castella and P. Degond [6], [5] for a deterministic way to generate scattering operators in the high frequency limit, R. Esposito, M. Pulvirenti, and A. Teta [8] as well as L. Erdös and H.T. Yau [7] for a stochastic approach to the latter problem, and at the numerical level, see J.D. Benamou [3] and the references therein.

Finally, we would like to mention that the classical method for deriving the high frequency limit of dispersive equations is through Eikonal equation (cf J.B. Keller and R. Lewis [12]). Clearly this approach is not enough to obtain the full result we prove here. Not only this method is limited by caustics, but also the source term can only be written using Fourier variables.

The outline of the paper is the following: we first present in §2 a formal derivation of the high frequency limit of Helmholtz equations and explain the argument which allows to obtain the source term in (1.4). The precise assumptions, apriori bounds and statements are given in §3, the proof of the main theorem, and of the condition at infinity, is given in §4.

2 High frequency limit of Helmholtz equation

In this Section, we give a formal derivation of the Liouville equation (1.4) from the Hemholtz equation (1.2). In fact, using the Wigner transform (subsection 1), we give another formulation of the Helmholtz equation. The limit itself follows after some treatment of the righthand side (subsection 2). The outgoing condition is treated in the last subsection. To simplify the calculations we take α_ε to be constant since it does not change the formalism.

2.1 Wigner transform

The Wigner Transform $f(x,\xi) \in \mathbb{R}$ of the function $u(x) \in \mathbb{C}$ is defined as follows. Doubling the variables, we denote

$$v(x,y) = u(x+\frac{\varepsilon}{2}y)\bar{u}(x-\frac{\varepsilon}{2}y), \tag{2.1}$$

$$f(x,\xi) = \mathcal{F}_{y\to\xi} v(x,y), \tag{2.2}$$

where the Fourier Transform is defined by

$$\widehat{u}(\xi) = \mathcal{F}u(\xi) = \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} e^{-iy\cdot\xi}u(y)dy. \tag{2.3}$$

and its inverse is then

$$\mathcal{F}^{-1}w(x) = \int_{\mathbb{R}^3} e^{ix\cdot\xi}w(\xi)d\xi. \tag{2.4}$$

In order to compute the equation satisfied by the Wigner Transform $f_\varepsilon(x,\xi)$ of the solution u_ε to the Helmholtz equation (1.2), we notice that

$$\nabla_y\nabla_x v_\varepsilon = \frac{\varepsilon}{2}\Big[\Delta u_\varepsilon(x+\frac{\varepsilon}{2}y)\bar{u}(x-\frac{\varepsilon}{2}y) - u_\varepsilon(x+\frac{\varepsilon}{2}y)\Delta\bar{u}(x-\frac{\varepsilon}{2}y)\Big]$$

and thus we have

$$+\alpha_\varepsilon v_\varepsilon + i\nabla_y\cdot\nabla_x v_\varepsilon(x,y) + \frac{i}{2\varepsilon}\Big[n^2(x+\frac{\varepsilon}{2}y) - n^2(x-\frac{\varepsilon}{2}y)\Big]v_\varepsilon(x,y) = \tag{2.5}$$
$$= \sigma_\varepsilon(x,y)$$

where we set

$$\sigma_\varepsilon(x,y) := -i\frac{\varepsilon}{2}\Big[S_\varepsilon(x+\frac{\varepsilon}{2}y)\bar{u_\varepsilon}(x-\frac{\varepsilon}{2}y) - \bar{S_\varepsilon}(x-\frac{\varepsilon}{2}y)u_\varepsilon(x+\frac{\varepsilon}{2}y)\Big]\ . \tag{2.6}$$

Therefore, after a Fourier Transform of (2.5), we obtain the following transport equation on the Wigner Transform of u_ε

$$+\alpha_\varepsilon f_\varepsilon + \xi\cdot\nabla_x f_\varepsilon(x,\xi) + Z_\varepsilon(x,\xi) *_\xi f_\varepsilon(x,\xi) = Q_\varepsilon(x,\xi), \tag{2.7}$$

and the quantities Z_ε, Q_ε arising in this equation are given by

$$Q_\varepsilon(x,\xi) = \mathcal{F}_{y\to\xi}\sigma_\varepsilon(x,y), \tag{2.8}$$

$$Z_\varepsilon(x,\xi) = \frac{i}{2\varepsilon}\mathcal{F}_{y\to\xi}\Big[n^2(x+\frac{\varepsilon}{2}y) - n^2(x-\frac{\varepsilon}{2}y)\Big] \tag{2.9}$$

and formally we have,

$$Z_\varepsilon(x,\xi) \to_{\varepsilon\to 0} \frac{1}{2}\nabla_x n^2(x)\cdot\nabla_\xi\delta(\xi). \tag{2.10}$$

In the next subsection we discuss the most interesting term, namely, Q_ε.

2.2 The righthand side Q_ε

The source term $Q_\varepsilon(x,\xi)$ is the Fourier Transform of the source term $\sigma_\varepsilon(x,y)$ in (2.6). In order to study it, we define the complex valued function

$$w_\varepsilon(y) = \varepsilon u_\varepsilon(\varepsilon y). \tag{2.11}$$

It satisfies the rescaled Helmholtz equation

$$-i\varepsilon\alpha_\varepsilon w_\varepsilon + \Delta w_\varepsilon + n^2(\varepsilon x)w_\varepsilon = -S(y). \tag{2.12}$$

Therefore w_ε formally converges towards the outgoing solution w of

$$\Delta w + n^2(0)w = -S(y), \tag{2.13}$$
$$w = -[\Delta + n^2(0) - i0]^{-1}S,$$

which is explicitely given by

$$w(y) = -S(y) \star_y \frac{e^{-i|y|}}{|y|}\ , \tag{2.14}$$

since $n(0) = 1$. Of course there is a deep difficulty here which is explained in §3.

In order to study the limit of $\sigma_\varepsilon(x,y)$ in the distribution sense by using the convergence result (2.14), let us use two test functions $\varphi(x)$, $\psi(y) \in \mathcal{S}(\mathbb{R}^3)$. We have

$$\int \sigma_\varepsilon(x,y)\ \varphi(x)\ \psi(y)\ dxdy =$$
$$= \frac{-i}{2\varepsilon^3}\int\Big[S(\frac{x}{\varepsilon}+\frac{y}{2})\overline{w}_\varepsilon(\frac{x}{\varepsilon}-\frac{y}{2}) - \overline{S}(\frac{x}{\varepsilon}-\frac{y}{2})w_\varepsilon(\frac{x}{\varepsilon}+\frac{y}{2})\Big]\varphi(x)\psi(y)dxdy$$
$$= \frac{-i}{2}\int\Big[S(z)\overline{w}_\varepsilon(z-y)\varphi(\varepsilon z-\frac{\varepsilon y}{2}) - \overline{S}(z)w_\varepsilon(z+y)\varphi(\varepsilon z+\frac{\varepsilon y}{2})\Big]\psi(y)dzdy$$
$$\to \frac{-i}{2}\varphi(0)\int\Big[S(z)\overline{w}(z-y) - \overline{S}(z)w(z+y)\Big]\psi(y)dzdy.$$

In other words, we have formally obtained that (in $\mathcal{S}'(\mathbb{R}^3)$),

$$\sigma_\varepsilon(x,y) \to_{\varepsilon\to 0} \sigma(x,y) = \frac{-i}{2}\delta(x)\int\Big[S(z)\overline{w}(z-y) - \overline{S}(z)w(z+y)\Big]dz, \tag{2.15}$$

which, after a Fourier Transform gives (always in $\mathcal{S}'(\mathbb{R}^3)$),

$$Q_\varepsilon(x,\xi) \to_{\varepsilon\to 0} Q(x,\xi) = \delta(x)\mathcal{I}m\Big[\widehat{S}(\xi)\overline{\widehat{w}}(\xi)\Big]. \tag{2.16}$$

The explicit form of $\widehat{w}$ (see the Appendix), gives the form of the righthand side $Q(x,\xi)$ in the limiting Liouville Equation (1.4).

2.3 The condition at infinity

To conclude this Section, we indicate how to recover the outgoing condition at infinity in the limiting equation (1.4). Let us recall that for n constant at infinity, the Sommerfeld condition at infinity is written (roughly, see Bo Zhang [19] for more details)

$$\frac{x}{|x|} \cdot \nabla_x u_\varepsilon - \frac{i}{\varepsilon} n^2(x) u_\varepsilon \to 0, \quad \text{as } |x| \to \infty.$$

It can also be interpreted, after doubling the variables, in terms of $v_\varepsilon(x, y)$, as

$$\frac{x}{|x|} \cdot \nabla_y v_\varepsilon - i\, n^2(x) v_\varepsilon + O(\varepsilon) \to 0, \quad \text{as } |x| \to \infty,$$

which after Fourier Transform yields

$$\Big(\frac{x}{|x|} \cdot \xi - n^2(x)\Big) f + O(\varepsilon) \to 0, \quad \text{as } |x| \to \infty.$$

In the limit $\varepsilon \to 0$, we recover the condition at infinity in (1.4). Notice however that, in §3, we do not obtain the condition in such a way but in a weak sense to be precised. Notice also that the radiation condition for f can also be formally obtained from the fact that f is the limit of f_ε, where f_ε satisfies a transport equation of the type $-\alpha_\varepsilon f_\varepsilon + \xi \cdot \nabla_x f_\varepsilon + \cdots = \cdots$, where $\alpha_\varepsilon > 0$.

3 Precise results

In this Section, we state precisely our results on the high frequency limit. We begin with stating the assumptions and results (subsection 1). Then, we prove a first result (a priori bound on f_ε). The proof of the main Theorem which identifies the limit f is given in the next section.

3.1 Assumptions

We begin by stating our assumptions on the index n. They all allow a very low regularity for n. For instance they do not allow to use the Cauchy-Lipschitz theorem for uniqueness of trajectories to the ray system (1.6). They are mainly (but not only) concerned with the critical decay of $n^2(x)$ to a constant at infinity.

First of all, we cannot use the L^2 bounds which are not uniform in ε, both for the study of u_ε, and w_ε (see §2.3). Uniform bounds in ε can rather be obtained through Morrey-Campanato estimates. These are weighted L^2 norms which are space homogeneous (and thus uniform in ε), and have been used by [1], [2] and, for evolution dispersive equations, by several authors (see [13] and the references therein). For Helmholtz equations with a variable n they have been derived in [17], with a new direct method. The assumptions needed are the following

$$0 < n_{min} \leq n(x) \leq n_{max}, \tag{3.1}$$

$$4 \sum_{j \in \mathbb{Z}} \sup_{C(j)} \frac{(x \cdot \nabla n^2)_-}{n^2(x)} := \beta < 1, \tag{3.2}$$

where $C(j)$ is the annulus $\{2^j \leq |x| \leq 2^{j+1}\}$. This assumption implies that the bicharacteristics (1.6) disperse at infinity in x for long times.

Secondly, we need to recover the outgoing condition at infinity for f in the limit $\varepsilon \to 0$ from the radiation condition for f_ε. This requires a second set of assumptions

$$(1 + |x|)^{N_0} \ \ |\nabla n^2(x)| \in L^\infty(\mathbb{R}^3) \quad \text{for some } N_0 > 5\,, \tag{3.3}$$

$$\nabla_x n^2 \text{ is continuous.} \tag{3.4}$$

Note that the norm on $|\nabla n^2(x)|$ involved in (3.3) is much stronger than the one used in (3.2). We mention in this respect that, although (3.3) may be too stringent, assumption (3.2) is close to a sharp condition when dealing with uniform estimates in the Helmholtz equation, and we refer to [17].

We now come to the assumptions on the source term S in (1.2). With the assumptions (3.1) and (3.2), the following bound holds for the solution to the scaled Helmholtz equation (1.2),

$$||u_\varepsilon||_M := \left[\sup_{R>0} \frac{1}{R} \int_{|x|\le R} |u_\varepsilon|^2 dx\right]^{1/2} \le C(\beta) N(S) \tag{3.5}$$

$$N(S) := \sum_{j\in\mathbb{Z}} \left[2^{j+1} \int_{C(j)} |S|^2 dx\right]^{1/2}.$$

This estimate is proved in [17] for $\varepsilon = 1$ but it also holds for all $\varepsilon \le 1$ thanks to its appropriate space homogeneity and the scaling in (1.2). Also notice that due to the oscillations in the solution, we have

$$\varepsilon ||\nabla u_\varepsilon||_M \le C(\beta) N(S).$$

Let us notice for later purposes that the function space,

$$B = \{u \ s.t. \ \frac{1}{R} \int_{|x|\le R} |u(x)|^2 dx \le C, \ \forall R > 0\}$$

is a dual space (see [1] for instance). Its norm is in fact the dual of the norm N used in (3.5). This leads to making the following assumption on the source term S in (1.2),

$$N(S) < \infty \ . \tag{3.6}$$

In fact, while computing the rigorous limit of the right-hand-side Q_ε in (2.7) - See (2.16) -, we will need a stronger norm on S, namely (See 3.9 for the definition of γ),

$$\int_{x\in\mathbb{R}^3} (1+|x|^2)^{N_1} \ |S|^2(x) \ dx \ < \infty \ , \quad \text{for some} \ \ N_1 > 1/2 + (3\gamma/\gamma+1) \ . \tag{3.7}$$

The key assumption we will need (and the main restriction to the present results) is concerned with the radiation condition for w_ε in (2.12) in the limit $\varepsilon \to 0$. More precisely, it is easy to prove - See Lemma 4.1.1 - that, up to extracting subsequences, the solution w_ε to (2.12) weakly converges to *some* solution to (2.13). However, we are not able to prove that it converges towards *the* outgoing condition to (2.13), as it is given by (2.14). The difficulty here is related to the fact that the refraction index $n^2(\varepsilon x)$ in (2.12) goes to $n^2(\infty)$ as x goes to infinity for any fixed $\varepsilon > 0$, whereas the limiting refraction index in (2.13) is constant equal to $n^2(0) \ne n^2(\infty)$ when $\varepsilon = 0$. Therefore, we shall assume that the following convergence holds, as $\varepsilon \to 0$

The weak limit of the solution w_ε to (2.12) is (3.8)
is the outgoing solution w given by (2.14).

Finally, and for sake of completeness, we now write down the technical assumption we need on the regularizing parameter α_ε in the Helmholtz equation (1.2), namely,

$$\alpha_\varepsilon \ge \varepsilon^\gamma \ , \qquad \text{for some } \gamma > 0. \tag{3.9}$$

We refer to §4.5 for this particular point.

We are now ready to state our results. The first step in the derivation of the equation on f, is to obtain bounds which allow to extract convergent subsequences from the family f_ε.

Theorem 3.1.1 . *Under the assumptions (3.1)-(3.2), for all sources S satisfying $N(S) < \infty$, and for any number $\lambda > 0$, the family of Wigner transforms f_ε of u_ε is bounded in the Banach space $X_\lambda^\star$ below and, extracting a subsequence, converges weak-$\star$ to a nonnegative, locally bounded measure f such that*

$$\sup_{R>0} \frac{1}{R} \int_{|x|\le R} \int_{\xi\in\mathbb{R}^3} f(x,\xi) \ dx \ d\xi \le C(\beta) \ N(S)^2. \tag{3.10}$$

The Banach space $X_\lambda^\star$ is defined as the dual space of the set X_λ of functions $\widehat{\varphi}(x,\xi)$ such that $\varphi(x,y) := \mathcal{F}_{\xi\to y}\big(\widehat{\varphi}(x,\xi)\big)$ satisfies,

$$\int_{\mathbb{R}^3} \sup_{x\in\mathbb{R}^3} (1+|x|+|y|)^{1+\lambda}\ |\varphi(x,y)|\ dy\ < \infty\ . \tag{3.11}$$

Notice that this bound on f is sharp since, for $n=1$ and $|\widehat{S}|^2=1$ on the sphere, the measure f is known (see Ap. 1) and $\int_{\mathbb{R}^3} f(x,\xi)\ d\xi = C/|x|^2$.

We can now deduce the transport equation for f.

Theorem 3.1.2 . *Under the assumptions of Theorem 3.1.1, and (3.3), (3.4), (3.7), (2.14), the measure f satisfies the transport equation (1.4) with outgoing condition (1.5) at infinity in the sense that for all functions R such that $R(x,\xi)\in\mathcal{D}(\mathbb{R}^6\backslash\{\xi=0\})$, and $g(x,\xi)=\displaystyle\int_0^{+\infty} R(x-\xi\, t,\xi)\ dt$, we have*

$$\int_{\mathbb{R}^6} \nabla\frac{n^2(x)}{2}\cdot\nabla_\xi\ g(x,\xi)\ f(x,\xi)\ dx\ d\xi + \frac{1}{(4\pi)^2}\int_{S^2}\widehat{S}(\xi)\ g(0,\xi)\ d\xi = \tag{3.12}$$

$$= \int_{\mathbb{R}^6} R(x,\xi)\ f(x,\xi)\ dx\ d\xi\ .$$

This last equation (3.12) is a duality formulation of (1.4) testing it against the solution of the ingoing solution to

$$\xi\cdot\nabla_x g = R, \quad g(x,\xi)=0 \quad \text{for } x\cdot\xi\geq 0 \text{ and } |x|\to\infty\ .$$

3.2 Bounds on the Wigner Transform

In this subsection we prove the Theorem 3.1.1. It follows the spirit of the proof of the corresponding bounds in [14]. We first observe that the bound (3.5) on u_ε readily gives,

$$\|\langle x\rangle^{-\frac{1}{2}-0} u_\varepsilon(x)\|_{L^2(\mathbb{R}^3)} \leq C\ \|u_\varepsilon\|_M \leq C\ N(S)\ , \tag{3.13}$$

where

$$\langle x\rangle := (1+|x|^2)^{1/2}\ , \tag{3.14}$$

and $1+0$ denotes any number close to 1 and larger than 1.

Using (3.13) gives therefore,

$$\left|\int_{\mathbb{R}^6} v_\varepsilon(x,y)\ \varphi(x,y)\ dx\ dy\right| \leq$$

$$\leq \int_{\mathbb{R}^6} \frac{|u|(x+\frac{\varepsilon}{2}y)}{\langle x+\frac{\varepsilon}{2}y\rangle^{\frac{1}{2}+0}}\ \frac{|\bar{u}|(x-\frac{\varepsilon}{2}y)}{\langle x-\frac{\varepsilon}{2}y\rangle^{\frac{1}{2}+0}}\ \langle x+\frac{\varepsilon}{2}y\rangle^{\frac{1}{2}+0}\ \langle x-\frac{\varepsilon}{2}y\rangle^{\frac{1}{2}+0}\ |\varphi|(x,y)\ dx\ dy$$

$$\leq C\ N(S)^2 \int_{\mathbb{R}^3} \sup_{x\in\mathbb{R}^3} \langle |x|+|y|\rangle^{1+0}\ |\varphi(x,y)|\ dy\ .$$

Hence we have

$$\left|\int_{\mathbb{R}^3} f_\varepsilon(x,\xi)\ \widehat{\varphi}(x,\xi)\ dx\ d\xi\right| \leq \int_{\mathbb{R}^6} |v_\varepsilon|(x,y)\ |\varphi|(x,y)\ dx\ dy$$

$$\leq C\ N(S)^2 \int_{\mathbb{R}^3} \sup_{x\in\mathbb{R}^3} \langle |x|+|y|\rangle^{1+0}\ |\varphi(x,y)|\ dy\ . \tag{3.15}$$

We deduce from this bound that the family f_ε is bounded in the Banach space $X_\lambda^\star$ ($\lambda > 0$). ¿From this, we deduce that we may extract from f_ε a subsequence which converges weak-$\star$ to a non-negative measure f (see [18], [14] for the non-negativity). Moreover we still deduce from (3.15) that

$$\left| \int_{\mathbb{R}^6} f(x,\xi)\ \widehat{\varphi}(x,\xi)\ dx\ d\xi \right| \leq C\ N(S)^2 \int_{\mathbb{R}^3} \sup_{x\in\mathbb{R}^3} \langle |x| + |y| \rangle^{1+0}\ |\varphi(x,y)|\ dy\ .$$

We also deduce the bound (3.10) using for instance the family (here χ denotes the indicator function) $\varphi(x,y) = \dfrac{1}{\mu^{3/2}} e^{-|y|^2/\mu}\ \dfrac{1}{R}\chi(|x| \leq R)$ and letting μ tend to zero. This concludes the proof of the a priori bounds on f, and of Theorem 3.1.1.

4 Proof of Theorem 3.1.2

Now, we wish to prove Theorem 3.1.2. For this we need to prove two results. Firstly, we need to prove that the weak-$\star$ limit f of f_ε (obtained in §3) is a distributional solution to (1.4). Secondly, we need to identify the radiation condition for f, i.e. prove (3.12). As the first point is an easy consequence of the proof we give for the radiation condition, we will simply skip it and concentrate on the proof of (3.12). We divide its proof into five steps; we first introduce preliminary estimates, then we give a duality form of the main term $\langle Z_\varepsilon *_\xi f_\varepsilon, g_\varepsilon \rangle$, which we estimate in a separate subsection, and finally prove its convergence. The fifth and last step is devoted to proving the convergence of the term $\langle Q_\varepsilon, g_\varepsilon \rangle$.

4.1 Preliminary observations.

Let us recall some bounds.

Lemma 4.1.1 *Consider the solutions w_ε to (2.12). The families w_ε and ∇w_ε are bounded in B. Therefore w_ε converges in the weak-$\star$ topology of B to a solution to (2.13).*

Proof. The bounds in B are again mere applications of the bound in [17],

$$\|\nabla w_\varepsilon\|_M, \|w_\varepsilon\|_M \leq C\ N(S). \tag{4.16}$$

¿From (4.16) it readily follows that, up to extracting subsequences, w_ε weakly converges in B towards some solution w to (2.13). Thanks to assumption (3.7), we identify w as given by formula (2.14). ■

Next, we consider a test function R as in Theorem 3.1.2 (i.e. R belongs to $C_c^\infty(\mathbb{R}^6)$ and its support does not meet $\{\xi = 0\}$), and introduce g the ingoing solution to the transport equation

$$\xi \cdot \nabla_x g(x,\xi) = R(x,\xi), \tag{4.17}$$

(see Theorem 3.1.2). Also, we introduce g_ε, the solution to,

$$-\alpha_\varepsilon g_\varepsilon + \xi \cdot \nabla_x g_\varepsilon = R(x,\xi), \tag{4.18}$$

which is given, using the notation $\omega = \xi/|\xi|$, by

$$g_\varepsilon = - \int_{s=0}^{+\infty} \exp(-\alpha_\varepsilon |\xi|^{-1} s)\ \frac{1}{|\xi|} R(x - \omega s, \xi)\ ds. \tag{4.19}$$

In the sequel, we need the following bound on the test function g_ε solution to (4.18).

Lemma 4.1.2 *For all $M \geq 0$, the following bound holds,*

$$|\widehat{g}_\varepsilon(x,y)| \leq C\ \frac{\langle x \rangle^M \wedge \alpha_\varepsilon^{-M}}{\langle y \rangle^M}\ , \tag{4.20}$$

where $.\wedge.$ denotes the infimum of two numbers, $\langle x \rangle = (1 + |x|^2)^{1/2}$ and C denotes a constant depending on R and M.

Proof. Since the source term R is compactly supported, say in a ball of *fixed* radius r_0, we first observe that the variable s over which the integration carries in (4.19) does not range in the full interval $[0, +\infty[$, but in some interval centered at $|x|$, say $[|x| - r_0, |x| + r_0]$. In particular, $s \sim |x|$ for large values of $|x|$.
Now we take a multi-index a of length $|a| \leq M$. We write,

$$y^a \widehat{g}_\varepsilon(x,y) = y^a \mathcal{F}_{\xi\to y}(g(x,\xi))$$

$$= \mathcal{F}_{\xi\to y}\Big(\int_{s=0}^{+\infty} [i\partial_\xi]^a \, [\exp(-\alpha_\varepsilon |\xi|^{-1} s) \frac{1}{|\xi|} \, R(x - \omega s, \xi)] \Big)$$

$$= \mathcal{F}_{\xi\to y}\Big(\int_{s=0}^{+\infty} \sum_{b,c,d,e} C(a,b,c,d,e) \exp(-\alpha_\varepsilon |\xi|^{-1} s) \; [i\partial_\xi]^b [-\alpha_\varepsilon |\xi|^{-1} s] \times [i\partial_\xi]^c [-s\omega] \times$$

$$\times \big([i\partial_x]^d [i\partial_\xi]^e \frac{1}{|\xi|} R \big)(x - \omega s, \xi) \Big).$$

Here, the sum $\sum_{b,c,d,e}$ carries over muli-indices of length $\leq M$ and the coefficients $C(a,b,c,d,e)$ simply come from applying the chain rule together with the derivation of products. Now we use that that $s \sim |x|$ for large x, together with the facts that R is compactly supported with a support which does not meet $\{\xi = 0\}$. This readily gives the inequality

$$|y^a \widehat{g}_\varepsilon(x,y)| \leq C \; \exp(-\alpha_\varepsilon \langle x \rangle) \; \langle x \rangle^M$$

(up to modifying α_ε by a constant factor) hence the Lemma. ∎

4.2 Duality form of the equation on f_ε.

As an obvious consequence of (2.7), (4.18), we obtain the duality form of the equation on f_ε

$$\langle f_\varepsilon, R \rangle = -\langle Q_\varepsilon, g_\varepsilon \rangle - \langle Z_\varepsilon *_\xi f_\varepsilon, g_\varepsilon \rangle. \tag{4.21}$$

The terms Z_ε and Q_ε are defined through (2.9), (2.6). Here, $\langle .,. \rangle$ denotes the L^2 scalar product product between functions on $\mathbb{R}^6_{x,\xi}$. The fact that these duality products are well defined is proved below.
In view of (4.21), proving the radiation condition (3.12) in Theorem 3.1.2 is therefore equivalent to proving that, in the limit $\varepsilon \to 0$, the following holds true,

$$\langle f, R \rangle = -\langle Q, g \rangle - \langle Z *_\xi f, g \rangle \;, \tag{4.22}$$

where f is the weak-$*$ limit of f_ε, g is defined in Theorem 3.1.2, and Q, Z are given by (2.10), (2.16). This is done in the next subsection.
Before ending this subsection, we write down two usefull formulae for Z_ε and Q_ε. Firstly, we have,

$$\langle Z_\varepsilon *_\xi f_\varepsilon, g_\varepsilon \rangle = \int_{\mathbb{R}^6} v_\varepsilon(x,y) i \frac{n^2(x + \frac{\varepsilon}{2} y) - n^2(x - \frac{\varepsilon}{2} y)}{\varepsilon} \; \widehat{g}_\varepsilon(x,y) \tag{4.23}$$

$$= \int_{-1}^{1} \int_{\mathbb{R}^6} v_\varepsilon(x,y) \; iy \cdot \nabla n^2(x + \frac{\varepsilon \theta y}{2}) \; \widehat{g}_\varepsilon(x,y) dx \; dy \; d\theta$$

$$= \int_{-1}^{1} \int_{\mathbb{R}^6} \Psi_\varepsilon(x,y,\theta) \; dx \; dy \; d\theta,$$

with

$$\Psi_\varepsilon(x,y,\theta) = u_\varepsilon(x + \frac{\varepsilon}{2} y) \bar{u}_\varepsilon(x - \frac{\varepsilon}{2} y) \; iy \cdot \nabla n^2(x + \frac{\varepsilon \theta y}{2}) \; \widehat{g}_\varepsilon(x,y). \tag{4.24}$$

Also,

$$\langle Q_\varepsilon, g_\varepsilon \rangle = Re \int_{\mathbb{R}^6} w_\varepsilon(x+y) S(x) \widehat{g}_\varepsilon(\varepsilon [x + \frac{y}{2}], y) \; dx \; dy \; . \tag{4.25}$$

4.3 Bounds.

In this section, we prove that the quantities (4.23), (4.25) are well defined, and we pass to the limit in (4.23), (4.25) in the next subsection. To do so, we decompose the integral $\int_{\mathbb{R}^6} \cdots$ in (4.23) into the following sets,

$$A_\varepsilon = \{x \in \mathbb{R}^3 \ , \ |\varepsilon^{1-0} y| \leq 1\} \ , \ B_\varepsilon = \{|x| \geq |\varepsilon y| \ , \ |\varepsilon^{1-0} y| \geq 1\} \ , \tag{4.26}$$

$$C_\varepsilon = \{|x| \leq |\varepsilon y| \ , \ |\varepsilon^{1-0} y| \geq 1\} \ .$$

On each of these sets, the method is to first take the L^1_x norm of the product $u_\varepsilon(\cdots)\bar{u}_\varepsilon(\cdots)$ thanks to the bound (3.13) on u_ε, and then to evaluate the remaining integral in y. Assumption (3.3) together with Lemma 4.1.2 allow indeed to obtain the desired integrability in y over each of these sets. Notice that the bounds below, could be derived as well with the same sets but defined with $|\varepsilon y|$ rather than $|\varepsilon^{1-0} y|$; however, we need these sets for the limit in the next subsection.

We now come to the details.

* On A_ε, starting from (4.23), (4.24) and using Lemma 4.1.2, we write,

$$|\Psi_\varepsilon(x, y, \theta)| \leq C \frac{|u_\varepsilon|(x + \cdots)}{\langle x + \cdots \rangle^{1/2+0}} \frac{|\bar{u}_\varepsilon|(x - \cdots)}{\langle x - \cdots \rangle^{1/2+0}} \langle x \rangle^{1+0} \langle x \rangle^{-N_0} \frac{\langle x \rangle^M}{\langle y \rangle^{M-1}} .$$

Then, we use $M = 4 + 0$ and $N_0 > M + 1$ and therefore, first performing the Cauchy-Schwarz inequality in x, then integrating in y, we obtain

$$\int_{A_\varepsilon} |\Psi_\varepsilon(x, y, \theta)| \leq C \|u_\varepsilon\|_M^2 \ \|\langle x \rangle^{N_0} \nabla_x n^2(x)\|_{L^\infty(\mathbb{R}^3)} \int_{|y| \leq \varepsilon^{-1+0}} \langle y \rangle^{1-M} \, dy \tag{4.27}$$

and thus,

$$\int_\theta \int_{A_\varepsilon} |\Psi_\varepsilon(x, y, \theta)| \leq C \|u_\varepsilon\|_M^2 \ \|\langle x \rangle^{N_0} \nabla_x n^2(x)\|_{L^\infty(\mathbb{R}^3)} \ .$$

* On B_ε we write,

$$|\Psi_\varepsilon(x, y, \theta)| \leq C \frac{|u_\varepsilon|(x + \varepsilon y/2)}{\langle x + \varepsilon y/2 \rangle^{1/2+0}} \frac{|\bar{u}_\varepsilon|(x - \cdots)}{\langle x - \cdots \rangle^{1/2+0}} \langle x \rangle^{1+0} \ |y| \ \langle x \rangle^{-N_0} \ \frac{\langle x \rangle^M}{\langle y \rangle^M} .$$

Again, we choose $N_0 > M + 1 + 0$ and we first perform the Cauchy-Schwarz inequality in x. This yields

$$\int_{B_\varepsilon} |\Psi_\varepsilon(x, y, \theta)| \leq$$

$$\leq C \|u_\varepsilon\|_M^2 \ \ \|\langle x \rangle^{N_0} \nabla_x n^2(x)\|_{L^\infty(\mathbb{R}^3)} \int_{|y| \geq \varepsilon^{-1+0}} \frac{1}{\langle y \rangle^{M-1}} \tag{4.28}$$

$$\leq C \|u_\varepsilon\|_M^2 \ \|\langle x \rangle^{N_0} \nabla_x n^2(x)\|_{L^\infty(\mathbb{R}^3)} \ o(1) \ ,$$

where $o(1) \rightarrow_{\varepsilon \to 0} 0$, upon choosing $M = 4 + 0$.

* On C_ε we argue exactly as above and, since εy dominates x now, we obtain

$$\int_{C_\varepsilon} |\Psi_\varepsilon(x, y, \theta)| \leq$$

$$\leq C \|u_\varepsilon\|_M^2 \ \|\nabla_x n^2(x)\|_{L^\infty(\mathbb{R}^3)} \int_{|y| \geq \varepsilon^{-1+0}} \langle \varepsilon y \rangle^{1+0} \ |y| \ \frac{\langle \varepsilon y \rangle^M \wedge \alpha_\varepsilon^{-M}}{\langle y \rangle^M} \ ,$$

and it remains to control (with $z = \varepsilon y$)

$$\varepsilon^{M-4} \int_{\mathbb{R}^3} \langle z \rangle^{1+0} \ |z| \ \frac{\langle z \rangle^M \wedge \alpha_\varepsilon^{-M}}{\langle z \rangle^M} \ dz.$$

We now use, from (3.9), that for some for some $\gamma > 0$, $\alpha_\varepsilon \geq \varepsilon^\gamma$. Accordingly, the above integral is upper bounded by

$$\varepsilon^{M-4-\gamma(5+0)} \int_{\mathbb{R}^3} \langle z \rangle^{1+0} \, |z| \, \frac{\langle z \rangle^M \wedge 1}{\langle z \rangle^{M-1}} \, dz,$$

and it remains to choose $M > 5\gamma + 4$. Putting these bounds together gives again,

$$\int_\theta \int_{C_\varepsilon} |\Psi_\varepsilon(x,y,\theta)| \leq C \|u_\varepsilon\|_M^2 \, \|\nabla_x n^2(x)\|_{L^\infty(\mathbb{R}^3)} \, o(1) \,, \tag{4.29}$$

where $o(1) \to_{\varepsilon\to 0} 0$.

As a conclusion, the above computations show the following statement:

For *any* $N_0 > 5$, we have,

$$|\langle Z_\varepsilon *_\xi f_\varepsilon, g_\varepsilon \rangle| \leq C \, \|u_\varepsilon\|_M^2 \, \|\langle x \rangle^{N_0} \nabla_x n(x)\|_{L^\infty} \,. \tag{4.30}$$

4.4 Convergence of $\langle Z_\varepsilon * f_\varepsilon, g_\varepsilon \rangle$.

We decompose $\langle Z_\varepsilon * f_\varepsilon, g_\varepsilon \rangle$ into the following form,

$$\begin{aligned}
&\langle Z_\varepsilon * f_\varepsilon, g_\varepsilon \rangle = \\
&\int_{\theta=-1}^1 \int_{\mathbb{R}^6} u_\varepsilon(x+\frac{\varepsilon y}{2}) \bar{u}_\varepsilon(x-\frac{\varepsilon y}{2}) \, iy \, \cdot \left[\nabla n^2(x+\frac{\varepsilon\theta y}{2}) - \nabla_x n^2(x)\right] \widehat{g}_\varepsilon(x,y) \, dx \, dy \, + \\
&+ \int_{\theta=-1}^1 \int_{\mathbb{R}^6} \widehat{f}_\varepsilon(x,y) \, iy \cdot \nabla_x n^2(x) \left[\widehat{g}_\varepsilon(x,y) - \widehat{g}(x,y)\right] \, dx \, dy \, + \\
&+ \int_{\theta=-1}^1 \int_{\mathbb{R}^6} \widehat{f}_\varepsilon(x,y) \, iy \cdot \nabla_x n^2(x) \widehat{g}(x,y) \, dx \, dy \\
&:= I_\varepsilon + II_\varepsilon + III_\varepsilon \,.
\end{aligned}$$

* The most obvious term is III_ε. Indeed, the test function $y \cdot \nabla_x n^2(x) \widehat{g}(x,y)$ involved in the definition of III_ε clearly belongs to X_λ for any $\lambda > 0$ sufficiently close to zero since,

$$\begin{aligned}
&\int_{\mathbb{R}^3} \sup_{x\in\mathbb{R}^3} \langle |x| + |y| \rangle^{1+0} \left| y \cdot \nabla_x n^2(x) \widehat{g}(x,y) \right| \, dy \;\leq \\
&\leq C \int_{\mathbb{R}^3} \sup_{x\in\mathbb{R}^3} \langle |x| + |y| \rangle^{1+0} \, \langle x \rangle^{-N_0} \, \frac{\langle x \rangle^M}{\langle y \rangle^{M-1}} \, dy \\
&\leq C \int_{\mathbb{R}^3} \sup_{x\in\mathbb{R}^3} \langle x \rangle^{1+M-N_0+0} \, \langle y \rangle^{-M+1} + C \int_{\mathbb{R}^3} \sup_{x\in\mathbb{R}^3} \langle x \rangle^{M-N_0} \, \langle y \rangle^{2+0-M} \\
&\leq C
\end{aligned}$$

up to taking $M = 4+0$ in the first term and $M = 5+0$ in the second, and since $N_0 > 5$. Here we used (3.13), (3.3) and (4.20).
Since f_ε converges weak-$*$ in X_λ^* (for any $\lambda > 0$), this establishes the convergence,

$$III_\varepsilon \to \langle f, \nabla_x n^2 \cdot \nabla_\xi g \rangle \,.$$

* We now come to the proof that I_ε vanishes as $\varepsilon \to 0$.
For this we again decompose the integral $\int_{\mathbb{R}^6} \cdots$ defining I_ε according to the sets A_ε, B_ε, C_ε introduced in (4.26). We have already proved in the previous subsection that the sets B_ε and C_ε give a vanishing contribution to I_ε as $\varepsilon \to 0$ (see the terms $o(1)$).
It remains therefore to estimate the contribution of the set A_ε, namely,

$$\int_{\theta=-1}^1 \int_{|y|\leq\varepsilon^{-1+0}} \int_{x\in\mathbb{R}^3} u_\varepsilon(\cdots) \, \bar{u}_e(\cdots) \, iy \, \left[\nabla_x n^2(x+\frac{\varepsilon\theta y}{2}) - \nabla_x n^2(x)\right] \widehat{g}_\varepsilon(x,y) \, dx \, dy$$

$$\leq C\ \|u_\varepsilon\|_M^2 \sup_{|y|\leq\varepsilon^{-1+0}} \sup_{x\in\mathbb{R}^3} \langle x\rangle^{N_0}\ |\nabla_x n^2(x+\varepsilon y) - \nabla_x n^2(x)| \to_{\varepsilon\to 0} 0\ ,$$

thanks to assumptions (3.3) and (3.4).

* We now come to the study of II_ε.
Firsly, by reproducing the method of proof of Lemma 4.1.2, we may write that, for any $M \geq 0$,

$$\langle y\rangle^M \left|\widehat{g}_\varepsilon(x,y) - \widehat{g}(x,y)\right| = \left|\int_{s=0}^{+\infty} \mathcal{F}_{\xi\to y}\langle i\partial_\xi\rangle^M\ [\exp(-\alpha_\varepsilon|\xi|^{-1}s)-1]\ [\frac{1}{|\xi|}\ R(x-\omega s,\xi)]\ ds\right.$$

$$\leq C\ \langle x\rangle^M\ |\exp(-\alpha_\varepsilon\langle x\rangle)-1|\ + C\ \alpha_\varepsilon\ \langle x\rangle^M\ \exp(-\alpha_\varepsilon\langle x\rangle)\ . \tag{4.31}$$

As in §4.3, we may therefore estimate, thanks to (4.31),

$$\left|\ II_\varepsilon\ \right| = \left|\ \int_{\mathbb{R}^6} u_\varepsilon(x+\frac{\varepsilon y}{2})\bar{u}_\varepsilon(x-\frac{\varepsilon y}{2})\ y\cdot\nabla_x n(x)\ \Big[g_\varepsilon(x,y)-g(x,y)\Big]\ dx\ dy\ \right|$$

$$\begin{aligned}\leq C\ \|u_\varepsilon\|_M^2 \int_{\mathbb{R}^3} \sup_{x\in\mathbb{R}^3} \langle |x|+|y|\rangle^{1+0}\ \langle x\rangle^{-N_0}\ \frac{\langle x\rangle^M}{\langle y\rangle^{M-1}} \times \\ \times\Big[\ |\exp(-\alpha_\varepsilon\langle x\rangle)-1| + \alpha_\varepsilon\exp(-\alpha_\varepsilon\langle x\rangle)\Big]\ dy\ .\end{aligned} \tag{4.32}$$

Here we have used assumption (3.3) together with (3.13).
As we did while estimating III_ε, we readily observe that the assumption $N_0 > 5$ implies that the function

$$\sup_x \langle |x|+|y|\rangle^{1+0}\ \langle x\rangle^{-N_0}\ \frac{\langle x\rangle^M}{\langle y\rangle^{M-1}}$$

is integrable in the y variable. Therefore, by the dominated convergence theorem, the estimate (4.32) implies that

$$II_\varepsilon \to_{\varepsilon\to 0} 0\ .$$

4.5 Convergence of $\langle Q_\varepsilon, g_\varepsilon\rangle$.

This step is essentially a reformulation of the method used in the paragraphs §4.3 and §4.4 above.
Using Lemma 4.1.1 together with formula (4.25), we easily bound, using assumption (3.7) and Lemma 4.1.2,

$$|\langle Q_\varepsilon, g_\varepsilon\rangle| \leq C \int_{\mathbb{R}^6} \frac{|w_\varepsilon|(x+y)}{\langle x+y\rangle^{\frac12+0}}\ \langle x+y\rangle^{\frac12+0}\ \langle x\rangle^{N_1}|S|(x)\ \langle x\rangle^{-N_1}\ \frac{\langle\varepsilon(x+\frac{y}{2})\rangle^M\wedge\alpha_\varepsilon^{-M}}{\langle y\rangle^M}$$

$$\leq C \int_{\mathbb{R}^3} \sup_{x\in\mathbb{R}^3} \langle |x|+|y|\rangle^{\frac12+0}\ \langle x\rangle^{-N_1}\ \frac{\langle\varepsilon(|x|+|y|)\rangle^M\wedge\alpha_\varepsilon^{-M}}{\langle y\rangle^M}\ ,$$

upon using the Cauchy-Schwarz inequality in x.
We now distinguish the cases $|x| \geq |y|$, and $|x| \leq |y|$. The term stemming from the case $|x| \geq |y|$ gives a contribution which is estimated by,

$$C \int_{\mathbb{R}^3} \langle y\rangle^{-M} \sup_{x\in\mathbb{R}^3} \langle x\rangle^{\frac12+0-N_1}\ [\ \langle\varepsilon x\rangle^M\wedge\alpha_\varepsilon^{-M}\ ]\ dy$$

$$\leq C\ \varepsilon^{-\gamma M}\ \varepsilon^{-(\gamma+1)(\frac12+0-N_1)} \int_{\mathbb{R}^3} \langle y\rangle^{-M}\ dy\ \leq C\ ,$$

upon taking $M = 3+0$, and $N_1 > \frac12 + \frac{3\gamma}{\gamma+1}$. Also the contribution of the term stemming from the case $|x| \leq |y|$ is easily bounded by

$$C \int_{\mathbb{R}^3} \langle y\rangle^{\frac12+0}\ \frac{\langle\varepsilon y\rangle^M\wedge\alpha_\varepsilon^{-M}}{\langle y\rangle^M}\ dy \to_{\varepsilon\to 0} 0\ ,$$

thanks to the same argument leading to (4.29). This establishes that $|\langle Q_\varepsilon, g_\varepsilon\rangle|$ is uniformly bounded in ε. More precisely, we may write,

$$|\langle Q_\varepsilon, g_\varepsilon\rangle| \leq C\ \|w_\varepsilon\|_M\ \|\langle x\rangle^{N_1} S(x)\|_{L^2}\ , \tag{4.33}$$

for *any* $N_1 > \frac{3\gamma}{\gamma+1} + \frac{1}{2}$.

On the more, in order to compute the actual limit of $\langle Q_\varepsilon, g_\varepsilon\rangle$ in ε, we may write,

$$\langle Q_\varepsilon, g_\varepsilon\rangle = \int_{x,y} w_\varepsilon(x+y)\ S(x)\ \Big[\widehat{g}_\varepsilon(\varepsilon(x+\frac{y}{2}), y) - \widehat{g}_\varepsilon(0,y)\Big] +$$

$$+ \int_{x,y} w_\varepsilon(x+y)\ S(x)\ \Big[\widehat{g}_\varepsilon(0,y) - \widehat{g}(0,y)\Big]\ +\ \int_{x,y} w_\varepsilon(x+y)\ S(x)\ \widehat{g}(0,y)$$

$$:= I_\varepsilon + II_\varepsilon + III_\varepsilon\ .$$

Reasonning as §4.4 above, it is straightforward to deduce from (4.33) together with assumption (3.7) and Lemma 4.1.2 that,

$$I_\varepsilon \to_{\varepsilon\to 0} 0\ ,\ II_\varepsilon \to_{\varepsilon\to 0} 0\ .$$

Also, using the weak convergence of w_ε as stated in (3.8) readily implies,

$$III_\varepsilon \to_{\varepsilon\to 0} \int_{x,y} w(x+y)\ S(x)\ \widehat{g}(0,y)\ .$$

This ends the proof of convergence of Q_ε.

¿From the results above, together with formula (4.21), we readily deduce (4.22) by taking the limit $\varepsilon \to 0$. This proves the radiation condition (3.12) in Theorem 3.1.2.

Ap. A1. Explicit formula in three dimensions

In order to give an explicit example of the above theory, we consider in three dimensions, the particular case $n = 1, S = \delta, \alpha_\varepsilon = 0$. Although it does not enter our assumptions (mainly because S is too singular here) it contains the main effects for the high frequency limit. We have

$$u_\varepsilon = \frac{e^{i|x|/\varepsilon}}{4\pi|x|}. \tag{A1.1}$$

Also, for $\varepsilon = 1$ we have

$$\mathcal{F}u_1 = \frac{1}{(2\pi)^3}[pv\frac{1}{1-|\xi|^2} + i\frac{\pi}{2}\delta(|\xi|=1)]\ .$$

This is easily seen by a Fourier Transform of the Helmholtz equation (1.2) as α_ε vanishes. We can also compute its Wigner Transform. Firstly, we have

$$v_\varepsilon(x,y) = \frac{1}{(4\pi)^2}\frac{e^{iy\cdot x/|x|}}{|x|^2}\ +\ O(\varepsilon).$$

The limiting transport equation is therefore

$$\xi\cdot\nabla_x f = -\frac{1}{(4\pi)^2}\delta(x)\delta(|\xi|=1),$$

whose solution, with the outgoing condition at infinity, is given by

$$f(x,\xi) = \frac{1}{(4\pi)^2}\frac{\delta(\xi + x/|x|)}{|x|^3} = \frac{1}{(4\pi)^2}\int_0^\infty\ \delta(x+\xi s)\ \delta(|\xi|=1)\ ds\ .$$

In particular, it is a locally (in x) bounded measure, but not a globally bounded measure, since the mass of $\{x = x_0\} \times \mathbb{R}^3_\xi$ is equal to $(4\pi)^{-2}|x_0|^{-2}$.

Ap. A2. The 1D case

In the one dimensional case the formulas differ somewhat. We write the Helmholtz equation

$$\Delta u_\varepsilon + \left(\frac{n(x)}{\varepsilon}\right)^2 u_\varepsilon = -\frac{2}{\varepsilon} S_\varepsilon(x), \quad x \in \mathbb{R}, \tag{A2.1}$$

together with the outgoing condition at infinity. Following the three dimensonal case (Section 2), we now obtain the geometrical optics equations

$$\xi \cdot \nabla_x f + \frac{1}{2}\nabla_x n^2(x) \cdot \nabla_\xi f = \delta(x)[\delta(\xi+1) + \delta(\xi-1)]. \tag{A2.2}$$

Indeed, we compute, in the special case $n = 1$, $S_\varepsilon = \delta$,

$$u_\varepsilon^{sing} = ie^{i|x|/\varepsilon}. \tag{A2.3}$$

Its Fourier Transform is, for $\varepsilon = 1$,

$$\mathcal{F}u_1^{sing} = \frac{1}{2\pi}\Big[pv\frac{2}{1-|\xi|^2} - i\pi[\delta(\xi+1) + \delta(\xi-1)]\Big]. \tag{A2.4}$$

Its Wigner Transform is simply

$$f_\varepsilon^{sing}(x,\xi) = -\delta(\xi - x/|x|) \ + \ O(\varepsilon). \tag{A2.5}$$

Again we check that its limit, as ε vanishes, satisfies the geometrical optics equation (A2.2), or in other words

$$\delta(\xi - x/|x|) = \int_0^{+\infty} \delta(x - \xi s)[\delta(\xi+1) + \delta(\xi-1)]ds.$$

Acknowledgment This work has been partially supported by TMR project ‘Conservation Laws’ of E.U.:HCL

ERBFMRXCT 960033 and by the TMR project ‘Asymptotic Methods in Kinetic Theory” of E.U.:ERBFMBXCT 970157

References

[1] S. Agmon, L. Hörmander, Asymptotic properties of solutions of differential equations with simple characteristics, J. Analyse Math., **30** (1976), 1–38.

[2] J.A. Barcelo, A. Ruiz, L. Vega, Weighted estimates for Helmholtz equation and some applications, J. of Funct. Anal. , **150** (1997), 356–382.

[3] J.-D. Benamou, Direct computation of multi-valued phase-space solutions of Hamilton-Jacobi equations, to appear in Comm. Pure and Appl. Math.

[4] J.-D. Benamou, F. Castella, B. Perthame, O. Runborg, High frequency limit in the Helmholtz equation with a general source, in preparation.

[5] F. Castella , On the derivation of a quantum Boltzmann equation from the periodic Von-Neumann equation, Math. Mod. An. Num. **33**, N. 2 (1999), 329–350.

[6] F. Castella, P. Degond, From the Von-Neumann equation to the Quantum Boltzmann equation in a deterministic framework, Preprint Université de Rennes 1 and C. R. Acad. Sci., t. 329, sér. I (1999), 231–236.

[7] L. Erdös, H.T. Yau, Linear Boltzmann equation as scaling limit of the quantum Lorentz gas, Preprint (1998).

[8] R. Esposito, M. Pulvirenti, A. Teta, The Boltzmann Equation for a one-dimensional Quantum Lorentz gas, Preprint (1998).

[9] I. Gasser, P. Markowich, B. Perthame, Dispersion and moments lemma revisited, to appear in J. Diff. Eq.

[10] P. Gérard, Microlocal defect measures, Comm. Partial Diff. Equations **16** (1991), 1761–1794.

[11] P. Gérard, P.A. Markowich, N.J. Mauser, F. Poupaud, Homogeneisation limits and Wigner transforms, Comm. pure and Appl. Math., **50** (1997), 321–357.

[12] J.B. Keller, R. Lewis, Asymptotic methods for partial differential equations: the reduced wave equation and Maxwell's equation, in Surveys in applied mathematics, eds J.B. Keller, D.McLaughlin and G. Papanicolaou, Plenum Press, New York, 1995.

[13] C. Kenig, G. Ponce, L. Vega, Small solutions to nonlinear Schroedinger equations, Annales de l'I.H.P., **10**, (1993), 255–288.

[14] P.-L. Lions, T. Paul, Sur les mesures de Wigner, Revista Matemática Iberoamericana, **9 (3)** (1993), 553–618.

[15] P.L.Lions, B.Perthame, Lemmes de moments, de moyenne et de dispersion. C. R. Acad. Sc t.314 (série I) (1992), 801–806.

[16] G. Papanicolaou, L. Ryzhik, Waves and Transport. IAS/ Park City Mathematics series. Volume 5 (1997).

[17] B. Perthame, L. Vega, Morrey-Campanato estimates for Helmholtz equations. To appear in J. Funct. Anal.

[18] L. Tartar, H-measures, a new approach for studying homogenisation, oscillations and concentration effects in partial differential equations, Proc. Roy. Soc. Ed., **115 A** (1990), 193–230.

[19] Bo Zhang, Radiation condition and limiting amplitude principle for acoustic propagators with two unbounded media, Proc. Roy. Soc. Ed., **128 A** (1998), 173–192.

Unit´e de recherche INRIA Lorraine, Technopôle de Nancy-Brabois, Campus scientifique,
615 rue du Jardin Botanique, BP 101, 54600 VILLERS LÈS NANCY
Unit´e de recherche INRIA Rennes, Irisa, Campus universitaire de Beaulieu, 35042 RENNES Cedex
Unit´e de recherche INRIA Rhône-Alpes, 655, avenue de l'Europe, 38330 MONTBONNOT ST MARTIN
Unit´e de recherche INRIA Rocquencourt, Domaine de Voluceau, Rocquencourt, BP 105, 78153 LE CHESNAY Cedex
Unit´e de recherche INRIA Sophia-Antipolis, 2004 route des Lucioles, BP 93, 06902 SOPHIA-ANTIPOLIS Cedex

Éditeur
INRIA, Domaine de Voluceau, Rocquencourt, BP 105, 78153 LE CHESNAY Cedex (France)
http://www.inria.fr
ISSN 0249-6399