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On the finite volume reformulation of the mixed finite elements method on triangles

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Abstract: We analyse the finite volume reformulation of the triangular mixed finite element approximation for the porous flow equation, as proposed in [10] [9]. We show that the finite volumes are obtained by aggregation of finite elements (usually one, sometimes two or more), that the matrix of the finite volume equations is regular, but generally not symmetrical, and that the finite volume formulation is algebraically equivalent to the mixed approximation. The finite volume matrix becomes symmetrical in the stationary case, and positive definite when the triangulation satisfies the Delaunay condition.

Key-words: mixed finite elements, finite volumes, flow in porous media

(Résumé : tsvp)

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Sur la reformulation volumes finis de l'approximation par éléments finis mixtes triangulaires

Résumé : Nous analysons dans ce travail la reformulation par volumes finis de l'approximation mixte sur des triangles pour l'équation des écoulements en milieux poreux, introduite par [10] [9]. Nous montrons que les volumes finis sont obtenus en agrégeant un ou plusieurs éléments finis, que la matrice des équations de volume fini est régulière, mais généralement non symétrique, et que cette formulation volume fini est algébriquement équivalente à la formulation mixte. Dans le cas stationnaire, la matrice en volumes finis devient symétriques, et définie positive lorsque la triangulation satisfait le critère de Delaunay.

Mots-clé : éléments finis mixtes, volumes finis, écoulement en milieux poreux.

1 Introduction

We consider in this paper the numerical resolution of the elliptic equation:

$$\left\{\begin{array}{l} -\nabla(a\nabla u) + bu = f \quad \text{in } \Omega \\ \\ u = u_e \text{ on } \partial\Omega_D \\ \\ -a\dfrac{\partial u}{\partial \nu} = g \text{ on } \partial\Omega_N \end{array}\right. \tag{1}$$

where Ω is a bounded, polygonal open set of $I\!R^2, \partial\Omega_D$ and $\partial\Omega_N$ is a partition of the boundary $\partial\Omega$ of Ω corresponding to Dirichlet and Neumann conditions, and ν is the outer normal to $\partial\Omega$.

This work was originally motivated by efficiency consideration for the computation of transient and stationary flow in porous media. If one uses for example an implicit time discretization, the corresponding equations are

$$\left\{\begin{array}{lll} c\dfrac{u^n - u^{n-1}}{\Delta t} & -\nabla(a\nabla u^n) + bu^n = f^n \text{ in } \Omega, & n = 1, 2, ..., \\ \\ u^n = u_e^n & \text{on } \partial\Omega_D, \quad n = 1, 2, ..., & \\ \\ -a\dfrac{\partial u^n}{\partial \nu} = g^n & \text{on } \partial\Omega_N, \quad n = 1, 2, ..., & \\ \\ u^0 = u_o & \text{on } \Omega & \end{array}\right. \tag{2}$$

So we see that

- $\underline{b = 0 \text{ in (1) corresponds to the stationary or steady state case}}$ $c = 0$ in (2),
- $\underline{b > 0 \text{ in (1) corresponds to the transient case}}$ $c > 0$ in (2) (substitute b by $\dfrac{c}{\Delta t}$ and f by $f + \dfrac{c}{\Delta t}u^n$),

and from now on we shall consider only equation (1).

In the context of flow through porous media, the numerical resolution of (1) by mixed finite elements has received a growing attention in the last years [5] [7] [4] : the separate approximation of the piezometric head u and the Darcy velocity $\vec{q} = -a\nabla u$ it provides allows for a precise determination of flow lines and propagation of contaminants.

But the development of the mixed approach have been hampered by the computational cost involved. When lowest-order Raviart-Thomas elements [8] are used, which is almost always the case in practical applications, the resolution of (1) leads to a system with one scalar unknown Tu_E per edge E (the notation Tu_E stands for "trace of u on E" and represents the piezometric head on E). This can be a major drawback for large problems, compared to finite difference and finite volume schemes which lead to one unknown u_K per cell K.

So attempts to reconcile the precision of mixed approximations and the efficiency of finite differences and finite volumes have been made by various authors. The first results were all based on the use of a quadrature formula to diagonalize the elemental matrices of the mixed approximation. This worked nicely on rectangular meshes, where numerical quadrature made mixed approximation equivalent to finite differences (see for example [2]). These results were extended to the case of tringular meshes in [1] , but the diagonalization of the elemental matrix by numerical quadrature appears to be a precise approximation only when the triangle has three sharp angles, and the corresponding schemes with one unknown u_K per triangle do not seem to be used in practice.

Recently, a purely algebraic transformation of the mixed approximation on a triangular mesh has been proposed, which allows to rewrite the equations in term of only one unknow H_K per triangle K, where:

$$H_K = \sum_{E \subset \partial K} \pi_{K,E} Tu_E \tag{3}$$

(H_K is different from the mixed elemental unknown u_K). This triangulation was introduced first in [10] for stationary problems, and extended to transient problem in [9]. It was shown in these papers that the resulting system for the H_K unknown was non-symmetrical in the transient case, and symmetrical (positive definite under additional assumptions) in the stationary case. All the calculations however were formal, in the sense that for example no attention was paid to check that the denominators in the formula were non zero, or to what happened to the final system of equations when some of the coefficients vanished.

So the objective of this paper is to give a rigorous presentation and a slight generalization of this material.
After recalling in detail the properties of the lowest-order Raviart-Thomas approximation on one triangle K in section 1, we introduce in section 2 the new unknown H_K by requiring that the formulas

$$Q_{K,E} = \xi_{K,E} H_K - \delta_{K,E} T u_E + \gamma_K, \quad E \subset \partial K \tag{4}$$

produce on K the same Dirichlet-to-Neumann map as the original mixed approximation. We give formula for $\pi_{K,E}, \xi_{K,E}, \delta_{K,E}$ and γ_K both in the case where all three off-diagonal entries of the 3×3 mixed Dirichlet-to-Neumann matrix are non zero (we call such a K regular), and in the case where some off-diagonal entries vanish (we call such a K singular). We show that, when $(\mathcal{T}_h, \quad h > 0)$ is a regular family of triangulation of Ω, it is always possible, provided h is small enough, to ensure that H_K is a linear interpolate/extrapolate of the $Tu_E, \ E \subset \partial K$ by chosing $\sum_{E \subset \partial K} \pi_{K,E} = 1$, and that the $\delta_{K,E}$ in (4) are non-zero.

In section 3 , we build up the "finite volume" equations

$$BH = F \tag{5}$$

where $H = (H_V, \ V \in \mathcal{V}_h)$, and H_V is the unknown associated to the finite volume V. Most of the times, a finite volume V will coincide with one triangular element K (and then $H_V = H_K$!), but it can also be made of two elements K and L sharing a singular edge F, that is an edge such that:

$$\delta_{K,F} + \delta_{L,F} = 0 \text{ or } \frac{1}{\delta_{K,F}} + \frac{1}{\delta_{L,F}} = 0 \tag{6}$$

(and then H_V is linked to H_K and H_L by simple affine relations). For simplicity, we limit ourselves in this paper to the case where a finite volume V is made of one or two elements, but in general a finite volume could be made of an arbitrary number of elements. The main findings in section 3 are that i) the finite volume equation (5) is algebraically equivalent to the lowest order Raviart-Thomas mixed approximation, and ii) the matrix B is regular, under technical hypothesis which are automatically satisfied, for a regular family of triangulation, when h is small enough. Notice that the number of finite unknows H_V in (5) can be significantly smaller than the number of elements K if many finite volumes are made of two elements !

In section 4, we specify the above results to the stationary case where $b = 0$. In this case, the finite volume equation (5) is always equivalent to the Raviart-Thomas mixed approximation, with a finite volume matrix B which is both regular and symmetrical. Then we give a necessary condition:

$$\sum_{E\subset\partial V, E\not\subset\partial\Omega_N} \delta_{V,W} > 0 \quad \forall V \in \mathcal{V}_h \tag{7}$$

and a sufficient condition

$$\delta_{V,W} > 0 \quad \forall V \in \mathcal{V}_h, \forall E \subset \partial V, E \not\subset \partial\Omega_N \tag{8}$$

for B to be positive definite, where:

- if $E \subset \partial V$ is an interior edge, then $E = K \cap L$, and

$$\frac{1}{\delta_{V,W}} = \frac{1}{\delta_{K,E}} + \frac{1}{\delta_{L,E}} = \frac{1}{2}\left(\frac{\text{cotan}\,\theta_{K,E}}{a_K} + \frac{\text{cotan}\,\theta_{L,E}}{a_L}\right), \tag{9}$$

- if $E \subset \partial V$ is a boundary edge, then $E = K \cap \partial\Omega$, and

$$\frac{1}{\delta_{V,W}} = \frac{1}{\delta_{K,E}} = \frac{1}{2}\frac{\text{cotan}\,\theta_{K,E}}{a_K}. \tag{10}$$

In the above formula, $\theta_{K,E}$ denotes the angle of K opposite to the edge E. When the diffusion coefficient a is constant, we see from (6) (9) that one finite volume V will be made all elements K of $\mathcal{T}_h$ which have their vertices on a same circle, and from (8) (9) that the matrix B will be (symmetric) positive definite as soon as the triangulation $\mathcal{T}_h$ satisfies the Delaunay criterion, i.e.: for any K of $\mathcal{T}_h$, the opposite vertices of its three neighbours are outside the circle circumvented to K.

2 The lowest order mixed Raviart-Thomas approximation over one element

Let us consider a triangular element K with vertices a_i $i = 1, 2, 3$, and denote by E_i, $i = 1, 2, 3$ the opposite edges. In the mixed-hybrid formulation of the mixed approximation, the solution u to equation (1) is approximated, over K, by the following quantities:

$$\begin{array}{lcl} u_K \in \mathbb{R} & \simeq & \text{mean value of } u \text{ over the element } K \\ Tu_{K,i} \in \mathbb{R} & \simeq & \text{mean value of } u \text{ over the edge } E_i, \quad i = 1, 2, 3 \\ \vec{q}_K \in H_K & \simeq & \text{approximation of } \vec{q} = -a\nabla u \text{ over } K \end{array} \tag{11}$$

where H_K is the lowest-order Raviart-Thomas space [8]. A basis of H_K is given by:

$$\vec{w}_{K,i}(x) = \frac{1}{2|K|}(x - a_i) \quad i = 1, 2, 3. \tag{12}$$

and $\vec{q}_K$ writes

$$\vec{q}_K = \sum_{i=1}^{3} Q_{K,i}\vec{w}_{K,i} \tag{13}$$

where $Q_{K,i}$ denotes the flow leaving K through the i^{th} edge. We introduce now local matrix notations on element K:

$$\left\{\begin{array}{l} A_K = [A_{K,i,j}] \quad \text{with } A_{K,i,j} = \displaystyle\int_K \vec{w}_{K,i} \cdot \vec{w}_{K,j}, \\ \text{(symmetric, positive definite elemental Raviart-Thomas matrix)} \\ D = [1 \quad 1 \quad 1] \text{ (elemental divergence matrix)} \\ Q_K = \begin{bmatrix} Q_{K,1} \\ Q_{K,2} \\ Q_{K,3} \end{bmatrix} \quad , Tu_K = \begin{bmatrix} Tu_{K,1} \\ Tu_{K,2} \\ Tu_{K,3} \end{bmatrix} \\ \text{(elemental flow and edge pressure vectors).} \end{array}\right. \tag{14}$$

When the domain Ω is covered by a single element K, the lowest order Raviart-Thomas mixed approximation to the elliptic problem (1) is:

$$D\, Q_K + b_K |K| u_K = Q_{s,K}, \tag{15}$$

$$A_K Q_K = a_K (u_K D^T - Tu_K), \tag{16}$$

with the boundary conditions:

$$Tu_{K,i} = u_{e,i} \quad \text{if edge } i \text{ is part of } \partial\Omega_D \tag{17}$$

$$Q_{K,i} + Q_{e,i} = 0 \quad \text{if edge } i \text{ is part of } \partial\Omega_N, \tag{18}$$

where we have used the notations:

$$\left\{\begin{array}{ll} |K| = & \text{area of } K, \\ a_K > 0 & \text{approximation of } a \text{ on } K, \\ b_K \geq 0 & \text{approximation of } b \text{ on } K, \\ Q_{s,K} = \displaystyle\int_K f & \text{source flow rate injected in } K \\ Q_{e,i} = \displaystyle\int_{E_i} g & \text{source flow rate injected through edge } E_i \\ u_{ei} \in I\!R & \text{approximation of } u_e \text{ on the edge } E_i. \end{array}\right. \tag{19}$$

Even in the very simple case of Dirichlet boundary conditions ($\partial\Omega_D = \partial\Omega$) where both $Q_{s,K}$ and Tu_K are known, one can notice that the matrix for the resolution of the one-element approximation (15), (16) is:

$$\begin{bmatrix} \frac{1}{a_K}A_K & -D^T \\ -D & -b_K|K| \end{bmatrix} \tag{20}$$

which is symmetric but not positive definite. The elemental Raviart-Thomas matrix A_K satisfies some nice properties, which we recall now (we refer to [3] and [6] for the proofs).

Lemma 2.1 *For any non-degenerated triangular element K, the matrix A_K is invertible, and:*

$$A_K^{-1} = C_K + \frac{1}{3l_K}D^T D \tag{21}$$

where (see figure 2 for the notations) the elements of the 3×3 matrix C_K are:

$$\begin{cases} C_{K,i,i} = 2\dfrac{|E_i|}{hi} = 2(cotan\ \theta_{K,j} + \ cotan\ \theta_{K,k}) \\ i = 1,2,3 \quad , i,j,k \ all\ different \end{cases} \tag{22}$$

$$\begin{cases} C_{K,i,j} = -2\ cotan\ \theta_{K,k} \\ i,j = 1,2,3 \quad i,j,k\ all\ different \end{cases} \tag{23}$$

and where $l_K > 0$ is a dimensionless shape coefficient of K, defined by:

$$l_K = \frac{1}{4|K|^2}\int_K x \sum_{i=1}^{3}(x - a_i) = \frac{3}{8}\frac{\rho_K^2}{|K|} \tag{24}$$

where ρ_K is the radius of giration of K defined by:

$$\frac{1}{2}\rho_K^2|K| = \int_K |x-g|^2,\ where\ g =\ gravity\ center\ of\ K. \tag{25}$$

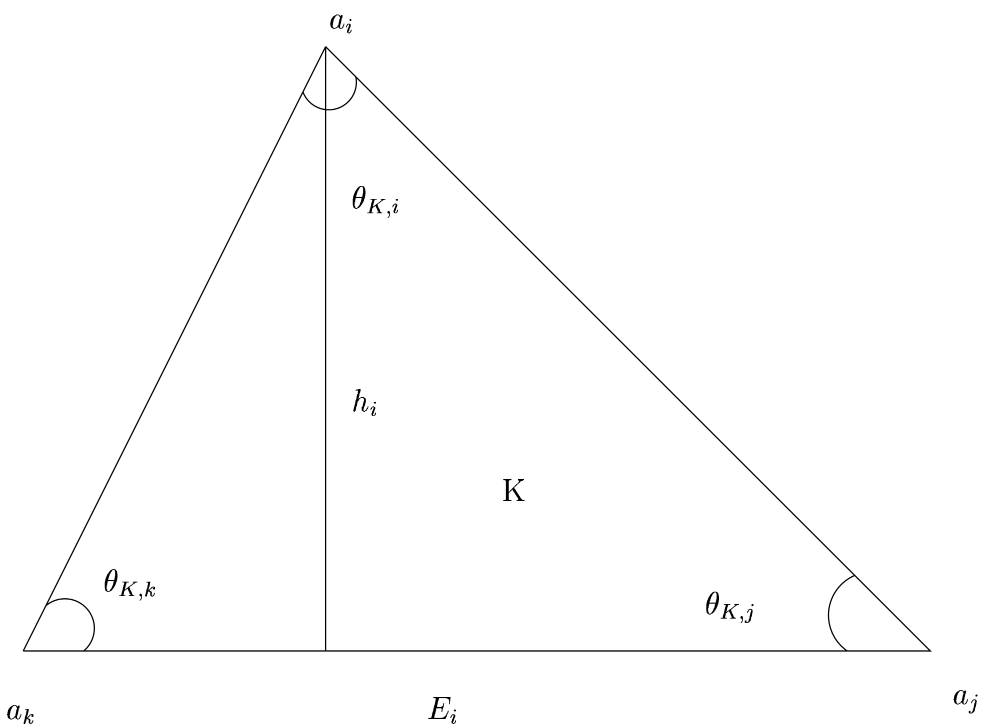

Figure 1: Notations for Lemma 2.1

Lemma 2.2 *The coefficients C_{Kij} of C_K satisfy, for a non-degenerated triangle K (we drop the index K):*

$$C_{ii} + C_{ij} + C_{ik} = 0 \quad (i, j, k \textit{ all different}), \tag{26}$$

$$C_{11} + C_{22} + C_{33} = 48l, \tag{27}$$

$$C_{12} + C_{23} + C_{31} = -24l, \tag{28}$$

$$C_{12}C_{13} + C_{21}C_{23} + C_{31}C_{32} = 4, \tag{29}$$

$$C_{ii}C_{jj} - C_{ij}^2 = 4 \quad (i, j, k \textit{ all different}), \tag{30}$$

$$C_{jk} \leq \frac{C_{ii}}{4} - \frac{4}{C_{ii}} \quad (i, j, k \textit{ all different}), \tag{31}$$

Moreover, the shape coefficient l satisfies always:

$$l \geq \sqrt{3}/12 \quad ie \quad 48l^2 \geq 1, \tag{32}$$

and for a given l, the achievable off-diagonal C_{jk}'s satisfy:

$$|C_{jk} + 8l| \leq \frac{4}{3}\sqrt{3(48l^2 - 1)} \quad j, k = 1, 2, 3, j \neq k. \tag{33}$$

The equality holds in (31) (32) and (33) when K is equilateral.

Lemma 2.3

$$DA_K = l_K D \tag{34}$$

$$A_K^{-1} D^T = l_K^{-1} D^T \tag{35}$$

We can now use lemma 2.1 and 2.3 to compute the Dirichlet-to-Neumann map of the one-element approximation (15), (16), i.e. the formula which expresses the flux vector Q_K in term of the edge-value vector Tu_K:

Proposition 2.1 *The mixed Dirichlet-to-Neumann map on the element K is given by:*

$$Q_K = -a_K \left\{ C_K + \frac{\nu_K}{3l_K} D^T D \right\} Tu_K + (1-\nu_K) Q_{s,K} D^T/3 \tag{36}$$

where $0 \leq \nu_K < 1$ is defined by:

$$\frac{\nu_K}{1-\nu_K} = \frac{b_K l_K |K|}{3a_K} = \frac{1}{8}\rho_K^2 \frac{b_K}{a_K} \tag{37}$$

where l_K is the shape coefficient of K defined in (24) and ρ_K its radius of giration defined in (25).

Proof : Premultiplying (16) by D and using (34) of lemma 2.3 gives:

$$l_K D Q_K = 3a_K u_K - a_K D T u_K$$

Plugging this expression for DQ_K into the balance equation (15) gives:

$$\left(\frac{3a_K}{l_K} + b_K |K| \right) u_K = \frac{a_K}{l_K} DTu_K + Q_{s,K}.$$

Substituting again the above expression for u_K in the constitutive equation (16) gives:

$$A_K Q_K = a_K \left\{ \frac{\frac{a_K}{l_K} DTu_K + Q_{s,K}}{\frac{3a_K}{l_K} + b_K |K|} D^T - Tu_K \right\},$$

which involves only Q_K and Tu_K. Then solving this equation for Q_K gives, using formula (35) of lemma 2.3 :

$$Q_K = a_K \left\{ \frac{\frac{1}{3} DTu_K + \frac{l_K}{3a_K} Q_{s,K}}{1 + \frac{b_K l_K |K|}{3a_K}} \frac{1}{l_K} D^T - A_K^{-1} Tu_K \right\}$$

which is (36) once A_K^{-1} is expresses by formula (21) of lemma 2.1 and $b_k l_K |K|/(3a_K)$ is expressed in term of ν_K using (37).

□

From proposition 2.1 we see that, in the stationary case where $b_K = 0$ and hence $\nu_K = 0$, formula (36) reduces to

$$Q_K = -a_K C_K T u_K + Q_{s,K} D^T/3 \tag{38}$$

which shows that the linear part $a_K C_K$ of the Dirichlet-to-Neumann map depends only on the geometry of K (up to the scalar coefficient a_K of course).

On the contrary, in the non stationary case where $b_K > 0$ and hence $0 < \nu_K < 1$, we see from (36) that the linear part of the Dirichlet-to-Neumann map depends both on the geometry of K (through C_K) and on the material properties ratio b_K/a_K (through ν_K). Its limit value is given by

$$C_K + \frac{\nu_K}{3l_K} D^T D \longrightarrow A_K^{-1} \quad \text{when} \quad \frac{b_K}{a_K} \longrightarrow \infty \quad ie \quad \nu_K \longrightarrow 1. \tag{39}$$

Corollary 2.1 *The balance equation on K writes:*

$$DQ_K + \mu_K DTu_K = (1 - \nu_K) Q_{s,K} \tag{40}$$

where we have set

$$\mu_K = a_K \frac{\nu_K}{l_K} = (1 - \nu_K) \frac{b_K |K|}{3} \tag{41}$$

Proof : Formula (40) is obtained by premultiplying (36) by D and using the fact that $DC_K = 0$ (property (26) of lemma 2.2).

□

The formulation (40) of the balance equation in terms of the edge unknowns Q_K and Tu_K only will be our starting point for the determination of the finite volume equations in section 3.

3 Definition of the new unknown H_K associated to element K

The idea is as follows: we want to define on the element K a new scalar unknown H_K such that

$$\left\{ \begin{array}{l} \text{the flux } Q_{K,i} \text{ leaving } K \text{ through the edge } E_i \\ \text{can be expressed in term of } H_K \text{ and } Tu_i \text{ only:} \\ Q_{K,i} = \xi_{K,i} H_K - \delta_{K,i} Tu_{K,i} + \gamma_{K,i} \quad i = 1,2,3 \end{array} \right. \tag{42}$$

$$\left\{ \begin{array}{l} \text{the unknown } H_K \text{ is a linear combination} \\ \text{of the edge-values } Tu_i \quad i = 1,2,3: \\ H_K = \displaystyle\sum_{j=1}^{3} \pi_{K,j} Tu_{K,j} \end{array} \right. \tag{43}$$

Condition (42) will be the key to the elimination of the Q_K and Tu_K unknowns, thus leading to a finite volume type systems of equations written in term of the sole H_K unknowns (see section 4).

Condition (43) will ensume that the Dirichelet-to-Neumann map associated to the new formulation is affine, which is a first necessary step.

All the calculations in this section are relative to one given element K. So we shall drop the K indexes in the rest of this section.
The mixed-hybrid Dirichlet-to-Neumann map (36) rewrites, with simpler notations:

$$Q_i = -a \sum_{j=1}^{3} \alpha_{ij} Tu_j + (1-\nu) Q_s/3 \quad i = 1,2,3 \tag{44}$$

where we have defined

$$\alpha_{ij} = C_{ij} + \nu/3l \quad \forall i,j = 1,2,3. \tag{45}$$

From (22) of lemma 2.1, we know that the diagonal coefficients $\alpha_{ii} \quad i = 1,2,3$ are all strictly positive. But the off-diagonal elements $\alpha_{ii} \quad i \neq j$ can be negative, zero or positive.

The existence of an H satisfying (42), (43) depends on wether some of the off-diagonal α_{ij} vanish or not:

Definition 3.1 *Let $K =$ triangle and $\nu \in [0,1[$ be given. We shall say that*

- *(K,ν) is a regular element if and only if K is non-degenerated, and all off-diagonal α's are non-zero:*

$$\alpha_{ij} \neq 0 \quad \forall i,j = 1,2,3\,, \quad i \neq j \tag{46}$$

- *(K,ν) is a singular element if and only if K is non-degenerated, and at least one of the off-diagonal α's vanishes. We shall say that (K,ν) is a singular element of type $k = 1, 2$ or 3 if exactly k off-diagonal α's vanish.*

Notice that singular elements of type 2 and 3 can occur only if $\nu > 0$. A singular element of type 2 is necessarily isosceles, and a singular element of type 3 is necessarily equilateral.

Theorem 3.1 *Let (K,ν) be a regular element. Then the set of coefficients π_i, ξ_i, δ_i and $\gamma_i \quad i = 1,2,3$ such that (42), (43) hold is non-void. It is completely described by the following formula:*

$$\pi_i = \frac{\alpha_{ij}\alpha_{ik}}{\bar{\sigma}} \quad , i,j,k \textit{ all different}, \tag{47}$$

$$\xi_i = -a\frac{\bar{\sigma}}{\alpha_{jk}} \quad , i,j,k \textit{ all different}, \tag{48}$$

$$\delta_i = a\left\{\alpha_{ii} - \frac{\alpha_{ij}\alpha_{ik}}{\alpha_{jk}}\right\} \quad , i,j,k \textit{ all different}, \tag{49}$$

$$\gamma_i = \gamma \stackrel{def}{=} (1-\nu)Q_s/3 \tag{50}$$

where

$$\bar{\sigma} \neq 0 \tag{51}$$

can be chosen arbitrarily.

Proof : The Dirichlet-to-Neumann map associated to (42) , (43) is:

$$Q_i = \xi_i \sum_{j=1}^{3} \pi_j T u_j - \delta_i T u_i + \gamma_i \quad i = 1, 2, 3. \tag{52}$$

We write now the equations to be satisfed by $\xi_i, \delta_i, \gamma_i, \pi_i \quad i = 1, 2, 3$ for (52) to coincide with the mixed Dirichelet-to-Neumann map (44):

i) equality of the constant terms:

$$\gamma_i = (1 - \nu) Q_s / 3 \quad i = 1, 2, 3$$

which is (50).

ii) equality of the coefficients of $T u_j$ in $Q_i, \quad i \neq j$ ("off-diagonal terms"):

$$\xi_i \pi_j = \alpha_{ij} \quad i, \quad j = 1, 2, 3, \quad i \neq j \tag{53}$$

This rewrites as:

$$\left\{ \begin{array}{ll} \xi_1 \pi_2 = \xi_2 \pi_1 & = -a\alpha_{12} \\ \xi_1 \pi_3 = \xi_3 \pi_1 & = -a\alpha_{13} \\ \xi_2 \pi_3 = \xi_3 \pi_2 & = -a\alpha_{23} \end{array} \right. \tag{54}$$

As (K, ν) is a regular element, none of the numbers $\xi_i, \pi_i i = 1, 2, 3$ can be zero. Hence the first and second equation imply

$$\frac{\pi_2}{\pi_3} = \frac{\xi_2}{\xi_3} = \frac{\alpha_{12}}{\alpha_{13}}.$$

We get similary from the first and third

$$\frac{\pi_1}{\pi_3} = \frac{\xi_1}{\xi_3} = \frac{\alpha_{12}}{\alpha_{23}},$$

and finally from the second and third:

$$\frac{\pi_1}{\pi_2} = \frac{\xi_1}{\xi_2} = \frac{\alpha_{13}}{\alpha_{23}}.$$

This implies that the vectors $(\pi_1\pi_2\pi_3)$ and $(\xi_1\xi_2\xi_3)$ are necessarily proportional. Hence we can search without restriction $\pi_i, \xi_i, \quad i = 1, 2, 3$ under the form

$$\pi_i = \mu\xi_i \quad i = 1, 2, 3 \quad (\mu \neq 0) \tag{55}$$

where the proportionality coefficient μ is still to be determined. Condition (55) implies that the three left equalities are satisfied in (54). Then the three right equalities rewrite:

$$\left\{ \begin{array}{ll} \xi_1\xi_2 & = -\mu^{-1} a\alpha_{12} \\ \xi_2\xi_3 & = -\mu^{-1} a\alpha_{23} \\ \xi_3\xi_1 & = -\mu^{-1} a\alpha_{13} \end{array} \right. \tag{56}$$

Multiplying two of the equations and dividing by the third gives

$$\left\{ \begin{array}{ll} \xi_1^2 & = -\mu^{-1} a\alpha_{12}\alpha_{13}/\alpha_{23} \\ \xi_2^2 & = -\mu^{-1} a\alpha_{12}\alpha_{23}/\alpha_{13} \\ \xi_3^2 & = -\mu^{-1} a\alpha_{23}\alpha_{13}/\alpha_{12} \end{array} \right. \tag{57}$$

The right hand sides of (57) have always the same sign, whatever the individual signs of α_{12}, α_{23} and α_{31} are. Hence (57) will admit solutions as soon as μ is chosen such that

$$\text{sign } (\mu) = - \text{ sign } (\alpha_{12}\alpha_{23}\alpha_{31}).$$

Without loss of generality, we can satisfy this by setting:

$$\mu = -\frac{1}{a}\alpha_{12}\alpha_{23}\alpha_{31}/\bar{\alpha}^4 \tag{58}$$

where $\bar{\alpha} > 0$ is still to be chosen. Whith this choice of μ, (57) rewrites:

$$\xi_i^2 = a^2\bar{\alpha}^4/(\alpha_{jk})^2, \quad i, j, k \text{ all different.}$$

Hence ξ_i is necessarily of the form:

$$\xi_i = \varepsilon_i a\bar{\alpha}^2/\alpha_{jk}, \quad i, j, k \text{ all different.} \tag{59}$$

where the $\varepsilon_i = \pm 1$ have to be chosen in such a way that the original equations (56) are satisfied. Substituting (59) into (56) shows that this will be the case as soon as

$$\varepsilon_i \varepsilon_j = +1 \text{ for } i, j = 1, 2, 3 \text{ with } i \neq j$$

which implies that all the ε_i's have the same value (either $+1$ or -1). Hence we choose:

$$\varepsilon_i = \varepsilon, \quad i = 1, 2, 3$$

where ε takes either the value $+1$ or the value -1. Finally the formula giving ξ_i is

$$\xi_i = -a\varepsilon\bar{\alpha}^2/\alpha_{jk} \quad i, j, k \text{ all different,} \tag{60}$$

which is (48) with $\bar{\sigma} = \varepsilon\bar{\alpha}^2$. Then (55) (58) give

$$\pi_i = \varepsilon\alpha_{ij}\alpha_{ik}/\bar{\alpha}^2, \quad i, j, k \text{ all different,} \tag{61}$$

which is (47) with again $\bar{\sigma} = \varepsilon\bar{\alpha}^2$.

iii) equality of the coefficient of Tu_i in Q_i ("diagonal terms"):

$$\xi_i\pi_i - \delta_i = -a\alpha_{ii} \quad i = 1, 2, 3$$

which gives exactly (49) when ξ_i and π_i are substituted using (60) and (61).

This ends the proof of theorem 3.1. □

Before we discuss the choice of the constant σ in theorem 3.1, we devote two lemmas to properties of π_i, ξ_i and δ_i:

Lemma 3.1 *Let (K, ν) be a regular elements and π_i, ξ_i and δ_i determined by theorem 3.1 for some constant $\bar{\sigma} \neq 0$. Then one has:*

$$\sum_{j=1}^{3} \pi_j = \frac{\sigma}{\bar{\sigma}} \tag{62}$$

$$\sum_{j=1}^{3} \xi_j = -a \frac{\sigma \bar{\sigma}}{\alpha_{12}\alpha_{23}\alpha_{31}} \tag{63}$$

$$\xi_i = \pi_i \sum_{j=1}^{3} \xi_j = \delta_i - \mu \quad i = 1, 2, 3 \tag{64}$$

where σ is defined by:

$$\sigma = \sum_{i=1}^{3} \alpha_{ij}\ \alpha_{ik} = 4 - 16\nu + \frac{\nu^2}{3l^2} \tag{65}$$

Proof : Properties (62) and (63) result directly from the definitions (47) (48) of π_j and ξ_j. The first equality in (64) is obtained by combining (47) (48) with (63) and the definition of σ (first equality in (65)). To proove the second equality in (64), notice that, given any three numbers Tu_i $i = 1, 2, 3$, the Q_i given by (52) coincide with the ones given by the mixed-hybrid Dirichelet-to-Neumann map (44), and hence satisfy the balance equation (40). Then substituting in (40) for the Q_i using (52) givens:

$$\sum_{i=1}^{3} \left\{ \pi_i \sum_{j=1}^{3} \xi_j + \mu - \delta_i \right\} Tu_i = 0, \tag{66}$$

which completes the proof of (64) as the Tu_i can take any values. The last equality in (65) results from the definition (45) of α_{ij} and the properties of C_{ij} started in lemma 2.2.

□

We investigate in the second lemma wether σ (and hence $\sum_j \pi_j$ and $\sum_j \xi_j$) and δ_i can vanish or not.

Lemma 3.2 *Let (K, ν) be a regular element with a shape coefficient l (see lemma 2.1 and 2.2) and a material property coefficient φ defined by:*

$$\varphi = \frac{b|K|}{3a} = \frac{\nu l^{-1}}{1-\nu}. \tag{67}$$

i) *Let σ be defined by (65), and $l_\sigma \geq \sqrt{3}/12$ by:*

$$3(48l_\sigma^2 - 1) = \left[\frac{8 - \sqrt{16 + \varphi^2}}{\varphi}\right]^2 . \tag{68}$$

Then:

$$\begin{cases} \sigma > 0 & \textit{for} \quad \sqrt{3}/12 \leq l < l_\sigma, \\ \sigma = 0 & \textit{for} \quad l = l_\sigma, \\ \sigma < 0 & \textit{for} \quad l_\sigma < l. \end{cases} \tag{69}$$

ii) *Suppose that $\sigma \neq 0$ (see above), and let $\delta_i \quad i = 1,2,3$ be determined by theorem 3.1 for the choice $\bar{\sigma} = \sigma \neq 0$, and $l_\delta \geq \sqrt{3}/12$ by:*

$$3(48l_\delta^2 - 1) = \left[\frac{16 - \sqrt{16 + 5\varphi^2}}{5\varphi}\right]^2 . \tag{70}$$

Then:

$$\begin{cases} \delta_i \neq 0 \; i = 1,2,3 & for \sqrt{3}/12 \leq l < l_\delta \\ \exists \tilde{K} \; with \; same \; l, \varphi & \\ s.t. \quad \tilde{\delta}_i = 0 \; for \; some \; i = 1,2,3 & for \; l_\delta \leq l \end{cases} \tag{71}$$

iii) *Both l_σ and l_δ decrease from $+\infty$ (flat degenerated triangle) for $\varphi = 0$ to $\sqrt{3}/12$ (equilateral triangle) for $\varphi = 4\sqrt{3}$, and then increase and tend to $1/6$ (for l_σ) or $1/3\sqrt{5}$ (for l_δ) when φ tends to infinity. They satisfy moreover*

$$\sqrt{3}/12 \leq l_\delta \leq l_\sigma \quad \forall \varphi > 0. \tag{72}$$

The proof of this lemma is given in Appendix 1. We turn now to the choice of the constant $\bar{\sigma} \neq 0$ in theorem 3.1. One would like to be able to interpret the new elemental unknown H as a linear interpolate/ extrapolate of the three edge values $Tu_i \quad i = 1,2,3$. This will be the case if we can choose

$$\bar{\sigma} = \sigma \text{ defined by (65)} \tag{73}$$

as this ensures by lemma 3.1 that the π_i sum up to one. But theorem 3.1 requires that $\bar{\sigma} \neq 0$, so the choice (73) will be possible only if $\sigma \neq 0$, which will be the case by lemma 3.2 as soon as φ and l satisfy

$$\sqrt{3}/12 \leq l < l_\sigma. \tag{74}$$

We summarize the situation for such a regular element in the next theorem:

Theorem 3.2 *Let (K,ν) be a regular element satisfying (74). Let $\pi_i, \xi_i, \delta_i, \gamma_i, i = 1,2,3$ be defined by theorem 3.1 with the choice (73) for $\bar{\sigma}$.*
Then (42) (43) hold, with:

$$0 < |\xi_i| < +\infty, \quad 0 \leq |\delta_i| < +\infty \quad i = 1,2,3, \tag{75}$$

$$\sum_{i=1}^{3} \pi_i = 1 \tag{76}$$

and β_i defined by

$$\beta_i = \delta_i/\xi_i \quad i = 1,2,3 \tag{77}$$

satisfies:

$$\beta_i = 1 + \mu/\xi_i \quad i = 1,2,3 \tag{78}$$

Notice that $\beta_i = 1$ in the stationary case, where $\varphi = \nu = \mu = 0$. This property is important, as it will ensure the symmetry of the final finite volume equations. It is also not true in general in the transient case.

We give now a reciprocal to theorem 3.2.

Corollary 3.1 *Let (K,ν) be a regular element satisfying (74). Let $\pi_i, \xi_i, \delta_i, \gamma_i$ $i = 1,2,3$ be defined by theorem 3.2 , and μ defined by (41).*
Let Q_i, Tu_i $i = 1,2,3$ and H be any numbers satisfying:

$$Q_i = \xi_i H - \delta_i T u_i + \gamma_i \quad i = 1,2,3, \tag{79}$$

$$\sum_{i=1}^{3}(Q_i + \mu T u_i) = (1-\nu)Q_s. \tag{80}$$

Then necessarily

$$H = \sum_{i=1}^{3} \pi_i T u_i \tag{81}$$

$$\left\{ \begin{array}{l} Q_i, Tu_i \; i = 1,2,3 \textit{ satisfy the mixed-hybrid} \\ \textit{Dirichlet-to-Neumann relation (44)} \end{array} \right. \tag{82}$$

Proof : Let Q_i, Tu_i $i = 1,2,3$ and H satisfy (79) (80). Elimination of $Q_1Q_2Q_3$ in these equations gives immediately

$$\left(\sum_{j=1}^{3} \xi_j\right) H = \sum_{i=1}^{3}(\delta_i - \mu) T u_i. \tag{83}$$

But from (74) implies by lemma 3.2 that $\sigma = \bar{\sigma} > 0$ that $\sum_j \xi_j \neq 0$.

Hence we can divide (83) by $\sum_j \xi_j$, which gives (81) using (64).

$\square$

We turn now to the case of singular elements, where some of the off-diagonal α_{ij}'s vanish.

The natural thing to do in that case is to add $\varepsilon \neq 0$ to ν to get all off-diagonal α_{ij} non-zero, apply theorem 3.2 and pass to the limit when $\varepsilon \longrightarrow 0$. But because in that case ξ and δ can simultaneously tend to infinity, the Dirichlet-to-Neumann formula (42) does not always hold in the limit, but rather its Neumann-to-Dirichlet counterpart:

$$Tu_i = \frac{H}{\beta_i} + \frac{\gamma_i - Q_i}{\delta_i} \quad i = 1,2,3. \tag{84}$$

The result of this approach is given in the next theorem for the case of singular elements of type 1, where one α_{ij} only vanishes (see definition 3.1); the proof of this theorem, and the results for singular elements of type 2 and 3, are given in Appendix 2.

Theorem 3.3 *Let (K, ν) be a singular element of type 1. Hence one of the off-diagonal α_{ij} vanishes, say $\alpha_{23} = 0$. Let $\pi_i, \xi_i, \delta_i, \gamma_i \quad i = 1, 2, 3$ be defined by:*

$$\left\{\begin{array}{llllll} \pi_1 & = 1 & \pi_2 & = 0 & \pi_3 & = 0, \\ \delta_1 & = \text{“}\infty'' & \delta_2 & = a\alpha_{22} & \delta_3 & = a\alpha_{33}, \\ \beta_1 & = 1 & \beta_2 & = -\dfrac{\alpha_{22}}{\alpha_{12}} & \beta_3 & = -\dfrac{\alpha_{33}}{\alpha_{13}}, \\ \gamma_1 & = & \gamma_2 & = & \gamma_3 & = (1-\nu)Q_s/3. \end{array}\right. \tag{85}$$

Then (84) (43) hold, with

$$0 < |\beta_i| < +\infty \quad , \quad 0 < |\delta_i| \leq +\infty \tag{86}$$

$$\sum_{i=1}^{3} \pi_i = 1 \tag{87}$$

and ξ_i defined by

$$\xi_i = \delta_i / \beta_i \tag{88}$$

satisfies

$$\beta_i = 1 + \mu / \xi_i \tag{89}$$

Once again, one sees that $\beta_i = 1$ in the stationary case. We give also a reciprocal for the case of singular elements:

Corollary 3.2 *Let (K, ν) be a singular element of type 1. Let $\pi_i, \xi_i, \delta_i, \gamma_i,\ i = 1, 2, 3$ be defined by theorem 3.3 , and μ defined by (41).*
Let $Q_i, Tu_i\ i = 1, 2, 3$ and H be any numbers satisfying

$$Tu_i = \frac{H}{\beta_i} + \frac{\gamma_i - Q_i}{\delta_i} \quad i = 1, 2, 3, \tag{90}$$

$$\sum_{i=1}^{3}(Q_i + \mu T u_i) = (1-\nu)Q_s. \tag{91}$$

Then necessarily Q_i, Tu_i $i = 1,2,3$ satisfy (81) and (82).

Proof : In a singular element of type 1, one of the δ_i is infinite, say for example $\delta_1 = \infty$, so that $\beta_1 = 1$ (see (85)). Hence (90) for $i = 1$ implies that

$$H = Tu_1 \tag{92}$$

which is (81) as $\pi_1 = 1$ and $\pi_2 = \pi_3 = 0$. Let us denote by $\tilde{Q}_i$ $i = 1,2,3$ the flux variables associated to Tu_i $i = 1,2,3$ by the mixed-hybrid Dirichlet-to-Neumann map (44). Then the $\tilde{Q}_i$ satisfy (91) (corollary 2.1) and (90) (because of (81) and theorem 3.3). But $\delta_i = a\alpha_{ii} \neq 0$ for $i = 2,3$, so that (90) can be solved univocally for Q_2, Q_3, and (91) for Q_1. Hence $Q_i = \tilde{Q}_i \quad i = 1,2,3$, and (82) is proved.

□

4 The finite volume formulation

We consider now the mixed approximation of the elliptic equation (1) over a triangular finite element mesh with lowest order Raviart-Thomas elements.

We denote by $\{\mathcal{T}_h, h > 0\}$ a family of triangulation of $\bar{\Omega}$ which is <u>regular</u>:

$$\exists l_{\max} \text{ such that } \forall h > 0, \forall K \in \mathcal{T}_h : \sqrt{3}/12 \leq l_K \leq l_{\max} \tag{93}$$

and <u>adaptated</u> to the boundary conditions:

$$\text{for any } E \in \mathcal{E}_h \text{ one has } \begin{cases} \text{either } E \text{ is an interior edge,} \\ \text{or } E \subset \partial\Omega_D, \\ \text{or } E \subset \partial\Omega_N \end{cases} \tag{94}$$

where we have set, for any $h > 0$:

$$\mathcal{E}_h = \{\text{edges } E \text{ of elements } K \text{ of } \mathcal{T}_h\}. \tag{95}$$

The mixed approximation to equation (1), and the equivalent mixed-hybrid formulation, can both be derived from the following set of equations:
Find $(u_k, Tu_K, Q_K) \in I\!R^7, K \in \mathcal{T}_h$, such that:

on all $K \in \mathcal{T}_h$:

$$\begin{cases} DQ_K + b_K|K|u_K = Q_{s,K}, \\ A_K Q_K = a_K(u_K D^T - Tu_K). \end{cases} \tag{96}$$

on all interior edges $E = K \cap L$:

$$\begin{cases} Tu_{K,E} = Tu_{L,E}, \\ Q_{K,E} + Q_{L,E} = 0. \end{cases} \tag{97}$$

on all Dirichlet edges $E \subset \partial\Omega_D$:

$$Tu_{K,E} = u_{e,E}. \tag{98}$$

on all Neumann edges $E \subset \partial\Omega_N$:

$$Q_{K,E} + Q_{e,E} = 0. \tag{99}$$

Elimination of the edge-unknowns $Tu_{K,E}$ leads to the original mixed approximation, and elimination of the element unknowns u_K and the edge fluxes $Q_{K,E}$ leads to the equivalent mixed-hybrid formulation (see [2]).
So we know that equation (96) thru (99) admit a unique solution (u_K, Tu_K, Q_K), $K \in \mathcal{T}_h$.

In order to limit the complexity of the forthcoming discussion of the finite volume formulation, we shall make a few simplifying assumptions. We give here the two first:

$$\text{all singular elements are of type 1,} \tag{100}$$

$$\sigma_K \neq 0 \quad \forall K \in \mathcal{T}_h. \tag{101}$$

As $(\mathcal{T}_h, h > 0)$ is a regular family of triangulation, it follows from (67) and (93) that $h \longrightarrow 0$ implies $\nu \longrightarrow 0$. Hence (100) and (101) will be automatically satisfied if h is small enough, as we check now:

- we see first from lemma 2.2 that:

$$C_{ij} = C_{ik} \leq -\frac{1}{6l(1+\sqrt{1-\frac{1}{48l^2}})} < 0 \quad \text{if } K \text{ isosceles,}$$
$$C_{ij} = C_{ik} = C_{kj} = -\frac{2}{\sqrt{3}} < 0 \qquad \text{if } K \text{ equilateral,}$$

  so that there will be no singular element of type 2 or 3 as soon as

$$0 \leq \nu < \frac{1}{2(1+\sqrt{1-\frac{1}{48l_{\max}^2}})}, \tag{102}$$

  in which case (100) will be satisfied.

- lemma 3.2 part i) implies that (101) will hold as soon as h is chosen small enough for $\varphi_K = b_K|K|/3a_K$ to produce by (68) on each regular K an $l_{\sigma,K} > l_{\max}$ defined in (93). On a singular K where, say, $\alpha_{jk} = 0$, one always has $\sigma_K = \alpha_{ij}\alpha_{ik} \neq 0$, and (101) is always satisfied.

We can now apply theorem 3.2 on all regular elements K of $\mathcal{T}_h$, and theorem 3.3 on the remaining singular elements. The coefficients $\pi_{K,E}, \xi_{K,E}, \beta_{K,E}, \delta_{K,E}$ and $\gamma_{K,E}$ defined in this way on all edges E of all elements K satisfy:

$$0 < |\xi_{K,E}| < +\infty,\ 0 \leq |\delta_{K,E}| < +\infty \quad \forall E \subset \partial K \text{ if } K \text{ is regular,} \tag{103}$$

$$0 < |\beta_{K,E}| < +\infty,\ 0 < |\delta_{K,E}| \leq +\infty \quad \forall E \subset \partial K \text{ if } K \text{ is singular,} \tag{104}$$

$$\sum_{E\subset\partial K} \pi_{K,E} = 1 \tag{105}$$

$$\delta_{K,E} = \xi_{K,E}\beta_{K,E} \quad \forall E \subset \partial K, \tag{106}$$

$$\beta_{K,E} = 1 + \mu_K/\xi_{K,E} \quad \forall E \subset \partial K. \tag{107}$$

We can now define, on each $K \in \mathcal{T}_h$, the elemental quantity H_K by

$$H_K = \sum_{E \subset \partial K} \pi_{K,E} T u_E, \tag{108}$$

which, as we know from theorems 3.2 and 3.3 , are linked to Tu_E and $Q_{K,E}$ by:

$$Q_{K,E} = \xi_{K,E} H_K - \delta_{K,E} T u_E + \gamma_K \quad \forall E \subset \partial K \text{ if } K \text{ is regular,} \tag{109}$$

$$T u_E = \frac{H_K}{\beta_{K,E}} + \frac{\gamma_K - Q_{K,E}}{\delta_{K,E}} \quad \forall E \subset \partial K \text{ if } K \text{ is singular.} \tag{110}$$

Now the elemental quantities H_K are defined on each element K. The next step consists in establishing the equations satisfied by these new quantities.
We shall start for this from the balance equation on each K in the form of corollary 2.1, which we recall here:

$$\sum_{E \subset \partial K} (Q_{K,E} + \mu_K T u_E) = (1 - \nu_K) Q_{s,K} \; \forall K \in \mathcal{T}_h. \tag{111}$$

Our goal will be achieved if we can express on all edges $E \in \mathcal{T}_h$, the quantity $Q_{K,E} + \mu_K T u_E$ in term of H_K on the adjacent elements and possibly the boundary conditions. The possibility of doing this will depend on wether or not E is a "regular edge" in the following sense:

Definition 4.1 *An edge $E \subset \mathcal{T}_h$ is called singular if and only if:*

$$\left\{ \begin{array}{l} \displaystyle\sum_{K \supset E} \delta_{K,E} = 0 \quad \mathit{or} \sum_{K \supset E} 1/\delta_{K,E} = 0 \\ \textit{(whichever makes sense)} \end{array} \right. \tag{112}$$

and one of the following properties holds:

$$\left\{ \begin{array}{ll} E = K \cap L & \textit{is interior,} \\ \textit{or:} & \\ E = K \cap \partial\Omega_D & \textit{and } K \textit{ is singular,} \\ \textit{or:} & \\ E = K \cap \partial\Omega_N & \textit{and } K \textit{ is regular.} \end{array} \right. \tag{113}$$

All other edges are called regular.

We recall that $|\delta_{K,E}|$ can range from zero for a regular element - see (103) - to infinity for a singular element - see (104).
We make here a third simplifying assumption to the effect of limiting again the complexity of the final equations:

$$\left\{ \begin{array}{l} \text{on all interior edges } E = K \cap L, \text{ there is at most} \\ \text{one zero } \delta_{T,E}, \quad T = K \text{ or } L. \end{array} \right. \tag{114}$$

We do not eliminate the case where $\delta_{K,E} = \delta_{L,E} = \infty$ (which can happen if K and L are singular), as this does not incur additional complexity in the discussion, and corresponds for stationary problems to the pratically interesting case of a rectangular mesh where each rectangle is split into two triangles.

Once again, we see from part ii) of lemma 3.2 that (114) will hold as soon as h is small enough for $\varphi_K = b_K|K|/a_K$ to produce by (70) on each regular K an $l_{\delta,K} > l_{\max}$ defined in (93).

The important feature of a regular edge E is that one can always retrieve both Tu_E and $Q_{K,E}$ from the neighbouring H-values and /or the boundary conditions.

Lemma 4.1 *Let $E \in \mathcal{T}_h$ be a regular edge.*

i) if $E = K \cap L$ is an interior edge:

$$\left\{ \begin{array}{ll} (\delta_{K,E} + \delta_{L,E})Tu_E & = \xi_{K,E}H_K + \gamma_K + \xi_{L,E}H_L + \gamma_L \\ \\ (\delta_{K,E} + \delta_{L,E})Q_{K,E} & = \delta_{L,E}(\xi_{K,E}H_K + \gamma_K) - \delta_{K,E}(\xi_{L,E}H_L + \gamma_L) \end{array} \right.$$

$$\text{... } \textit{if } K \textit{ and } L \textit{ are regular,} \tag{115}$$

$$\left\{ \begin{array}{ll} \left(1 + \dfrac{\delta_{K,E}}{\delta_{L,E}}\right) Tu_E & = \dfrac{1}{\delta_{L,E}}(\xi_{K,E}H_K + \gamma_K) + \dfrac{H_L}{\beta_{L,E}} + \dfrac{\gamma_L}{\delta_{L,E}} \\ \left(1 + \dfrac{\delta_{K,E}}{\delta_{L,E}}\right) Q_{K,E} & = \xi_{K,E}H_K + \gamma_K - \delta_{K,E}\left(\dfrac{H_L}{\beta_{L,E}} + \dfrac{\gamma_L}{\delta_{L,E}}\right) \end{array} \right.$$

... if K is regular and L singular, (116)

$$\left\{\begin{array}{ll}\left(\dfrac{1}{\delta_{K,E}}+\dfrac{1}{\delta_{L,E}}\right)Tu_E & =\frac{1}{\delta_{L,E}}\left(\dfrac{H_K}{\beta_{K,E}}+\dfrac{\gamma_K}{\delta_{K,E}}\right)+\frac{1}{\delta_{K,E}}\left(\dfrac{H_L}{\beta_{L,E}}+\dfrac{\gamma_L}{\delta_{L,E}}\right)\\ \left(\dfrac{1}{\delta_{K,E}}+\dfrac{1}{\delta_{L,E}}\right)Q_{K,E} & =\dfrac{H_K}{\beta_{K,E}}+\dfrac{\gamma_K}{\delta_{K,E}}-\left(\dfrac{H_L}{\beta_{L,E}}+\dfrac{\gamma_L}{\delta_{L,E}}\right)\end{array}\right.$$

... if K and L are singular, (117)

ii) $E = K \cap \partial\Omega_D$ is a Dirichlet boundary edge *:*

$$\left\{\begin{array}{l}Tu_E = u_{e,E},\\ Q_{K,E} = \xi_{K,E}H_K + \gamma_K - \delta_{K,E}u_{e,E},\end{array}\right.$$

(with the convention that ξ is replaced by δ/β if K is singular). (118)

iii) $E = K \cap \partial\Omega_N$ is a Neumann boundary edge *:*

$$\left\{\begin{array}{l}\delta_{K,E}Tu_E = \xi_{K,e}H_K + \gamma_K + Q_{e,E},\\ Q_{K,E} + Q_{e,E} = 0\end{array}\right.$$

(with the convention that ξ is replaced by δ/β if K singular) (119)

The proof follows immediately from (109) (110) and the definition 4.1 of a regular edge. Of course, it follows immediately from lemma 4.1 that $Q_{K,E} + \mu_K Tu_E$ can also be expressed in terms of the H unknows and the boundary conditions on any regular edge:

Lemma 4.2 *Let $E \in \mathcal{E}_h$ be a given regular edge.*

i) if $E = K \cap L$ is an interior edge *:*

$$Q_{K,E} + \mu_K Tu_E = \frac{\mu_K + \delta_{L,E}}{\delta_{K,E}+\delta_{L,E}}(\xi_{K,E}H_K + \gamma_K) + \frac{\mu_K - \delta_{K,E}}{\delta_{K,E}+\delta_{L,E}}(\xi_{L,E}H_L + \gamma_L), \tag{120}$$

ii) if $E = K \cap \partial\Omega_D$ is a Dirichlet boundary edge *:*

$$Q_{K,E} + \mu_K T u_E = \xi_{K,E} H_K + \gamma_K + (\mu_K - \delta_{K,E}) u_{e,E}, \tag{121}$$

iii) if $E = K \cap \partial\Omega_N$ is a Neumann boundary edge *:*

$$Q_{K,E} + \mu_K T u_E = \frac{\mu_K}{\delta_{K,E}} \{\xi_{k,E} H_K + \gamma_K + Q_{e,E}\} - Q_{e,E}, \tag{122}$$

... with the convention that, in the above formula, $\xi_{T,E}$ is replaced by $\delta_{T,E}/\beta_{T,E}$ if $T = K$ and / or L happens to be singular.

Because of the possible presence of singular edges, it will not be possible to write the balance equation (111) in terms of the H unknowns and boundary conditions on all elements K of $\mathcal{T}_h$. Once again, in order to limit the complexity of the final H equations, we make a fourth, but non-essential, assumption:

$$\forall K \in \mathcal{T}_h, K \text{ contains at most one singular edge.} \tag{123}$$

We clear first the case of elements $K \in \mathcal{T}_h$ having one singular edge F on $\partial\Omega$:

if $F = K \cap \partial\Omega_D$, then necessarily K is singular, and $\delta_{K,F} = \infty$ (definition 4.1). But (85) implies that

$$\beta_{K,F} = 1 \quad 0 < |\xi_{K,E}| < +\infty \quad \forall E \subset \partial K, E \neq F \tag{124}$$

Hence we get from (110) (98) that

$$H_K = u_{e,F}. \tag{125}$$

if $F = K \cap \partial\Omega_N$, then necessarily K is regular, and $\delta_{K,F} = 0$ (definition 4.1). Notice that this can happen only in the transient case (if the problem is stationary, one has $\nu_K = \mu_K = 0$, so necessarily $\delta_{K,F} = \xi_{K,F} \neq 0$ by theorem 3.2).

But then we get from (99) (109) that

$$\xi_{K,F} H_K + \gamma_K + Q_{e,F} = 0, \tag{126}$$

and H_K is known on K.

So we shall call boundary elements (BE) the above elements:

$$K \text{ is a } B.E. \Longleftrightarrow K \cap \partial\Omega \text{ is a singular edge,} \tag{127}$$

which have the property that:

$$H_K \text{ is known on all Boundary Elements.} \tag{128}$$

Hence we are led to define:

$$\tilde{\mathcal{T}}_h = \{K \in \mathcal{T}_h | K \text{ is not a } BE\}, \tag{129}$$

$$\tilde{\Omega} = \cup_{K \in \tilde{\mathcal{T}}_h} K \tag{130}$$

and to partition $\partial\tilde{\Omega}$ into

$$\partial\tilde{\Omega} = \partial\tilde{\Omega}_D \cup \partial\tilde{\Omega}_N \cup \partial\tilde{\Omega}_{BE} \tag{131}$$

So we are left with searching for $(H_K, K \in \tilde{\mathcal{T}}_h)$. Because of the possible presence of singular edges in $\tilde{\mathcal{T}}_h$, which now are necessarily interior to $\tilde{\Omega}$, we are led to introduce finite volumes V, which will aggregate the elements on each side of a singular edge:

Definition 4.2 *A finite volume V is made of*

- *either one element $K \in \tilde{\mathcal{T}}_h$ with no singular edge*
- *or two elements K and L of $\tilde{\mathcal{T}}_h$ sharing a common singular edge F.*

We call $\mathcal{V}_h$ the set of all finite volumes V in $\tilde{\Omega}$.

We associate now to each finite volume $V \in \mathcal{V}_h$ a finite volume unknown H_V and a finite volume equation:

i) if V is made of a single element $K \in \tilde{\mathcal{T}}_h$, we define simply H_V by :

$$H_V = H_K. \tag{132}$$

Then by definition 4.2, all edges in ∂K are regular, and we decide that:

$$(FV-1)\left\{\begin{array}{l}\text{the finite volume equation on } V \text{ is obtained}\\ \text{by substituting in the balance equation (111)}\\ Q_{K,E}+\mu_K T u_E \text{ by its expressions given in lemma 4.2.}\end{array}\right. \tag{133}$$

ii) if V is made of two-elements $K, L \in \tilde{\mathcal{T}}_h$ sharing a common singular edge $F = K \cap L$, we have, by definition 4.1:

$$\delta_{K,F}+\delta_{L,F}=0 \quad \text{or} \quad \frac{1}{\delta_{K,F}}+\frac{1}{\delta_{L,F}}=0 \tag{134}$$

whichever makes sense according to (103) (104). When K and L are both regular, (134) cannot be satisfied with $\delta_{K,F}=\delta_{L,F}=0$ because of hypothesis (114).
When K or L or both are singular, then at least one of the $\delta_{K,F}$ or $\delta_{L,F}$ is non-zero because of (104).
Hence in all cases one has

$$\delta_{K,F}=-\delta_{L,F}\neq 0. \tag{135}$$

When K and L are regular, we get from (134) and (109) that:

$$\xi_{K,F}H_K+\gamma_K+\xi_{LF}H_L+\gamma_L=0$$

which rewrites, after division by $\delta_{K,F}\neq 0$:

$$\frac{H_K}{\beta_{K,F}}+\frac{\gamma_K}{\delta_{K,F}}=\frac{H_L}{\beta_{L,F}}+\frac{\gamma_L}{\delta_{L,F}}, \tag{136}$$

where $\beta_{T,F}=\delta_{T_iF}/\xi_{T,F}\quad T=K,L$ satisfies, because of (135) (103):

$$0<|\beta_{T,F}|<+\infty. \tag{137}$$

When K or L or both become singular, one checks easily using (104) that (136) (137) remain valid.

This leads to define the finite volume unknown H_V on $V = K \cup L$ by:

$$H_V = \frac{H_K}{\beta_{K,F}} + \frac{\gamma_K}{\delta_{K,F}} = \frac{H_L}{\beta_{L,F}} + \frac{\gamma_L}{\delta_{L,F}} \tag{138}$$

Because of (137), the above formula defines a one-to-one relation between H_V and H_K, as weel as between H_V and H_L.
We turn now to the choice of the finite volume equation on $V = K \cup L$. As F is singular, it is now impossible to express $Q_{T,F} + \mu_T T u_F$ for $T = K, L$ using lemma 4.2. These expression will always involve one edge variable ! From (109) (110) and (135) one finds easily that

$$\left\{ \; Q_{T,F} + \mu_T T u_F = \mu_T \left(\frac{H_T}{\beta_{T,F}} + \frac{\gamma_T}{\delta_{T,F}} \right) + \left(1 - \frac{\mu_T}{\delta_{T,F}} \right) Q_{T,F} \right. \tag{139}$$

(this formula is valid for $0, 1$ or 2 singular elements among K, L).
The only way to eliminate the edge value $Q_{T,F}$ will be to combine properly the balance equations (111) on K and L: we multiply first (111) written on K, by $1 - \dfrac{\mu_L}{\delta_{L,F}}$, then multiply (111), written on L by $1 - \dfrac{\mu_K}{\delta_{K,F}}$, and add the resulting equations. This gives:

$$\begin{aligned} &\left(1 - \frac{\mu_L}{\delta_{L,F}}\right) \sum_{\substack{E \subset \partial K \\ E \neq F}} (Q_{K,E} + \mu_K T u_E) + \left(1 - \frac{\mu_K}{\delta_{K,F}}\right) \sum_{\substack{E \subset \partial L \\ E \neq F}} (Q_{L,E} + \mu_L T u_E) \\ &+ (\mu_K + \mu_L) H_V = \left(1 - \frac{\mu_L}{\delta_{L,F}}\right) (1 - \nu_{s,K}) Q_{s,K} + \left(1 - \frac{\mu_K}{\delta_{K,F}}\right) (1 - \nu_{s,L}) Q_{s,L} \end{aligned} \tag{140}$$

The edges $E \subset \partial K \cup \partial L$ with $E \neq F$ are regular. Hence:

$$(FV-2) \left\{ \begin{array}{l} \text{the finite volume equation on } V = K \cup L \text{ is obtained by substituting in equation (140),} \\ Q_{K,E} + \mu_K T u_E \text{ and } Q_{L,E} + \mu_L T u_E \text{ by their expressions given in lemma 4.2} \end{array} \right. \tag{141}$$

We can now state the main theorem of this section.

Theorem 4.1 *Under hypothesis (100) (101) (114) and (123), the finite volume equation*

$$BH = F \quad where \quad H = (H_V, V \in \mathcal{V}_h) \tag{142}$$

defined by $(FV-1)(FV-2)$ *is equivalent to the lowest order triangular Raviart-Thomas approximation (96) - (99). The matrix* B *is regular, but not symmetric in general.*

Proof : We have already proven that, if $(u_K, Tu_K, Q_K, K \in \mathcal{T}_h)$ is a solution of (96) (99), then $(H_V, V \in \mathcal{V}_h)$ defined by (108) (132) (138) satisfies (142). Conversely, let $(H_V, V \in \mathcal{V}_h)$ be a solution of (142). We construct first edge values Tu_E and $Q_{K,E}$ for all edges $E \in \mathcal{E}_h$ elements $K \in \mathcal{T}_h$ such that $\partial K \supset E$, in such a way that

$$\begin{cases} \text{formula (109) (if } K \text{ is regular) or (110)} \\ \text{(if } K \text{ is singular) holds on } E \text{ for all } K \text{ such that } \partial K \supset E \end{cases} \tag{143}$$

We first define, on all boundary elements, H_K by (125) (if K is singular) or (126) (if K is regular). Then we define Tu_E and $Q_{K,E}$ as follows:

- **on a regular edge** , we define Tu_E and $Q_{T,E}$ for $\partial T \supset E$ using lemma 4.1. It is immediate to check that (143) holds in that case.

- **on a singular interior edge** $F = K \cap L$, we already know $Tu_E, Q_{KE} = -Q_{LE}$ on all edges $E \subset \partial(K \cup L) = \partial V$, where V is the two elements finite volume associated to the singular edge F.

We search for $Q_{K,F} = -Q_{L,F}$ and Tu_F which satisfy two conditions:

i) we want first that (143) holds both for $T = K$ and L. But we have seen in (135) that $\delta_{K,F} = -\delta_{L,F} \neq 0$, so that (109) can always be rewritten in the for of (110). Hence (143) is equivalent, using the definition (135) of H_V, to:

$$\frac{Q_{T,F}}{\delta_{T,F}} + Tu_F = H_V, \quad \forall T = K, L. \tag{144}$$

ii) we want also that formula (52) be satisfied on both K and L:

$$Q_{T,F} + \mu_T T u_F + \sum_{\substack{E \subset \partial T \\ E \neq F}} (Q_{T,E} + \mu_T T u_E) = (1 - \nu_T) Q_{s,T} \quad \forall T = K, L. \tag{145}$$

The two equations (144) for $T = K$ and L are obviously dependant, as $Q_{K,F} = -Q_{L,F}$ and $\delta_{K,F} = -\delta_{L,F}$. Then, multiplying (145) on K by $1 - \dfrac{\mu_L}{\delta_{L,F}}$ and on L by $1 - \dfrac{\mu_K}{\delta_{K,F}}$ and adding produces equation (144) multiplied by $\mu_K + \mu_L$ when the finite volume equation (140) is used. Hence the four equations (144) (145) will be satisfied as soon as (144) and (145) are satisfied on one element, say K.

But theorem 3.2 (if K is regular) or 3.3 (if K is singular) show that $\dfrac{\mu_K}{\delta_K} - 1 = -\dfrac{1}{\beta_K} \neq 0$. Hence equations (144) (145) on K have a unique solution $Q_{K,F} = -Q_{L,F}$ and $T u_F$.

- **On a singular edge** $F = K \cap \partial\Omega_D$, necessarily K is singular, with $\delta_{K,F} = \infty$ and $\beta_{K,F} = 1$, and H_K is defined by (125). So we define:

$$T u_F = u_{e,F}, \tag{146}$$

and $Q_{K,F}$ by requiring that formula (52) is satisfied on K:

$$Q_{K,F} + \mu_K T u_F + \sum_{\substack{E \subset \partial K \\ E \neq F}} (Q_{K,E} + \mu_K T u_E) = (1 - \nu_K) Q_{s,K}. \tag{147}$$

Then (146) , $\delta_{K,F} = \infty$ and $\beta_{K,F} = 1$ imply that (125) reduces to (110), so that (143) is satisfied on F.

- **On a singular edge** $F = K \cap \partial\Omega_N$, necessarily K is regular with $\delta_{K,F} = 0, \mu_K > 0$ (see before (126), and H_K is defined by (126).

Hence we can define $Q_{K,F}$ by

$$Q_{K,F} = -Q_{e,F}, \tag{148}$$

and Tu_F by requiring that (52) holds on K , ie by equation (147). Then (148) and $\delta_{K,F} = 0$ imply that (126) reduces to (109), so that (143) is satisfied on F.

So we have constructed edge values Tu_E and $Q_{K,E}$ on all edges E which satisfy (143). These edge values satisfy also equation (52) on all elements K of $\mathcal{T}_h$:

- because of the finite volume equation $(FV-1)$ on all K with no singular edge,
- because of (145) on all K with a singular edge interior to Ω ,
- because of (147) on all K with a boundary singular edge.

So we can apply corollary 3.1 on regular elements K and 3.2 on singular elements, which proves that

$$\left\{\begin{array}{l} \forall K \in \mathcal{T}_h, Q_{K,E} \text{ and } Tu_E, E \subset \partial K \text{ satisfy} \\ \text{the mixed-hybrid Dirichlet-to-Neumann relation (44).} \end{array}\right. \tag{149}$$

But by construction

$$\left\{\begin{array}{l} Q_{K,E} \text{ and } Tu_E, E \subset \partial\Omega \text{ satisfy the Dirichlet} \\ \text{and Neumann condition on } \partial\Omega_D \text{ and } \partial\Omega_N \end{array}\right. \tag{150}$$

Hence $Q_{K,E}$ and $Tu_E, E \in \mathcal{E}_h$ coincide with the unique solution of the mixed-hybrid equations (96) - (99). The elemental value u_K can be defined afterwards on each K in the usual way by the formula:

$$u_K = \frac{l_K}{3a_K}(1-\nu_K)Q_{s,K} + \frac{1}{3}(1-\nu_K)\sum_{E\subset\partial K} Tu_{K,E} \quad \forall K \in \mathcal{T}_h. \tag{151}$$

This proves the equivalence of the mixed-hybrid approximation and the new formulation.
It remains to check that the matrix B in (142) is regular. So let $H_K^j, K \in \mathcal{T}_h,\quad j = 1,2$ be two solutions of (142), and set

$$\Delta H_K = H_K^1 - H_K^2 \quad \forall K \in \tilde{\mathcal{T}}_h. \tag{152}$$

As the edge values $Q_{K,E}, Tu_E, E \in \mathcal{E}_h$ associated to H_K^1 and H_K^2 are the same (the mixed-hybrid equations (96) - (99) have a unique solution !), we see that:

on each regular edge E , lemma 4.1 implies:

i) if $E = K \cap L$ **is interior, then**

$$\left\{ \begin{array}{l} \xi_{K,E}\Delta H_K + \xi_{L,E}\Delta H_L = 0 \\ \\ \delta_{LE}\xi_{K,E}\Delta H_K - \delta_{K,E}\xi_{L,E}\Delta H_L = 0 \end{array} \right. \Longrightarrow \left\{ \begin{array}{l} \Delta H_K = \Delta H_L = 0 \\ \\ \text{as } \xi_{K,E}\xi_{L,E}(\delta_{K,E} + \delta_{L,E}) \neq 0 \end{array} \right.$$

$$\text{... if } K \text{ and } L \text{ are regular,} \tag{153}$$

$$\left\{ \begin{array}{l} \dfrac{\xi_{K,E}}{\delta_{L,E}}\Delta H_K + \dfrac{\Delta H_L}{\beta_{L,E}} = 0 \\ \\ \xi_{K,E}\Delta H_K - \delta_{K,E}\dfrac{\Delta H_L}{\beta_{L,E}} = 0 \end{array} \right. \Longrightarrow \left\{ \begin{array}{l} \Delta H_K = \Delta H_L = 0 \\ \\ \text{as } \dfrac{\xi_{K,E}}{\beta_{L,E}}\left(1 + \dfrac{\delta_{K,E}}{\delta_{L,E}}\right) \neq 0 \end{array} \right.$$

$$\text{... if } K \text{ is regular and } L \text{ is singular,} \tag{154}$$

$$\left\{ \begin{array}{l} \dfrac{1}{\delta_{L,E}}\dfrac{\Delta H_K}{\beta_{K,E}} + \dfrac{1}{\delta_{K,E}}\dfrac{\Delta H_L}{\beta_{L,E}} = 0 \\ \\ \dfrac{\Delta H_K}{\beta_{K,E}} - \dfrac{\Delta H_L}{\beta_{L,E}} - \dfrac{\Delta H_L}{\beta_{L,E}} = 0 \end{array} \right. \Longrightarrow \left\{ \begin{array}{l} \Delta H_K = \Delta H_L = 0 \\ \\ \text{as } \dfrac{1}{\beta_{K,E}\beta_{L,E}}\left(\dfrac{1}{\delta_{K,E}} + \dfrac{1}{\delta_{L,E}}\right) \neq 0 \end{array} \right.$$

$$\text{... if } K \text{ and } L \text{ are singular.} \tag{155}$$

ii) if $E = K \cap \partial\Omega$ is a boundary edge, then necessarily $0 < |\delta_{K,E}| < +\infty$ and

$$\xi_{K,E}\Delta H_K = 0 \Longrightarrow \Delta H_K = 0 \text{ as } 0 < |\xi_{K,E}| < +\infty$$

$$\text{... if } K \text{ is regular,} \tag{156}$$

$$\delta_{K,E}\frac{\Delta H_K}{\beta_{K,E}} = 0 \Longrightarrow \left\{ \begin{array}{l} \Delta H_K = 0 \text{ as } 0 < |\delta_{K,E}| < +\infty \\ \text{and } 0 < |\beta_{K,E}| < +\infty \end{array} \right.$$

$$\text{... if } K \text{ is singular,} \tag{157}$$

on each singular edge $F = K \cap L$ **interior to** Ω , we get from (144) that $\Delta H_V = 0$, and hence from (138) that

$$\Delta H_K = \Delta H_L = 0. \tag{158}$$

This ends the proof of theorem 4.1.

□

5 The stationary case

We consider in this section the case where $b = 0$ in equation (1), which corresponds to the case of an elliptic, time independant diffusion equation. Hence

$$b_K = \varphi_K = \nu_K = \mu_k = 0 \quad \forall K \in \mathcal{T}_h. \tag{159}$$

Hence the coefficicents $\alpha_{K,E,F}$ of the mixed-hybrid Dirichlet-to-Neumann relation (44) are given, using lemma 2.1 and the notations of figure 1 with E_1, E_2, E_3 replaced by: E, F, G:

$$\alpha_{K,F,G} = C_{K,F,G} = -2 \text{ cotan } \theta_{K,E}, \quad E, F, G \text{ all different,} \tag{160}$$

$$\alpha_{K,E,E} = C_{K,E,E} = 2(\text{cotan } \theta_{K,F} + \text{cotan } \theta_{K,G}), \quad E, F, G \text{ all different.} \tag{161}$$

Singular elements of type 1 correspond to triangles K having one right angle. There are no singular elements of type 2 and 3, as a triangle cannot have more than one right angle ! Hence (100) is always satisfied.
Also lemma 3.1 with $\nu_K = 0$ implies

$$\sigma_K = \sum_{E,F,G \text{ all different}} C_{K,E,F} C_{K,E,G} = 4 \tag{162}$$

for any regular K. But (29) in lemma 2.2 shows that (162) holds also if K is singular. Hence hypothesis (101) is always satisfied.

Then ($\pi_{K,E}, \xi_{K,E}, \delta_{K,E}, \beta_{K,E}$ and $\gamma_K, K \in \mathcal{T}_h, E \subset \partial K$) defined by theorems 3.2 and 3.3 satisfy:

$$\pi_{K,E} = \frac{C_{K,E,F}C_{K,E,G}}{4} = \text{ cotan } \theta_{K,G} + \text{ cotan } \theta_{K,F}, \tag{163}$$

$$\sum_{E \subset \partial K} \pi_{K,E} = 1, \tag{164}$$

$$\xi_{K,E} = \delta_{K,E} = -\frac{4a_K}{C_{K,F,G}} = \frac{2a_K}{\text{cotan } \theta_{K,E}}, \tag{165}$$

$$0 < |\xi_{K,E}| = |\delta_{K,E}| \leq +\infty, \tag{166}$$

$$\beta_{K,E} = \delta_{K,E}/\xi_{K,E} = 1, \tag{167}$$

$$\gamma_K = Q_{s,K}/3. \tag{168}$$

So we get from (165) or (166) that $\delta_{K,E} \neq 0$ for any $K \in \mathcal{T}_h$ and $E \subset \partial K$. Hence (114) is always satisfied, as there is no edge on which $\delta_{K,E} = 0$. Also, by definition 4.1 we see that on any regular interior edge $E = K \cap L$, one has $\dfrac{1}{\delta_{K,E}} + \dfrac{1}{\delta_{L,E}} \neq 0$. Hence we are led to define:

$$\left\{ \begin{array}{l} \forall E \in \mathcal{E}_h \text{ with } E \text{ interior and regular,} \\ \dfrac{1}{\delta_{K,L}} = \dfrac{1}{\delta_{K,E}} + \dfrac{1}{\delta_{L,E}} = \dfrac{1}{2}\left(\dfrac{\text{cotan } \theta_{K,E}}{a_K} + \dfrac{\text{cotan } \theta_{L,E}}{a_L}\right). \end{array} \right. \tag{169}$$

which satisfies

$$0 < |\delta_{K,L}| < +\infty \quad E \text{ interior and regular.} \tag{170}$$

In the stationary case considered here, singular edges are more directly linked to the geometry:

$$F \subset \partial\Omega \text{ is singular } \Longleftrightarrow \theta_{K,E} = \pi/2 \tag{171}$$

$$F = K \cap L \text{ with } a_K = a_L \Longleftrightarrow K \text{ and } L \text{ have the same circumcentre.} \tag{172}$$

So even in the constant coefficient case $a_K = a \; \forall K$, it is possible for an element K to have more than one singular edge ! It is still possible, in this case to write a finite volume formulation equivalent to the mixed-hybrid formulation. But then a finite volume V will possibly be made of more than two elements K, and the definition of H_V will be slightly more complex.
So in order to keep things simple, we shall continue to make the non-essential hypothesis (123) that each K has at most one singular edge..

Before we define the finite volumes V, we have first to eliminate from the list of unknowns the Boundary Elements (defined in (127)) where H_K is directly given by the boundary conditions. From (171) we see that a Boundary Element K is necessarily singular, as it has a right angle opposite to the singular boundary edge F, which implies that F is necessarily in $\partial\Omega_D$. Hence we define H_K on all Boundary Elements by (125), i.e.:

$$H_K = u_{e,F} \quad \forall K = B.E.. \tag{173}$$

The finite volumes $V \in \mathcal{V}_h$ are then made of one or two non-boundary elements according to definition (97).

The definitions (132) (138) of the finite volume unknows reduce now to:

$$H_V = H_K \qquad\qquad \text{if } V = K, \tag{174}$$

$$H_V = H_K + \frac{\gamma_K}{\delta_{K,F}} = H_L + \frac{\gamma_L}{\delta_{L,F}} \qquad \text{if } V = K \cup L. \tag{175}$$

Then the finite volume equations (FV-1) (FV-2) or (142) simplify to:

$$\begin{aligned}
&\sum_{\substack{E \subset \partial V \\ E \text{ interior}}} \delta_{K,L} \left\{ \left(H_K + \frac{\gamma_K}{\delta_{K,E}} \right) - \left(H_L + \frac{\gamma_L}{\delta_{L,E}} \right) \right\} \\
&+ \sum_{E \subset \partial V \cap \partial\Omega_D} \delta_{K,E} \left\{ \left(H_K + \frac{\gamma_K}{\delta_{K,E}} \right) - u_{e,E} \right\} \\
&- \sum_{E \subset \partial V \cap \partial\Omega_N} Q_{e,E} = \sum_{T \subset V} Q_{s,T} \quad \forall\, V \in \mathcal{V}_h,
\end{aligned} \tag{176}$$

where it is understood that K is the element of V which contains E, and , when E is interior, L is the element exterior to V which contains E.

Theorem 5.1 *In the stationary case, the finite volume equation (142) is defined, under hypothesis (123), by (174)-(176), and the matrix B is symmetric and regular.*

A necessary condition for B to be positive definite is:

$$\sum_{E \subset \partial V, E \not\subset \partial\Omega_D} \delta_{V,W} > 0 \quad \forall V \in \mathcal{V}_h, \tag{177}$$

and a sufficient condition is:

$$\delta_{V,W} > 0 \quad \forall V \in \mathcal{V}_h, \forall E \subset \partial V, E \not\subset \partial\Omega_N, \tag{178}$$

where $\delta_{V,W}$ is defined by:

$$\delta_{V,W} = \delta_{K,L} \quad \textit{if } E = K \cap L \subset \partial V \textit{ is interior}, \tag{179}$$

$$\delta_{V,W} = \delta_{K,E} \quad \textit{if } E = K \cap \partial\Omega_D \subset \partial V \textit{ is a Dirichlet edge}. \tag{180}$$

Proof : We know from theorem 4.1 that B is regular, and see from (174)-(176) that it is symmetric. In order to investigate its definite positiveness, we evaluate:

$$
\begin{aligned}
< BH, H > &= \sum_{V \in \mathcal{V}_h} (BH)_V H_V \\
&= \sum_{V \in \mathcal{V}_h} \left\{ \sum_{\substack{E \subset \partial V \\ E \text{ interior}}} \delta_{V,W}(H_V - H_W) + \sum_{E \subset \partial V \cap \partial \Omega_D} \delta_{V,E} H_V \right\} H_V,
\end{aligned}
$$

which rewrites after rearranging the summations:

$$
< BH, H >= \sum_{E \text{ interior and regular}} \delta_{V,W}(H_V - H_W)^2 + \sum_{E \subset \partial\Omega_D \text{ and regular}} \delta_{V,E} H_V^2. \tag{181}
$$

The necessary condition (177) is obtained by requiring that $< BH, H > > 0$ for $H_V = 1$ and $H_W = 0$ for $V \neq W$; then (178) implies obviously that $< BH, H > > 0$ for $H \neq 0$, and hence that B is positive definite.

□

When the coefficient a is constant, condition (178) amount to require that the triangulation satisfies the Delaunay condition.

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Appendix A: Proof of lemma 2.4

Proof of part i) We want to study the sign of σ defined in (65). But from (37) and (64) we see that

$$\nu = l\varphi/(1 + l\varphi). \tag{182}$$

Substitution into σ gives:

$$\sigma = \frac{-36l^2\varphi^2 - 24l\varphi + (\varphi^2 + 12)}{3(1 + l\varphi)^2}. \tag{183}$$

The denominator is always strictly positive, and the numerator is, for given φ, a second order polynomial in l with a single positive roor l_γ given by

$$l_\gamma = \frac{-2 + \sqrt{16 + \varphi^2}}{6\varphi}. \tag{184}$$

Hence

$$l_\gamma - \sqrt{3}/12 = \frac{-4 + 2\sqrt{16 + \varphi^2} - \sqrt{3}\varphi}{12\varphi}$$

$$l_\gamma - \sqrt{3}/12 = \frac{(\varphi - 4\sqrt{3})^2}{12\varphi(2\sqrt{16 + \varphi^2} + 4 + \varphi\sqrt{3})} \tag{185}$$

Changing $\sqrt{3}$ into $-\sqrt{3}$ gives:

$$l_\gamma + \sqrt{3}/12 = \frac{(\varphi + 4\sqrt{3})^2}{12\varphi(2\sqrt{16 + \varphi^2} + 4 - \varphi\sqrt{3})} \tag{186}$$

Multiplying (185) and (186) gives:

$$l_\gamma^2 - 1/48 = \left[\frac{\varphi^2 - 48}{12\varphi(8 + \sqrt{16 + \varphi^2})}\right]^2 \tag{187}$$

which gives (70) after multiplying by the conjugate quantity.

Proof of part ii) We want to study under which conditions δ_i defined by (49) can vanish.
From (64) we see that

$$\delta_i = 0 \Longleftrightarrow \xi_i + \mu = 0 \tag{188}$$

ie, using definition (48) of ξ_i and (68) of μ:

$$\frac{\bar{\sigma}}{\alpha_{jk}} = \frac{\nu}{l}.$$

But $\bar{\sigma} = \sigma = 4 - 16\nu + \dfrac{\nu^2}{3l^2}$ and $\alpha_{jk} = C_{jk} + \nu/3l$ so the equation rewrites:

$$4 - 16\nu + \frac{\nu^2}{3l^2} = \frac{\nu}{l}\left(C_{jk} + \frac{\nu}{3l}\right),$$

i.e.:

$$4 - 16\nu = \frac{\nu}{l} C_{jk}.$$

Subsituting ν by $l\varphi/(1 + l\varphi)$ gives

$$4 - 12l\varphi = \varphi C_{jk} \tag{189}$$

But we know exactly by lemma 2.2 the range of C_{jk} which can be achieved in triangles with a given shape coefficient l:

$$|C_{jk} + 8l| \leq \frac{4}{3}\sqrt{3(48l^2 - 1)}. \tag{190}$$

Hence we see that (189) will have a solution for some K if and only if the value of C_{jk} given by (189) falls into the range of C_{jk} given by (190). Hence δ_i will possibly vanish if and only if:

$$\begin{aligned} \left|\frac{4 - 12l\varphi}{\varphi} + 8l\right| &\leq \frac{4}{3}\sqrt{3\,(48l^2 - 1)}, \\ (1 - l\varphi)^2 &\leq \frac{\varphi^2}{3}(48l^2 - 1), \end{aligned}$$

i.e.:

$$15l^2\varphi^2 + 2l\varphi - \left(1 + \frac{\varphi^2}{3}\right) \geq 0. \tag{191}$$

This second degree equation admits a unique positive root l_δ given by:

$$l_\delta = \frac{-1 + \sqrt{16 + 5\varphi^2}}{15\varphi}. \tag{192}$$

Hence the δ_i will possibly vanish if and only if $l \geq \delta$, which is (71). Then (70) is obtqined by transformation of (192).

Proof of part iii) is straightforward by comparing the right-hand sides of (68) and (70).

Appendix B: The case of a singular element

Let (K, ν) be a singular element. We define then $\pi_i, \beta_i, \delta_i, \gamma_i$ $i = 1, 2, 3$ as follows:

$$\gamma_i = \gamma \stackrel{\text{def}}{=} (1 - \nu)Q_s/3 \quad i = 1, 2, 3 \tag{193}$$

and:

i) for an element of type 1, say $\alpha_{23} = 0$

$$\left\{ \begin{array}{llllll} \pi_1 & = 1 & \pi_2 & = 0 & \pi_3 & = 0, \\ \delta_1 & = \text{“}\infty\text{''} & \delta_2 & = a\alpha_{22} & \delta_3 & = a\alpha_{33}, \\ \beta_1 & = 1 & \beta_2 & = -\dfrac{\alpha_{22}}{\alpha_{12}} & \beta_3 & = -\dfrac{\alpha_{33}}{\alpha_{13}}, \end{array} \right. \tag{194}$$

ii) for an element of type 2, say $\alpha_{23} = \alpha_{13} = 0$ (hence necessarily $\nu > 0$ and K isosceles):

$$\left\{\begin{array}{lllll} \pi_1 = & \pi_2 & = \dfrac{1}{2} & \pi_3 & = 0 \\ \delta_1 = & \delta_2 & = a(\tilde{\alpha} - \alpha_{12}) & \delta_3 & = a\alpha_{33}, \\ \beta_1 = & \beta_2 & = \dfrac{\alpha_{12} - \tilde{\alpha}}{\alpha_{12}} & \beta_3 & = \text{“}\infty\text{''}, \end{array}\right. \tag{195}$$

where we have set $\tilde{\alpha} = \alpha_{11} = \alpha_{22}$,

iii) for an element of type 3, where $\alpha_{12} = \alpha_{23} = \alpha_{31} = 0$ (here necessarily $\nu > 0$ and K is equilateral):

$$\left\{\begin{array}{llll} \pi_1 & = \pi_2 & = \pi_3 & = 1/3, \\ \delta_1 & = \delta_2 & = \delta_3 & = a\tilde{\alpha}, \\ \beta_1 & = \beta_2 & = \beta_3 & = \text{“}\infty\text{''}, \end{array}\right. \tag{196}$$

where we have set $\tilde{\alpha} = \alpha_{11} = \alpha_{22} = \alpha_{33}$.
We check now that $\pi_i, \beta_i, \delta_i, \gamma_i \quad i = 1, 2, 3$ defined in (194) for a singular element of type 1 satisfy the properties stated in theorem 3.3.

We derive first equations (194) We know that

$$\alpha_{12} \neq 0, \quad \alpha_{13} = 0, \quad \text{but } \alpha_{23} = 0.$$

for $\varepsilon \neq 0$ small enough, we replace ν by $\nu^\varepsilon = \nu + \varepsilon$. hence α_{12}, α_{13} and α_{23} are replaced by:

$$\alpha_{12}^\varepsilon = \alpha_{12} + \varepsilon \neq 0, \quad \alpha_{13}^\varepsilon = \alpha_{13} + \varepsilon \neq 0, \quad \alpha_{23}^\varepsilon = \varepsilon \neq 0.$$

Moreover, (65) shows that

$$\sigma = \alpha_{12}\alpha_{13} \neq 0$$

and σ can be replaced by

$$\sigma_\varepsilon = 4 - 16\nu^\varepsilon + \frac{\nu_\varepsilon^2}{3l^2} \to \sigma = \alpha_{12}\alpha_{13} \text{ when } \varepsilon \to 0.$$

So we can apply theorem 3.2 to define $\pi_i^\varepsilon, \delta_i^\varepsilon, \beta_i^\varepsilon$ by:

$$\begin{aligned}
\pi_1^\varepsilon &= \alpha_{12}^\varepsilon \alpha_{13}^\varepsilon / \sigma^\varepsilon, \\
\pi_2^\varepsilon &= \alpha_{12}^\varepsilon \varepsilon / \sigma^\varepsilon, \\
\pi_3^\varepsilon &= \alpha_{13}^\varepsilon \varepsilon / \sigma^\varepsilon, \\
\delta_1^\varepsilon &= a \left(\alpha_{11}^\varepsilon - \frac{\alpha_{12}^\varepsilon \alpha_{13}^\varepsilon}{\varepsilon} \right), \\
\delta_2^\varepsilon &= a \left(\alpha_{22}^\varepsilon - \frac{\alpha_{12}^\varepsilon \varepsilon}{\alpha_{13}^\varepsilon} \right), \\
\delta_3^\varepsilon &= a \left(\alpha_{33}^\varepsilon - \frac{\alpha_{13}^\varepsilon \varepsilon}{\alpha_{12}^\varepsilon} \right), \\
\beta_1^\varepsilon &= (\alpha_{12}^\varepsilon \alpha_{13}^\varepsilon - \alpha_{11}^\varepsilon \varepsilon) / \sigma^\varepsilon, \\
\beta_2^\varepsilon &= (\alpha_{12}^\varepsilon \varepsilon - \alpha_{22}^\varepsilon \alpha_{13}^\varepsilon) / \sigma^\varepsilon, \\
\beta_3^\varepsilon &= (\alpha_{13}^\varepsilon \varepsilon - \alpha_{33}^\varepsilon \alpha_{12}^\varepsilon) / \sigma^\varepsilon.
\end{aligned}$$

Passing to the limit in the above equations leads to define $\pi_i, \delta_i, \beta_i \quad i = 1, 2, 3$ by formulas (194).

We prove now that (79) (43) hold. Let $Tu_i, Q_i \quad i = 1, 2, 3$ satisfy the mixed Dirichlet-to-Neumann map (44). Then, for $\varepsilon \neq 0$, we define:

$$Q_i^\varepsilon = -a \sum_{i=1}^{3} \alpha_{ij}^\varepsilon T u_j + (1 - \nu^\varepsilon) Q_s / 3 \quad i = 1, 2, 3, \tag{197}$$

so that

$$Tu_i, Q_i^\varepsilon \quad i = 1, 2, 3 \text{ satisfy the perturbed Dirichlet-to-Neumann map} \tag{198}$$

and of course

$$Q_i^\varepsilon \longrightarrow Q_i \text{ when } \varepsilon \longrightarrow 0, \quad i = 1, 2, 3. \tag{199}$$

But from theorem 3.2 we know, as $\alpha_{ij}^\varepsilon \neq 0, i \neq j$ for ε small enough, that Tu_i, Q_i^ε satisfy (42) and (43). But δ_i^ε and $\beta_i^\varepsilon \neq 0$ for ε small enough, so (42) (43) rewrite:

$$\begin{cases} Tu_i = \dfrac{H^\varepsilon}{\beta_i^\varepsilon} + \dfrac{\gamma^\varepsilon - Q_i^\varepsilon}{\delta_i^\varepsilon}, \quad i = 1, 2, 3, \\ H^\varepsilon = \displaystyle\sum_{i=1}^{3} \pi_i^\varepsilon Tu_i. \end{cases} \tag{200}$$

When $\varepsilon \to 0$, we can pass to the limit in (189), as we know that $\pi_i^\varepsilon \to \pi_i, \delta_i^\varepsilon \to \delta_i, \beta_i^\varepsilon \to \beta_i, \gamma_i^\varepsilon \to \gamma_i$, and $Q_i^\varepsilon \to Q_i$ (the fact that $\delta_i^\varepsilon \to \delta_i =$ "∞" means simply that, in the limit, the last term disappears from the first equation of (189) for $i = 1$). This proves that Tu_i, Q_i $i = 1, 2, 3$ satisfy (84) and (43).

The remaining equalities (87) (66) are obtained by passing to the limit in (76) (78) written for $\pi_i^\varepsilon, \beta_i^\varepsilon$ and ξ_i^ε.

References

[1] J. Baranger, J.F. Maitre, and F. Oudin. Application de la théorie des éléments finis mixtes à l'étude d'une classe de schémas aux volumes-différences finis pour les problèmes elliptiques. *C.R. Acad. Sci. Paris*, I(319):401, 1994.

[2] G. Chavent and J. Roberts. A unified physical presentation of mixed, mixed-hybrid finite elements and standard finite difference approximations for the determination of velocities in waterflow problems. *Adv. Water Resour.*, 14(6):329, 1991.

[3] Cordes and Kinzelbach. Comment on "Application of the mixed-hybrid finite approximation in a groundwater flow model: luxury or necessity" by Mosé et al. *Water Resourc.*, 32(6):1905–1909, 1996.

[4] C. Cordes and M.Putti. Triangular mixed finite elements versus finite differences and finite volumes in groundwater flow modeling. In *Computational Methods in Subsurface Flow and Transport Problems*, page 61, Cancun, Mexico, 1996. Comput. Mech., Southampton.

[5] L.J. Durlofsky. Accuracy of mixed and control volume finite element approximations to Darcy velocity and related quantities. *Water Resour. Res.*, 30(4):965, 1994.

[6] Maitre. proof of formula for a_k^{-1}.

[7] R. Mose, P. Siegel, P. Ackerer, and G. Chavent. Application of the mixed hybrid finite element approximation in a groundwater flow model: luxury or necessity ? *Water Resour. Res.*, 30(11):3001, 1994.

[8] P.A. Raviart and J.M. Thomas. A mixed hybrid finite element method for the second order elliptic problems. In *Lectures Notes in Mathematics: Mathematical Aspects of the Finite Element Methods*. Springer Verlag, New-York, 1977.

[9] A. Younès, R. Mose, P. Ackerer, and G. Chavent. A New Formulation of the Mixed Finite Element Method for Solving Elliptic and Parabolic PDE with Triangular Elements. *Journal of Computational Physics*, 149:148–167, 1999.

[10] A. Younès, R. Mose, P. Ackerer, and G. Chavent. Une résolution par les éléments finis mixtes à une inconnue par maille. *C.R. Acad. Sci. Paris*, 328(I):623–626, 1999.

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