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Abstract: In the Black-Scholes option pricing paradigm it is assumed that the market-maker designs a continuous-time hedge. This is not realistic from a practical point of view. We introduce trading restrictions in the Black-Scholes model in the sense that hedging is only allowed a given number of times-only the number is fixed, the market-maker is free to choose the (stopping) times and hedge ratios. We identify the strategy which minimizes the variance of the tracking error for a given initial value of the portfolio. The minimal variance is shown to be the solution to a sequence of optimal stopping problems. Existence and uniqueness is proved. We design a lattice algorithm with complexity N^3 (N being the number of lattice points) to solve the corresponding discrete problem in the Cox-Ross-Rubinstein setting. The convergence of the scheme relies on a viscosity solution argument. Numerical results and dynamic simulations are provided.

Key-words: Discrete hedging, Black-Scholes model, Option pricing, Dynamic programming, Optimal stopping, Stochastic integral, Viscosity solutions.

(Résumé : tsvp)

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Couverture Optimale dans le modèle de Black-Scholes pour un Nombre de Transactions donné.

Résumé : Dans l'approche traditionnelle du modèle de Black-Scholes, le market-maker doit pour obtenir une couverture parfaite se couvrir de manière continue. Ceci est évidemment impossible en pratique. On introduit des restrictions de couverture dans le modèle de Black-Scholes: on suppose que le market-maker ne peut se couvrir qu'un nombre fixé de fois, à des instants aléatoires de son choix. On identifie la stratégie qui minimize la variance de l'erreur de couverture. On montre que la variance minimale est solution d'une suite de problèmes d'arrêts optimaux. On donne un algorithme de complexité N^3 (où N est le nombre de points du maillage) pour résoudre le problème correspondant discrêt dans le modèle de Cox-Ross-Rubinstein. On montre la convergence par des arguments de solution de viscosité. On compare les résultats numériques obtenus avec la stratégie de couverture à intervalles de temps réguliers.

Mots-clé : Couverture discrète, Modèle de Black-Scholes, Evaluation d'options, Programmation dynamique, Arrêt optimal, Intégrale stochastique, Solutions de viscosité.

1 Introduction

1.1 Effective trading strategies

For obvious reasons, continuous trading is a mathematical abstraction from a practical viewpoint. In the presence of transaction costs for example the investor would like to hedge as little as possible. In fact even if there is no transaction costs nor liquidity restrictions, an investor can and will obviously in practice follow a discrete trading strategy at stopping times.

From standard theory on stochastic integrals, we know that the perfect continuous-time hedge can be arbitrarily well approximated by such effective strategies, at least in a weak (L^2 or probability) sense-the most natural approximation being the Riemann approximation of the stochastic integral which corresponds to deterministic hedging times. Let us cite Henrotte ([7]), Zhang ([13]), among others, who have studied the variance of the replication error in this context.

However it does not seem natural to hedge at deterministic times regardless of the changes in the stock prices or hedging ratios. Well-known strategies from an empirical point of view are hedging when the underlying stock moves from a prescribed amount (latent Brownian motion in the terminology of Prigent-Renault-Scaillet [11]), or when the delta moves, and so on..to our opinion these strategies, or more generally stopping times hedging strategies have not been thoroughly investigated yet and many interesting questions remain open.

In this paper, we consider the problem of selecting the best hedging times and ratios given a fixed number of trading times. As a criteria we take the variance of the replication error. Although the criteria of variance is questionable, it displays nice features which lead to easier mathematical treatment. Moreover we work under the martingale measure. This should be seen as a first stage in the study of variance optimal strategies. Note that the optimal price of the option for this criteria is obviously the Black-Scholes price, so that we assume that the hedger of the option trades at this price.

In other terms: given a fixed number N of rebalancing, what is the smallest variance one can reach? The problem will be to determine the optimal hedging dates and the optimal portfolios at these dates. Let us now set the problem in a stochastic control framework. We would also like to mention an alternative approach to the same question, using the theory of vector-valued stopping times, by Trabelsi and Trad [12].

1.2 Minimal variance hedging given N rebalancing

We consider the hedge ratios and hedging times as a control parameter. Let $(\tau_1^n, \tau_2^n, \dots, \tau_n^n)$ denote the rebalancing (stopping) times and $\left(\delta_{\tau_1^n}, \delta_{\tau_2^n}, \dots, \delta_{\tau_n^n}\right)$ the corresponding (adapted) hedging ratios chosen by the investor. In order to apply Dynamic Programming techniques, we shall consider an investor who initiates his strategy at time t, the value of the underlying being $S_t = x$, with the selling of the option and an initial hedge of an amount α of stocks. Let V_T denote the value of the portfolio at time T, $\varphi(x)$ the payoff of the option, $c(t, x)$ the Black-Scholes price, $F = (\mathcal{F}_t)_{0 \le t \le T}$ the standard augmentation of the natural filtration of the Brownian motion, $\mathcal{T}_{t,T}$ the set of all stopping times of the filtration F which satisfy $t \leq \tau \leq T$ with probability one, E_t the conditional expectation with respect to F. Let also denote $\tilde{A}_t$ the quantity A_t discounted at time 0 that is $A_t e^{rt}$. The associated variance of the tracking error is then:

$$\begin{aligned} I(t,x,\alpha,\mathcal{P}_n) &\equiv E_t\left[\{\tilde{V}_T - \tilde{\varphi}(S_T^{t,x})\}^2\right] \\ &= E_t\left[\{\tilde{c}(t,S_t^{t,x}) + \alpha(\tilde{S}_{\tau_1^n}^{t,x} - \tilde{S}_t^{t,x}) + \sum_{i=1}^{n} \delta_{\tau_i^n}(\tilde{S}_{\tau_{i+1}^n}^{t,x} - \tilde{S}_{\tau_i^n}^{t,x}) - \tilde{\varphi}(S_T^{t,x})\}^2\right] \end{aligned}$$

where by convention $\tau_{n+1}^n = T$ and $\mathcal{P}_n = \left(\tau_1^n, \tau_2^n, ..., \tau_n^n, \delta_{\tau_1^n}, \delta_{\tau_2^n}, ..., \delta_{\tau_n^n}\right)$

In the expression above the process $(S_s^{t,x}, s \geq t)$ is the solution of the equation

$$\begin{cases} dS_s = rS_s ds + \sigma S_s dW_s \text{ for } s \geq t \\ S_t = x \text{ (and } S_u = x \text{ for } u \leq t) \end{cases}$$

The problem at hand is to characterize the optimal cost function v_n given the initial hedge:

$$v_n(t,x,\alpha) \equiv \inf_{\mathcal{P}_n} I(t,x,\alpha,\mathcal{P}_n) \tag{1}$$

and to find, if some, $\mathcal{P}_n^*$ minimizing $I(t,x,\alpha,\mathcal{P}_n)$ i.e

$$v_n(t,x,\alpha) = I(t,x,\alpha,\mathcal{P}_n^*)$$

and the optimal cost function

$$v_n^*(t,x) = \inf_{\alpha} v_n(t,x,\alpha)$$

At first glance this is a highly non-standard stochastic control problem since it involves a vector of stopping times. Standard techniques encompass the case with a single stopping time, and also (impulse control), the case where the number of stopping times is infinite.

The natural idea is to use Dynamic Programming to reduce our problem to a sequence of standard optimal stopping problems. Indeed since we work under the risk-neutral probability and with the square function, $I(t,x,\alpha,\mathcal{P}_n)$ splits naturally in the first local replication error until the next hedging time and the error starting afresh at that date:

Set

$$L = \tilde{c}\left(t, S_t^{t,x}\right) + \alpha\left(\tilde{S}_{\tau_1^{n+1}}^{t,x} - \tilde{S}_t^{t,x}\right) + \sum_{i=1}^{n+1} \delta_{\tau_i^{n+1}}\left(\tilde{S}_{\tau_{i+1}^{n+1}}^{t,x} - \tilde{S}_{\tau_i^{n+1}}^{t,x}\right) - \tilde{\varphi}\left(S_T^{t,x}\right)$$

Then

$$E_t\left[L^2\right] = E_t\left[\{\tilde{c}\left(t, S_t^{t,x}\right) + \alpha(\tilde{S}^{t,x}_{\tau_1^{n+1}} - \tilde{S}_t^{t,x}) - \tilde{c}\left(\tau_1^{n+1}, S^{t,x}_{\tau_1^{n+1}}\right)\}^2 + \{\tilde{c}\left(\tau_1^{n+1}, S^{t,x}_{\tau_1^{n+1}}\right) + \delta_{\tau_1^{n+1}}(\tilde{S}^{t,x}_{\tau_2^{n+1}} - \tilde{S}^{t,x}_{\tau_1^{n+1}}) + \sum_{i=2}^{n+1} \delta_{\tau_i^{n+1}}(\tilde{S}^{t,x}_{\tau_{i+1}^{n+1}} - \tilde{S}^{t,x}_{\tau_i^{n+1}}) - \tilde{\varphi}(S_T^{t,x})\}^2\right]$$

So it is natural to conjecture the following Dynamic Programming relation:

$$v_{n+1}(t,x,\alpha) = \inf_{\tau \in \mathcal{T}_{t,T}} E_t\left[\{\tilde{c}\left(t, S_t^{t,x}\right) + \alpha(\tilde{S}_\tau^{t,x} - \tilde{S}_t^{t,x}) - \tilde{c}\left(\tau, S_\tau^{t,x}\right)\}^2 + v_n^*(\tau, S_\tau^{t,x})\right] \quad \text{(DP)}$$

or yet

$$v_{n+1}(t,x,\alpha) = -\{\tilde{c}\left(t, S_t^{t,x}\right) - \alpha\tilde{S}_t^{t,x}\}^2 + \inf_{\tau \in \mathcal{T}_{t,T}} E\left[\{\tilde{c}\left(\tau, S_\tau^{t,x}\right) - \alpha\tilde{S}_\tau^{t,x}\}^2 + v_n^*(\tau, S_\tau^{t,x})\right]$$

Observe now that v_n^* being given, we face a standard optimal stopping problem. The solution to such a stopping time problem is well known (cf [9]).

We will show in section 4 that the value function v_n is given by the solution of the following sequence of variational inequalities on $[0,T] \times \mathbb{R}^{+*} \times K$:

$$\begin{cases} \frac{\partial v_n}{\partial t}(t,x,\alpha) + Av_n(t,x,\alpha) + \sigma^2(xe^{-rt})^2(\Delta(t,x) - \alpha)^2 \geq 0 \\ v_n(t,x,\alpha) \leq \inf_\delta v_{n-1}(t,x,\delta) \\ (v_n(t,x,\alpha) - \inf_\delta v_{n-1}(t,x,\delta))(\frac{\partial v_n}{\partial t}(t,x,\alpha) + Av_n(t,x,\alpha) + \sigma^2(xe^{-rt})^2(\Delta(t,x) - \alpha)^2) = 0 \\ v_n(T,x,\alpha) = 0 \end{cases}$$

where A is the differential operator associated to S_t given by $Av\left(t,x\right) = rx\frac{\partial v}{\partial x}\left(t,x\right) + \frac{\sigma^2}{2}x^2\frac{\partial^2 v}{\partial x^2}\left(t,x\right)$, $\Delta(t,x) = \frac{\partial c}{\partial x}(t,x)$ (i.e. the Black-Scholes delta) and K is a compact set where Δ takes its values (this is not a restriction as we shall see below). Notice than once $v_1, v_2, ..., v_n$ have been found, $\tau_1^n, \tau_2^n, ..., \tau_n^n$ can be constructed in the same way as done for optimal stopping problems.

1.3 Some properties of the solution

1.3.1 Optimal hedge ratios

If we know the dates of trading, it is easy to find the optimal deltas:

$$\begin{aligned} E\left[\left\{\tilde{V}(T) - \tilde{\varphi}(S_T)\right\}^2\right] &= E\left[\left\{\sum_{i=0}^{n} \int_{\tau_i^*}^{\tau_{i+1}^*} \left(\Delta\left(t, S_t\right) - \delta_{\tau_i^*}\right) d\tilde{S}_t\right\}^2\right] \\ &= \sum_{i=0}^{n} E\left[\left\{\int_{\tau_i^*}^{\tau_{i+1}^*} \left(\Delta\left(t, S_t\right) - \delta_{\tau_i^*}\right) d\tilde{S}_t\right\}^2\right] \end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=0}^{n} E\left[\int_{\tau_i^*}^{\tau_{i+1}^*} (\Delta(t,S_t) - \delta_{\tau_i^*})^2 \sigma^2 \tilde{S}_t^2\, dt\right] \\
&= \sum_{i=0}^{n} E\left[E_{\tau_i^*}(\int_{\tau_i^*}^{\tau_{i+1}^*} \Delta^2(t,S_t)\,\sigma^2 \tilde{S}_t^2 dt)\right. \\
&\qquad -2\delta_{\tau_i^*} E_{\tau_i^*}(\int_{\tau_i^*}^{\tau_{i+1}^*} \Delta(t,S_t)\,\sigma^2 \tilde{S}_t^2\, dt) \\
&\qquad \left. +\delta_{\tau_i^*}^2 E_{\tau_i^*}(\int_{\tau_i^*}^{\tau_{i+1}^*} \sigma^2 \tilde{S}_t^2\, dt)\right]
\end{aligned}
$$

This entails

$$
\delta_{\tau_i^*} = \frac{E_{\tau_i^*}\left[\int_{\tau_i^*}^{\tau_{i+1}^*} \Delta(t,S_t)\,\sigma^2 \tilde{S}_t^2\, dt\right]}{E_{\tau_i^*}\left[\int_{\tau_i^*}^{\tau_{i+1}^*} \sigma^2 \tilde{S}_t^2\, dt\right]}
$$

which shows by the mean value theorem that $\delta_{\tau_i^*} = \Delta(u_i, S_{u_i})$ for some random u_i between τ_i^* and τ_{i+1}^*.

Now if we assume that φ is a $k-$Lipschitz function, then $|\Delta(u_i, S_{u_i})| \leq k$, so that we can restrict ourselves to $[-k,k]$, or better yet to the range of the function Δ. In the case of a Call, the interval $[0,1]$ is chosen, and $[-1,0]$ for a Put.

1.3.2 Limit as $N \to \infty$

Obviously, the hedging error goes to zero as the number of hedges goes to infinity since it is true for the deterministic case $h = \frac{T-t}{N}$, $t_i = (i-1)h$. Moreover in this case we know the convergence rate. Therefore

$$
v_N(0,x,\alpha) \leq R_N = E\left[\left\{\tilde{c}(0,S_0) + \Delta(0,S_0)(\tilde{S}_{t_2} - \tilde{S}_0) + \sum_{i=2}^{N} \Delta(t_i,S_{t_i})(\tilde{S}_{t_{i+1}} - \tilde{S}_{t_i}) - \tilde{\varphi}(S_T)\right\}^2\right]
$$

with

$$
\lim_{N\to\infty} N R_N = \frac{e^{-2rT}}{2} E\left[\int_0^T e^{-2rt} S_t^4 \sigma^4 (\frac{\partial^2 c}{\partial x^2}(t,S_t))^2\, dt\right]
$$

where $\frac{\partial^2 c}{\partial x^2}$ is the gamma of the option. This result was shown by Zhang ([13]).

2 Time-dependent optimal stopping

In this section, we recall known results on the optimal stopping problem when the value function is time-dependent. The main objective is to establish the required properties of the value function in order to well-define the function v_n by the recursive Dynamic Programming equation (DP). We study the following optimal stopping problem:

$$u(t,x,\alpha) = \inf_{\tau \in \mathcal{T}_{t,T}} E_t\left[\Phi(\tau, S_\tau^{t,x}, \alpha)\right] \tag{2}$$

where $\Phi : [0,T] \times \mathbb{R}^+ \times K \to \mathbb{R}$ where K is a compact set of (say) $\mathbb{R}$.

We introduce the following hypotheses:

$(H1)$ Φ is continuous, non negative on $[0,T] \times \mathbb{R}^+ \times K$
$(H2)$ $\Phi(t,x,\alpha) \leq (1+L)\left(x^2+\alpha^2\right)$ on $[0,T] \times \mathbb{R}^+ \times K$

2.1 Existence and uniqueness

The first result is a theorem of existence and uniqueness for the optimal stopping problem.

Theorem 1 *([9]) Suppose $(H1)$ and that for every (t,x,α) the family of random variables*

$$\left\{\Phi(\tau, S_\tau^{t,x}, \alpha),\ \tau \in \mathcal{T}_{t,T}\right\}$$

is uniformly integrable. Set

$$D = \{(t,x) \in [0,T] \times \mathbb{R}^+ \,/\, u(t,x,\alpha) < \Phi(t,x,\alpha)\}$$

and $\tau_D = inf\{t \leq u \leq T \,/\, (u, S_u^{t,x}) \notin D\}$. Then (existence)

$$u(t,x,\alpha) = E_t\left[\Phi(\tau_D, S_{\tau_D}^{t,x}, \alpha)\right] \tag{3}$$

and τ_D is an optimal stopping time. Moreover (uniqueness) if a stopping time τ satisfies (3) then $\tau_D \leq \tau$ almost surely.

Under $(H2)$, the property of uniform integrability is satisfied.

2.2 Properties of the value function

In order to connect the function u and the optimal hedging problem of section (1.2) we shall need the following property:

Proposition 1 *Under $(H1)$, let:*

$$V(t) = essinf_{\tau \in \mathcal{T}_{t,T}} E\left[\phi(\tau, S_\tau, \alpha) \mid \mathcal{F}_t\right]$$

then $V(t) = u(t, S_t, \alpha)$ a.s.

Proof We adapt the proof given in [8] in the case where Φ depends only on x. We define the filtration $(\mathcal{F}_{t,s})_{t\leq s\leq T}$ where $\mathcal{F}_{t,s}$ is generated by the increments $W_u - W_t$, $t \leq u \leq s$ and $\bar{\mathcal{T}}_{t,T}$ the set of all stopping times of this filtration. Recall that the two σ-algebras $\mathcal{F}_t$ and $\mathcal{F}_{t,s}$ are independent and $\mathcal{F}_s = \mathcal{F}_t \vee \mathcal{F}_{t,s}$ for $s \in [t,T]$. We note $\hat{\mathcal{T}}_{t,T}$ the set of all stopping time τ of $\mathcal{T}_{t,T}$ such that $\tau = \sum_n I_{A_n}\tau_n$ where (A_n) are $\mathcal{F}_t$-measurable and form a partition of Ω and τ_n a sequence of stopping time of $\bar{\mathcal{T}}_{t,T}$. Note that $\bar{\mathcal{T}}_{t,T} \subset \hat{\mathcal{T}}_{t,T} \subset \mathcal{T}_{t,T}$ and that if $\tau \in \mathcal{T}_{t,T}$ then there exists a sequence (τ_n) of $\hat{\mathcal{T}}_{t,T}$ such that $\tau_n \to \tau$ a.s. Then we can deduce the following equalities:

$$V(t) = essinf_{\tau\in\hat{\mathcal{T}}_{t,T}} E\left[\phi(\tau, S_\tau, \alpha) \mid \mathcal{F}_t\right]$$

and

$$u(t,x,\alpha) = \inf_{\tau\in\hat{\mathcal{T}}_{t,T}} E\left[\phi(\tau, S_\tau^{t,x}, \alpha)\right]$$

If $\tau \in \hat{\mathcal{T}}_{t,T}$ with $\tau = \sum_n I_{A_n}\tau_n$ where $(A_n)$are $\mathcal{F}_t$-measurable and form a partition of Ω and τ_n a sequence of stopping time of $\bar{\mathcal{T}}_{t,T}$, we have:

$$\begin{aligned}
E\left[\phi(\tau, S_\tau, \alpha) \mid \mathcal{F}_t\right] &= E\left[\sum_n I_{A_n}\phi(\tau_n, S_{\tau_n}, \alpha) \mid \mathcal{F}_t\right] && (4)\\
&= \sum_n I_{A_n} E\left[\phi(\tau_n, S_{\tau_n}, \alpha) \mid \mathcal{F}_t\right] && (5)\\
&\geq \inf_n E\left[\phi(\tau_n, S_{\tau_n}, \alpha) \mid \mathcal{F}_t\right] && (6)
\end{aligned}$$

We already know that:

$$V(t) \leq essinf_{\tau\in\bar{\mathcal{T}}_{t,T}} E\left[\phi(\tau, S_\tau, \alpha) \mid \mathcal{F}_t\right]$$

then we obtain:

$$V(t) = essinf_{\tau\in\bar{\mathcal{T}}_{t,T}} E\left[\phi(\tau, S_\tau, \alpha) \mid \mathcal{F}_t\right]$$

On an another hand

$$\begin{aligned}
E\left[\phi(\tau, S_\tau^{t,x}, \alpha)\right] &= E\left[\sum_n I_{A_n}\phi(\tau_n, S_{\tau_n}^{t,x}, \alpha)\right]\\
&= \sum_n E\left[I_{A_n}\phi(\tau_n, S_{\tau_n}^{t,x}, \alpha)\right]\\
&= \sum_n P(A_n) E\left[\phi(\tau_n, S_{\tau_n}^{t,x}, \alpha)\right]\\
&\geq \inf_n E\left[\phi(\tau_n, S_{\tau_n}^{t,x}, \alpha)\right]
\end{aligned}$$

where we used the independence of τ_n and $\mathcal{F}_t$. It follows that

$$u(t,x,\alpha) = \inf_{\tau \in \bar{\mathcal{T}}_{t,T}} E\left[\phi(\tau, S_\tau^{t,x}, \alpha)\right]$$

The optimal stopping time depends on t and x, and is independent of $\mathcal{F}_t$.

For all stopping time $\tau \in \bar{\mathcal{T}}_{t,T}$, we can write :

$$E\left[\phi(\tau, S_\tau, \alpha) \mid \mathcal{F}_t\right] = E\left[\phi(\tau, S_\tau^{t,S_t}, \alpha) \mid \mathcal{F}_t\right]$$

τ is independent of $\mathcal{F}_t$ so we have:

$$E\left[\phi(\tau, S_\tau^{t,S_t}, \alpha) \mid \mathcal{F}_t\right] = J_{(\tau)}^{t,S_t}$$

where $J_{(\tau)}^{t,x} = E\left[\phi(\tau, S_\tau^{t,x}, \alpha)\right]$ and then $V(t) = essinf_{\tau \in \bar{\mathcal{T}}_{t,T}} J_{(\tau)}^{t,S_t}$.

We note that for t fixed, the family of functions $x \to J_{(\tau)}^{t,x}$ (indexed by τ) is equicontinuous because:

$$|J_{(\tau)}^{t,x} - J_{(\tau)}^{t,y}| \le E\left[\sup_{t \le s \le T} |\phi(s, S_s^{t,x}, \alpha) - \phi(s, S_s^{t,y}, \alpha)|\right]$$

We can use the following lemma and the essential inf can be written as an inf on a countable family.

Lemma 1 *([8]) Let $(\psi_j)_{j\in J}$ be an equicontinuous set of functions from $\mathbb{R}^+$ to $\mathbb{R}$ such that: $\sup_{j\in J}\psi_j(x) < \infty$ for all x. Then, Ψ defined by: $\Psi(x) = \sup_{j\in J}\psi_j(x)$ is continuous. Moreover, there exists a countable subset J_0 of J such that: $\Psi(x) = \sup_{j\in J_0}\psi(x)$, for all x.*

Under the above hypotheses we also have:

Proposition 2 *Under $(H1)$ and $(H2)$, u is continuous on $[0,T] \times \mathbb{R}^{+*} \times K$.*

Proof Let (t_1, x_1, β) and $(t_2, x_2, \alpha) \in [0,T] \times \mathbb{R}^{+*} \times K$, with $t_1 < t_2$ then:

$$\begin{aligned} u(t_2,x_2,\alpha) - u(t_1,x_1,\beta) &= \inf_{\tau \in \mathcal{T}_{t_2,T}} E\left[\phi(\tau, S_\tau^{t_2,x_2}, \alpha)\right] - \inf_{\tau \in \mathcal{T}_{t_2,T}} E\left[\phi(\tau, S_\tau^{t_1,x_1}, \beta)\right] \\ &+ \inf_{\tau \in \mathcal{T}_{t_2,T}} E\left[\phi(\tau, S_\tau^{t_1,x_1}, \beta)\right] - \inf_{\tau \in \mathcal{T}_{t_1,T}} E\left[\phi(\tau, S_\tau^{t_1,x_1}, \beta)\right] \end{aligned}$$

First we note that

$$\begin{aligned} &\left|\inf_{\tau \in \mathcal{T}_{t_2,T}} E\left[\phi(\tau, S_\tau^{t_2,x_2}, \alpha)\right] - \inf_{\tau \in \mathcal{T}_{t_2,T}} E\left[\phi(\tau, S_\tau^{t_1,x_1}, \beta)\right]\right| \\ \le \quad & E\left[\sup_{t_2 \le s \le T}\left|\phi(s, S_s^{t_2,x_2}, \alpha) - \phi(s, S_s^{t_1,x_1}, \beta)\right|\right] \end{aligned}$$

Let $Y_t = \phi(t, S_t^{t_1,x_1}, \beta)$. Then

$$\inf_{\tau \in \mathcal{T}_{t_1,T}} E\left[Y_\tau\right] = \inf_{\tau \in \mathcal{T}_{t_1,T}} E\left[Y_\tau 1\left(\tau \leq t_2\right) + Y_\tau 1\left(\tau > t_2\right)\right]$$

and $Y_{t_2} = Y_{t_2} 1\left(\tau \leq t_2\right) + Y_{t_2} 1\left(\tau > t_2\right)$,
we obtain

$$\begin{aligned}
\inf_{\tau \in \mathcal{T}_{t_1,T}} E\left[Y_\tau + Y_{t_2}\right] &= \inf_{\tau \in \mathcal{T}_{t_1,T}} E\left[Y_{\tau \wedge t_2} + Y_{\tau \vee t_2}\right] \\
&\geq \inf_{\tau \in \mathcal{T}_{t_1,T}} E\left[Y_{\tau \wedge t_2}\right] + \inf_{\tau \in \mathcal{T}_{t_1,T}} E\left[Y_{\tau \vee t_2}\right] \\
&\geq \inf_{\tau \in \mathcal{T}_{t_1,t_2}} E\left[Y_\tau\right] + \inf_{\tau \in \mathcal{T}_{t_2,T}} E\left[Y_\tau\right]
\end{aligned}$$

therefore we get (for the second part)

$$\begin{aligned}
0 &\leq \inf_{\tau \in \mathcal{T}_{t_2,T}} E\left[Y_\tau\right] - \inf_{\tau \in \mathcal{T}_{t_1,T}} E\left[Y_\tau\right] \\
&= \inf_{\tau \in \mathcal{T}_{t_2,T}} E\left[Y_\tau\right] - \inf_{\tau \in \mathcal{T}_{t_1,T}} E\left[Y_\tau\right] \\
&\quad + E\left[Y_{t_2}\right] - E\left[Y_{t_2}\right] \\
&\leq \inf_{\tau \in \mathcal{T}_{t_2,T}} E\left[Y_\tau\right] + E\left[Y_{t_2}\right] \\
&\quad - \inf_{\tau \in \mathcal{T}_{t_1,t_2}} E\left[Y_\tau\right] - \inf_{\tau \in \mathcal{T}_{t_2,T}} E(Y_\tau) \\
&= E\left[Y_{t_2}\right] - \inf_{\tau \in \mathcal{T}_{t_1,t_2}} E\left[Y_\tau\right] \\
&\leq E\left[\sup_{t_1 \leq s \leq t_2} \left|\phi(s, S_s^{t_1,x_1}, \beta) - \phi(t_2, S_{t_2}^{t_1,x_1}, \beta)\right|\right]
\end{aligned}$$

So we conclude

$$\begin{aligned}
\left|u(t_2, x_2, \alpha) - u(t_1, x_1, \beta)\right| &\leq E\left[\sup_{t_2 \leq s \leq T} \left|\phi(s, S_s^{t_2,x_2}, \alpha) - \phi(s, S_s^{t_1,x_1}, \beta)\right|\right] \\
&+ E\left[\sup_{t_1 \leq s \leq t_2} \left|\phi(s, S_s^{t_1,x_1}, \beta) - \phi(t_2, S_{t_2}^{t_1,x_1}, \beta)\right|\right]
\end{aligned}$$

By dominated convergence using $(H2)$, the uniform continuity on compact sets of ϕ, and the fact that the applications

$$(s, t_i, x_i) \mapsto S_s^{t_i,x_i}(\omega)$$

are continuous for ω fixed, we get the result as soon as

$$E\left[\sup_{0 \leq t \leq s \leq T, x \in K} \left(S_s^{t,x}\right)^2\right] < \infty$$

where K is a compact of $\mathbb{R}^+$, which is clear.

In the sequel, more precisely in order to deal with the Dynamic Programming equation (DP), we will need to look at the infimum of u with respect to α, (t,x) being fixed. The following easy lemma gives us all what we need:

Lemma 2 *[6] Let v be a continuous function on $\mathbb{R}^m \times A$ where A is a compact, then v^* defined by*

$$v^*(x) = \inf_{\alpha \in A} v(x,\alpha)$$

is continuous on $\mathbb{R}^m$. Moreover, there exists a measurable function $x \to \alpha^*(x)$ such that $v^*(x) = v(x, \alpha^*(x))$.

3 Reduction to a sequence of optimal stopping problems

Now we have at our disposal all the required machinery. We can proceed to the proof of the Dynamic Programming equation (DP). Before that, we apply the preceeding result to show that the equation (DP) well-defines the corresponding sequence of functions.

3.1 The Dynamic Programming Equation

We define by recurrence the following quantities:

$$\begin{aligned} u_0(t,x,\alpha) &= E_t\left[\left\{\widetilde{c}(t,x) + \alpha\left(\tilde{S}_T^{t,x} - \tilde{S}_t^{t,x}\right) - \tilde{\varphi}\left(S_T^{t,x}\right)\right\}^2\right] \\ u_0^*(t,x) &= \inf_{\alpha} u_0(t,x,\alpha) \end{aligned}$$

and for $n \geq 0$

$$u_{n+1}(t,x,\alpha) =$$
$$\inf_{\tau \in \mathcal{T}_{t,T}} E_t\left[\left\{\widetilde{c}(t,x) + \alpha\left(\tilde{S}_\tau^{t,x} - \tilde{S}_t^{t,x}\right) - \widetilde{c}\left(\tau, S_\tau^{t,x}\right)\right\}^2 + u_n^*\left(\tau, S_\tau^{t,x}\right)\right]$$

$$u_{n+1}^*(t,x) = \inf_{\alpha} u_{n+1}(t,x,\alpha) \tag{7}$$

We will show:

Proposition 3 *(i) The function $(t,x,\alpha) \mapsto u_n(t,x,\alpha)$ is continuous.*

(ii) Let for all $0 \leq k \leq n$, $\alpha_k^*(t,x)$ a measurable function given by lemma 2 such that $u_k^*(t,x) = u_k(t,x,\alpha_k^*(t,x))$. Then there exists some measurable functions

$$\begin{aligned} [0,T] \times \mathbb{R}_+^* \times \Omega &\rightarrow [0,T] \\ (t,x,\omega) &\mapsto \tau_k^n(t,x,\omega) \end{aligned}$$

such that:

(*a*) *For all* (t,x) , $\tau_k^n(t,x,\omega)$ *is a stopping time of* $\overline{\mathcal{T}}_{t,T}$ *and*

$$t \le \tau_1^n(t,x,\omega) \le ... \le \tau_n^n(t,x,\omega) \le T$$

(*b*) *For all* (t,x) ,

$$u_n^*(t,x) = \\ E_t\left[\left\{\widetilde{c}(t,x) + \sum_{k=0}^{n} \alpha_{n-k}^*\left(\tau_k^n, \tilde{S}_{\tau_k^n}^{t,x}\right)\left(\tilde{S}_{\tau_{k+1}^n}^{t,x} - \tilde{S}_{\tau_k^n}^{t,x}\right) - \widetilde{\varphi}\left(\tilde{S}_T^{t,x}\right)\right\}^2\right] \tag{8}$$

with the convention $\tau_0^n(t,x,\omega) = t$, $\tau_{n+1}^n(t,x,\omega) = T$.

Proof

(i) and (ii) are checked for $n=0$. Suppose (i) and (ii) are true for n. By lemma 2, the function $u_n^*(t,x)$ is continuous and moreover

$$0 \le u_n^*(t,x) \le u_{n-1}^*(t,x) \le u_0^*(t,x) \le L\left(1+x^2\right)$$

indeed $u_n(t,x,\alpha) \le u_{n-1}^*(t,x)$ taking $\tau = t$ in (7). Moreover, $u_0^*(t,x) \le L\left(1+x^2\right)$ is obvious. Taking into account that $|\widetilde{c}(t,x)| \le L'(1+x)$ the family

$$\left\{\{\widetilde{c}(t,x) + \alpha\left(\tilde{S}_\tau^{t,x} - \tilde{S}_t^{t,x}\right) - \widetilde{c}\left(\tau, S_\tau^{t,x}\right)\}^2 + u_n^*\left(\tau, S_\tau^{t,x}\right)\right\}_{\tau\in\mathcal{T}_{t,T}}$$

is uniformly integrable and by proposition 2 the function $u_{n+1}(t,x,\alpha)$ defined by (7) is continuous. Let $\alpha_{n+1}^*(t,x)$ a measurable function given by lemma 2 such that $u_{n+1}^*(t,x) = u_{n+1}\left(t,x,\alpha_{n+1}^*(t,x)\right)$. We have by definition

$$u_{n+1}^*(t,x) = \\ \inf_{\tau\in\mathcal{T}_{t,T}} E\left[\left\{\widetilde{c}(t,x) + \alpha_{n+1}^*(t,x)\left(\tilde{S}_\tau^{t,x} - \tilde{S}_t^{t,x}\right) - \widetilde{c}\left(\tau, S_\tau^{t,x}\right)\right\}^2 + u_n^*\left(\tau, S_\tau^{t,x}\right)\right]$$

By theorem 1 there exists a stopping time τ^* of $\overline{\mathcal{T}}_{t,T}$ such that

$$u_{n+1}^*(t,x) = \\ E\left[\left\{\widetilde{c}(t,x) + \alpha_{n+1}^*(t,x)\left(\tilde{S}_{\tau^*}^{t,x} - \tilde{S}_t^{t,x}\right) - \widetilde{c}\left(\tau^*, S_{\tau^*}^{t,x}\right)\right\}^2 + u_n^*\left(\tau^*, S_{\tau^*}^{t,x}\right)\right]$$

Moreover

$$\tau^* = \inf\left\{u \ge t \,/\, u_{n+1}^*\left(u, S_u^{t,x}\right) \ge u_n^*\left(u, S_u^{t,x}\right)\right\} \wedge T$$

with the convention $\inf \emptyset = \infty$.

Let us show now (ii). We set

$$\tau_1^{n+1}(t,x,\omega) = \inf\left\{u \geq t \,/\, u_{n+1}^*\left(u, S_u^{t,x}\right) \geq u_n^*\left(u, S_u^{t,x}\right)\right\} \wedge T$$

Then the function

$$(t,x,\omega) \mapsto \tau_1^{n+1}(t,x,\omega)$$

is measurable: indeed (3.1) writes

$$\tau_1^{n+1}(t,x,\omega) = \inf\left\{u \geq t \,/\, F(t,x,u,\omega) \geq 0\right\} \wedge T$$

where $F(t,x,u,\omega)$ is a measurable function and continuous in the last 2 arguments. Then

$$\left\{\tau_1^{n+1}(t,x,\omega) > r\right\} = \cap_{q<r,q \text{ rationnel}} \left\{F(t,x,q,\omega) < 0\right\}$$

is a measurable set.

We set for $n+1 \geq k \geq 2$

$$\tau_k^{n+1}(t,x,\omega) = \tau_1^{n+1}(t,x,\omega) + \tau_{k-1}^n\left(\tau_1^{n+1}(t,x,\omega), S_{\tau_1^{n+1}}^{t,x}, \Theta_{\tau_1^{n+1}}(\omega)\right)$$

(where Θ is the usual shift operator) and we check that τ_k^{n+1} are stopping times which verify $(ii)(a)$.

Then we have

$$\begin{aligned}
& E\left[\left\{\widetilde{c}(t,x) + \sum_{k=0}^{n+1} \alpha_{n+1-k}^*\left(\tau_k^{n+1}, S_{\tau_k^{n+1}}^{t,x}\right)\left(\tilde{S}_{\tau_{k+1}^{n+1}}^{t,x} - \tilde{S}_{\tau_k^{n+1}}^{t,x}\right)\right.\right. \\
& \left.\left. -\widetilde{\varphi}\left(S_T^{t,x}\right)\right\}^2\right] \\
= \; & E\left[E\left[\left\{\widetilde{c}(t,x) + \sum_{k=0}^{n+1} \alpha_{n+1-k}^*\left(\tau_k^{n+1}, S_{\tau_k^{n+1}}^{t,x}\right)\left(\tilde{S}_{\tau_{k+1}^{n+1}}^{t,x} - \tilde{S}_{\tau_k^{n+1}}^{t,x}\right)\right.\right.\right. \\
& \left.\left.\left. -\widetilde{\varphi}\left(S_T^{t,x}\right)\right\}^2 \mid F_{\tau_1^{n+1}}\right]\right] \\
= \; & E\left[\left\{\widetilde{c}(t,x) + \alpha_{n+1}^*(t,x)\left(\tilde{S}_{\tau_1^{n+1}}^{t,x} - \tilde{S}_t^{t,x}\right) - \widetilde{c}\left(\tau_1^{n+1}, S_{\tau_1^{n+1}}^{t,x}\right)\right\}^2\right. \\
& +E\left[\left\{\widetilde{c}\left(\tau_1^{n+1}, S_{\tau_1^{n+1}}^{t,x}\right) + \sum_{k=1}^{n+1} \alpha_{n+1-k}^*\left(\tau_k^{n+1}, S_{\tau_k^{n+1}}^{t,x}\right)\left(\tilde{S}_{\tau_{k+1}^{n+1}}^{t,x} - \tilde{S}_{\tau_k^{n+1}}^{t,x}\right)\right.\right. \\
& \left.\left. -\widetilde{\varphi}\left(S_T^{t,x}\right)\right\}^2 \mid F_{\tau_1^{n+1}}\right]\Bigg]
\end{aligned}$$

by orthogonality of $\tilde{S}_{\tau_{k+1}^{n+1}} - \tilde{S}_{\tau_k^{n+1}}$ under the risk-neutral probability.

Now

$$\tau_k^{n+1}(t,x,\omega) = \tau_1^{n+1}(t,x,\omega) + \tau_{k-1}^n\left(\tau_1^{n+1}(t,x,\omega), S_{\tau_1^{n+1}}^{t,x}, \theta_{\tau_1^{n+1}}\omega\right)$$

whence

$$\begin{aligned}
& E\left[\left\{\widetilde{c}\left(\tau_1^{n+1}, S^{t,x}_{\tau_1^{n+1}}\right) + \sum_{k=1}^{n+1} \alpha^*_{n+1-k}\left(\tau_k^{n+1}, S^{t,x}_{\tau_k^{n+1}}\right)\left(\tilde{S}^{t,x}_{\tau_{k+1}^{n+1}} - \tilde{S}^{t,x}_{\tau_k^{n+1}}\right)\right.\right. \\
& \left.\left. -\widetilde{\varphi}\left(S_T^{t,x}\right)\right\}^2 \mid F_{\tau_1^{n+1}}\right] \\
= \; & U\left(\tau_1^{n+1}, S^{t,x}_{\tau_1^{n+1}}\right)
\end{aligned}$$

by the strong Markov property by noticing that

$$\begin{aligned}
S^{t,x}_{\tau_k^{n+1}}(\omega) &= S^{t,x}_{\tau_1^{n+1}+\tau_{k-1}^n \circ \Theta_{\tau_1^{n+1}(\omega)}}(\omega) \\
&= S^{t,x}_{\tau_{k-1}^n} \circ \Theta_{\tau_1^{n+1}}(\omega)
\end{aligned}$$

where

$$\begin{aligned}
U(t,z) &= E\left[\left\{\widetilde{c}(t,z) + \sum_{k=1}^{n+1} \alpha^*_{n+1-k}\left(\tau_{k-1}^n(t,z), S^{t,z}_{\tau_{k-1}^n}\right) \times \left(\tilde{S}^{t,z}_{\tau_k^n} - \tilde{S}^{t,z}_{\tau_{k-1}^n}\right) - \widetilde{\varphi}\left(\tilde{S}_T^{t,z}\right)\right\}^2\right] \\
&= u_n^*(t,z)
\end{aligned}$$

according to (8). The result follows.

3.2 Proof of the Dynamic Programming equation

It remains to show that u_n is the desired function, that is to prove the equality between u_n and the function v_n which is defined in (1).
We introduce the family $\{U_n^\alpha(\rho)\}_\rho$ of random variables indexed by $\rho \in \mathcal{T}_{0,T}$, defined by:

$$U_0^\alpha(\rho) = E\left[\{\tilde{c}(\rho, S_\rho) + \alpha(\tilde{S}_T - \tilde{S}_\rho) - \tilde{\varphi}(S_T)\}^2 \mid \mathcal{F}_\rho\right].$$

and for $n \geq 1$

$$\begin{aligned}
U_n^\alpha(\rho) &\equiv essinf_{\tau \in \mathcal{T}_{\rho,T}} E\left[\{\tilde{c}(\rho, S_\rho) + \alpha(\tilde{S}_\tau - \tilde{S}_\rho) - \tilde{c}(\tau, S_\tau)\}^2 + u^*_{n-1}(\tau, S_\tau) \mid \mathcal{F}_\rho\right] \\
&= -\left\{\tilde{c}(\rho, S_\rho) - \alpha\tilde{S}_\rho\right\}^2 + essinf_{\tau \in \mathcal{T}_{\rho,T}} E\left[\{\tilde{c}(\tau, S_\tau) - \alpha\tilde{S}_\tau\}^2 + u^*_{n-1}(\tau, S_\tau) \mid \mathcal{F}_\rho\right]
\end{aligned} \tag{10}$$

We define by $\widehat{U}_n^\alpha(.)$ the largest RCLL sub-martingale which is less than $\{\tilde{c}(t, S_t) - \alpha\tilde{S}_t\}^2 + u^*_{n-1}(t, S_t)$.

Since the path of $\{\tilde{c}(.,S_.)-\alpha\tilde{S}_.\}^2+u^*_{n-1}(.,S_.)$ are continuous, it is known that for every stopping time ρ of $\mathcal{T}_{0,T}$,

$$\widehat{U}^\alpha_n(\rho)=\left\{\tilde{c}(\rho,S_\rho)-\alpha\tilde{S}_\rho\right\}^2+U^\alpha_n(\rho)\text{ a.s}$$

Moreover, by Proposition 1, we have: $U^\alpha_n(t)=u_n(t,S_t,\alpha)$.

We also introduce another family $\{V^\alpha_n(\rho)\}_\rho$ of random variables indexed by $\rho\in\mathcal{T}_{[0,T]}$, defined by:

$$V^\alpha_0(\rho)=E\left[\{\tilde{c}(\rho,S_\rho)+\alpha(\tilde{S}_T-\tilde{S}_\rho)-\tilde{\varphi}(S_T)\}^2/\mathcal{F}_\rho\right].$$

and for $n\geq 1$

$$V^\alpha_n(\rho)\equiv essinf_{\substack{\tau_1,\ldots,\tau_n\ \in\mathcal{T}_{\rho,T}\\ \delta_{\tau_1},\ldots,\delta_{\tau_n}}}\ E\left[\{\tilde{c}(\rho,S_\rho)+\alpha(\tilde{S}_{\tau_1}-\tilde{S}_\rho)+\sum_{i=1}^n\delta_{\tau_i}(\tilde{S}_{\tau_{i+1}}-\tilde{S}_{\tau_i})-\tilde{\varphi}(S_T)\}^2\mid\mathcal{F}_\rho\right]$$

We will show the following theorem:

Theorem 2 $U^\alpha_n(t)=V^\alpha_n(t)$ *a.s*

Proof

The equality is true for n=0. We suppose $U^\alpha_n(t)=V^\alpha_n(t)$ a.s.

First we show $V^\alpha_{n+1}(t)\geq U^\alpha_{n+1}(t)$ a.s.

Let $\mathcal{P}_{n+1}=\left(\tau_1,\tau_2,\ldots,\tau_{n+1},\delta_{\tau_1},\delta_{\tau_2},\ldots,\delta_{\tau_{n+1}}\right)$ a control, then:

$$\begin{aligned}
& E\left[\left\{\widetilde{c}(t,S_t)+\alpha(\tilde{S}_{\tau_1}-\tilde{S}_t)+\sum_{k=1}^{n+1}\delta_{\tau_{k+1}}\left(\tilde{S}_{\tau_{k+1}}-\tilde{S}_{\tau_k}\right)-\widetilde{\varphi}(S_T)\right\}^2\mid\mathcal{F}_t\right]\\
= & E\left[E\left[\left\{\widetilde{c}(t,S_t)+\alpha(\tilde{S}_{\tau_1}-\tilde{S}_t)+\sum_{k=1}^{n+1}\delta_{\tau_k}\left(\tilde{S}_{\tau_{k+1}}-\tilde{S}_{\tau_k}\right)-\widetilde{\varphi}(S_T)\right\}^2/\mathcal{F}_{\tau_1}\right]\mid\mathcal{F}_t\right]\\
= & E\left[\left\{\widetilde{c}(t,S_t)+\alpha(\tilde{S}_{\tau_1}-\tilde{S}_t)-\widetilde{c}(\tau_1,S_{\tau_1})\right\}^2\right.\\
& +E\left[\left\{\widetilde{c}(\tau_1,S_{\tau_1})+\sum_{k=1}^{n+1}\delta_{\tau_k}\left(\tilde{S}_{\tau_{k+1}}-\tilde{S}_{\tau_k}\right)-\widetilde{\varphi}(S_T)\right\}^2/\mathcal{F}_{\tau_1}\right]\mid\mathcal{F}_t\Bigg]\\
\geq & E\left[\left\{\widetilde{c}(t,S_t)+\alpha(\tilde{S}_{\tau_1}-\tilde{S}_t)-\widetilde{c}(\tau_1,S_{\tau_1})\right\}^2\right.\\
& +essinf_{\substack{\tau_2,\ldots,\tau_{n+1}\ \in\mathcal{T}_{\tau_1,T}\\ \delta_{\tau_2},\ldots,\delta_{\tau_{n+1}}}}\ E\left[\left\{\widetilde{c}(\tau_1,S_{\tau_1})+\sum_{k=1}^{n+1}\delta_{\tau_k}\left(\tilde{S}_{\tau_{k+1}}-\tilde{S}_{\tau_k}\right)-\widetilde{\varphi}(S_T)\right\}^2\mid\mathcal{F}_{\tau_1}\right]\mid\mathcal{F}_t\Bigg]
\end{aligned}$$

$$\begin{aligned}
&\geq E\left[\left\{\widetilde{c}(t,S_t)+\alpha(\tilde{S}_{\tau_1}-\tilde{S}_t)-\widetilde{c}(\tau_1,S_{\tau_1})\right\}^2+u_n^*(\tau_1,S_{\tau_1})\mid\mathcal{F}_t\right]\\
&\geq U_{n+1}^{\alpha}(t)
\end{aligned}$$

and then

$$V_{n+1}^{\alpha}(t)\geq U_{n+1}^{\alpha}(t) \text{ a.s.}$$

To show the other inequality, we use the last proposition. For every (t,x), there exists stopping times $\tau_{k+1}^{n+1}(t,x,\omega)$ $(1\leq k\leq n+1)$ of $\overline{\mathcal{T}}_{t,T}$, with $t\leq\tau_1^{n+1}(t,x,\omega)\leq\ldots\leq\tau_{n+1}^{n+1}(t,x,\omega)\leq T$ such that

$$\begin{aligned}
u_{n+1}(t,x,\alpha) &= E\left[\left\{\widetilde{c}(t,x)+\alpha(\tilde{S}^{t,x}_{\tau_1^{n+1}}-\tilde{S}^{t,x}_t)+\sum_{k=1}^{n+1}\delta_{\tau_{k+1}^{n+1}}\left(\tilde{S}^{t,x}_{\tau_{k+1}^{n+1}}-\tilde{S}^{t,x}_{\tau_k^{n+1}}\right)-\widetilde{\varphi}\left(S_T^{t,x}\right)\right\}^2\right]\\
&= E\left[\left\{\widetilde{c}(t,x)+\alpha(\tilde{S}_{\tau_1^{n+1}}-\tilde{S}_t)+\sum_{k=1}^{n+1}\delta_{\tau_{k+1}^{n+1}}\left(\tilde{S}_{\tau_{k+1}^{n+1}}-\tilde{S}_{\tau_k^{n+1}}\right)-\widetilde{\varphi}(S_T)\right\}^2\mid S_t\right]
\end{aligned}$$

This yields $U_{n+1}^{\alpha}(t)=u_{n+1}(t,S_t,\alpha)\geq V_{n+1}^{\alpha}(t)$ a.s.

Finally we have proved

$$\begin{aligned}
u_{n+1}(t,x,\alpha) &= \inf_{\tau\in\mathcal{T}_{t,T}} E\left[\left\{\tilde{c}(t,x)+\alpha\left(\tilde{S}_{\tau}^{t,x}-\tilde{S}_t^{t,x}\right)-\tilde{c}\left(\tau,S_{\tau}^{t,x}\right)\right\}^2+u_n^*(\tau,S_{\tau}^{t,x})\right]\\
&= \inf_{\mathcal{P}_{n+1}} E\left[\left\{\widetilde{c}(t,x)+\alpha(\tilde{S}^{t,x}_{\tau_1}-\tilde{S}^{t,x}_t)+\sum_{k=1}^{n+1}\delta_{\tau_{k+1}}\left(\tilde{S}^{t,x}_{\tau_{k+1}}-\tilde{S}^{t,x}_{\tau_k}\right)-\widetilde{\varphi}\left(S_T^{t,x}\right)\right\}^2\right]
\end{aligned}$$

such reducing the N stopping times problem to a sequence of optimal stopping problems.

4 Variational inequality

In this section we derive the infinitesimal form of (DP).

4.1 Formal Derivation

4.1.1 First method

We give a first method to get formally the variational inequality using the definition of v_n^{α}.

- If we trade immediately, the cost of such a decision is:

$$v_{n-1}^{\delta}(t,x)$$

hence we can state the condition

$$v_n^{\alpha}(t,x) \leq \inf_{\delta} v_{n-1}^{\delta}(t,x)$$

- If we decide not to trade during a small interval of time [t,t+δ] and then act optimally in the future, the resulting cost is:

$E\left[\{\tilde{c}_t + \alpha(\tilde{S}_{t+\delta} - \tilde{S}_t) - \tilde{c}_{t+\delta}\}^2 \mid \mathcal{F}_t\right] + E\left[v_n^{\alpha}(t+\delta, S_{t+\delta}) \mid \mathcal{F}_t\right]$

By definition of v_n^{α}, we have:

$$v_n^{\alpha}(t,s) \leq E\left[\{\tilde{c}_t + \alpha(\tilde{S}_{t+\delta} - \tilde{S}_t) - \tilde{c}_{t+\delta}\}^2 \mid \mathcal{F}_t\right] + E\left[v_n^{\alpha}(t+\delta, S_{t+\delta}) \mid \mathcal{F}_t\right]$$

Using Ito, we get

$$E\left[\{\tilde{c}_t + \alpha(\tilde{S}_{t+\delta} - \tilde{S}_t) - \tilde{c}_{t+\delta}\}^2 \mid \mathcal{F}_t\right] + E\left[\int_t^{t+\delta} (\frac{\partial v_n^{\alpha}}{\partial t}(u,S_u) + A v_n^{\alpha}(u,S_u))\, du / \mathcal{F}_t\right] \geq 0$$

where A is the characteristic operator of S_t, $Av\,(t,x) = rx \frac{\partial v}{\partial x}\,(t,x) + \frac{\sigma^2}{2} x^2 \frac{\partial^2 v}{\partial x^2}\,(t,x)$.

We divide by δ, and we let $\delta \to 0$.

By noticing

$$\begin{aligned} E\left[\{\tilde{c}_t + \alpha(\tilde{S}_{t+\delta} - \tilde{S}_t) - \tilde{c}_{t+\delta}\}^2 \mid \mathcal{F}_t\right] &= E\left[\{\int_t^{t+\delta} (\alpha - \Delta(u,S_u))\, d\tilde{S}_u\}^2 / \mathcal{F}_t\right] \\ &= E\left[\int_t^{t+\delta} (\alpha - \Delta(u,S_u))^2 \sigma^2 \tilde{S}_u^2\, du \mid \mathcal{F}_t\right] \end{aligned}$$

we get

$$\frac{\partial v_n^{\alpha}}{\partial t}(t,S_t) + A v_n^{\alpha}(t,S_t) + \sigma^2 \tilde{S}_t^2 (\Delta(t,S_t) - \alpha)^2 \geq 0$$

We finally notice that one or the other of inequalities must be an equality, since either we trade immediately or not. Therefore we have:

$$(v_n^{\alpha}(t,x) - \inf_{\delta} v_{n-1}^{\delta}(t,x))(\frac{\partial v_n^{\alpha}}{\partial t}(t,x) + A v_n^{\alpha}(t,x) + \sigma^2 (x e^{-rt})^2 (\Delta(t,x) - \alpha)^2) = 0$$

Then, we have to solve the following system:

$$\left\{\begin{array}{c} \frac{\partial v_n^\alpha}{\partial t}(t,x) + Av_n^\alpha(t,x) + \sigma^2(xe^{-rt})^2(\Delta(t,x)-\alpha)^2 \geq 0 \\ v_n^\alpha(t,x) \leq \inf_\delta v_{n-1}^\delta(t,x) \\ (v_n^\alpha(t,x) - \inf_\delta v_{n-1}^\delta(t,x))(\frac{\partial v_n^\alpha}{\partial t}(t,x) + Av_n^\alpha(t,x) + \sigma^2(xe^{-rt})^2(\Delta(t,x)-\alpha)^2) = 0 \\ v_n^\alpha(T,x) = 0 \end{array}\right.$$

4.1.2 Second method

Since v_n^α is the value function of an optimal stopping problem, it is known that v_n^α is the solution of variational inequalities.

4.1.3 Comments

The knowledge of the function v_n^α makes it possible to determine the optimal policy in the following way: the domain [0,T] $\times\mathbb{R}^+$ is divided into two regions:

- The continuation region $D = \{(t,x) \in [0,T] \times \mathbb{R}^+ / v_n^\alpha(t,x) < \inf_\delta v_{n-1}^\delta(t,x)\}$
- Its complement, the exercice region $\mathcal{E} = D^C = \{(t,x) \in [0,T] \times \mathbb{R}^+ / v_n^\alpha(t,x) = \inf_\delta v_{n-1}^\delta(t,x)\}$

If the region $\mathcal{E}$ is reached, the best policy is to make a transaction, the trader chooses the δ which minimizes the cost v_{n-1}^δ. The boundary between the two regions is the optimal stopping boundary.

If we define $\widehat{\tau_1} = inf\{u \geq t/(u,S_u) \notin D\}$ and $\widehat{\delta_1} = \widehat{\delta}(\widehat{\tau_1}, S_{\widehat{\tau_1}})$ the optimal δ in the problem $\inf_\delta v_{n-1}^\delta$, then $\widehat{\tau_1}$ is the first optimal time of trading and $\widehat{\delta_1}$ the associated transaction .

By iterating this procedure, we can construct step by step the optimal politicy.

4.2 Viscosity solutions

The notion of viscosity solutions was first introduced by Crandall and Lions [5] to solve problems related to first-order Hamilton-Jacobi equations. For a general overview of the theory, we refer to the "user's guide" by Crandall-Ishii-Lions [4].

In this section, we will show that given v_{n-1}, the value function v_n^α is the unique viscosity solution of the variational inequalities:

$$\left\{\begin{array}{c} \frac{\partial v_n^\alpha}{\partial t}(t,x) + Av_n^\alpha(t,x) + \sigma^2(xe^{-rt})^2(\Delta(t,x)-\alpha)^2 \geq 0 \\ v_n^\alpha(t,x) \leq \inf_\alpha v_{n-1}^\alpha(t,x) \\ (v_n^\alpha(t,x) - \inf_\delta v_{n-1}^\alpha(t,x)(\frac{\partial v_n^\alpha}{\partial t}(t,x) + Av_n^\alpha(t,x) + \sigma^2(xe^{-rt})^2(\Delta(t,x)-\alpha)^2) = 0 \\ v_n^\alpha(T,x) = 0 \end{array}\right. \tag{11}$$

4.2.1 Existence

We now give the definition of viscosity solution in our problem.

Definition 1 *Let W a continuous function on $[0,T] \times \mathbb{R}^+$ such that $W(T,x) = 0$.*
W is a viscosity solution of the system if and only if

i) $\forall \phi \in C^{1,2}([0,T] \times \mathbb{R}^+)$, and $(t_0, s_0) \in [0,T] \times \mathbb{R}^+$ such that $\phi \geq W$ and $\phi(t_0,s_0) = W(t_0,s_0)$ we have:

$$min(\frac{\partial \phi}{\partial t}(t_0,s_0) + A\phi(t_0,s_0) + \sigma^2(s_0 e^{-rt_0})^2(\Delta(t_0,s_0) - \alpha)^2, -\phi(t_0,s_0) + v^*_{n-1}(t_0,s_0)) \geq 0$$

ii) $\forall \phi \in C^{1,2}([0,T] \times \mathbb{R}^+)$, and $(t_0, s_0) \in [0,T] \times \mathbb{R}^+$ such that $\phi \leq W$ and $\phi(t_0,s_0) = W(t_0,s_0)$ we have:

$$min(\frac{\partial \phi}{\partial t}(t_0,s_0) + A\phi(t_0,s_0) + \sigma^2(s_0 e^{-rt_0})^2(\Delta(t_0,s_0) - \alpha)^2, -\phi(t_0,s_0) + v^*_{n-1}(t_0,s_0)) \leq 0$$

Theorem 3 *The value function v_n^α is a viscosity solution of the system.*

Proof We already proved that v_n^α is continuous and $v_n^\alpha(T,x) = 0$.
We divide our proof in two steps.
1) We first show that v_n^α is a viscosity supersolution:
Let $\phi \in C^{1,2}([0,T] \times \mathbb{R}^+)$, and $(t_0,s_0) \in [0,T] \times \mathbb{R}^+$ such that $\phi \geq v_n^\alpha$ and $\phi(t_0,s_0) = v_n^\alpha(t_0,s_0)$, we want to show that:

$$min(\frac{\partial \phi}{\partial t}(t_0,s_0) + A\phi(t_0,s_0) + \sigma^2(s_0 e^{-rt_0})^2(\Delta(t_0,s_0) - \alpha)^2, -\phi(t_0,s_0) + v^*_{n-1}(t_0,s_0)) \geq 0$$

We set

$$D = \{(t,x) \in [t_0,T] \times \mathbb{R}^+ / v_n^\alpha(t,x) < v^*_{n-1}(t,x)\}$$

and

$$\tau_{op} = inf\{u \geq t_0 / (u, S_u) \notin D\}$$

We know that τ_{op} is the smallest optimal stopping time.
$\forall \tau$ stopping time $\geq t_0$, we have:

$$v_n^\alpha(t_0,s_0) + \{\tilde{c}_{t_0} - \alpha s_0 e^{-rt_0}\}^2 \leq E\left[v_n^\alpha(\tau, S_\tau) + \{\tilde{c}_\tau - \alpha \tilde{S}_\tau\}^2 \mid S_{t_0} = s_0\right]$$

$$\begin{aligned}
\phi(t_0,s_0)=v_n^\alpha(t_0,s_0) &\leq -\{\tilde{c}_{t_0}-\alpha s_0 e^{-rt_0}\}^2+E\left[v_n^\alpha(\tau,S_\tau)+(\tilde{c}_\tau-\alpha\tilde{S}_\tau)^2 \mid S_{t_0}=s_0\right]\\
&\leq -\{\tilde{c}_{t_0}-\alpha s_0 e^{-rt_0}\}^2+E\left[\phi(\tau,S_\tau)+(\tilde{c}_\tau-\alpha\tilde{S}_\tau)^2 \mid S_{t_0}=s_0\right]\\
&= -\{\tilde{c}_{t_0}-\alpha s_0 e^{-rt_0}\}^2+E\left[\phi(t_0,s_0)+\int_{t_0}^{\tau}(\frac{\partial\phi}{\partial t}+A\phi)(u,S_u)\,du\right.\\
&\qquad \left.+\{\tilde{c}_\tau-\alpha\tilde{S}_\tau\}^2 \mid S_{t_0}=s_0\right]
\end{aligned}$$

We get

$$0\leq -\{\tilde{c}_{t_0}-\alpha s_0 e^{-rt_0}\}^2+E\left[\{\tilde{c}_\tau-\alpha\tilde{S}_\tau\}^2 \mid S_{t_0}=s_0\right]+E\left[\int_{t_0}^{\tau}(\frac{\partial\phi}{\partial t}+A\phi)(u,S_u)\,du \mid S_{t_0}=s_0\right]$$

We set $\tau=\tau\wedge(t_0+\epsilon)$.
Dividing by $E\left[\tau\wedge(t_0+\epsilon)\mid S_{t_0}=s_0\right]$ and let $\epsilon\to 0$, and noticing

$$\begin{aligned}
E\left[\{\tilde{c}_\tau-\alpha\tilde{S}_\tau\}^2-\{\tilde{c}_{t_0}-\alpha s_0 e^{-rt_0}\}^2 \mid S_{t_0}=s_0\right] &= E\left[\{\tilde{c}_\tau-\tilde{c}_{t_0}-\alpha(\tilde{S}_\tau-s_0e^{-rt_0})\}^2 \mid S_{t_0}=s_0\right]\\
&= E\left[\{\int_{t_0}^{\tau}(\Delta(u,S_u)-\alpha)\,d\tilde{S}_u\}^2 \mid S_{t_0}=s_0\right]\\
&= E\left[\int_{t_0}^{\tau}(\Delta(u,S_u)-\alpha)^2\sigma^2\tilde{S}_u^2\,du \mid S_{t_0}=s_0\right]
\end{aligned}$$

yields

$$0\leq\frac{\partial\phi}{\partial t}(t_0,s_0)+A\phi(t_0,s_0)+\sigma^2(s_0e^{-rt_0})^2(\Delta(t_0,s_0)-\alpha)^2$$

and then v_n^α is a viscosity supersolution.

2) We turn now to the subsolution case.
Let $\phi\in C^{1,2}([0,T]\times\mathbb{R}^+)$, and $(t_0,s_0)\in[0,T]\times\mathbb{R}^+$ such that $v_n^\alpha\geq\phi$ et $v_n^\alpha(t_0,s_0)=\phi(t_0,s_0)$. We want to show that

$$min(\frac{\partial\phi}{\partial t}(t_0,s_0)+A\phi(t_0,s_0)+\sigma^2(s_0e^{-rt_0})^2(\Delta(t_0,s_0)-\alpha)^2,-\phi(t_0,s_0)+v_{n-1}^*(t_0,s_0))\leq 0$$

If $(t_0,s_0)\notin D$ then $v_n^\alpha(t_0,s_0)=v_{n-1}^*(t_0,s_0)$, and then $\phi(t_0,s_0)=v_{n-1}^*(t_0,s_0)$, and the inequality hold.
If $(t_0,s_0)\in D$ then $v_n^\alpha(t_0,s_0)<v_{n-1}^*(t_0,s_0)$. Let τ a stopping time $<\tau_{op}$, we have:

$$\begin{aligned}
\phi(t_0,s_0)=v_n^\alpha(t_0,s_0) &= -\{\tilde{c}_{t_0}-\alpha s_0 e^{-rt_0}\}^2 + E\left[v_n^\alpha(\tau,S_\tau)+\{\tilde{c}_\tau-\alpha\tilde{S}_\tau\}^2 \mid S_{t_0}=s_0\right]\\
&\geq -\{\tilde{c}_{t_0}-\alpha s_0 e^{-rt_0}\}^2 + E\left[\phi(\tau,S_\tau)+\{\tilde{c}_\tau-\alpha\tilde{S}_\tau\}^2 \mid S_{t_0}=s_0\right]\\
&= -\{\tilde{c}_{t_0}-\alpha s_0 e^{-rt_0}\}^2 + E\left[\phi(t_0,s_0)+\int_{t_0}^{\tau}(\frac{\partial\phi}{\partial t}+A\phi)(u,S_u)\,du\right.\\
&\qquad \left.+\{\tilde{c}_\tau-\alpha\tilde{S}_\tau\}^2 \mid S_{t_0}=s_0\right]
\end{aligned}$$

We obtain

$$0\geq -\{\tilde{c}_{t_0}-\alpha s_0 e^{-rt_0}\}^2 + E\left[\{\tilde{c}_\tau-\alpha\tilde{S}_\tau\}^2 \mid S_{t_0}=s_0\right] + E\left[\int_{t_0}^{\tau}(\frac{\partial\phi}{\partial t}+A\phi)(u,S_u)\,du \mid S_{t_0}=s_0\right]$$

We set $\tau=\tau\wedge(t_0+\epsilon)$.
We divide by $E\left[\tau\wedge(t_0+\epsilon)\mid S_{t_0}=s_0\right]$ and we let ϵ go to 0, we obtain:

$$\frac{\partial\phi}{\partial t}(t_0,s_0)+A\phi(t_0,s_0)+\sigma^2(s_0e^{-rt_0})^2(\Delta(t_0,s_0)-\alpha)^2\leq 0$$

This shows that v_n is a viscosity subsolution.

4.2.2 Uniqueness

We have to show now that v_n is the unique viscosity solution of the variational inequality. It is known ([4]) that if the domain is bounded then the system has a unique solution.

We recall:

$$\begin{aligned}
v_n^\alpha(t,x) &= -\{\tilde{c}_t-\alpha x e^{-rt}\}^2+\inf_{\tau_1\geq t}E\left[\{\tilde{c}_{\tau_1}-\alpha\tilde{S}_{\tau_1}\}^2+v_{n-1}^*(\tau_1,S_{\tau_1})\mid S_t=x\right]\\
&= -\{\tilde{c}_t-\alpha x e^{-rt}\}^2+\inf_{\tau_1\geq t}E\left[\Phi_n^\alpha(\tau_1,S_{\tau_1})\mid S_t=x\right]
\end{aligned}$$

where $\Phi_n^\alpha(t,x)=\{\tilde{c}_t-\alpha x e^{-rt}\}^2+v_{n-1}^*(t,x)$.

v_n is a continuous function and the family $\{v_n^\alpha(\tau,S_\tau),\tau$ stopping time $\}$ is uniformly integrable.

We first show the following result:

Proposition 4 *Let U a domain in $[0,T]\times\mathbb{R}^+$*

$$\tau_U=inf\{u\geq t/(u,S_u)\notin U\}$$

(by convention $inf\emptyset=T$)

a) We define w(t,x)=-$\{\tilde{c}_t - \alpha x e^{-rt}\}^2 + E\left[\Phi(\tau_U, S_{\tau_U}) \mid S_t = x\right]$
We suppose w continuous on $[0,T] \times \mathbb{R}^+$.
Then w is a viscosity solution of the equation

$$\left\{ \begin{array}{l} \frac{\partial w}{\partial t}(t,x) + Aw(t,x) + \sigma^2 (xe^{-rt})^2 (\Delta(t,x) - \alpha)^2 = 0 \quad in\ U \\ w(T,x) = 0 \end{array} \right. \tag{12}$$

Moreover, w satisfies the condition: the family

$$\{w(\tau, S_\tau), \tau \le \tau_U\} \tag{13}$$

is uniformly integrable.
b)If u is a viscosity solution of 12 with the boundary values $u = v^*_{n-1}$ *on* ∂U, *and satisfies 13, then u=w.*

Proof a)We first show that w is a viscosity solution of 12.
First, we have $w(T,x) = 0$ and w is continuous.
We want to prove that w is a viscosity supersolution:
Let $\phi \in C^{1,2}(U)$ and $(t_0, s_0) \in U$ such that $\phi \ge w$ on U and $\phi(t_0,s_0) = w(t_0,s_0)$, we want to show that

$$\frac{\partial \phi}{\partial t}(t_0,s_0) + A\phi(t_0,s_0) + \sigma^2 (s_0 e^{-rt_0})^2 (\Delta(t_0,s_0) - \alpha)^2 \ge 0$$

Let τ a stopping time $< \tau_U$, by the Markov property we have:

$$\begin{aligned}
\phi(t_0,s_0) = w(t_0,s_0) &= E\left[w(\tau,S_\tau) \mid S_{t_0} = s_0\right] - \{\tilde{c}_{t_0} - \alpha s_0 e^{-rt_0}\}^2 + E\left[\{\tilde{c}_\tau - \alpha \tilde{S}_\tau\}^2 \mid S_{t_0} = s_0\right] \\
&\le E\left[\phi(\tau,S_\tau) \mid S_{t_0} = s_0\right] - \{\tilde{c}_{t_0} - \alpha s_0 e^{-rt_0}\}^2 + E\left[\{\tilde{c}_\tau - \alpha \tilde{S}_\tau\}^2 \mid S_{t_0} = s_0\right] \\
&= E\left[\phi(t_0,s_0) + \int_{t_0}^{\tau} (\frac{\partial \phi}{\partial t} + A\phi)(u,S_u)\, du \mid S_{t_0} = s_0\right] - \{\tilde{c}_{t_0} - \alpha s_0 e^{-rt_0}\}^2 \\
&\quad + E\left[\{\tilde{c}_\tau - \alpha \tilde{S}_\tau\}^2 \mid S_{t_0} = s_0\right]
\end{aligned}$$

hence:

$$\frac{\partial \phi}{\partial t}(t_0,s_0) + A\phi(t_0,s_0) + \sigma^2 (s_0 e^{-rt_0})^2 (\Delta(t_0,s_0) - \alpha)^2 \ge 0$$

and then w is a viscosity supersolution.
Let's turn now to the subsolution case.
Let $\phi \in C^{1,2}(U)$ et $(t_0,s_0) \in U$ such that $\phi \le w$ on U and $\phi(t_0,s_0) = w(t_0,s_0)$, we have to show that

$$\frac{\partial \phi}{\partial t}(t_0,x_0) + A\phi(t_0,s_0) + \sigma^2 (s_0 e^{-rt_0})^2 (\Delta(t_0,s_0) - \alpha)^2 \le 0$$

Let τ a stopping time $< \tau_U$, by the Markov property we have:

$$\begin{aligned}
\phi(t_0, s_0) = w(t_0, s_0) &= E\left[w(\tau, S_\tau) \mid S_{t_0} = s_0\right] - \{\tilde{c}_{t_0} - \alpha s_0 e^{-rt_0}\}^2 + E\left[\{\tilde{c}_\tau - \alpha \tilde{S}_\tau\}^2 \mid S_{t_0} = s_0\right] \\
&\geq E\left[\phi(\tau, S_\tau) \mid S_{t_0} = s_0\right] - \{\tilde{c}_{t_0} - \alpha s_0 e^{-rt_0}\}^2 + E\left[\{\tilde{c}_\tau - \alpha \tilde{S}_\tau\}^2 \mid S_{t_0} = s_0\right] \\
&= E\left[\phi(t_0, s_0) + \int_{t_0}^{\tau} (\frac{\partial \phi}{\partial t} + A\phi)(u, S_u)\, du \mid S_{t_0} = s_0\right] \\
&\quad -\{\tilde{c}_{t_0} - \alpha s_0 e^{-rt_0}\}^2 + E\left[\{\tilde{c}_\tau - \alpha \tilde{S}_\tau\}^2 \mid S_{t_0} = s_0\right]
\end{aligned}$$

hence:

$$(\frac{\partial \phi}{\partial t} + A\phi)(t_0, s_0) + \sigma^2 (s_0 e^{-rt_0})^2 (\Delta(t_0, s_0) - \alpha)^2 \leq 0$$

and then w is a viscosity subsolution .

(2) is clear.

b) By theorem 8.2 in [4], we know that if U is bounded there is a unique viscosity solution. To prove the result in the general case , we consider :

$$U(N) = \{s \in U; |s| < N\}$$

Let v a viscosity solution of (1) and (3), and define $v_N(.) = v(.)|_{U(\bar{N})}$.

Then obviously , v_N is a viscosity solution of $\frac{\partial u}{\partial t}(t,x) + Au(t,x) + \sigma^2 \tilde{s}^2 (\Delta(t,x) - \alpha)^2 = 0$ in U(N) and u(T,.)=0 and u(.)=v_N(.) on $\partial U(N)$.

Since U(N) is bounded, by uniqueness and a), we have:

$$v_N(t,x) = E\left[v_N(\tau_{U(N)}, S_{\tau_{U(N)}}) + \{\tilde{c}_{\tau_{U(N)}} - \alpha \tilde{S}_{\tau_{U(N)}}\}^2 - \{\tilde{c}_t - \alpha x e^{-rt}\}^2 \mid S_t = x\right]$$

Since $v_N(.) = v(.)$ on $U(\bar{N})$, and $v(.) = v^*_{n-1}(.)$ on ∂U and $\tau_{U(N)} \to \tau_U < \infty$ when $N \to \infty$, we have:

$$\begin{aligned}
v(t,x) &= \lim_{N \to \infty} v_N(t,x) \\
&= \lim_{N \to \infty} [E\left[v_N(\tau_{U(N)}, S_{\tau_{U(N)}}) \mid S_t = x\right] + E\left[\{\tilde{c}_{\tau_{U(N)}} - \alpha \tilde{S}_{\tau_{U(N)}}\}^2 \mid S_t = x\right]] \\
&\quad -\{\tilde{c}_t - \alpha x e^{-rt}\}^2 \\
&= E\left[\Phi(\tau_U, S_{\tau_U}) \mid S_t = x\right] - \{\tilde{c}_t - \alpha x e^{-rt}\}^2 \\
&= w(t,x)
\end{aligned}$$

Lemma 3 *Let v a viscosity supersolution of:*

$$\begin{cases} min(\frac{\partial v}{\partial t} + Av + \sigma^2(xe^{-rt})^2(\Delta(t,x) - \alpha)^2, -v + v^*_{n-1}) = 0 \\ v(T,x) = 0 \end{cases}$$

then for every (t,x) $v(t,x) \leq v^*_{n-1}(t,x)$.

Proof Let (t,x), V a neighbourhood of (t,x), then $\exists(\bar{t},\bar{s})$ maximum of v in $\bar{V}$. We can find $\phi \in C^{1,2}([0,T] \times \mathbb{R}^+)$ such that $\phi \geq v$ et $\phi(\bar{t},\bar{s}) = v(\bar{t},\bar{s})$ then

$$v(\bar{t},\bar{s}) \leq v^*_{n-1}(\bar{t},\bar{s})$$

Since V can be arbitrary small, we conclude by continuity that

$$v(t,x) \leq v^*_{n-1}(t,x)$$

If v is a viscosity solution we set

$$A = A_v = \{(t,x) \in [0,T] \times \mathbb{R}^+ / v(t,x) < v^*_{n-1}(t,x)\}$$

Theorem 4 *(Uniqueness) Let v a viscosity solution of:*

$$\begin{cases} min(\frac{\partial v}{\partial t} + Av + \sigma^2(xe^{-rt})^2(\Delta(t,x) - \alpha)^2, -v + v^*_{n-1})(t,x) = 0 \\ v(T,x) = 0 \end{cases}$$

with the property that $\{v(\tau, S_\tau), \tau$ *stopping time* $\}$ *is uniformly integrable.*
Then $v = v^\alpha_n$.

Proof We first observe that v is a viscosity solution of

$$\begin{cases} \frac{\partial v}{\partial t} + Av + \sigma^2(xe^{-rt})^2(\Delta(t,x) - \alpha)^2 = 0 \text{ in } A \\ v(T,x) = 0 \\ v(t,x) = v^*_{n-1}(t,x) \text{ on } \partial A \end{cases}$$

then with the last proposition

$$v(t,x) = -(\tilde{c}_t - \alpha x e^{-rt})^2 + E\left[\Phi(\tau_A, S_{\tau_A}) \mid S_t = x\right]$$

so that $v^\alpha_n(t,x) \leq v(t,x)$
To get the opposite inequality, we set:

$$S(N) = \{s \in \mathbb{R}^+ / |s| < N\}$$

and

$$v_N(t,.) = v(t,.)|S(\bar{N})$$

then v_N is a viscosity solution of

$$\left\{\begin{array}{l} min(\frac{\partial v_N}{\partial t} + Av_N + \sigma^2(xe^{-rt})^2(\Delta(t,x)-\alpha)^2, -v_N + v^*_{n-1}) = 0 \,\mathrm{in}\, [0,T[\times S(N) \\ v_N(T,.) = 0 \,\mathrm{in}\, S(N) \\ v_N(t,.) = v(t,.)|\partial S(N) \end{array}\right.$$

By a comparison theorem (th 8.2),

$$v_N(t,x) \leq u_N(t,x) \,\mathrm{in}\, [0,T] \times S(N)$$

where u_N is the unique viscosity of the system with $u_N(t,.) = v^*_{n-1}(t,.)$ on $\partial S(N)$. Then (by theorem 2),

$$v_N(t,x) \leq \inf_{\tau \leq \tau_{S(N)}} E\left[\Phi(\tau, S_\tau) \mid S_t = x\right] - (\tilde{c}_t - \alpha x e^{-rt})^2$$

We let N go to infinity and we get:

$$v(t,x) \leq v^\alpha_n(t,x)$$

5 The discretized problem

We give numerical results obtained in the Cox-Ross-Rubinstein model. We solve in this setting the corresponding Dynamic Programming equation.

5.1 Formulation of the problem in the binomial model

Among the N possible dates of trading, the hedger will decide to hedge only n $(< N)$ times. So he can not any longer duplicate the payoff by constructing a self-financing strategy. His goal is to minimize the variance of the tracking error under the risk neutral probability.

In the first part of the paper, we have shown that the value function v_n is solution of an optimal stopping problem:

$$v_n(t,x,\alpha) = -(\tilde{c}_t - \alpha x e^{-rt})^2 + \inf\nolimits_{\tau \in \mathcal{T}_{t,T}} E[\{\tilde{c}\left(\tau, S^{t,x}_\tau\right) - \alpha \tilde{S}^{t,x}_\tau\}^2 + v^*_{n-1}(\tau, S^{t,x}_\tau)]$$

where $\mathcal{T}_{t,T}$ is the set of all stopping time which satisfy $t \leq \tau \leq T$. Recall that the optimal stopping time which realizes the minimum is given by

$$\tau^*_t = \min(u \in \{t, ..., T\} / v^*_{n-1}(u, S^{t,x}_u) \leq v_n(u, S^{t,x}_u, \alpha))$$

If n is fixed, an application of the Bellman principle reduces the optimal stopping problem to a recursive procedure to find the value function v_n. Then, we have the following corollary:

Corollary 1 *Let $V_t^{n,\alpha}, t \leq T$ be the nonnegative adapted process defined recursively by $V_T^{n,\alpha} = 0$ and for $t \leq T-1$*

$$V_t^{n,\alpha} = min(E\left[V_{t+1}^{n,\alpha} \mid \mathcal{F}_t\right] + E\left[\{\tilde{c}\left(t+1, S_{t+1}^{t,S_t}\right) - \alpha\tilde{S}_{t+1}^{t,S_t}\}^2 \mid \mathcal{F}_t\right] - (\tilde{c}_t - \alpha\widetilde{S}_t)^2, V_t^{n-1,*})$$

then the variance of the tracking error v_n is given at time t by:

$$v_n(t, S_t, \alpha) = V_t^{n,\alpha}$$

Moreover, the first optimal date of trading (after t) is given by:

$$\tau_t^* = \min(u \in \{t, ..., T\}/V_u^{n-1,*} \leq V_u^{n,\alpha})$$

The above result gives the following algorithm: using $V_T^{n,\alpha} = 0$, we can compute the variance of the tracking error at every node at time T. Using the Dynamic Programming equation, we can compute the error at time $T-1$ by simply compare at every node, the value of the error if traded, with the value if not traded and set the error value at that node equal to the smaller of the two. If the minimal of the two values is $V_{T-1}^{n-1,*}$ it is optimal to trade. We can then apply this procedure at every node at every time step, working backwards throughout the tree.

The scheme at hand is of complexity $n * N^3$: we solve for each resolution of (DP) a family of optimal stopping problem (complexity N^2) for every level α (which is discretized with N levels). There are n such steps.

5.2 Convergence to the continuous limit

We will now prove the convergence of our numerical scheme to the continuous time limit studied above. We follow the viscosity solutions approach of Barles and Souganidis ([1]).

5.2.1 The convergence result

We can rewrite the equation in the general form

$$G(x, t, u, \frac{\partial u}{\partial t}, Du, D^2u) = 0 \tag{14}$$

We will consider numerical approximations of the form

$$(\Delta t, \Delta x, n, j, u_j^n, \tilde{u}) = 0 \tag{15}$$

We assume that the scheme satisfies the following assumptions:

Stability For all $(\Delta t, \Delta x)$, there exits a solution $\tilde{u}$ of 15 with a bounded independent of $(\Delta t, \Delta x)$.

Consistency For any smooth function ϕ and for any $(x,t) \in \bar{\Omega} \times [0,T]$ we have

$$\liminf_{\substack{\Delta t, \Delta x \to 0 \\ (n\Delta t, j\Delta x) \to (x,t) \\ \xi \to 0}} \frac{S(\Delta t, \Delta x, n, j, \phi + \xi, \tilde{\phi} + \xi)}{\rho(\Delta t, \Delta x)} \geq G(x, t, \phi(x,t), \frac{\partial \phi}{\partial t}(x,t), D\phi(x,t), D^2\phi(x,t))$$

and

$$\limsup_{\substack{\Delta t, \Delta x \to 0 \\ (n\Delta t, j\Delta x) \to (x,t) \\ \xi \to 0}} \frac{S(\Delta t, \Delta x, n, j, \phi + \xi, \tilde{\phi} + \xi)}{\rho(\Delta t, \Delta x)} \leq G(x, t, \phi(x,t), \frac{\partial \phi}{\partial t}(x,t), D\phi(x,t), D^2\phi(x,t))$$

for some function $\rho(\Delta t, \Delta x) > 0$.

Monotonicity

$$S(\Delta t, \Delta x, n, j, u_j^n, \tilde{u}) \leq S(\Delta t, \Delta x, n, j, v_j^n, \tilde{v})$$

if $\tilde{u} \geq \tilde{v}$ and if $u_j^n = v_j^n$ for all $\Delta t, \Delta x > 0, n, j$, and $\tilde{u}$ and $\tilde{v} \in R^n$.

Strong uniqueness If the locally bounded upper semi continuous (resp. lower semi continuous) function is a subsolution (resp. supersolution) of 14 then

$$u \leq v \text{ in } \tilde{\Omega}.$$

The result is the following:

Theorem 5 *Under the above assumptions, the solution $\tilde{u}$ of the scheme converges as $\Delta t, \Delta x \to 0$ uniformly on each compact subset of $\bar{\Omega}$ to the unique viscosity solution of the equation 14.*

5.2.2 Convergence of the binomial scheme

We argue by induction on n. We suppose that v_{n-1} converges uniformly on each compact to the unique viscosity solution of the corresponding equation. The binomial scheme satisfies clearly the properties required for the convergence: the scheme is obviously monotone, stable and we have the strong uniqueness property. For the consistency we note, slightly changing notations:

$$\begin{aligned}\min(\quad & -f(t,x,\alpha) + E_t\left[f(t+\Delta t, S_{t+\delta t}^{t,x}, \alpha)\right] + \\ E_t & \left[\{\tilde{c}\left(t+\delta t, S_{t+\delta t}^{t,x}\right) - \alpha \tilde{S}_{t+\delta t}^{t,x}\}^2\right] - (\tilde{c}_t - \alpha \widetilde{S}_t)^2, v_{n-1}^*(t,x,\alpha,\delta t) - f(t,x,\alpha) \quad) = 0\end{aligned}$$

Since by the recurrence hypothesis for every (t,x,α), $v_{n-1}^*(t,x,\alpha,\delta t) \to v_{n-1}^*(t,x,\alpha)$, it is easily seen that the limiting equation is 11.

5.3 Numerical results

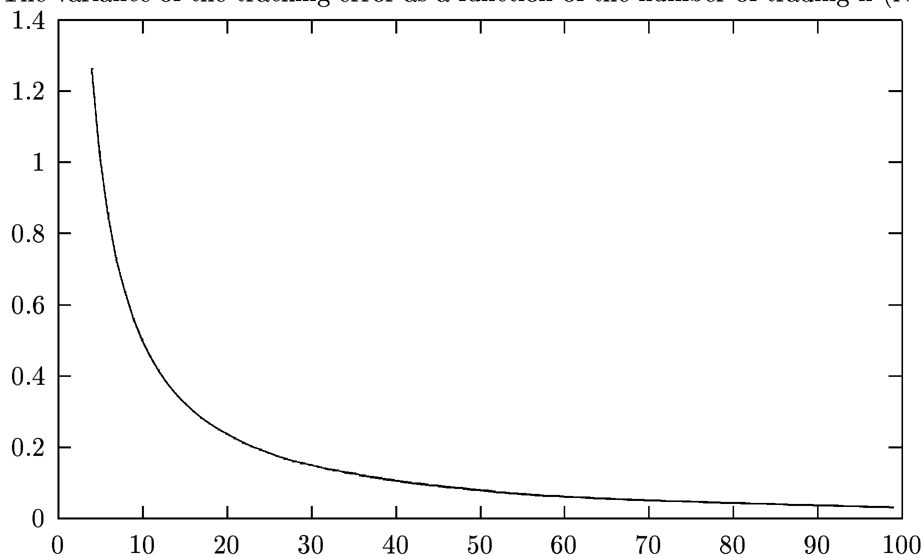
5.3.1 Tracking error in the binomial scheme

The following figure plots the variance of the tracking error as a function of the number of rebalancing. This figure illustrates the rapid decrease of the tracking error as the number of trading increases.

The chosen parameters are the following:

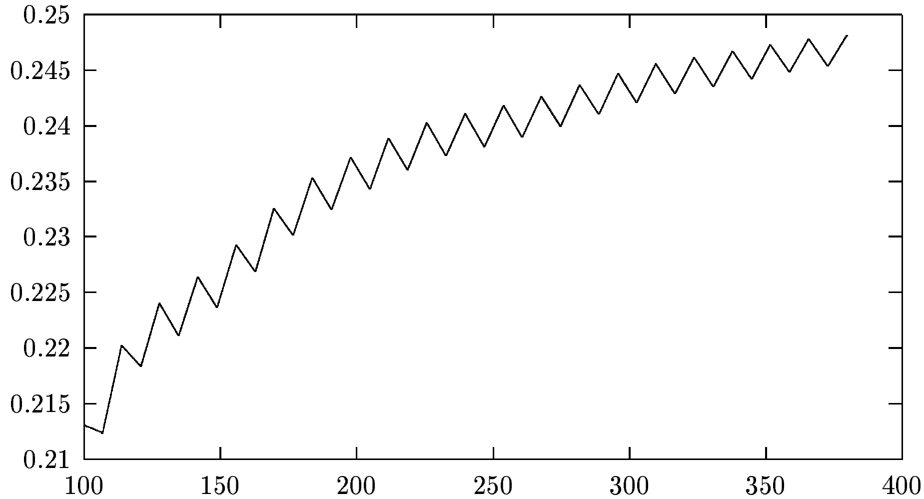
$$S_0 = 100, K = 100, r = 0, \sigma = 0.2, T = 0.333, \varphi(S_T) = (S_T - K)^+$$

The variance of the tracking error as a function of the number of trading n (N=200)



In the next plot, we fix the number of hedging, and we draw the error as a function of the number of time steps.

The variance of the tracking error as a function of the number of time steps N (n=20)



5.3.2 Comparison with deterministic strategies

In the following table, we compare the variance of the tracking error in our model with the error obtained in the deterministic case.

Number of hedging times	Error for optimal stopping times	Error for deterministic times
5	1.015422	3.325936
10	0.500272	1.662968
15	0.324128	1.108645
20	0.236881	0.831484
25	0.183150	0.665187
30	0.149072	0.554323
35	0.125143	0.475134
40	0.105763	0.415742
45	0.091259	0.369548
50	0.078915	0.332594

Number of hedging times	Error for optimal stopping times	Error for deterministic times
55	0.068568	0.302358
60	0.061146	0.277161
65	0.055649	0.255841
70	0.051079	0.237567
75	0.047032	0.221729
80	0.043331	0.207871
85	0.039877	0.195643
90	0.036608	0.184774
95	0.033471	0.175049
100	0.031035	0.166297

6 Conclusion

In the Black-Scholes model, we consider the problem of L^2 hedging of a standard European contingent claim when the number of trading dates is fixed. We show that the optimal variance is the solution to a sequence of optimal stopping problems and we have identified the optimal strategy. We design a lattice algorithm to solve the corresponding problem in the Cox-Ross-Rubinstein setting. Using viscosity solutions methods we prove the convergence of the algorithm. Numerical results are given, and the table shows the interesting result that to get the same error as in the standard equally sampled deterministic case we need to hedge about 3 times less.

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