Modelling the Dissipation Equation in Supersonic Turbulent Mixing Layers with High Density Gradients

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Abstract: In this work, the ability of a $k-\varepsilon$ model, to compute mixing layers with significant density gradients, is discussed. It is shown that as in boundary layers, deficiencies appear for compressible flows, even if the $k-\varepsilon$ model is improved by compressibility corrections. A modification of the σ_ε coefficient is proposed to compute efficiently compressible mixing layers with high density variations. This new formulation leads to a separate modelling of the subsonic and supersonic effects. The results are presented on mixing layer computations with density ratios between 1/7 (velocity/density co-gradients) and 7 (velocity/density cross-gradients), and the convective Mach number evolves from 0.3 to 1. They show a clear improvement of the capability of the $k-\varepsilon$ model to accurately compute high speed mixing layers.

Key-words: Mixing layer, Supersonic Turbulent flows, $k-\varepsilon$ model

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Modélisation de l'équation de la dissipation dans les couches mélangées supersoniques en présence de forts gradients de masse volumique

Résumé : Ce travail étudie la capacité du modèle $k - \varepsilon$, à calculer des couches de mélange en présence de forts gradients de masse volumique. On montre que comme dans les couches limites, le modèle a des difficultés à reproduire les expériences même si le modèle est amélioré par des corrections de compressibilité. On propose une modification du cœfficient σ_ε pour corriger ces défauts. Cette nouvelle formulation permet de séparer la modélisation des effets subsonique et supersonique. On présente les résultats sur des calculs de couche de mélange ayant des rapports de masse volumique entre 1/7 et 7 et un nombre de Mach convectif entre 0.3 et 1. Ces résultats montrent une amélioration nette des capacités du modèle $k - \varepsilon$, à calculer des couches de mélange à grande vitesses

Mots-clés : Couches de mélange, Ecoulements turbulents supersoniques, modèle $k - \varepsilon$

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Nomenclature

a	Local speed of sound
k	Kinetic energy of turbulence
p	Mean Static pressure
q	Coefficient of proportionality (between $\mathcal{P}$ and ε_t)
$r = u_2/u_1$	Velocity ratio
$s = \rho_2/\rho_1$	Density ratio
u_i	Velocity components (u, v)
x	Horizontal coordinate
y	Vertical coordinate
C_p	Specific heat at constant pressure
$C_{\varepsilon_1} = 1.44, C_{\varepsilon_2} = 1.92$	ε-equation coefficients
E	Mean Total energy
M	Mach number
$M_c = (u_1 - u_2)/(a_1 + a_2)$	Convective Mach number
$M_t = \sqrt{2k}/a$	Turbulent Mach number
$\mathcal{P}$	Turbulence production term
$Pr = 0.72$	Laminar Prandtl number
$Pr_t = 0.7$	Turbulent Prandtl number
S_{ij}	Mean rate of strain tensor
T	Mean static temperature
U_c	Convection velocity of large structures

δ	Mixing layer thickness
$\delta_0^{'}$	Incompressible spreading rate
$\delta^{'}$	Compressible spreading rate
ΔU	Velocity difference between the external streams
ε	Solenoidal dissipation rate
ε_c	Compressible dissipation rate
$\varepsilon_t = \varepsilon + \varepsilon_c$	Total dissipation rate
λ	Molecular conductivity coefficient
λ_t	Turbulent conductivity coefficient
μ	Dynamic viscosity
μ_t	Eddy viscosity
$\nu = \mu/\rho$	Kinematic viscosity
ρ	Mean density
Φ	Dimensionless rate of spread

$_1$ and $_2$	Subscripts for the faster and slower flows

1 Introduction

It is well known that the standard $k-\varepsilon$ model fails to predict highly compressible flows, like supersonic mixing layers. In such high speed flows, from a modelling point of view, two aspects have to be considered : a density variation and a Mach number effect.
In the early '90 some models, called *compressibility corrections* [1], [2], [3], have been developed to cure the deficiencies due to Mach number effects, appearing mainly in the poor prediction of the decrease of the spreading rate of compressible mixing layers with standard models. Compressibility can act on several ways in turbulence modelling : it can change the dissipation rate and the pressure dilatation term in the $k-\varepsilon$ equation. It may alter the anisotropy of the Reynolds stress and the diffusion rate of kinetic energy. Models of the first generation have underlined the possible importance of the first two effects. Recent developments suggest that other mechanisms can be involved. For instance, the compressibility correction models do not take in account the density gradient influence and usually have been calibrated on uniform density test-cases. The present paper is focused on density gradient effects combined with the influence of the Mach number.
Works on density variation effects in supersonic flows are still recent and have been essentially focused on boundary layers [4], [5]. A major contribution done in [4] was to show that the k-ε model fails to reproduce the logarithmic law because of density variations. The analysis of [4] concluded that the ε-equation is responsible for this problem.
The cure proposed in [4], is to account for density gradients through coefficients σ_ε or C_{ε_1}. Following the same idea, Aupoix and Viala [5] suggested to modify the diffusion term of the ε-equation, so that density gradients are counterbalanced.
It is examined in the present paper to which extent such ideas can be applied to free shear flows like mixing layers. The initial choice was to use a model of $k-\varepsilon$ type. Considering the properties of turbulence near the center of a self-similar mixing layer, it will be shown that the relation between σ_ε and the other coefficients of the model is not the same as in the equilibrium zone of a boundary layer. A relation independent of the models will be proposed, in which the relative importance of the phenomena involving Mach number and density gradients appears clearly. To illustrate the importance of this question, and as the models are rather sensitive to the values of the constants, results from empirical origin will be injected into the formulation to find necessary conditions for turbulent closure to reproduce the trends observed in simple flows like supersonic free shear layers.

2 Governing equations and the $k-\varepsilon$ model

For turbulent flows, governing equations are obtained from the averaged Navier-Stokes equations. The modelling of compressible flow equations is often made with two kinds of averages : velocity components (u_i), temperature (T) and total energy (E) are Favre mass-weighted averaged while a Reynolds average is used for density (ρ) and pressure (p). To express the unknown correlations, a turbulence model is required. The closure for Reynolds stresses is given by Boussinesq's assumption : $-\rho\widetilde{u_i''u_j''} = \mu_t S_{ij} - 2/3\rho k\delta_{ij}$ with $\mu_t = C_\mu \dfrac{\rho k^2}{\varepsilon}$ the eddy viscosity, $k = \frac{1}{2}\widetilde{u_i''u_i''}$ the turbulent kinetic energy and $S_{ij} = \dfrac{\partial u_i}{\partial x_j} + \dfrac{\partial u_j}{\partial x_i} - \dfrac{2}{3}\dfrac{\partial u_k}{\partial x_k}\delta_{ij}$ the rate of strain. The use of the $k-\varepsilon$ model [6] results in the following set of equations :

$$\left\{\begin{array}{l}
\dfrac{\partial \rho}{\partial t} + \dfrac{\partial \rho u_i}{\partial x_i} = 0 \\[2ex]
\dfrac{\partial \rho u_j}{\partial t} + \dfrac{\partial\left(\rho u_i u_j + p\delta_{ij}\right)}{\partial x_i} = \dfrac{\partial}{\partial x_i}\left[\left(\mu+\mu_t\right) S_{ij} - \dfrac{2}{3}\rho k \delta_{ij}\right] \\[2ex]
\dfrac{\partial E}{\partial t} + \dfrac{\partial u_i\left(E+p\right)}{\partial x_i} = \dfrac{\partial}{\partial x_i}\left[\left(\mu+\mu_t\right) S_{ij} u_j + \left(\lambda+\lambda_t\right)\dfrac{\partial T}{\partial x_i} + \dfrac{\mu_t}{\sigma_k}\dfrac{\partial k}{\partial x_i} - \dfrac{2}{3}\rho k u_i\right] \\[2ex]
\dfrac{\partial \rho k}{\partial t} + \dfrac{\partial \rho u_i k}{\partial x_i} = \dfrac{\partial}{\partial x_i}\left[\left(\mu + \dfrac{\mu_t}{\sigma_k}\right)\dfrac{\partial k}{\partial x_i}\right] + \mathcal{P} - \rho\varepsilon \\[2ex]
\dfrac{\partial \rho \varepsilon}{\partial t} + \dfrac{\partial \rho u_i \varepsilon}{\partial x_i} = \dfrac{\partial}{\partial x_i}\left[\left(\mu + \dfrac{\mu_t}{\sigma_\varepsilon}\right)\dfrac{\partial \varepsilon}{\partial x_i}\right] + C_{\varepsilon_1}\dfrac{\varepsilon}{k}\mathcal{P} - C_{\varepsilon_2}\rho\dfrac{\varepsilon^2}{k}
\end{array}\right. \tag{1}$$

with $\mathcal{P} = -\rho\widetilde{u_i''u_j''}\dfrac{\partial u_i}{\partial x_j}$, the production term of k and E the total energy defined as : $E = \rho(C_v T + 1/2 u_i u_i + k)$. The model is quite standard in compressible turbulence modelling. For details and explanations concerning this model, the reader can refer to the survey article of Gatski [7].

3 Density gradients modelling

3.1 The ε-equation

According to Huang *et al.* [4], the current modelling of ε-equation presents an incompatibility with the existence of high density gradients. The reason is that the standard $k-\varepsilon$ model uses a set of modelling constants which remains unchanged whatever the flow : the usual values of these constants, as they are employed in the present work are as follows :

$$C_{\varepsilon 1} = 1.44 \quad , \quad C_{\varepsilon 2} = 1.92 \quad , \quad \sigma_\varepsilon = 1.3 \quad , \quad \sigma_k = 1.0 \quad , \quad C_\mu = 0.09$$

For constant density flows, the local equilibrium condition in the wall region of a boundary layer leads to the following relation:

$$\frac{\sigma_\varepsilon\left(C_{\varepsilon 2} - C_{\varepsilon 1}\right)}{\kappa^2}\sqrt{C_\mu} = 1 \tag{2}$$

For variable density boundary layers, relation (2) *is not verified*, and density gradients have to be taken in account. If gradients of σ_ε and if the diffusion of k are negligible, the alternate following relation can be derived,

$$\frac{\sigma_\varepsilon\left(C_{\varepsilon 2}-C_{\varepsilon 1}\right)}{\kappa^2}\sqrt{C_\mu}=1+F\left(\frac{y}{\rho}\frac{\partial\rho}{\partial y},\frac{y^2}{\rho}\frac{\partial^2\rho}{\partial y^2}\right)$$

showing that density gradient effects enter into the compatibility relationship between modelling constants. According to Huang *et al.* [4] this explains the difficulties of the k-ε model to reproduce the logarithmic law of a compressible boundary layer. In such flows, the term $F\left(y/\rho\partial\rho/\partial y, y^2/\rho\partial^2\rho/\partial y^2\right)$ cannot be neglected. The same idea is used in the present work to study mixing layers with large density variations. However, self similar mixing layers do not obey equilibrium conditions, diffusion cannot be neglected and ε varies along the flow. Therefore, it may be suspected that the analysis of wall flows is not valid for free shear flows. This is examined in the next section.

3.2 Description of the density gradient and compressible turbulence corrections

Our aim is to propose a relation between the ε-equation coefficients, in order to properly describe incompressible mixing layers with density gradients, and then to determine if such new values are compatible with the existing compressibility models. We chose to keep constant $C_\mu, C_{\varepsilon 1}, C_{\varepsilon 2}, \sigma_k$ and to express σ_ε in terms of other coefficients to obtain a relation that takes into account subsonic effects (density variations) and supersonic effects (compressibility) altogether. We consider the following k and ε equations :

$$\left\{\begin{aligned} \rho\frac{Dk}{Dt} &= \frac{\partial}{\partial y}\left(\frac{\mu_t}{\sigma_k}\frac{\partial k}{\partial y}\right)+\mathcal{P}-\rho\varepsilon_t+\overline{p'd'} \\ \rho\frac{D\varepsilon}{Dt} &= \frac{\partial}{\partial y}\left(\frac{\mu_t}{\sigma_\varepsilon}\frac{\partial \varepsilon}{\partial y}\right)+C_{\varepsilon 1}\frac{\varepsilon}{k}\mathcal{P}-C_{\varepsilon 2}\frac{\varepsilon}{k}\rho\varepsilon \end{aligned}\right. \tag{3}$$

where $\overline{p'd'}$ is the pressure-dilatation term and ε_t is the total dissipation.

Early compressible models [1] [3] [2] include *explicit* terms of compressibility like the *pressure dilatation* correlation and the *dilatation dissipation* term ($\epsilon_c = 4/3\overline{\mu(\nabla.u'')^2}$) which is part of the total dissipation rate. The recent developments in compressible models [8], [9] suggest that the dilatation dissipation varies like M_t^4 and therefore in many supersonic turbulent flows where $M_t << 1$ this contribution to dissipation may be neglected. For weak compressibility, the pressure dilatation term varies like M_t^2, but is negligible when production balances dissipation. A consequence is that in many situations, this term may be weak too. The main effect of compressibility, as suggested by the numerical simulations of Sarkar [10] and of Freund *et al* [11] can therefore be a decrease of the Reynolds stress tensor isotropy. It may be underlined that the experimental attempts to check this hypothesis are technically difficult; there is no clear indication of changes in the anisotropy in [12], [13] while the measurements in [14] and [15] suggest that such an effect is important. Models representing this effect are not yet available. The physical guesses used by the existing models of the first generation are certainly questionable. However, as they were tuned to compute supersonic mixing layers with constant density, they have probably some ability to represent globally the effect of compressibility on turbulence in mixing layers. More precisely, it may be shown in a simple case how an effect attributed to dissipation may represent an alteration of production. For example, consider the equilibrium zone of a compressible boundary layer in which

$$\mathcal{P} = \varepsilon_t \quad \text{or} \quad -\widetilde{u''v''}\frac{\partial U}{\partial y} = \varepsilon + \varepsilon_c$$

If we express the dilatation dissipation in the form $\varepsilon_c = \varepsilon f(M_t)$, the previous equality can be rewritten as:

$$\sqrt{C_\mu} k \frac{\partial U}{\partial y}[1 - f(M_t)\frac{\varepsilon}{\mathcal{P}}] = \varepsilon$$

If $\varepsilon/\mathcal{P}$ is not far from 1, the final relation shows that $f(M_t)$ which was interpreted as a compressible dissipation correction, can be understood as a correction to C_μ. In mixing layers, the equilibrium condition is not fulfilled, even at low speed. However, it is found from Wygnanski and Fiedler's work [16] that $\varepsilon = 0.7$ or $0.8\mathcal{P}$. Assuming that the proportionality constant is not dramatically altered by compressibility, the same analysis can be used, at least qualitatively. In the examples presented here, existing models although questionable in their interpretation will be used to explore their consequences in the calculations. They involve the pressure-dilatation term $\overline{p'd'}$ and the total dissipation ε_t. According to Sarkar *et al.* [1] or Zeman [2], this term can be decomposed into a solenoidal part (noted ε, which verifies the above ε-equation) and a compressible part (which needs to be modeled). Many models (for example [1], [3], [2]) have the form : $\varepsilon_t = F(M_t) \times \varepsilon$, with $F(M_t) = 1 +$ function(M_t). Moreover, the definition of the eddy viscosity has to be reconsidered. The usual low-speed form is $C_\mu \rho k^2/\varepsilon$. The value of C_μ identified as $(-\widetilde{u''v''}/k)^2$ derives from equilibrium considerations. The equilibrium can now be expressed as $\mathcal{P} = \varepsilon_t$ in which $\varepsilon_t = \varepsilon + \varepsilon_c$. This modification leads to the relation :

$$\nu_t = C_\mu \rho k^2/(\varepsilon + \varepsilon_c)$$

This shows that the total dissipation should be used in this relation if the relation $C_\mu = (-\widetilde{u''v''}/k)^2$ is still used. If dilatation-dissipation is small, this refinement makes no large difference; however, with the existing models, ε_c can have significant values, and some caution should be taken.

In order to check the generality of relationships between the constants of the models, self- similar mixing layers in the vicinity of the maximum of turbulent kinetic energy are considered. The main idea is to find conditions to ensure that a correct balance in the equation for ε in the zone where the terms are significant, namely in the center of the layer. It is recalled that in such flows, self-similarity implies that the peak value of turbulent kinetic energy is constant along the flow, and that the locus of the maximum of k is a streamline.

Moreover, streamlines are straight lines. Chosing this particular streamline as longitudinal axis of coordinate implies:

$$\frac{Dk}{Dt} = 0 \quad \text{and} \quad \frac{\partial k}{\partial y} = 0 \quad \text{for} \quad x = 0$$

In general, it is found in experiments (See for example [16] for subsonic reference) that, within measurement accuracy, ν_t is approximately constant at the vicinity of the maximum of k. This approximation will be used in the present derivation. It implies that the maximum of friction is located at the inflection point of the mean velocity profile.

As suggested in [15] and although dissipation measurements are difficult, it is assumed that in the center of the mixing layer, production and dissipation are proportional. With such an assumption, and considering the definition of $\nu_t = C_\mu k^2/\varepsilon$ and $\varepsilon = q\mathcal{P}$ it is straightforward to show that $(\tau/k)^2 = C_\mu/q$ where τ is the turbulent friction.

Therefore, proportionality of production and dissipation implies that the maximum of friction $\partial\tau/\partial y$ is located on the axis where $\partial k/\partial y = 0$. The hypothesis of a weak gradient of ν_t implies that the maximum of ε occurs at the same point, which is the inflection point of the mean velocity profile, too.

Injecting these properties into (3) yields:

$$\begin{cases} \dfrac{\mu_t}{\sigma_k}\dfrac{\partial^2 k}{\partial y^2} + \mathcal{P} - \rho\varepsilon_t + \overline{p'd'} = 0 \\ \\ \rho\dfrac{D\varepsilon}{Dt} = \dfrac{\mu_t}{\sigma_\varepsilon}\dfrac{\partial^2\varepsilon}{\partial y^2} + C_{\epsilon 1}\dfrac{\varepsilon}{k}\mathcal{P} - C_{\varepsilon 2}\dfrac{\varepsilon}{k}\rho\varepsilon \end{cases} \tag{4}$$

If $\mathcal{P}$ and ε are proportional when self-similarity is achieved, it is clear that ε should decrease along x. The derivative $D\varepsilon/Dt$ is approximated by:

$$U\frac{\partial\varepsilon}{\partial x} \sim U_c\frac{\partial}{\partial x}q\mathcal{P}$$

From the definition of the vorticity thickness δ, $\partial U/\partial y = \Delta U/\delta$ along the streamline considered here, and using the relation proposed by Papamoschou and Roshko [24] for the spreading rate $d\delta/dx$:

$$\frac{d\delta}{dx} = \frac{1}{2}\delta'_{ref}\frac{\Delta U}{U_c}\Phi(M_c)$$

where δ'_{ref} is the spreading rate of the subsonic half-jet with constant density, it is easy to show that:

$$U\frac{\partial\varepsilon}{\partial x} \sim -\frac{1}{2}\varepsilon\frac{\partial U}{\partial y}\delta'_{ref}\Phi(M_c)$$

Introducing these approximations into (4), eliminating diffusion of turbulent energy between the two equations, the following relation is obtained:

$$\sigma_\varepsilon = \frac{\sigma_k(q-1) - \dfrac{\sigma_k}{\tau/k}\dfrac{\overline{p'd'}}{\rho k}(\dfrac{\partial U}{\partial y})^{-1} + \nu_t \dfrac{\partial^3 U}{\partial y^3}(\dfrac{\partial U}{\partial y})^{-2}\dfrac{k}{\tau}}{\dfrac{q}{F(M_t)}C_{\varepsilon_2} - C_{\varepsilon_1} - \dfrac{1}{2}\delta'_{ref}\dfrac{\Phi(M_c)}{\tau/k}}$$

A first important difference with the case of boundary layers can be seen: the density gradient does not appear explicitly. The reason is that the term $\partial\rho/\partial y$ is associated with $\partial k/\partial y$ or $\partial\varepsilon/\partial y$, which are zero along the considered streamline. Two other elements are present. Firstly, the term of the numerator involving σ_k results from the diffusion term of the k equation. It vanishes when $q = 1$ i.e $\mathcal{P} = \varepsilon$. Secondly, the last term of the denominator originates from the longitudinal variations of dissipation. The consequence is that the relationship valid in the log zone of a boundary layer cannot be suitable for the center of a mixing layer, where the diffusion term is maximum. It can be seen that compressibility can act explicitly through pressure-dilatation, F and Φ, and implicitly through the influence of Mach number on ν_t. It is clear that many parameters have to be taken in account. This expression is strictly valid at the vicinity of the maximum of k. However, we wanted to ensure that a correct balance in the equation for ε is obtained, in the part of the flow where the different source terms have significant values. For this reason, conditions in the center of the layer were defined. It is expected that the resulting formula represents an acceptable assumption over the central third or half of the layer. It is certainly not valid near the edge, where gradients of density, of k and of ε are present. However, as the overall level of the source terms is weaker in this zone, the consequence of using a relation with constant σ_ε is probably of minor importance. A constraint forcing the model to reproduce the correct level of dissipation and kinetic energy was found by imposing an empirical input to specify ν_t. Following [17], the eddy viscosity can be expressed as a function of the mixing layer parameters:

$$\nu_t = \frac{1}{2}K(s, r, M_c, M_g)\delta'_{ref}\Delta u\delta\Phi(M_c).$$

Replacing the velocity gradient by $\Delta U/\delta$ and expressing the third derivative as $\dfrac{\partial^3 u}{\partial y^3} = f(s, r, M_c, , M_g)\dfrac{\Delta u}{\delta^3}$, the final expression, which will be used in the examples presented in a next section, is obtained :

$$\sigma_\varepsilon = 2F\frac{\sigma_k\left(q-1\right)\dfrac{\tau}{k} - \sigma_k\dfrac{\overline{p'd'}}{\rho k}\dfrac{\delta}{\Delta U} + \dfrac{\delta'_{ref}\Phi K f}{2}}{2q\dfrac{\tau}{k}C_{\varepsilon 2} - F\left(2\dfrac{\tau}{k}C_{\varepsilon 1} + \delta'_{ref}\Phi\right)} \tag{5}$$

In expression (5), $F(M_t)$ and $\overline{p'd'}$ depend on the choice of the compressibility model, so that there are only two free parameters left : q and the product Kf. For subsonic flows, the experiments of Wygnanski and Fiedler [16] suggest for q a value close to 0.7. In the present work, it was found for incompressible mixing layers that a value of 0.75 is well fitted to $k - \varepsilon$ calculations. This value was kept for compressible calculations. $Kf = K(s, r, M_c, M_g) \times f(s, r, M_c, M_g)$ contains the influence of the density on σ_ε, through the parameter s. From the analysis of self-similar mixing layers and from experiments [18],[19], it is reasonable to assume for $M_c \leq 1$ that the product Kf is almost constant with the convective Mach number. As a consequence, the following procedure has been retained. Firstly, incompressible case computations (section 5.2) are used to determine acceptable values of σ_ε to predict the correct spreading rate. The values of Kf are deduced (see table 2). Secondly, compressible case computations (section 5.3) are performed. The compressibility correction proposed in [20] is considered : $\varepsilon_c = \alpha \; M_t^2$, $\alpha = 0.5$ with $F(M_t) = 1 + \alpha M_t^2$ and $\overline{p^{'} d^{'}} = 0$. The value of α has been calibrated on a mixing layer with uniform density ($s = 1$). It will be shown that this value has to be reconsidered for flows with variable density ($s >> 1$ or $s << 1$).

4 Numerical method

In this study, the numerical method is based on a mixed finite element / finite volume spatial approximation, and the MUSCL technique is used to obtain a second order spatial accuracy. The equations (1) can be re-written in a divergence form as :

$$\frac{\partial W}{\partial t} + \nabla . F(W) = \nabla . R(W) + \nabla . S(W) + \Omega(W)$$

where

- $W = (\rho, \rho u, \rho v, E^{'}, \rho k, \rho \varepsilon)^t$ is the vector of the conservative variables and $E^{'}$ denotes the modified total energy :

$$E^{'} = \frac{p^{'}}{\gamma - 1} + \frac{1}{2}\rho(u^2 + v^2) \quad \text{with} \quad p^{'} = p + \frac{2}{3}\rho k$$

$$E = \frac{p}{\gamma - 1} + \frac{1}{2}\rho(u^2 + v^2) + \rho k \quad \Rightarrow \quad E^{'} = E + \beta \rho k \text{ with } \beta = -1 + \frac{2}{3(\gamma - 1)}$$

- F represent the hyperbolic flux functions, R and S are the laminar and turbulent diffusive flux functions, and Ω is the source term due to the k-ε model.

The discrete equations are obtained from a finite-volume scheme applied to hyperbolic fluxes and a finite-element method applied to the diffusive and source terms. For the finite-element method, the terms are evaluated on each triangle with a classical P1-Galerkin formulation. For the finite-volume method, the fluxes are calculated at the interface of two cells (see figure 1). A Roe's approximate Riemann solver is used to integrate the mean flow variables $(\rho, \rho u, \rho v, E^{'})$, while a multi-components flux as proposed by B. Larrouturrou [21] is used for the turbulent ones $(\rho k, \rho \varepsilon)$, in order to preserve positivity.

Such fluxes have a first-order spatial accuracy and a second-order scheme is obtained with the MUSCL approach. The results were obtained on a 4840 point rectangular mesh. Open boundary conditions are approximated by Steger-Warming flux [22] in the inlet and outlet sections. A slip condition ($\vec{V}.\vec{n} = 0$) is applied to the superior and inferior boundaries of the computational domain. An implicit first-order preconditioning time advancing method is employed, and the steady state is considered as being reached when the residual on total energy normalized by the first-step value is 10^{-6}. Technical details on the numerical method employed in this study can be found in [23] or [24]. For each test-case, it has been verified that the computational domain has a mesh refined enough to have a solution independent of the discretization step.

5 Results and discussion

5.1 Validation test-cases

Essentially two kinds of models are examined : the standard $k-\varepsilon$ model (with $\sigma_\varepsilon = 1.3$) and its modified version with variable σ_ε (relation (5)). They are applied to mixing layers with large density gradients, in subsonic and supersonic conditions. The investigated convective Mach numbers range from values close to 0 to near 1. The validation test-cases include velocity/density co-gradients and cross-gradients ($s = 1/7$, $s = 1$ and $s = 7$). Furthermore, the same velocity ratio (a moderate value of $r = 0.65$) was chosen for all the studied flows. Characteristics of the flows are given in table 1.

5.2 Incompressible mixing layer results

The aim of this section is the adaptation of the $k-\varepsilon$ model to low speed mixing layers with variable density; Relation (5), in which Kf is an unknown will be examined. If we consider the peculiarities of the present test cases, expression (5) can be simplified by noting that in this case, $\overline{p'd'} = 0$, $F(M_t) = 1$ and $\Phi(M_c) = 1$. Under these conditions, relation (5) has been estimated by :

$$\sigma_\varepsilon = 1 - \frac{4}{3}Kf \tag{6}$$

The spreading rate $\delta'_{0\,exp}$ is given by the following semi-empirical relation [25].

$$\delta'_{0\,exp} = \frac{0.15}{2} \times \frac{(1 - r)(1 + \sqrt{s})}{(1 + r \times \sqrt{s})} \tag{7}$$

In figure 2, the corresponding curve is referenced by "*experiment*" ; the results of the standard $k-\varepsilon$ model with constant σ_ε (σ_ε=1.3) are denoted by "*k-eps*", and "*rel. compa.*" refers to computations with variable σ_ε deduced from the compatibility relation (6). Figure 2 clearly shows the influence of the density ratio s on the spreading rate, even for these incompressible flows. We can see that the spreading rate is well predicted by the $k-\varepsilon$ model if the density is uniform ($s = 1$). But, performances deteriorate when $s \neq 1$ and the discrepancies on computed δ_0' can reach 20 % . Moreover, the standard $k-\varepsilon$ model predicts the wrong trend for the variations of δ_0' versus s : we remark that the experimental curve indicates an increase of the spatial growth rate with s while the $k-\varepsilon$ model gives the opposite behavior. Values of Kf are determined to minimize the differences, and it can be seen in figure 2 that negligible discrepancies between computations and experiments are now obtained. As a matter of fact, the new σ_ε coefficient forces the incompressible mixing layer to reproduce the subsonic effects, consequently it controls in a consistent way the density influence. In the following, values indicated in table 2 will be utilized. In this table, the value of Kf for $s=1$ has been determined to have σ_ε=1.3. The values for $s=1/7$ and $s=7$ have been obtained numerically to recover the correct variations of the spreading rate. In such conditions, σ_ε ranges from 0.53 (for $s=7$) to 4.33 (for $s=1/7$). As pointed out by Cazalbou *et al.* [26], the regularity of the solutions at the outer edges implies some conditions on σ_k and σ_ε :

$$\begin{cases} \dfrac{\sigma_\varepsilon}{2} < \sigma_k < 2 \\ 2\,\sigma_k - 1 \leq \sigma_\varepsilon \end{cases}$$

The first conditions imply that turbulent quantities vanish at a finite distance, with a balance between transport and diffusion near the edge. The second one is imposed to avoid a velocity profile with an infinite slope at the boundary.

For $s=1/7$, the first condition is not obeyed, while the case $s=7$ violates the second one. However, no particular problem appeared with the shape of the velocity profile at the vicinity of the external flows, perhaps because the mesh size is large in this zone. It may be remarked that this weakness of the proposed model is not inherent to the formulation of the problem itself (compatibility between coefficients and density gradients in a given flow), but is due to an oversimplification in our application. Relations (5) and (6) are valid only in the vicinity of the maximum of k, but with the simple assumptions used in the present work, they were applied in the whole section. In an improved version of the model, an adaptation could be proposed to recover the standard value of σ_ε near the edges, as done in [27] for computations of boundary layers.

5.3 Compressible mixing layer results

5.3.1 Procedure

With the previous results, we have only considered the influence of density on the spreading rate. Now, it has to be coupled to the Mach number effect. To make comparisons with experimental data, we use the normalized spreading rate defined by δ'/δ_0' and the Langley consensus curve $\delta'/\delta_0' = \Phi(M_c)$ which was deduced from a compilation of experimental data [4]. In many numerical works (see [28] or [29]), the incompressible spreading rate δ_0' is obtained from another computation with the same model at a low convective Mach number and keeping the same values of ratios r and s. This value, δ_{0num}', was retained for normalization and $\delta_{num}'/\delta_{0num}'$ was compared to the Langley curve $\Phi(M_c)$. This procedure is correct provided it has been previously checked that δ_{0num}' matches correctly the low speed correlation. It will be shown here that a poor prediction of the low speed flows with variable density can affect the compressible turbulence model and may alter the conclusion on the model efficiency.

5.3.2 Computations with the standard $k-\varepsilon$ model associated with the SEHK corrections

The Langley curve puts in evidence the influence of the convective Mach number on the spreading rate regardless of the other parameters that govern the flow. For a given experiment with spreading rate $\delta^{'}$, in the Langley curve representation, the density gradient influence is confined in the normalization by the incompressible spreading rate $\delta^{'}_{0}$. Therefore, the sole comparison of numerical results with the Langley curve is not sufficient to assess the capability of a model to accurately compute a given mixing layer : the discrepancy with the experiment can be totally hidden in the normalization by the incompressible spreading rate $\delta^{'}_{0}$.
An example of such a mismatch is given in figures 3 and 4. In figure 3, the results of computations with the standard $k-\varepsilon$ model ($\sigma_\varepsilon = 1.3$) associated with $F(M_t) = 1 + 0.5{M_t}^2$ are shown. $\delta^{'}$ is normalized by the value of the incompressible rate of spread given by the same numerical code. The evolution of $\delta^{'}_{0num}$ is not excellent but gives the expected overall trend. Moreover, the same values are found for $s = 7$ and $s = 1/7$. Figure 4 displays the same results with $\delta^{'}$ normalized by $\delta^{'}_{0exp}$. It now appears clearly that the results for $s = 1/7$ and $s = 7$ do not collapse together : Compressibility is no more represented by the convective Mach number alone. Moreover, none of the computed trend is satisfactory and the actual value of the spreading rate can be in error by 30 %. In fact, such a model has serious difficulties to compute accurately high speed mixing layers with density gradient.

5.3.3 Computations with variable σ_ϵ and compressibility corrections.

The case with variable σ_ϵ and compressible turbulence modelling is now examined. Computations were performed with relation (5) into which the values of Kf have been determined by the procedure described in sections 3.2 and 5.2. These computations were associated with a compressibility model of the SEKH type : $F(M_t) = 1 + \alpha M_t^2$. Since now, σ_ε depends on M_c, s and r, it is expected that the value of the constant $\alpha = 0.5$ has to be revised. It was found that this value predicted rather poorly the spreading rate for $s = 1/7$, but that the value $\alpha = 1.66$ gives rather satisfactory results for $\delta^{'}/\delta^{'}_{0}$, see Table 3 and Figure 5. In figure 5, the spreading rate $\delta^{'}$ is referred to $\delta^{'}_{0exp}$ (relation 7); using $\delta^{'}_{0num}$ with this model would have provided practically the same result, since $\delta^{'}_{0exp} \sim \delta^{'}_{0num}$ (figure 2). The determination of α was made for $s = 1/7$. However figure 5 shows that computations for $s = 7$ provide practically the same value for $\Phi(M_c)$: The convective Mach number can be used as a compressibility parameter for the spreading rate as found in the experiments. To sum up this paragraph, it may be underlined that with the present solution a very sound behavior is found : discrepancies on spreading rate remain weak for all the computed Mach numbers and the results obtained for the two density ratios are very close to each other and remain in agreement with the Langley curve.

6 Conclusion

The present work investigates the performances of the $k-\varepsilon$ model in compressible mixing layers with high density variations. It is shown that the standard k-ε model, even if coupled to a compressibility correction, gives poor predictions of spreading rates ; the compressible turbulence modelling can certainly be improved but the deficiencies increase for compressible mixing layers in the presence of large density ratios. To cure this problem, a relationship redefining the σ_ε coefficient has been proposed to take into account both density and compressibility influence. This relation for σ_ε ensures compatibility between the density variations and possible compressibility effects. A basic idea of this study is the necessity to have an efficient model for subsonic flows with density variation, to properly compute supersonic flows. We have shown that when this is achieved, convenient results can be obtained for the computation of mixing layers with high density variation even in supersonic flows. It has also been shown that the values of the constants for the compressible modeling depend also on the density variations and therefore cannot be tuned in flows with constant density.
The present model can certainly be improved by allowing a dependence of the σ_ε coefficient on local variables, instead of defining it in terms of the global characteristics of the mixing layer, or by re-examining the assumptions made to derive the relationship defining σ_ε. However, this work has shown that density gradients effects cannot be ignored in compressible mixing layer and that the compressibility corrections alone are not sufficient to accurately compute these flows in the presence of high density ratio between the two streams.

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7 Figures

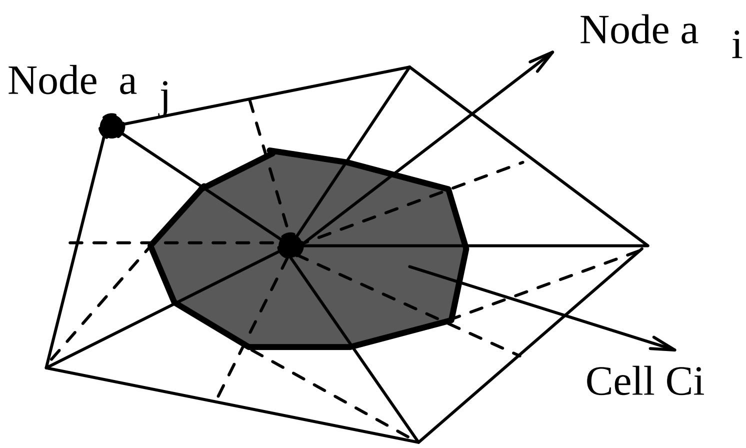

Figure 1: Control cell C_i

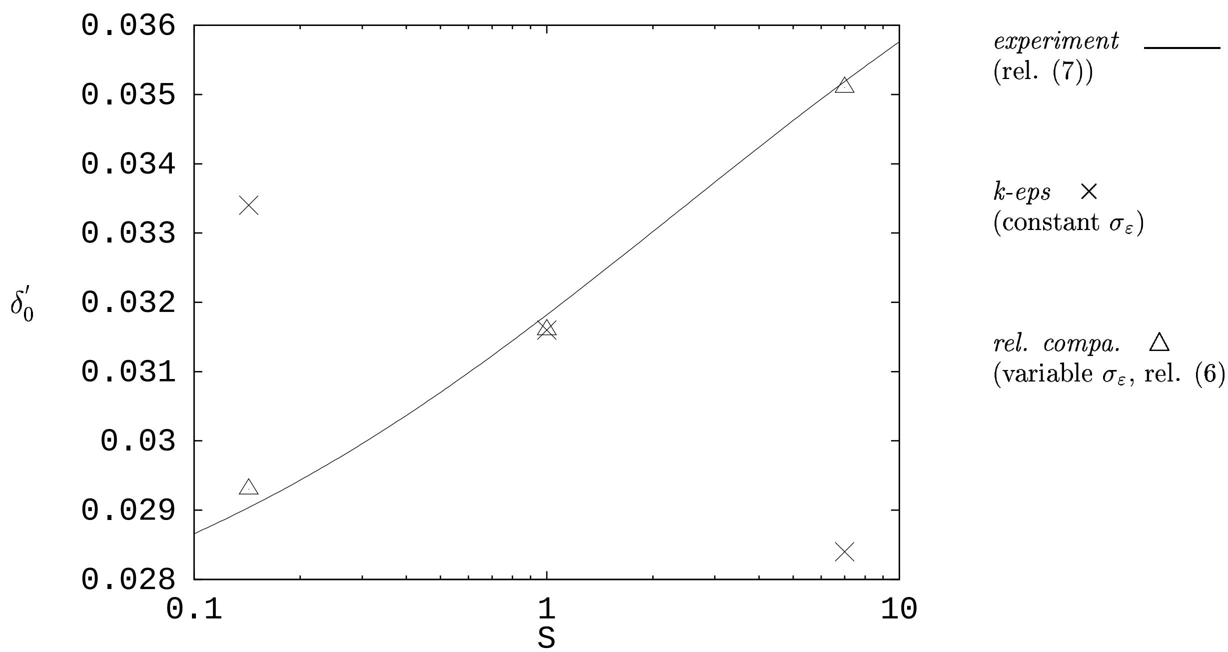

Figure 2: Comparisons of the incompressible spreading rates ($r = 0.65$)

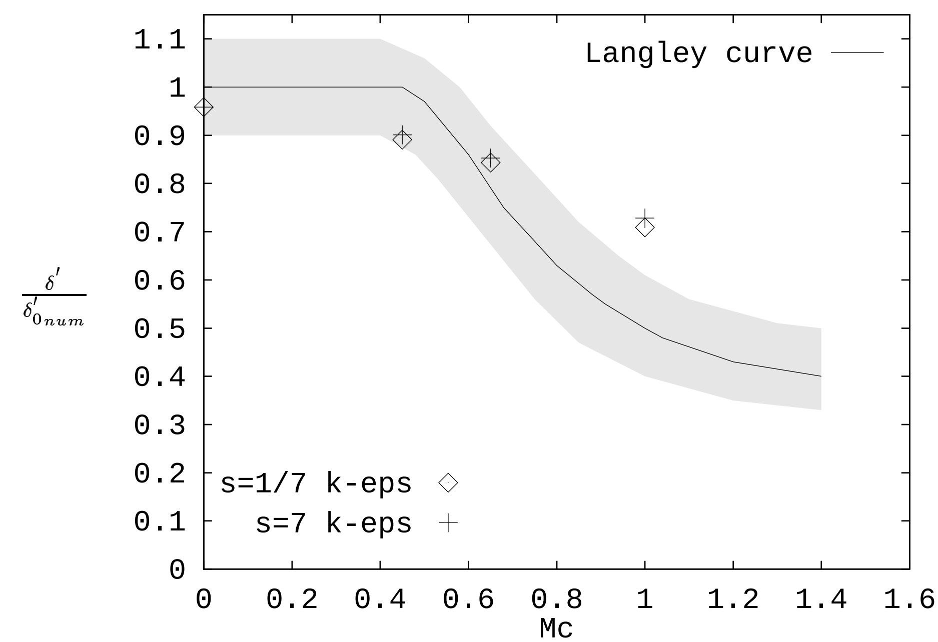

Figure 3: Compressible spreading rates : results of $k-\varepsilon$ model with $\sigma_\varepsilon = 1.3$ and SEHK compressibility model

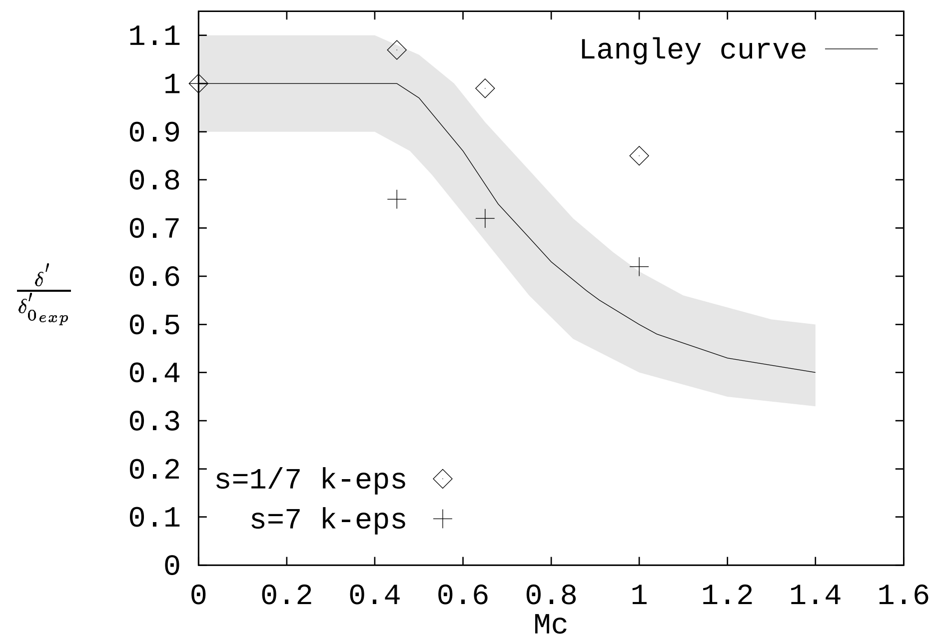

Figure 4: Compressible spreading rates : results of $k - \varepsilon$ model

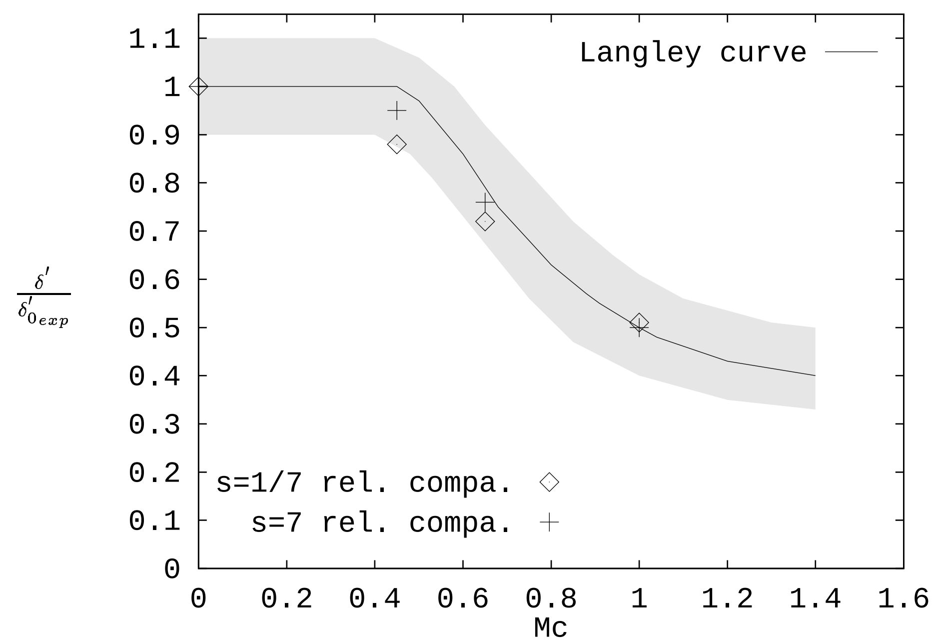

Figure 5: Compressible spreading rates : results of the modified σ_ε with $\alpha = 1.66$ – "rel.compa" indicates that the compatibility relation (5) has been used to determine σ_ε

8 Tables

$M_c \simeq 0$	$M_c = 0.45$	$M_c = 0.65$	$M_c = 1.00$
$s = 1/7$, $s = 1$ & $s = 7$	$s = 1/7$ & $s = 7$	$s = 1/7$ & $s = 7$	$s = 1/7$ & $s = 7$
$r = 0.65$	$r = 0.65$	$r = 0.65$	$r = 0.65$

Table 1: Characteristics of the validation cases for mixing layers

s	1/7	1	7
Kf	-2.5	-0.225	0.35
σ_ε	4.33	1.3	0.53

Table 2: Kf and corresponding σ_ε values versus s

M_c	δ'_{exp} rel. (7)	δ'_{num} (discrepancy in %) *k-eps* (σ_ε=1.3, α=0.5)	*rel. compa.* ($\sigma_\varepsilon \neq 1.3$, α=1.66)
0.45	0.0291	0.0310 (+ 6.5%)	0.0256 (- 12.0%)
0.65	0.0224	0.0287 (+28.1%)	0.0209 (- 6.7%)
1.00	0.0146	0.0248 (+69.9%)	0.0150 (+ 2.7%)

Table 3: s=1/7 : Comparison of the compressible spreading rates

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