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► **To cite this version:**

Hervé Brönnimann, Olivier Devillers. The union of Unit Balls has Quadratic Complexity, even if They all Contain the Origin. RR-3758, INRIA. 1999. <inria-00072904>

**HAL Id: inria-00072904**

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Submitted on 24 May 2006

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*The union of unit balls has quadratic complexity,  
even if they all contain the origin*

Hervé Brönnimann — Olivier Devillers

**N° 3758**

Septembre 1999

THÈME 2



*Rapport  
de recherche*



# The union of unit balls has quadratic complexity, even if they all contain the origin

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Thème 2 — Génie logiciel  
et calcul symbolique  
Projet Prisme

Rapport de recherche n° 3758 — Septembre 1999 — 8 pages

**Abstract:** We provide a lower bound construction showing that the union of unit balls in  $\mathbb{R}^3$  has quadratic complexity, even if they all contain the origin. This settles a conjecture of Sharir.

**Key-words:** Computational geometry, Union of balls, Geometric example

# Une union de boules unités d'intersection non vide a une complexité quadratique

**Résumé :** Nous proposons une construction d'un ensemble de boules de  $\mathbb{R}^3$  dont l'union a un nombre quadratique de faces et d'arêtes. Ces boules ont la particularité d'avoir le même rayon et d'avoir une intersection non vide. Cette note démontre une conjecture énoncé par M. Sharir.

**Mots-clés :** Géométrie algorithmique, Union de boules, Exemple géométrique

## 1 Introduction

The union of a set of  $n$  balls in  $\mathbb{R}^3$  has quadratic complexity  $\Theta(n^2)$ , even if they all have the same radius. All the already known constructions have balls scattered around, however, and Sharir posed the problem whether a quadratic complexity could be achieved if all the balls (of same radius) contained the origin.

In this note, we show a construction of  $n$  unit balls, all containing the origin, whose union has complexity  $\Theta(n^2)$ . As a trivial observation, we observe that the centers are arbitrarily close to the origin in our construction. In fact, if the centers are forced to be at least pairwise  $\varepsilon$  apart, for some constant  $\varepsilon > 0$ , then no more than  $O(\frac{1}{\varepsilon^3})$  can meet in a single point, and hence the union has complexity at most  $O(\frac{1}{\varepsilon^3}n) = O_\varepsilon(n)$ . It is an interesting open question what a condition should be so that the union have subquadratic complexity and yet the balls have arbitrarily close centers.

By contrast, the *intersection* of  $n$  balls can have quadratic complexity if their radii are not constrained, but the complexity is linear if all the radii are the same [2]. Similarly, the convex hull of  $n$  balls can have also quadratic complexity [1], but that complexity is linear if they all have the same radius.

## 2 Construction

Let  $m$  and  $k$  be any integers. We define two families of unit balls: the first consists of  $k$  unit balls whose centers lie on a small vertical segment; the second consists of  $m$  unit balls whose centers lie on a small circle under the segment. (See Figure 3.) We show below that their union has quadratic  $O(km)$  complexity.

**The balls  $B_1 \dots B_k$ .** We denote by  $B(p, r)$  the ball centered at  $p$  and of radius  $r$ . Let  $n = k + m$  and  $P_i$  denote the point of coordinates  $(0, 0, (i - 1)/n^4)$ , and  $B_i = B(P_i, 1)$ , for  $i = 1, \dots, k$ . It is clear that the boundary of  $\cup_{1 \leq i \leq k} B_i$  consists of two hemispheres belonging to  $B_1$  and  $B_k$  linked by a narrow cylinder of height less than  $k/n^4 \leq 1/n^3$ . This cylinder contains all the circles  $\partial B_i \cap \partial B_{i+1}$  for  $i = 1, \dots, k - 1$ . (See Figure 1.)

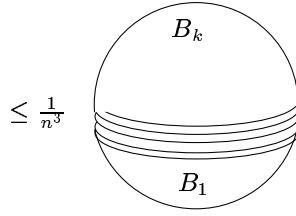


Figure 1: The union  $\cup_{1 \leq i \leq k} B_i$ .

**The balls  $B_{k+1} \dots B_{k+m}$ .** Let  $R$  be the point of coordinates  $(x, 0, z)$  with

$$x = \frac{2n^2 - 4}{n^4}, \quad z = -\frac{2n^2 - 4}{n^3}.$$

(Any values satisfying the constraints  $P_k H < 1$  in (1) and  $\ell < \frac{2}{n}$  in (2) below would do.) We define  $\theta$  as the rotation around the  $z$ -axis of angle  $2\pi/m$ , and for each  $j = 1, \dots, m$ ,  $R_{k+j} = \theta^{j-1}(R)$  and  $B_{k+j} = B(R_{k+j}, 1)$ .

### 3 Analysis

By our choice of  $x$  and  $z$ , we prove below that the boundaries of  $B_{k+1}$  and of the union  $\cup_{i=1}^k B_i$  depicted in Figure 1 meet along a curve  $\gamma$  which satisfies the two claims below. The situation is depicted on Figure 2.

**Claim 1** *The curve  $\gamma$  intersects all the balls  $B_i$  for  $i = 0, \dots, k, \dots$*

**Claim 2** *The portion of  $\gamma$  which does not belong to  $B_1$  (equivalently, which belongs to the union  $\cup_{i=2}^k B_i$ ) is contained in an angular sector of angle at most  $2\pi/m$ .*

From claim 2, we conclude that the portion of  $\gamma$  which does not belong to  $B_1$  is contained in the boundary of the union of the  $n = k + m$  balls. From claim 1, we conclude that the portion of  $\gamma$  which does not belong to  $B_1$  has complexity  $\Omega(k)$ . From claim 2, that it is contained in a small angular sector, hence appears completely on the boundary of the union of the  $n = k + m$  balls, and it is replicated  $m$  times, once for each of the balls  $B_j$ ,  $j = 1, \dots, m$ . It

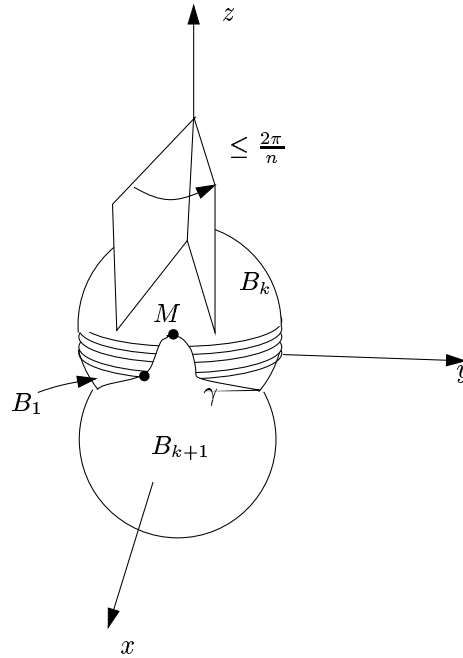


Figure 2: The union  $\bigcup_{1 \leq i \leq k} B_i \cup B_{k+1}$ . The curve  $\gamma$  consists of a portion that belongs to  $B_1$  and of another portion which is contained in a dihedral sector of angle less than  $\pi/m$ .

follows that the union of all the balls  $B_i$  for  $i = 1, \dots, k + m$  has quadratic complexity  $\Omega(km)$ . Moreover, all the balls contain the origin. The union of the  $n$  balls is depicted on Figure 3.

The proofs involve only elementary geometry and trigonometry. The situation is depicted in Figure 4 and 5. Figure 4 depicts a section in the  $xz$ -plane of the spheres  $\partial B_i$  and  $\partial B_{k+1}$  and the point  $M$ , the highest point of intersection of the bounding spheres. The point  $M$  is also depicted on Figure 2.

**Proof of Claim 1.** It suffices to prove that  $M$  is higher than  $P_k$ , since then  $\gamma$  extends higher than  $P_k$  as well and passes through  $M$  by symmetry. The lowest point of  $\gamma$  belongs to  $B_1$  and is clearly below the origin. The two facts together prove that  $\gamma$  must intersect all the balls between  $B_1$  and  $B_k$ .



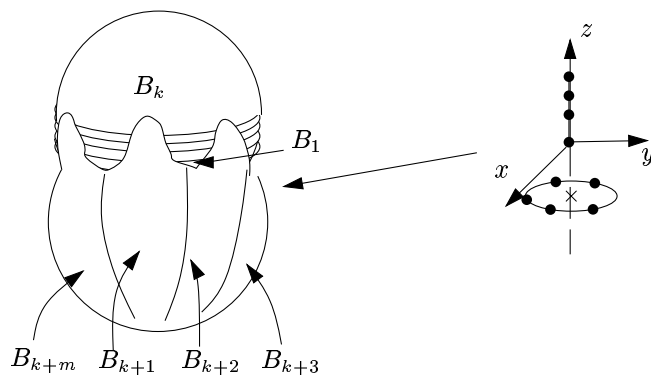


Figure 3: The union  $\cup_{1 \leq i \leq k+m} B_i$ . On the right, a blow-up of the centers.

Let  $H$  be the point in the  $xz$ -plane on the median bisector of  $R$  and  $P_k$ , with same  $z$ -ordinate as  $P_k$ . (See Figure 4.) In order to prove that  $M$  is higher than  $P_k$ , it suffices to prove that  $H$  belongs to  $B_k$ , since then  $M$  is farther along the bisector. The two triangles  $QP_kH$  and  $KRP_k$  have equal angles, hence they are similar. It follows that

$$P_k H = P_k R \frac{P_k Q}{R K} = \frac{P_k R^2}{2 R K} = \frac{x^2 + (z - z_k)^2}{2x}, \quad (1)$$

where  $z_k = \frac{k-1}{n^4}$ . For  $x$  and  $z$  as given in the construction, we have

$$P_k H = 1/16 \frac{-40 n^4 - 15 n^2 + 68 + 16 n^6 - 16 n^3 + 28 n}{n^4 (n^2 - 2)}$$

which is smaller than 1 for  $n \geq 2$ .

**Proof of Claim 2.** It is easy to see that the intersection of  $\gamma$  and a ball  $B_i$  ( $2 \leq i \leq k$ ) consists of at most two arcs of circle, any of which is monotone in angular coordinates around the  $z$ -axis, and that any such arc is entirely above the plane  $z = 0$ . Hence the intersections of  $\gamma$  with the  $xy$ -plane belong to  $B_1$  and  $B_{k+1}$ . It suffices to show that these intersections are at a distance  $\ell$  at most  $\frac{2}{n} \leq \sin \frac{\pi}{m}$  from the  $x$ -axis. (See Figure 5.)

In the  $xy$ -plane section,  $B_1$  is a unit circle, and  $B_{k+1}$  is a circle of radius  $r = \sqrt{1 - z^2}$  and center  $R'$  of coordinates  $(x, 0)$ . (Recall that the center of

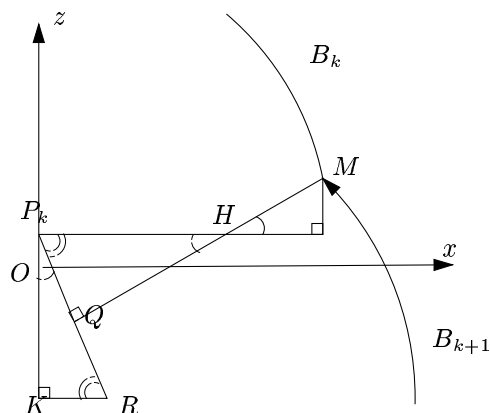


Figure 4: Figure for Claim 1.

$B_{k+1}$  has coordinates  $(x, 0, z)$ .) Hence  $\ell$  is the height of a triangle with base  $x$  and sides 1 and  $r < 1$ . It is elementary to compute that

$$\ell = \sqrt{1 - \left(\frac{z^2 + x^2}{2x}\right)^2}. \quad (2)$$

For our choice of  $x$  and  $z$ , this yields

$$\ell = \sqrt{\frac{2n^6 + 3n^4 - 4n^2 - 4}{n^8}}$$

which is smaller than  $2/n$  for  $n \geq 2$ .

**Acknowledgments.** Thanks to Micha Sharir for pointing out the problem to us. It was also pointed out that Alon Efrat might have a construction which leads to a quadratic lower bound as well. We have derived our construction independently.

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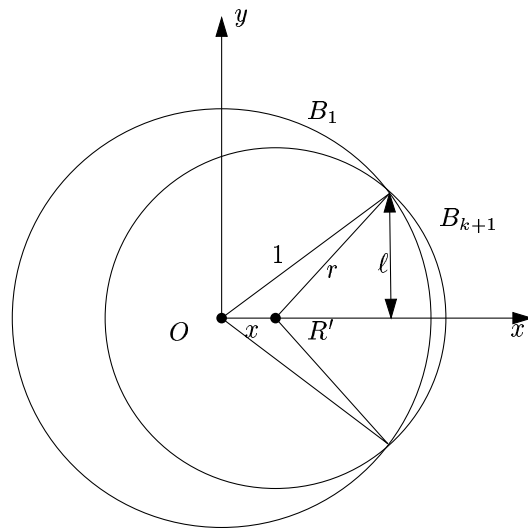


Figure 5: Figure for Claim 2.

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INRIA - Domaine de Voluceau - Rocquencourt, B.P. 105 - 78153 Le Chesnay Cedex (France)  
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ISSN 0249-6399