

# Design of Control Lyapunov Functions for Homogeneous Jurdjevic-Quinn Systems

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*Design of control Lyapunov functions for  
homogeneous Jurdjevic-Quinn systems*

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## Design of control Lyapunov functions for homogeneous Jurdjevic-Quinn systems

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Thème 4 — Simulation et optimisation  
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**Abstract:** This paper presents a method to design explicit control Lyapunov functions for affine and homogeneous systems that satisfy the so-called “Jurdjevic-Quinn conditions”. For these systems a positive definite function  $V_0$  is known that can only be made non increasing by feedback. We describe how a control Lyapunov function can be obtained via a deformation of this “weak” Lyapunov function. Some examples are presented, and the linear quadratic situation is treated as an illustration.

**Key-words:** Feedback stabilization, control Lyapunov function, Lyapunov design.

This paper is submitted to COCV.

## Construction de fonctions de Lyapunov contrôlées pour des systèmes homogènes de type Jurdjevic-Quinn

**Résumé :** Cet article présente une méthode permettant de construire explicitement des fonctions de Lyapunov contrôlées pour des systèmes affines et homogènes vérifiant les conditions dites “de Jurdjevic-Quinn”. Pour de tels systèmes, on connaît une fonction définie positive  $V_0$  et une loi de commande qui fait seulement décroître  $V_0$ , avec une dérivée qui peut s’annuler en certains points. Nous montrons comment il est possible d’obtenir une fonction de Lyapunov contrôlée via une déformation de cette fonction de Lyapunov “au sens large”. Quelques exemples sont présentés, et le cas linéaire quadratique est traité comme illustration.

**Mots-clés :** Stabilisation par feedback, fonction de Lyapunov contrôlée, stabilisation par Lyapunov.

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## 1. INTRODUCTION

In all this paper we consider a control affine system :

$$\dot{x} = f_0(x) + \sum_{k=1}^m u_k f_k(x) , \quad (1)$$

with state  $x \in \mathbb{R}^n$  and control  $(u_1, \dots, u_m) = u \in \mathbb{R}^m$ . The  $f_k$ 's are smooth vector fields in  $\mathbb{R}^n$ . Smooth means of class  $C^\infty$ .

For differentiable dynamical system without control  $\dot{x} = F(x)$ , “Lyapunov’s second theorem” states that existence of a positive definite and radially unbounded function that is decreasing along the trajectories of the system (such a function is called a Lyapunov function) implies asymptotic stability of its minimum. Asymptotic stability can therefore be proved by exhibiting such a function, this is the so-called “Lyapunov direct method” (see [11]). The scope of this method is general since the existence of a Lyapunov function is also necessary, see [15] for a proof with a right hand side that is only assumed to be continuous. However an unfortunate drawback is that there is no systematic way to build explicitly this Lyapunov function.

For control systems, the “Lyapunov design” of a stabilizing control law (see [3]), based on Lyapunov direct method, consists in designing a positive definite and radially unbounded function together with a continuous feedback law that assigns this function to be a Lyapunov function for the closed-loop system. Artstein’s theorem (see [2, 23, 3]) points out exactly which relations (control Lyapunov function, small control property, see below) a function has to satisfy in order to allow existence of a feedback law that makes it decrease.

**Definition 1.1** (assignable functions). Let  $V$  be a positive definite and radially unbounded function. We say that a continuous  $u : \mathbb{R}^n \mapsto \mathbb{R}^m$  assigns  $V$  to be a Lyapunov function for the closed loop system (1) if the derivative of  $V$  along the trajectories of the closed loop

system is negative definite, that means that for all  $x$  in  $\mathbb{R}^n \setminus \{0\}$  :

$$L_{f_0} V(x) + \sum_{k=1}^m u_k L_{f_k} V(x) < 0 .$$

**Theorem 1.2** (Artstein's Theorem). *Let  $V$  be a positive definite and radially unbounded function . There exists a continuous feedback that assigns  $V$  to be a Lyapunov function for the closed loop system (1) if and only if*

(a) *it is a control Lyapunov function (CLF) :*

$$\forall x \in \mathbb{R}^n \setminus \{0\}, \left. \begin{array}{l} L_{f_1} V(x) = 0 \\ \vdots \\ L_{f_m} V(x) = 0 \end{array} \right\} \implies L_{f_0} V(x) < 0 ; \quad (2)$$

(b) *it satisfies the small control property (SCP): for any  $\epsilon > 0$ , there is a  $\delta > 0$  such that for all  $x \in \mathbb{R}^n \setminus \{0\}$ ,*

$$\|x\| < \delta \implies \exists u \left\{ \begin{array}{l} \|u\| < \epsilon , \\ L_{f_0 + \sum_{k=1}^m u_k f_k} V(x) < 0 . \end{array} \right. \quad (3)$$

This is an existence result, but some explicit formulae have been given, like for instance :

**Proposition 1.3** (Sontag's Formula [23]). *If the function  $V$  is a control Lyapunov function that satisfies the small control property for the affine control system (1) then the control  $u = (u_1, \dots, u_m)$  defined by*

$$u_k = \begin{cases} -\frac{a + \sqrt{a^2 + (\sum_{i=1}^m b_i^2)^2}}{\sum_{i=1}^m b_i^2} b_k & \text{if } b_k \neq 0 , \\ 0 & \text{if } b_k = 0 . \end{cases} \quad (4)$$

where  $a = L_{f_0} V(x)$  and  $b_k = L_{f_k} V(x)$  for all  $k$  in  $\{1 \dots m\}$ , is a continuous and asymptotically stabilizing feedback law. Besides it assigns  $V$  to be a Lyapunov function.

In this paper we will not refer to this general expression because we will restrict our attention to homogeneous systems. In this particular case a simpler construction can be made (see Proposition 2.5 below) that moreover keeps the homogeneity of the closed loop system.

Many others methods than building a control Lyapunov function exists in the literature to design stabilizing feedback law. However, it is well known that a control Lyapunov function, when it is available, is a convenient tool for many reasons: for instance, the negative definite function  $\dot{V}$  allows one to quantify robustness to model errors or perturbations in the control,

see [7, section 3.3]. Also, it can be used to perform Lyapunov redesign to enhance robustness properties, see [14, section 5.5]. Finally, some simple back-stepping-like techniques for adding an integrator require a strict Lyapunov function (see [7]).

For these reasons it is interesting to obtain control Lyapunov functions for systems that may be asymptotically stabilized by other methods. Of course, the converse Lyapunov theorems (see for instance [15,11]) imply the *existence* of functions that are strictly decreasing along the solutions of the closed loop systems and finally control Lyapunov functions that are assigned by the same control law. However these theorems are far from giving an explicit construction of these functions. This work is an attempt to give an explicit design of control Lyapunov functions as precisely as possible when asymptotic stabilization can be solved via the so called “Jurdjevic-Quinn method”, also called “damping control”. A preliminary version was presented in [8].

Note that for linear systems, the above mentioned converse Lyapunov theorems can be made constructive, and yield a quadratic Lyapunov function, see section 6 for a discussion. For some special cases of nonlinear systems, explicit control Lyapunov functions have already been obtained in the literature, based on this quadratic Lyapunov function for the linear approximation : in [17, proof of lemma 2] for linear systems with saturations and in [18, proof of corollary 2.16] for cascaded systems with some growth assumptions. The present paper deals with more general affine control systems, although these results are not *stricco sensu* a consequence of ours since we assume homogeneity.

The paper is organized as follows. Section 2 is devoted to introduce some classical notations and properties about differential geometry and homogeneity. After making precise the class of systems we deal with in section 3, the result consisting in reshaping a Lyapunov function in the direction of a vector field is presented in section 4. After some comments on the result in section 5, we pay attention to three examples in the last sections: the linear case is studied in section 6 as a particular situation of the results, in section 7 the problem of stabilization of an underactuated rigid body is considered and in section 8 we deal with the so-called TORA system. Although this last system does not satisfy all the hypotheses of our results, the method leads to a control Lyapunov function for this system.

## 2. SOME NOTATIONS AND PROPERTIES ABOUT VECTOR FIELDS AND HOMOGENEITY

Let us precise our notations and recall basic properties. For an entire overview, the reader can refer to [13] and [24].

**Definition 2.1** (Flow of a vector field). Let  $X$  be a vector field on  $\mathbb{R}^n$ . We define  $\phi_X^t(x)$  by :

$$\begin{aligned} \frac{\partial}{\partial t}(\phi_X^t(x)) &= X(\phi_X^t(x)) , \\ \phi_X^0 &= x . \end{aligned} \tag{5}$$

In general,  $(t, x) \mapsto \phi_X^t(x)$  is defined on a neighborhood of  $\{t = 0\}$  in  $\mathbb{R}^{n+1}$ . If it is defined for all  $(t, x)$  in  $\mathbb{R}^{n+1}$ , the vector field  $X$  is called *complete*. For a fixed choice of  $x$ ,  $t \mapsto \phi_X^t(x)$



is a solution of  $\dot{x} = X(x)$ . For a fixed  $t$ ,  $x \mapsto \phi_X^t(x)$  is called the *flow* of  $X$  at time  $t$ , it is a diffeomorphism from an open set of  $\mathbb{R}^n$  onto its image (from  $\mathbb{R}^n$  onto  $\mathbb{R}^n$  if  $X$  is complete).

For any function  $h$ , we denote by  $L_X h$  its Lie derivative along  $X$ :

$$L_X h(x) = \left. \frac{d}{dt} \right|_{t=0} h(\phi_X^t(x)) . \quad (6)$$

**Proposition 2.2.** *Let  $X$  be a vector field on  $\mathbb{R}^n$  and  $\varphi$  a diffeomorphism onto  $\mathbb{R}^n$ . Then for all  $t$ ,  $\varphi \circ \phi_X^t \circ \varphi^{-1}$  is the flow at time  $t$  of a vector field denoted by  $\varphi * X$  and called the conjugate of  $X$  by  $\varphi$  :*

$$\phi_{\varphi * X}^t = \varphi \circ \phi_X^t \circ \varphi^{-1} . \quad (7)$$

In coordinates, the expression of  $\varphi * X$  is given by :

$$\varphi * X(y) = \varphi'(\varphi^{-1}(y))X(\varphi^{-1}(y)) . \quad (8)$$

In the sequel we will also use the notion of homogeneity. Let us define precisely what we mean by homogeneous function and homogeneous vector field.

**Definition 2.3.** For a fixed choice of coordinates on  $\mathbb{R}^n$ , and for  $n$  positive real numbers  $r_1, \dots, r_n$ ,

- (a) a dilation of weight  $r = (r_1, \dots, r_n)$  onto  $\mathbb{R}^n$  is a family of maps  $(\delta_\mu^r)$  defined, for all  $\mu > 0$  and for all  $x$  in  $\mathbb{R}^n$ , by :

$$\delta_\mu^r(x) = (\mu^{r_1}x_1, \dots, \mu^{r_n}x_n) ;$$

- (b) a function  $V : \mathbb{R}^n \mapsto \mathbb{R}$  is *homogeneous of degree  $\tau$*  with respect to the family of dilations  $(\delta_\mu^r)$  if and only if for all  $\mu > 0$ ,

$$V \circ \delta_\mu^r = \mu^\tau V ;$$

- (c) a vector field  $X$  is *homogeneous of degree  $\sigma$*  with respect to the family of dilations  $(\delta_\mu^r)$  if and only if for all  $\mu > 0$ ,

$$\delta_\mu^r * X = \mu^\sigma X . \quad (9)$$

In the coordinates of the dilation,  $X$  can be written  $X = \sum f_i \frac{\partial}{\partial x_i}$ , and  $X$  is homogeneous of degree  $\sigma$ , if and only if, each  $f_i$  is a homogeneous function of degree  $\sigma + r_i$ .

- (d) In the sequel  $\|\cdot\|$  denotes a function defined on  $\mathbb{R}^n$ , for a sufficient large  $p$ , by :

$$\|x\| = \left( \sum_{j=1}^n |x_j|^{\frac{p}{r_j}} \right)^{\frac{1}{p}}$$

This positive definite function is smooth outside the origin and homogeneous of degree 1 with respect to the family of dilations  $(\delta_\mu^r)$ . Such a function is called a *homogeneous norm*.

In the sequel  $r$  and the family of dilations  $(\delta_\mu^r)$  are fixed. “Homogeneous of degree  $d$ ” will replace “homogeneous of degree  $d$  with respect to  $(\delta_\mu^r)$ ”.

**Lemma 2.4.** *Let  $G$  be a complete vector field on  $\mathbb{R}^n$ .  $G$  is homogeneous of degree 0 if and only if the flow of  $G$  commutes with the family of dilations, i.e for all  $t$  in  $\mathbb{R}$  and all positive  $\mu$ ,*

$$\delta_\mu^r \circ \phi_G^t = \phi_G^t \circ \delta_\mu^r . \quad (10)$$

*Proof.* from (7) we have for all positive  $\mu$  and all  $t$  in  $\mathbb{R}$  :

$$\delta_\mu^r \circ \phi_G^t \circ (\delta_\mu^r)^{-1} = \phi_{\delta_\mu^r * G}^t .$$

Hence, from (9),  $G$  is homogeneous of degree 0 if and only if  $\delta_\mu^r \circ \phi_G^t \circ (\delta_\mu^r)^{-1} = \phi_G^t$  i.e. if and only if  $\phi_G^t$  and  $\delta_\mu^r$  commute.  $\square$

When all the vector fields  $f_i$  of the system (1) are homogeneous, and when a homogeneous control Lyapunov function is known for this system, the following result, proved in [3], gives a feedback control, simpler than Sontag’s one, that assigns  $V$  to be a Lyapunov function for the closed loop system. Moreover, this feedback law has the good property to keep the homogeneity of the closed loop system, that means that the right-hand side of the closed-loop system  $\dot{x} = f_0(x) + \sum_1^m u_k(x)f_k(x)$ , is homogeneous of the same degree as  $f_0$ .

**Proposition 2.5.** *Consider the affine control system (1). Assume that each  $f_k$  ( $k = 0 \dots m$ ) is a vector field homogeneous of degree  $c_k$ , with  $c_0 \geq 0$  and  $c_k < c_0$  ( $k = 1 \dots m$ ). Assume as well that  $V$  is a control Lyapunov function for the system, homogeneous of degree  $d$ . Then the feedback control, defined for all  $x$  in  $\mathbb{R}^n$  by :*

$$u_k(x) = -\alpha L_{f_k} V(x) \|x\|^{c_0 - 2c_k - d} \quad (11)$$

*is homogeneous of degree  $c_0 - c_k$  ( $k = 1 \dots m$ ). This makes the right-hand side of the closed-loop system  $\dot{x} = f_0(x) + \sum_{k=1}^m u_k(x)f_k(x)$  homogeneous of degree  $c_0$ , and assigns the control Lyapunov function  $V$  to be strictly decreasing if,  $T$  being the set  $\{x, \|x\| = 1, L_{f_0} V(x) \geq 0\}$ ,*

$$\alpha > \sup_{x \in T} \frac{L_{f_0} V(x)}{\sum_{k=1}^m ((L_{f_k} V(x))^2 \|x\|^{c_0 - 2c_k - d})} . \quad (12)$$

### 3. PROBLEM STATEMENT

In [12], Jurdjevic and Quinn noticed that a first integral of the drift vector field plus some controllability conditions allow one to derive smooth asymptotically stabilizing control laws. This method has been next generalized in many papers, as [10, 3, 20]. In this paper we call “Jurdjevic-Quinn type” systems, these that allow the use of this stabilization method.

Let us give the specific assumption we need here. Consider the affine control system (1). For a function  $V_0$  and for an integer  $L$ , define the set  $\mathcal{W}_L(V_0)$  to be :

$$\mathcal{W}_L(V_0) = \left\{ \begin{array}{l} x \in \mathbb{R}^n, L_{f_0} V_0(x) = L_{ad_{f_0}^i(f_k)} V_0(x) = 0 \\ k = 1 \dots m; i = 0 \dots L \end{array} \right\} . \quad (13)$$

Note that this set decreases as  $L$  increases.

**Assumption 3.1** (Jurdjevic-Quinn conditions). *We assume that a function  $V_0 : \mathbb{R}^n \rightarrow \mathbb{R}$  is known and has the following three properties : it is smooth, positive definite and radially unbounded ; it satisfies :*

$$\forall x \in \mathbb{R}^n, L_{f_0} V_0(x) \leq 0 ; \quad (14)$$

*it is such that there is an integer  $L$  such that*

$$\mathcal{W}_L(V_0) \text{ is reduced to } \{0\} . \quad (15)$$

**Remark 3.2.** For many control systems (see the examples in section 7 and 8), the natural drift vector field does not meet (14), but a preliminary feedback, by adding to it a linear combination of the control vector fields, gives an  $f_0$  satisfying (14). Further comments on the consequence of this preliminary feedback are given in section 5.

Following [12, 10, 20, 3], one may set :

$$u_k(x) = -L_{f_k} V_0(x) \quad (k = 1, \dots, m) . \quad (16)$$

The derivative of  $V_0$  along the closed-loop system is then given by,

$$\dot{V}_0 = L_{f_0} V_0 - \sum_{k=1}^m (L_{f_k} V_0)^2 .$$

This equation shows that  $\dot{V}_0$  is non positive and vanishes at points where the  $L_{f_k} V_0$  ( $k = 0 \dots m$ ) vanish. If such points exist, condition (15) together with LaSalle Invariance principle (see [11, 16]) allow one to conclude to asymptotic stability of the origin :

**Proposition 3.3.** *If system (1) satisfies assumption 3.1, then formula (16) defines a continuous asymptotically stabilizing feedback control law.*

However,  $V_0$  is clearly not a control Lyapunov function, because there are in general some points where all the  $L_{f_k} V_0$  ( $k = 0 \dots m$ ) vanish. In the sequel, we “re-shape” this function to build a control Lyapunov function.

**Remark 3.4.** In [18, proof of corollary 2.16], the case where  $f_0$  is a linear vector field and where the control vector fields depends on the control (systems are non affine in the control but it has a particular structure) is studied. A positive definite and radially unbounded function together with a feedback law that assigns it to be a strict Lyapunov function is obtained following the idea of [17, proof of lemma 2]. This explicit construction consists in finding a Lyapunov function for the linear approximation (via Lyapunov equation for instance, see section 6 equation (31)), and making a combination of this function and the above defined  $V_0$ . It is not clear how this approach might be generalized. We do not follow this track here.

**Remark 3.5.** Let us clarify the status of our assumptions with respect to the literature (and with the help of [21]). Denoting by  $\mathcal{L}$  the union of all the trajectories of  $\dot{x} = f_0(x)$  that remain in the set  $\{x \in \mathbb{R}^n, L_{f_k} V_0(x) = 0, k = 0, \dots, m\}$ , LaSalle’s invariance principle states that  $\mathcal{L} = \{0\}$  is a necessary and sufficient conditions for (16) to be an asymptotically stabilizing control law. All the assumptions made in the literature may be viewed as sufficient conditions for  $\mathcal{L} = \{0\}$ .

In the early works [12, 10], some simpler and more restrictive assumptions were made instead of assumption 3.1: (14) was replaced by  $L_{f_0} V_0 \equiv 0$  ( $V_0$  is a first integral of the drift vector field) and (15) was replaced by the “ad-condition” ( $rk\{ad_{f_0}^i f_k, 1 \leq k \leq m, i \geq 0\} = n$ ) plus the origin being the only critical point of  $V_0$ . These assumptions are rather restrictive, but they have the advantage of “decoupling” the properties of the control vector fields and these of  $f_0$ .

There has then been an effort to weaken these assumptions. By differentiation along the trajectories, it is clear that the set

$$\mathcal{W}_{gen} = \left\{ x \in \mathbb{R}^n, L_{f_0}^{i+1} V_0(x) = L_{f_0}^i L_{f_k} V_0(x) = 0 \right. \\ \left. k = 1 \dots m; i \in \mathbb{N} \right\},$$

contains  $\mathcal{L}$ , defined above, and in fact they coincide for real analytic vector fields. In [20] the stabilization result is stated under the condition  $\mathcal{W}_{gen} = \{0\}$ . It is noted there that, with

$$\mathcal{W}'_{gen} = \left\{ x \in \mathbb{R}^n, L_{f_0}^{i+1} V_0(x) = L_{ad_{f_0}^i(f_k)} V_0(x) = 0 \right. \\ \left. k = 1 \dots m; i \in \mathbb{N} \right\},$$

the condition  $\mathcal{W}_{gen} = \{0\}$  is equivalent to  $\mathcal{W}'_{gen} = \{0\}$ . Note that in general (without further assumption like  $L_{f_0} V_0$  identically zero)  $L_{ad_{f_0}^i(f_k)} V_0(x) = L_{f_0}^i L_{f_k} V_0(x)$  is true only for  $x$  in  $\mathcal{L}$ , so that the sets  $\mathcal{W}_{gen}$  and  $\mathcal{W}'_{gen}$  might not coincide if they are not reduced to zero.

The set  $\mathcal{W}_L(V_0)$  contains obviously  $\mathcal{W}'_{gen}$  and so our condition is stronger. However, in the very common situation where the drift vector field is conservative ( $L_{f_0} V_0 \equiv 0$ ), the knowledge of the integer  $L$  is our only restriction.

#### 4. RESHAPING LYAPUNOV FUNCTIONS VIA THE FLOW OF A VECTOR FIELD

For a complete vector field  $G$ , we define a function  $V_\lambda$  from  $V_0$  as follows :

$$V_\lambda = V_0 \circ \phi_G^\lambda, \quad (17)$$

where  $\phi_G^\lambda$  is the flow of the vector field  $G$ , see (5). We shall always consider a vector field  $G$  that is smooth except possibly at  $x = 0$ , with  $G(0) = 0$ .

In general, it is impossible to give an explicit expression of the flow  $\phi_G^\lambda$  with some usual functions. For the purpose of making our method really constructive, we may use for small  $\lambda$  (all our results will anyway be valid for small  $\lambda$  only), a function  $W_\lambda$  that is easier to compute and agrees with  $V_\lambda$  up to order 1 with respect to  $\lambda$  :

$$W_\lambda(x) = V_\lambda(x) + \lambda^2 a(\lambda, x), \quad (18)$$

where  $a$  is a smooth function on  $[0, +\infty) \times \mathbb{R}^n$  that satisfies for all positive  $\lambda$ ,  $a(\lambda, 0) = 0$ . A straightforward computation will convince the reader that the following  $W_\lambda$  can be put in the above form :

$$W_\lambda(x) = V_0(x + \lambda G(x)). \quad (19)$$

Note that this formula is explicit (does not involve solving any differential or algebraic equation) and that it depends on the chosen coordinates whereas the definition of  $V_\lambda$  in (17) is coordinate-free.

The function  $V_\lambda$  is identically equal to  $V_0$  when  $\lambda$  is equal to zero. Since  $L_{f_0} V_0$  vanishes at some points where  $L_{f_k} V_0$  ( $k = 1 \dots m$ ) vanish, and we want  $L_{f_0} V_\lambda$  to be negative at points where  $L_{f_k} V_\lambda$  ( $k = 1 \dots m$ ) vanish, it is natural to require that  $L_{f_0} V_\lambda$  be a decreasing function of  $\lambda$  for small values of  $\lambda$  and at points where  $L_{f_k} V_0$  ( $k = 1 \dots m$ ) vanish. we require the following slightly stronger property :

$$\forall x \in \mathbb{R}^n \setminus \{0\}, \left. \begin{array}{l} L_{f_0} V_0(x) = 0 \\ L_{f_1} V_0(x) = 0 \\ \vdots \\ L_{f_m} V_0(x) = 0 \end{array} \right\} \implies \left. \frac{d}{d\lambda} \right|_{\lambda=0} (L_{f_0} V_\lambda(x)) < 0 \quad (20)$$

The same discussion holds replacing  $V_\lambda$  with  $W_\lambda$  satisfying (18).

Reshaping  $V_0$  via the flow of a vector field  $G$  has the interesting advantage that it replaces the above difficult condition on  $V_\lambda$  given by equation (20) by a simpler condition on  $G$  as shown by the following lemma :

**Lemma 4.1.** For function  $V_\lambda$  and  $W_\lambda$  defined by (17) and (18), we have for all  $x \in \mathbb{R}$ ,

$$\left. \frac{d}{d\lambda} \right|_{\lambda=0} (L_{f_0} V_\lambda)(x) = \left. \frac{d}{d\lambda} \right|_{\lambda=0} (L_{f_0} W_\lambda)(x) = L_{f_0} L_G V_0(x) \quad (21)$$

*Proof.* Let the function  $\chi$  be defined by  $\chi(t, \lambda, x) = V_0(\phi_G^\lambda(\phi_{f_0}^t(x)))$ . From (6) and the definition (17) of  $V_\lambda$ , the left-hand side of (21) is equal to  $\frac{\partial^2 \chi}{\partial \lambda \partial t}(0, 0, x)$ . From (18), the expression with  $W_\lambda$  instead of  $V_\lambda$  is also equal to  $\frac{\partial^2 \chi}{\partial t \partial \lambda}(0, 0, x)$ . On the other hand, applying (6) twice (with  $X = G$  and  $X = f_0$ ), the right-hand side is equal to  $\frac{\partial^2 \chi}{\partial t \partial \lambda}(0, 0, x)$ . This proves (21) because the partial derivatives commute.  $\square$

Unfortunately, we are not able to prove that condition (20) is sufficient, in general, for  $V_\lambda$  to be a control Lyapunov function. This is however true for control affine systems which have some homogeneity properties :

**Theorem 4.2.** Assume that the control affine system (1) meets the Jurdjevic-Quinn conditions (assumption 3.1) with a function  $V_0$  that is homogeneous of degree  $d$ . Besides, assume that each  $f_k$  ( $k = 0 \dots m$ ) is homogeneous of degree  $c_k$  with  $c_k \geq 0$  and  $c_k < c_0$  ( $k = 1 \dots m$ ).

Let  $G$  be an homogeneous vector field of degree 0 which satisfies, for all  $x$  in  $\mathbb{R}^n \setminus \{0\}$ ,

$$\left. \begin{array}{l} L_{f_0} V_0(x) = 0 \\ L_{f_1} V_0(x) = 0 \\ \vdots \\ L_{f_m} V_0(x) = 0 \end{array} \right\} \implies L_{f_0} L_G V_0(x) < 0 \quad (22)$$

Then there exists a positive real  $\lambda_0$  such that for all  $\lambda$  satisfying  $0 < \lambda < \lambda_0$ ,  $V_\lambda$  is a homogeneous control Lyapunov function of degree  $d$ . The same property is satisfied by  $W_\lambda$  given by (19) in the coordinates of the dilation. Any  $W_\lambda$  that agrees with (18) also satisfies this property provided that it is homogeneous and definite positive.

Of course, in order to give an explicit method to build a control Lyapunov function, the previous result is not sufficient: one still needs to find a vector field  $G$  satisfying (22). The following result gives a “universal” way of designing such a vector field. Note that the above homogeneity requirement on  $G$  makes it difficult to construct such a vector field smooth at the origin.

**Theorem 4.3.** Assume that the control affine system (1) meets the Jurdjevic-Quinn conditions (assumption 3.1) with a function  $V_0$  that is homogeneous of degree  $d$ . Besides, assume that each  $f_k$  ( $k = 0 \dots m$ ) is homogeneous of degree  $c_k$  with  $c_k \geq 0$  and  $c_k < c_0$  ( $k = 1 \dots m$ ).

Then the vector field  $G$  defined by (23) and (24) is homogeneous of degree 0 and satisfies the condition (22) :

$$G = \sum_{i=0}^{L-1} \sum_{k=1}^m \lambda_{i,k} a d_{f_0}^i f_k, \quad (23)$$

with  $\lambda_{i,k}$  ( $i = 0, \dots, L-1; k = 1, \dots, m$ ) some functions defined by :

$$\begin{cases} \lambda_{i,k} &= \sum_{j=i}^{L-1} (-1)^{j-i+1} \frac{L_{ad_{f_0}^{(2j-i+1)}(f_k)} V_0}{(2V_0)^{\alpha_{j,k}}} , \\ \alpha_{j,k} &= \frac{(2j+1)c_0 + 2c_k + d}{2} . \end{cases} \quad (24)$$

Note that there are of course many other vector fields  $G$  satisfying (22), this is discussed in section 5.

The following lemma is used in the proof of theorem 4.2, but has some interest in itself.

**Lemma 4.4.** *If  $V_0$  is homogeneous of degree  $d$  and the vector field  $G$  is homogeneous of degree zero then, for all  $\lambda \geq 0$ , the functions  $V_\lambda$  defined by (17) and  $W_\lambda$  defined by (19) are homogeneous of degree  $d$ .*

Of course, for a general  $W_\lambda$  satisfying (18), homogeneity is not granted except if, for  $\lambda \leq 0$ ,  $x \mapsto a(\lambda, x)$  is homogeneous of the same degree as  $V_0$ .

*Proof.* Since  $G$  is homogeneous of degree 0, lemma 2.4 implies that  $\delta_\mu^r$  commutes with the flow of the vector field  $G$  at any positive time  $\lambda$ . Then for all  $\mu > 0$  and  $\lambda \geq 0$ ,

$$V_\lambda \circ \delta_\mu^r = V_0 \circ \phi_G^\lambda \circ \delta_\mu^r = V_0 \circ \delta_\mu^r \circ \phi_G^\lambda = \lambda^d V_0 \circ \phi_G^\lambda = \mu^d V_\lambda .$$

This precisely means that  $V_\lambda$  is homogeneous of degree  $d$ . The function  $W_\lambda = V_0 \circ (Id + \lambda G)$  is also homogeneous of degree  $d$  because :

$$W_\lambda \circ \delta_\mu^r = V_0 \circ (\delta_\mu^r + \lambda G \circ \delta_\mu^r) = V_0 \circ \delta_\mu^r \circ (Id + \lambda G) = \mu^d V_0 \circ (Id + \lambda G) .$$

This ends the proof of the lemma. □

*Proof of theorem 4.2.* We are going to give the proof that  $V_\lambda$  is a control Lyapunov function. The proof that  $W_\lambda$  satisfies the same property follows exactly along the same lines because only the properties of lemma 4.1 and 4.4 are used.

From lemma 4.4 the functions  $V_\lambda$  are homogeneous of degree  $d$ . Since each vector field  $f_k$  ( $k = 0 \dots m$ ) is homogeneous of degree  $c_k$ , each function  $L_{f_k} V_\lambda$  ( $k = 0 \dots m$ ) is homogeneous of degree  $c_k + d$ . Let

$$S = \{y \in \mathbb{R}^n, \|y\| = 1\}$$

be the homogeneous unit sphere. For any  $x \in \mathbb{R}^n \setminus \{0\}$ , there exists  $y \in S$  and a real number  $\mu > 0$  such that  $x = \delta_\mu(y)$ . Consequently, the homogeneity of each  $L_{f_k} V_\lambda$  implies that it is equivalent to prove that for all  $x \in \mathbb{R}^n$ ,  $L_{f_0} V_\lambda(x)$  is negative when each  $L_{f_k} V_\lambda(x)$  ( $k = 1 \dots m$ ) vanishes and to prove this property only for  $x$  in  $S$ .

To simplify the proof of this fact, let us introduce some notations :

$$\begin{aligned}
E &= \{x \in S, L_{f_0} L_G V_0(x) \geq 0\}, \\
F &= \{x \in S, L_{f_0} V_0(x) = 0\}, \\
\Gamma &= \{x \in S, L_{f_1} V_0(x) = \dots = L_{f_m} V_0(x) = 0\}, \\
\Gamma_\lambda &= \{x \in S, L_{f_1} V_\lambda(x) = \dots = L_{f_m} V_\lambda(x) = 0\}, \\
\Gamma' &= \Gamma \cap F = \{x \in S, L_{f_0} V_0(x) = L_{f_1} V_0(x) = \dots = L_{f_m} V_0(x) = 0\}.
\end{aligned}$$

With these notations, (22) is equivalent to

$$\Gamma' \cap E = \emptyset, \quad (25)$$

and from the above considerations,  $V_\lambda$  is a control Lyapunov function if and only if  $L_{f_0} V_\lambda$  is negative on the set  $\Gamma_\lambda$ .

Compactness of  $S$  and continuity of  $(x, \lambda) \mapsto L_{f_k} V_\lambda$  imply that any neighborhood of  $\Gamma$  in  $S$ , say for instance

$$N_q(\Gamma) = \{x \in S, d(x, \Gamma) \leq q\}, \quad q > 0,$$

contains the sets  $\Gamma_\lambda$ , for  $\lambda$  small enough (If not, there would exist a sequence outside that neighborhood that would converge to a point in  $\Gamma$ ). More precisely for any  $q > 0$  there exists  $\lambda_2(q) > 0$  s.t.,

$$0 < \lambda < \lambda_2(q), \Rightarrow \Gamma_\lambda \subset N_q(\Gamma). \quad (26)$$

To end the proof, it remains to prove that there exists two positive real numbers  $\lambda_0$  and  $q$  such that,

$$\left. \begin{array}{l} 0 < \lambda < \lambda_0 \\ x \in N_q(\Gamma) \end{array} \right\} \Rightarrow L_{f_0} V_\lambda(x) < 0, \quad (27)$$

First, note that from (25),  $\Gamma'$  and  $E$  are two disjoint compact sets, so that the distance between them is strictly positive. Let  $q'$  be such that  $0 < q' < d(E, \Gamma')$ , and define:

$$N_{q'}(\Gamma') = \{x \in S, d(x, \Gamma') \leq q'\}.$$

This set is compact and disjoint from  $E$ . Define the function  $\chi : \mathbb{R} \times \mathbb{R}^n \rightarrow (-\infty, 0]$  by  $\chi(\lambda, x) = L_{f_0} V_\lambda(x)$ . It is smooth away from  $x = 0$ , and in particular on  $[0, +\infty) \times N_{q'}(\Gamma')$ . From lemma 4.1,  $\frac{\partial \chi}{\partial \lambda}(0, x) = L_{f_0} L_G V_0(x)$  for all  $x$ , so that  $\frac{\partial \chi}{\partial \lambda}(0, x)$  is strictly negative for  $x$  in  $N_{q'}(\Gamma')$ . Compactness of  $N_{q'}(\Gamma')$  implies that :

$$\max_{x \in N_{q'}(\Gamma')} \frac{\partial \chi}{\partial \lambda}(0, x) = -\beta < 0.$$



Continuous differentiability of  $\chi$  on  $[0, +\infty) \times N_{q'}(\Gamma')$  implies that, for a certain  $\lambda_1 > 0$  :

$$(\lambda, x) \in [0, \lambda_1] \times N_{q'}(\Gamma') \implies \frac{\partial \chi}{\partial \lambda}(\lambda, x) = -\frac{\beta}{2} < 0 .$$

By integrating with respect to  $\lambda$  on the interval  $[0, \lambda_1]$  and taking into account the fact that, from (14),  $\chi(0, x) \leq 0$  for all  $x$ , one obtains that  $\chi(\lambda, x) < -\frac{\beta\lambda}{2}$  for  $(\lambda, x)$  in  $[0, \lambda_1] \times N_{q'}(\Gamma')$ . This proves:

$$\left. \begin{array}{l} 0 < \lambda < \lambda_1 \\ x \in N_{q'}(\Gamma') \end{array} \right\} \implies L_{f_0} V_\lambda(x) < 0 . \quad (28)$$

Let us now introduce the set  $\Gamma'' = \Gamma \cap \{x \in S, d(x, \Gamma') \geq \frac{q'}{2}\}$ . If  $\Gamma''$  is not empty, then, by their definition  $\Gamma''$  and  $F$  are two disjoint compact set, so that the distance between them is strictly positive. Pick  $q''$  such that  $0 < q'' < d(\Gamma'', F)$ , and define

$$N_{q''}(\Gamma'') = \{x \in S, d(x, \Gamma'') \leq q''\} .$$

The function  $L_{f_0} V_0$  is strictly negative on this compact set. Hence it has a strictly negative maximum and again, by continuity of the above function  $\chi$ , we deduce that for a certain  $\lambda_2 > 0$  :

$$\left. \begin{array}{l} 0 < \lambda < \lambda_2 \\ x \in N_{q''}(\Gamma'') \end{array} \right\} \implies L_{f_0} V_\lambda(x) < 0 . \quad (29)$$

If  $\Gamma''$  is empty then, setting  $N_{q''}(\Gamma'') = \emptyset$  by convention, the above is trivially true for any  $\lambda_2 > 0$  and  $q'' > 0$ .

Finally, let us define the two positive real numbers  $q$  and  $\lambda_0$  by  $q = \min(\frac{q'}{2}, q'')$  and  $\lambda_0 = \min(\lambda_1, \lambda_2)$ . Let  $\lambda$  be in  $(0, \lambda_0)$  and  $x$  be in  $N_q(\Gamma)$ . If  $d(x, \Gamma'') \leq q''$  then  $x$  belongs to  $N_{q''}(\Gamma'')$ . Since  $\lambda_0 \leq \lambda_2$ , equation (29) proves that  $L_{f_0} V_\lambda(x) < 0$ . If  $d(x, \Gamma'') > q''$  we have  $d(x, \Gamma) < d(x, \Gamma'')$ ; and, noticing that  $\Gamma = \Gamma'' \cup \{y \in \Gamma, d(y, \Gamma') \leq \frac{q'}{2}\}$ , we deduce that the points  $y$  in  $\Gamma$  such that  $d(x, y) = d(x, \Gamma)$  satisfy  $d(y, \Gamma') \leq \frac{q'}{2}$ . Then we have  $d(x, \Gamma') \leq d(x, y) + d(y, \Gamma') \leq q'$ . Consequently  $x$  is in  $N_{q'}(\Gamma')$ . Since  $\lambda_0 \leq \lambda_1$ , equation (28) proves that  $L_{f_0} V_\lambda(x) < 0$ . This proves (27) and completes the proof of the theorem.  $\square$

*Proof of theorem 4.3.* One can easily checks that  $G$  is homogeneous of degree 0. Let us proves that it satisfies (22). If  $G$  is given by (23), then  $[F_0, G] = \sum \sum \left( L_{f_0} \lambda_{i,k} ad_{f_0}^i f_k + \lambda_{i,k} ad_{f_0}^{i+1} f_k \right)$ .

Thus, using (24),

$$\begin{aligned}
L_{[f_0, G]}V_0(x) &= - \sum_{i=1}^L \sum_{k=1}^m \frac{(L_{ad_{f_0}^{i-1}}(f_k)V_0(x))^2}{(2V_0)^{\alpha_{i,k}}} \\
&+ \sum_{k=1}^m L_{f_0} \lambda_{0,k}(x) L_{f_k} V_0(x) \\
&+ L_{f_0} V_0(x) \left( \sum_{i=1}^L \sum_{k=1}^m \sum_{j=i}^{L-1} (-1)^{j-i} 2\alpha_{j,k} \frac{L_{ad_{f_0}^{2j-i+1}}(f_k)V_0(x) L_{ad_{f_0}^{i-1}}(f_k)V_0(x)}{(2V_0)^{\alpha_{j,k}+1}} \right)
\end{aligned}$$

for all  $x$  in  $\mathbb{R}^n$ . Considering (15) this implies, for all  $x$  in  $\mathbb{R}^n \setminus \{0\}$ ,

$$\left. \begin{array}{l} L_{f_0} V_0(x) = 0 \\ L_{f_1} V_0(x) = 0 \\ \vdots \\ L_{f_m} V_0(x) = 0 \end{array} \right\} \implies L_{[f_0, G]}V_0(x) < 0 .$$

To finish the proof recall that, since  $L_{f_0} V_0$  is non positive on  $\mathbb{R}^n$ , it is maximum at all points where it vanishes. Consequently, for all  $x$  such that  $L_{f_0} V_0(x) = 0$ , we have  $L_G L_{f_0} V_0(x) = 0$ , hence  $L_{[f_0, G]}V_0(x) = L_{f_0} L_G V_0(x)$ . It follows that condition (22) is satisfied by the vector field  $G$ .  $\square$

## 5. THE METHOD TO DESIGN CONTROL LYAPUNOV FUNCTIONS, SOME EXTENSIONS

Theorems (4.2) and (4.3) give a method to design a strict Lyapunov function for homogeneous systems that satisfy Jurdjevic-Quinn conditions (assumption 3.1) with a homogeneous function  $V_0$  :

- Step 1:** Design a vector field  $G$  that satisfies the condition (22). Theorem 4.3 gives a “universal” method to obtain it, but many other solutions  $G$  for (22) may be considered. They produce different control Lyapunov functions, possibly simpler, possibly “better”. See the the examples and remark below.
- Step 2:** Compute  $V_\lambda$  or  $W_\lambda$  according to (17), (18), or (19). Formula (19) always yields explicit control Lyapunov functions for  $\lambda$  positive and small enough. If it is possible to compute explicitly the flow of  $G$ , then one may also take  $V_\lambda$  given by (17). Finally other formulae than (19) are possible to satisfy (18). See (46) for instance.
- Step 3:** Design a feedback law that makes  $V_\lambda$  or  $W_\lambda$  strictly decreasing. Equations (11) and (12) give a possibility to obtain it.

**Remark 5.1.** It is possible that a preliminary feedback is necessary to transform the system in a Jurdjevic-Quinn type system (see remark 3.2). Then, if we denote by  $f_0$  the drift vector field for the original system, the assumptions are in fact met with  $\tilde{f}_0 = f_0 + \lambda_1 f_1 + \dots + \lambda_m f_m$

instead of  $f_0$ . Of course, this preliminary feedback is in general necessary : there is no reason for  $L_{\tilde{f}_0} V_0 \leq 0$  to imply  $L_{f_0} V_0 \leq 0$ , for instance. However, being a Control Lyapunov function is obviously feedback invariant : from definition given by equation (2), a function  $V$  is control Lyapunov function for the system defined by  $(f_0, f_1, \dots, f_m)$  if and only if it is a control Lyapunov function for the system defined by  $(\tilde{f}_0, f_1, \dots, f_m)$ . Hence our results yield control Lyapunov functions not only for the Jurdjevic-Quinn type system  $(\tilde{f}_0, f_1, \dots, f_m)$ , but also for the original system. Consequently, in step 3, the design of the feedback law can be made without taking into account the preliminary feedback. This feedback is used for constructing the CLF, but not in the control design itself, contrary to the Jurdjevic-Quinn design. See for instance the example in section 7 : formula (41) considers the original system (34) and does not take into account the preliminary feedback (35) that was necessary to design the control Lyapunov function  $W_\lambda$ .

**Remark 5.2** (On the homogeneity requirement). The assumption of homogeneity is not very natural. It is a convenient case where we can restrict our attention to the situation on the homogeneous sphere, take advantage of the compactness, and extend the result to all the space. This condition is of course not necessary. The TORA example (see section 8) is a good illustration of this: the model does not meet homogeneity hypothesis, but a control Lyapunov function is obtained by reshaping  $V_0$  in  $W_\lambda$  with a vector field  $G$  that meets condition (22).

**Remark 5.3** (On the choice of  $G$ ). Formula (23)-(24) gives a “universal” way to obtain a vector field  $G$  that satisfies condition (22). However, this equation is very undetermined, and it may be a good idea to choose another another solution than (23)-(24).

It is natural to write  $G$  as a linear combination of some  $ad_{f_0}^i f_k$ , as in (23), and translate (22) into equations for the coefficients  $\lambda_{i,k}$ . Note that the vector fields  $ad_{f_0}^i f_k$  ( $k = 1, \dots, m$ ,  $i = 0, \dots, L$ ) need not span the whole tangent space at all points. But despite this, (23) is convenient, and although it does not recover all possible vector fields, it allows to satisfy the negativity requirement in (22) because  $\mathcal{W}_L(V_0)$ , defined by (13), is reduced to zero.

It is easy to see that the form of equation (22) yields rather simple equations for the coefficients  $\lambda_{i,k}$ . Formula (24) has the advantage of singling out a solution basically obtained by taking zero for  $\lambda_{i,k}$  with the highest  $i$  (in (23)  $i$  “should” naturally range from 0 to  $L$  instead of  $L - 1$ ), but many other possibilities might be used. For instance the choices of the vector fields  $G$  in sections 7 and 8 for the rigid body and the TORA systems, are obtained writing explicite condition (22) and solving simple partial differentials equations.

## 6. THE LINEAR CASE

Let us just make a short visit in the linear world. Consider the linear system given by :

$$\dot{x} = Ax + Bu , \tag{30}$$

with  $x \in \mathbb{R}^n$  and  $u \in \mathbb{R}^m$ .

For a quadratic function  $V_0(x) = \frac{1}{2}x^T Q x$  (where  $Q$  is a symmetric positive definite matrix), the equations of the set  $\mathcal{W}_L(V_0)$  (see (13)) are given by :

$$x^T(A^T Q + Q A)x = (A^i B)^T Q x = 0, \quad i = 0 \dots n-1 .$$

It is well known that, from Cayley-Hamilton theorem, the equations for  $i \geq n$  are redundant. Assume that the system (30) satisfies the Jurdjevic-Quinn conditions defined by assumption 3.1 :

- (a) condition (14) becomes :  $A^T Q + Q A \leq 0$ ,
- (b) condition (15) becomes :  $rk\{A^T Q + Q A, B, AB, \dots, A^{n-1} B\} = n$ .

Note that if  $A^T Q + Q A$  is identically equal to zero then (a) is satisfied and (b) is equivalent to the controllability condition:  $rk\{B, AB, \dots, A^{n-1} B\} = n$ .

From proposition 3.3, the linear feedback  $u = -B^T Q x$  makes the origin of the closed-loop system

$$\dot{x} = (A - BB^T Q)x$$

asymptotically stable, although  $V_0$  is not assigned to be a Lyapunov function. For such systems the problem of finding a “strict” Lyapunov function can be solved via Lyapunov equation (see for instance [4]) : for each symmetric positive definite matrix  $M$  there exists one and only one positive definite matrix  $Q_1$  satisfying the linear equation :

$$(A - BB^T Q)^T Q_1 + Q_1(A - BB^T Q) = -M . \quad (31)$$

Our result gives an alternative method to tackle this problem, without solving this equation. Indeed, we may apply theorem 4.3 : formula (23) gives, by a simple computation :

$$G(x) = \sum_{k=0}^{n-2} A^k B \sum_{i=0}^{n-k-2} (-1)^i (A^{(k+2i+1)} B)^T Q x . \quad (32)$$

Then theorem 4.2 proves that  $V_\lambda$  (resp.  $W_\lambda$ ), defined by,

$$V_\lambda = \frac{1}{2}x^T e^{\lambda G^T} Q e^{\lambda G} x \quad \left( \text{resp. } W_\lambda = \frac{1}{2}x^T (Id + \lambda G^T) Q (Id + \lambda G)x \right) \quad (33)$$

is a control Lyapunov function for  $\lambda$  small enough.

Let us apply this method on the most simple example : a double integrator with a preliminary feedback that makes the system conservative :

$$\begin{cases} \dot{x}_1 &= x_2 , \\ \dot{x}_2 &= -x_1 + u . \end{cases}$$

This system satisfies the above conditions with  $Q = Id$ , i.e.  $V_0 = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2$ . Applying our method, we obtain  $G$  via formula (32) :

$$G(x) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} .$$

Then a control Lyapunov function is obtained via formula (33), with in this case  $e^{\lambda G} = Id + \lambda G$  :

$$V_\lambda(x) = \frac{1}{2}x_1^2 + \frac{1}{2}(x_2 + \lambda x_1)^2 .$$

Since  $\dot{V}_\lambda(x) = -\lambda x_1^2 + (x_2 + \lambda x_1)(u + \lambda x_2)$ , the function  $V_\lambda$  is a control Lyapunov function for all  $\lambda > 0$ . Note that controlling the first equation by  $x_2 = -\lambda x_1$  and using classical methods to “add an integrator” yields the same Lyapunov function.

## 7. EXAMPLE: CONTROL OF THE ANGULAR VELOCITY OF A RIGID BODY

The problem of stabilization of the velocity of a rigid body with two torques has been first solved in [4, 1]. In [19], Pascal Morin gives a homogeneous control law that allows him to prove asymptotic stability together with robustness towards general errors on the model. These articles prove asymptotic stability using Jurdjevic-Quinn methods. In this example, a control Lyapunov function is obtained via our method.

Consider a rigid body in an inertial reference frame. We note  $\omega_1, \omega_2$  and  $\omega_3$  the angular velocity components along the principal axes of inertia.  $j_1, j_2$  and  $j_3$  denotes the principal moments of inertia. The Euler equations yield the following model :

$$\begin{cases} \dot{\omega}_1 &= c_1 \omega_2 \omega_3 + \frac{\tau_1}{j_1} \\ \dot{\omega}_2 &= c_2 \omega_1 \omega_3 + \frac{\tau_2}{j_2} \\ \dot{\omega}_3 &= c_3 \omega_1 \omega_2 \end{cases} \quad (34)$$

with  $c_1 = \frac{j_2 - j_3}{j_1}$ ,  $c_2 = \frac{j_3 - j_1}{j_2}$ ,  $c_3 = \frac{j_1 - j_2}{j_3}$

A preliminary feedback is proposed in [19] to make the system satisfy the Jurdjevic-Quinn hypotheses.

$$\begin{cases} \tau_1 &= j_1(-c_1 \omega_2 \omega_3 + \beta c_3 \omega_1 \omega_2 + k_1) \\ \tau_2 &= j_2(-(c_2 + \mu c_3) \omega_1 \omega_3 + k_2) \end{cases} \quad (35)$$

We obtain the new system :

$$\dot{\omega} = f_0(\omega) + k_1 f_1 + k_2 f_2 \quad (36)$$

$$f_0 = \begin{bmatrix} \beta c_3 \omega_1 \omega_2 \\ -\mu c_3 \omega_1 \omega_3 \\ c_3 \omega_1 \omega_2 \end{bmatrix}, f_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, f_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

where  $k_1$  and  $k_2$  are the new controls of the system. This system is of Jurdjevic-Quinn type with the positive definite and radially unbounded function  $V_0$  defined by

$$V_0 = \frac{1}{2}(\omega_1 - \beta\omega_3)^2 + \frac{1}{2}\omega_2^2 + \frac{\mu}{2}\omega_3^2. \quad (37)$$

In fact it is easy to verify that the set  $\mathcal{W}_1(V_0)$  is reduced to  $\{0\}$ . The stabilizing control law given in [19] is obtained applying proposition 3.3 and modifying (16) in order to keep the homogeneity of the closed-loop system :

$$\begin{cases} k_1 &= -\alpha_1 |L_{f_1} V_0| L_{f_1} V_0, \\ k_2 &= -\alpha_2 |L_{f_2} V_0| L_{f_2} V_0. \end{cases} \quad (38)$$

The system (36) is a Jurdjevic-Quinn type system with a homogeneous non strict Lyapunov function  $V_0$ . In addition the vector fields  $f_0$ ,  $f_1$  and  $f_2$  are homogeneous of degree 1,  $-1$  and  $-1$ . As a consequence, we are able to design a strict Lyapunov function (this is new to our knowledge).

#### Design of $G$ .

One can use theorem 4.3 to design the vector field  $G$ , but the following simpler  $G$  also satisfies condition (22) (left to the reader) :

$$G(\omega) = \begin{bmatrix} 0 \\ \frac{\beta\mu c_3 \omega_3^2}{(2V_0)^{\frac{1}{2}}} \\ 0 \end{bmatrix}. \quad (39)$$

#### Design of the control Lyapunov function.

The flow of  $G$  does not have a simple expression , so we use formula (19). It yields :

$$W_\lambda = \frac{1}{2}(\omega_1 - \beta\omega_3)^2 + \frac{1}{2} \left( \omega_2 + \lambda \frac{\beta\mu c_3 \omega_3^2}{\sqrt{2V_0}} \right)^2 + \frac{\mu}{2}\omega_3^2. \quad (40)$$

One can check that  $W_\lambda$  is a control Lyapunov function if  $0 < \lambda < \frac{1}{\beta c_3}$ .

#### Design of the new control law.

Of course,  $W_\lambda$  is not in general a Lyapunov function for the control law defined by (35) and (38). One has to design a new control law that assigns  $W_\lambda$  to be decreasing. Moreover, as it is underlined in remark 5.1,  $W_\lambda$  is a control Lyapunov function for the original system. Then, equations (11) and (12) give the following expression :

$$\begin{cases} \tau_1 &= -\alpha \|\omega\| L_{f_1} W_\lambda \\ \tau_2 &= -\alpha \|\omega\| L_{f_2} W_\lambda \end{cases} \quad (41)$$

with  $\|\cdot\|$  a homogeneous norm and  $\alpha$  sufficiently large:

$$\alpha > \sup_{\omega \in T} \frac{c_1\omega_2\omega_3 \frac{\partial W_\lambda}{\partial x_1}(\omega) + c_2\omega_1\omega_3 \frac{\partial W_\lambda}{\partial x_2}(\omega) + c_3\omega_1\omega_2 \frac{\partial W_\lambda}{\partial x_3}(\omega)}{\left( \left( \frac{1}{j_1} \frac{\partial W_\lambda}{\partial x_1}(\omega) \right)^2 + \left( \frac{1}{j_2} \frac{\partial W_\lambda}{\partial x_2}(\omega) \right)^2 \right) \|\omega\|}$$

where  $T = \{\omega, \|\omega\| = 1, c_1\omega_2\omega_3 \frac{\partial W_\lambda}{\partial x_1}(\omega) + c_2\omega_1\omega_3 \frac{\partial W_\lambda}{\partial x_2}(\omega) + c_3\omega_1\omega_2 \frac{\partial W_\lambda}{\partial x_3}(\omega) \geq 0\}$ .

#### Robustness to misalignment of actuators.

There is no reason for these control laws to have better robustness or performance properties than these given in [19], but it gives a tool (the negative definite function  $\dot{W}_\lambda$ ) to *analyze* it. Let us quantify, for instance, the tolerable errors on actuators misalignment with the principal axes. We consider the initial system (34):

$$\dot{\omega} = X_0(\omega) + \tau_1 f_1 + \tau_2 f_2 \quad (42)$$

with  $(\tau_1, \tau_2)$  defined by the addition of the preliminary feedback and the control law we have made  $(k_1, k_2)$ .

Let  $(\varphi_1, \theta_1)$  and  $(\varphi_2, \theta_2)$  be angles between the position of the torques and the principal axes. That means the vector fields  $f_1$  and  $f_2$  of the system (42) are given by :

$$f_1 = \begin{bmatrix} \cos(\varphi_1) \\ \sin(\varphi_1)\cos(\theta_1) \\ \sin(\varphi_1)\sin(\theta_1) \end{bmatrix} \quad f_2 = \begin{bmatrix} \sin(\varphi_2)\cos(\theta_2) \\ \cos(\varphi_2) \\ \sin(\varphi_2)\sin(\theta_2) \end{bmatrix}$$

Then we have :

$$\dot{W}_\lambda = L_{f_0} W_\lambda + k_1 L_{\hat{f}_1} W_\lambda + k_2 L_{\hat{f}_2} W_\lambda + \tau_1 L_{f_1 - \hat{f}_1} W_\lambda + \tau_2 L_{f_2 - \hat{f}_2} W_\lambda$$

We defined three positive constants  $\beta_1$ ,  $\beta_2$  and  $\beta_3$  :

$$\begin{cases} \beta_1 &= - \sup_{\|\omega\|=1} (L_{f_0} W_\lambda + k_1 L_{\hat{f}_1} W_\lambda + k_2 L_{\hat{f}_2} W_\lambda) \\ \beta_2 &= \sup_{\|\omega\|=1} \tau_1 \left\| \frac{\partial W_\lambda}{\partial \omega} \right\| \\ \beta_3 &= \sup_{\|\omega\|=1} \tau_2 \left\| \frac{\partial W_\lambda}{\partial \omega} \right\| \end{cases}$$

Then we deduce :

$$\dot{W}_\lambda \leq -\|\omega\|^3 \left( \beta_1 - 2\beta_2 \left| \sin\left(\frac{\varphi_1}{2}\right) \right| - 2\beta_3 \left| \sin\left(\frac{\varphi_2}{2}\right) \right| \right)$$

Consequently, if  $\varphi_1$  and  $\varphi_2$  are small enough to satisfy the following equation,

$$2\beta_2 \left| \sin\left(\frac{\varphi_1}{2}\right) \right| + 2\beta_3 \left| \sin\left(\frac{\varphi_2}{2}\right) \right| < \beta_1$$

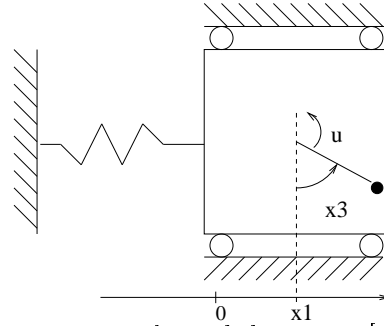
then the origin remains an asymptotically stable equilibrium.

## 8. EXAMPLE: THE TORA SYSTEM

Using our ideas to find some control Lyapunov function for this system was suggested by Rodolphe Sepulchre, from Université de Liège. The following is a summary of [9, section 4].

Let us consider the mechanical system called TORA (Translational Oscillator with Rotating Actuator) in [22] and RTAC (Rotational/Translational proof-mass Actuator) in [5, 6, 25]. It consists of a platform connected to a fixed frame of reference by a linear spring. The platform can oscillate without friction in the horizontal plane. On the platform, an eccentric rotating mass is actuated by a DC motor. The control of this rotating motion is used to dampen the translational oscillations of the platform. A precise description is given in [5].

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \frac{-x_1 + \varepsilon x_4^2 \sin x_3}{1 - \varepsilon^2 \cos^2 x_3} + \frac{-\varepsilon \cos x_3}{1 - \varepsilon^2 \cos^2 x_3} u \\ \dot{x}_3 &= x_4 \\ \dot{x}_4 &= \frac{\varepsilon \cos x_3 (x_1 - \varepsilon x_4^2 \sin x_3) + u}{1 - \varepsilon^2 \cos^2 x_3} \end{aligned}$$



In [22], this system is extensively used as an illustrative example, and the papers [5, 6, 25] also propose various control methods. The following is an illustration of our methods on this example to derive a (global) CLF. We do not propose new or “better” control laws.

Setting the new state variables  $z_1 = x_1 + \varepsilon \sin x_3$ ,  $z_2 = x_2 + \varepsilon x_4 \cos x_3$ ,  $z_3 = x_3$ ,  $z_4 = x_4$ , a preliminary feedback (see [22, 9] for more details) yields the following control affine system :

$$\dot{z} = f_0(z) + v f_1(z)$$

where  $f_0$  and  $f_1$  are, in the coordinates  $(z_1, z_2, z_3, z_4)$ ,

$$f_0 = \begin{bmatrix} z_2 \\ -z_1 + \varepsilon \sin z_3 \\ z_4 \\ \varepsilon x_1 \cos z_3 - k_1 z_3 \end{bmatrix} \quad f_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad (43)$$

Let us define the following positive definite and radially unbounded function  $V_0$  :

$$V_0(z) = \frac{1}{2} (z_1 - \varepsilon \sin z_3)^2 + \frac{1}{2} z_2^2 + \frac{k_1}{2} z_3^2 + \frac{1}{2} z_4^2 \quad (44)$$

with  $k_1$  some positive constant.



We have  $L_{f_0} V_0 \equiv 0$  and a computation of the  $L_{f_0}^j L_{f_1} V_0$  for  $j = 1, 2, 3$ , shows that this system satisfies the Jurdjevic-Quinn conditions (assumption 3.1); precisely,  $\mathcal{W}_3(V_0) = \{0\}$ .

Unfortunately we can not actually apply our method because this system does not meet the homogeneous assumptions of our theorems. Yet in the following we look for a complete vector field  $G$  satisfying condition (22), we give a function  $W_\lambda$  that satisfies formula (18), and we prove that it is a control Lyapunov function for system (43).

The first step consists in finding a vector field  $G$  that meets the condition (22). Instead of using formula (23)-(24), we find another simpler vector field  $G$  that satisfies this condition. Indeed, Let  $G$  be defined by :

$$G(x_1, z_2, z_3, z_4) = \begin{bmatrix} -\varepsilon^2 \rho(z_2) \cos^2 z_3 \\ 0 \\ 2\varepsilon \rho(z_2) \cos z_3 \\ -\varepsilon x_1 (1 + 2\rho'(z_2)) \cos z_3 + k_1 z_3 - 2\varepsilon z_4 \rho(z_2) \sin z_3 \end{bmatrix},$$

where  $\rho$  is a smooth function of  $z_2$  that is bounded, whose derivative is positive, bounded and such that :

$$z_2 \neq 0 \Rightarrow z_2 \rho(z_2) > 0 \quad (45)$$

A simple computation shows that this vector field satisfies condition (22) (see [9]). Following the idea of our method, we define  $W_\lambda$ . The following formula gives a  $W_\lambda$  that differs from (19), but satisfies (18) :

$$\begin{aligned} W_\lambda &= \frac{1}{2} \left( x_1 - \lambda \varepsilon^2 \rho(z_2) \cos^2 z_3 \right)^2 + \frac{1}{2} z_2^2 + \frac{k_1}{2} \left( z_3 + 2\lambda \varepsilon \rho(z_2) \cos z_3 \right)^2 \\ &+ \frac{1}{2} \left( (1 - 2\lambda \varepsilon \rho(z_2) \sin z_3) z_4 - \lambda \varepsilon x_1 (1 + 2\rho'(z_2)) \cos z_3 + \lambda k_1 z_3 \right)^2. \end{aligned} \quad (46)$$

Recall that  $L_{f_1} W_\lambda$  is simply  $\partial W_\lambda / \partial z_4$ . Hence

$$L_{f_1} W_\lambda = (1 - 2\lambda \varepsilon \rho(z_2) \sin z_3)^2 \xi_4, \quad (47)$$

$$\text{with } \xi_4 = z_4 + \lambda \frac{k_1 z_3 - \varepsilon x_1 (1 + 2\rho'(z_2)) \cos z_3}{1 - 2\lambda \varepsilon \rho(z_2) \sin z_3}. \quad (48)$$

This vanishes either when  $\xi_4 = 0$  or when  $1 - 2\lambda \varepsilon \rho(z_2) \sin z_3 = 0$ . As  $\rho$  is bounded we can take  $\lambda$  such that  $2\lambda \varepsilon \rho(z_2) < 1$ . Provided this is satisfied, a careful computation yields,

$$\begin{aligned}
\dot{W}_\lambda = & -\lambda \left[ k_1 z_3 - \varepsilon \cos z_3 \left( x_1 - \lambda \left( k_1 + \frac{\varepsilon^2}{2} \cos^2 z_3 \right) \rho(z_2) \right) \right]^2 \\
& - \lambda \varepsilon^2 \cos^2 z_3 \rho'(z_2) \left[ x_1 - \lambda \frac{\varepsilon^2}{2} \cos^2 z_3 \rho(z_2) \right]^2 \\
& - \lambda \varepsilon^2 \cos^2 z_3 \rho(z_2) \left[ z_2 - \lambda^2 \rho(z_2) \right. \\
& \quad \left. \left( \frac{\varepsilon^4}{4} (1 + \rho'(z_2)) \cos^4 z_3 + k_1 \varepsilon^2 \cos^2 z_3 + k_1^2 \right) \right] \\
& + \xi_4 \left[ \lambda R(\lambda, \varepsilon, x_1, z_2, z_3, z_4) + k_1 z_3 - \varepsilon x_1 \cos z_3 \right. \\
& \quad \left. \frac{(1 - 2\lambda\varepsilon\rho(z_2) \sin z_3)^2}{1 - \varepsilon^2 \cos^2 z_3} \left( \varepsilon \cos z_3 (x_1 - \varepsilon z_4^2 \sin z_3) + u \right) \right],
\end{aligned} \tag{49}$$

where  $R$  is some function, whose expression is somewhat lengthy but can easily be handled with a computer algebra system.

The three first terms depend only on  $x_1$ ,  $z_2$ ,  $z_3$ , and they are a negative definite function of these three variables since  $\rho'(z_2)$  is everywhere positive, and  $|\rho(z_2)| \leq a|z_2|$ . Note that the requirements on  $\rho$  are met for instance with  $\rho(z_2) = \frac{2}{\pi} \text{Arctan} z_2$ , and that, if  $k_1 = 1$  and  $\varepsilon = 0.1$ ,  $\lambda$  only has to be taken slightly less than 1.

Then equation (49) clearly proves that  $W_\lambda$  is a control Lyapunov function, and it satisfies the small control property since equation (49) allows one to derive very explicitly a continuous stabilizing control by making the last term non-positive, and negative when  $\xi_4 \neq 0$ .

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## REFERENCES

- [1] D. AEYELS, *Stabilization of a class of nonlinear systems by smooth feedback control*, Syst. & Control Lett., 5 (1985), pp. 289–294.
- [2] Z. ARTSTEIN, *Stabilization with relaxed control*, Nonlinear Analysis TMA, 7 (1983), pp. 1163–1173.
- [3] A. BACCIOTTI, *Local stabilizability of nonlinear control systems*, vol. 8 of Series on advances in mathematics for applied sciences, World Scientific, Singapore, River Edge, London, 1992.
- [4] R. W. BROCKETT, *Asymptotic stability and feedback stabilization*, in Differential Geometric Control Theory, R. W. Brockett, R. S. Millman, and H. J. Sussmann, eds., Basel-Boston, 1983, Birkhäuser.

- 
- [5] R. T. BUPP, D. S. BERNSTEIN, AND V. T. COPPOLA, *A benchmark problem for nonlinear control design*, Int. J. Robust and Nonlinear Cont., 8 (1998), pp. 307–310.
- [6] ———, *Experimental implementation of integrator back-stepping and passive nonlinear controllers on the RTAC testbed*, Int. J. Robust and Nonlinear Cont., 8 (1998), pp. 435–457.
- [7] J.-M. CORON, L. PRALY, AND A. R. TEEL, *Feedback stabilization of nonlinear system : Sufficient conditions and lyapunov and input-output techniques*, in Trends in Control, a European Perspective, A. Isidori, ed., Springer-Verlag, 1995.
- [8] L. FAUBOURG AND J.-B. POMET, *Strict control Lyapunov functions for homogeneous Jurdjevic-Quinn type systems*, in Nonlinear Control Systems Design Symposium (NOLCOS'98), H. Huijberts, H. Nijmeijer, A. van der Schaft, and J. Scherpen, eds., IFAC, July 1998, pp. 823–829.
- [9] L. FAUBOURG AND J.-B. POMET, *Design of control Lyapunov functions for “Jurdjevic-Quinn” systems*, in Stability and Stabilization of Nonlinear Systems, D. Aeyels et al., eds., Lect. Notes in Contr. & Inform. Sci., Springer-Verlag, 1999, to appear.
- [10] J.-P. GAUTHIER, *Structure des Systèmes non-linéaires*, Éditions du CNRS, Paris, 1984.
- [11] W. HAHN, *Stability of Motion*, vol. 138 of Grundlehren der Mathematischen Wissenschaften, Springer-Verlag, Berlin, New-York, 1967.
- [12] V. JURDJEVIC AND J. P. QUINN, *Controllability and stability*, J. of Diff. Equations, 28 (1978), pp. 381–389.
- [13] M. KAWSKI, *Homogeneous stabilizing feedback laws*, Control Th. and Adv. Technol., 6 (1990), pp. 497–516.
- [14] H. K. KHALIL, *Nonlinear Systems*, MacMillan, New York, Toronto, Singapore, 1992.
- [15] J. KURZWEIL, *On the inversion of lyapunov’s second theorem on stability of motion*, A.M.S. Translations, ser. II, 24 (1956), pp. 19–77.
- [16] J.-P. LASALLE, *Stability theory for ordinary differential equations*, J. of Diff. Equations, 4 (1968), pp. 57–65.
- [17] W. LIU, Y. CHITOUR, AND E. SONTAG, *Remarks on finite gain stabilizability of linear systems subject to input saturation*, in 32th IEEE Conf. on Decision and Control, San Antonio, USA, Dec. 1993, IEEE.
- [18] F. MAZENC, *Stabilisation de trajectoires, ajout d’intégration, commandes saturées*, thèse de doctorat, Ecole des Mines de Paris, Paris, France, 1989.
- [19] P. MORIN, *Robust stabilization of the angular velocity of a rigid body with two actuators*, Europ. J. of Control, 2 (1996), pp. 51–56.
- [20] R. OUTBIB AND G. SALLET, *Stabilizability of the angular velocity of a rigid body revisited*, Syst. & Control Lett., 18 (1992), pp. 93–98.
- [21] G. SALLET, *Historique des techniques de Jurdjevic-Quinn*, private communication.
- [22] R. SÉPULCHRE, M. JANKOVIĆ, AND P. V. KOKOTOVIĆ, *Constructive Nonlinear Control*, Comm. and Control Engineering, Springer-Verlag, 1997.
- [23] E. D. SONTAG, *Feedback stabilization of nonlinear systems*, in Robust control of linear systems and nonlinear control (vol. 2 of proceedings of MTNS'89), M. A. Kaashoek, J. H. van Schuppen, and A. Ran, eds., Basel-Boston, 1990, Birkhäuser, pp. 61–81.
- [24] M. SPIVAK, *A Comprehensive Introduction to Differential Geometry*, vol. 1, Publish or Perish, Houston (U.S.A.), second ed., 1979.
- [25] J. ZHAO AND I. KANELLAKOPOULOS, *Flexible back-stepping design for tracking and disturbance attenuation*, Int. J. Robust and Nonlinear Cont., 8 (1998), pp. 331–348.



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