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Mixed finite elements, strong symmetry and mass lumping for elastic waves

Eliane BECACHE, Patrick JOLY, Chrysoula TSOGKA

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†(eliane.becache@inria.fr)
‡(patrick.joly@inria.fr)
§(chrysoula.tsogka@inria.fr)
¶INRIA, Domaine de Voluceau-Rocquencourt, BP 105, F-78153 Le Chesnay Cédex

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Mixed finite elements, strong symmetry and mass lumping for elastic waves

Eliane BECACHE*, Patrick JOLY†, Chrysoula TSOGKA‡

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Abstract: We present here the continuation of our work on mixed finite elements for wave propagation problems. In a previous report, we constructed and analysed a new family of quadrangular (2D) or cubic (3D) mixed finite elements, for the approximation of the scalar anisotropic wave equation. This work is extended here to the elastic wave equation, including in the case of an anisotropic medium. These new elements present the specificity to enforce the symmetry of the stress tensor in a str ong way and lead to explicit schemes (via mass lumping), after time discretization. The convergence analysis of these mixed finite elements is not straightforward: neither the standard abstract theory nor the theory we developed for the scalar case can be applied. That is why we introduce a new abstract theory which allows to get error estimates.

Key-words: mixed finite elements, mass lumping, anisotropic elastic waves

(Résumé : tsvp)

* INRIA-rocquencourt. Eliane.Becache@inria.fr
† INRIA-rocquencourt. Patrick.Joly@inria.fr
‡ INRIA-rocquencourt. Chrysoula.Tsogka@inria.fr

Eléments finis mixtes, symétrie forte et condensation de masse pour les ondes élastiques

Résumé : Nous présentons ici la suite de nos travaux sur les éléments finis mixtes pour les problèmes de propagation d'ondes. Dans un précédent rapport, nous avons construit et analysé une nouvelle famille d'éléments finis mixtes quadrangulaires (2D) ou cubiques (3D) pour l'approximation de l'équation des ondes anisotropes scalaire. Ce travail est étendu ici à l'équation des ondes élastiques, en milieu anisotrope. Ces nouveaux éléments présentent la spécificité d'imposer la symétrie du tenseur des contraintes de façon forte et de conduire à des schémas explicites (condensation de masse) après discrétisation en temps. L'analyse de convergence de ces éléments finis mixtes n'est pas immédiate: on ne peut appliquer ni le théorie classique ni la théorie développée dans le cas scalaire. C'est pourquoi nous introduisons une nouvelle théorie abstraite qui nous permet d'obtenir des estimations d'erreur.

Mots-clé : éléments finis mixtes, condensation de masse, ondes élastiques anisotropes

Table of Contents

1 Introduction **3**

2 Presentation of the new mixed finite elements **4**
2.1 The model problem : the 2D elastic wave equation 4
2.1.1 The elastodynamic problem 5
2.1.2 Variational formulation in $\underline{\underline{X}}^{sym}$ 6
2.1.3 A relaxed variational formulation in $\underline{\underline{X}}$ 6
2.2 Presentation of the finite element in the lowest order 7
2.3 Extension to higher orders and mass lumping 11
2.4 Extension to 3D 13

3 Analysis of the new mixed finite element for an elliptic problem : the static elasticity case **14**
3.1 The elliptic problem in $\underline{\underline{X}}^{sym}$ and its approximation 14
3.2 The difficulties of the convergence analysis 15
3.3 A relaxed discrete elliptic problem in $\underline{\underline{X}}_h$ 16
3.4 An abstract result 17
3.5 Application to the elastic problem 24
3.5.1 Statement of the main results 24
3.5.2 Equivalence with the relaxed problem : Characterization of the space L_h 25
3.5.3 Proof of the main theorem : Verification of the conditions (H1), (H0), (H3), (H4) and (H4b) 29
3.5.4 Proof of the main theorem : Verification of the condition (H2) 31

4 Error estimates for the evolution problem **40**
4.1 The elliptic projection error 41
4.2 From error estimates for the elliptic problem to error estimates for the evolution problem 42

1 Introduction

We are concerned with the construction of an efficient numerical method for approximating the linear elastodynamic problem in complex media, such as anisotropic heterogeneous media with cracks of arbitrary shapes. The main application we have in mind is non-destructive testing, i.e. detecting the existence of cracks in a medium by studying the diffraction of a known incident wave by the medium. In such experiments the incident wave is a time pulse, which justifies a direct computation in the time domain. We have built a specific numerical method based on the following motivations:

- We would like to be able to perform large scale computations (i.e. for which the wavelength is small with respect to the size of the computational domain). To facilitate the implementation and reduce the computation time, we have chosen to use regular meshes, squares in 2D and cubes in 3D, for the space discretization and explicit schemes for the time discretization.

- To deal with the complex geometry of the crack without producing spurious numerical diffractions as in a staircase approximation (see [13]), we will use a fictitious domain method, c.f [13, 17, 18]. Thus, we have to treat the boundary condition on a crack as an essential condition, i.e. as an equality constraint in an adequate functional space, via the introduction of an appropriate Lagrange multiplier. The boundary condition on a crack is a free surface condition, which means that the normal stress on the crack is zero. This condition becomes an essential condition when we consider the mixed velocity-stress formulation for elastodynamics. The Lagrange multiplier can then be interpreted as the jump of the velocity across the crack.

- In most examples, the dimensions of the cracks are very small compared to the scale of the problem: we thus need to consider the elastic wave propagation in an unbounded domain. This can be done by using a new absorbing layer model based on the Perfectly Matched absorbing Layer (PML) introduced by Berenger for the 2D Maxwell problem, c.f [8, 9, 12, 22]. The extension of this model to elastodynamics is natural when using the mixed velocity-stress formulation, c.f [14].

These considerations justify to base our numerical method on the mixed velocity-stress variational formulation of the elastodynamics equations. However in this paper, we will not consider the presence of the crack, and we refer for that to a future paper. Moreover, for simplicity, we will consider in this paper an homogeneous Dirichlet boundary condition, the extension of the PML model to elastodynamics is described in [14].

In order to get a really explicit method after time discretization, we want our spacial discretization to be compatible with a mass lumping technique. Moreover, it is well known that the main difficulty encountered in trying to construct mixed finite element methods for the system of elasticity is finding a way to take into account the symmetry of the stress tensor. Several elements, in particular for plane elasticity, have been proposed in the literature [1, 25, 20]. In these papers a relaxed symmetry approach is used: the symmetry of the stress tensor is imposed in a weak sense via the introduction of a Lagrange multiplier. Another approach using spaces of symmetric stress tensors, based on composite elements, was introduced in [19]. None of these however are adapted to mass lumping.

For this reason, we have introduced mixed finite elements (inspired by Nédelec's second family [21]), which appear to be new in spite of their simplicity. These new elements have two basic characteristics; they allow mass lumping and they use spaces of symmetric stress tensors.

The first part of this paper (§2) is devoted to the description of these elements in two and three space dimensions. In §2.1, we define some notation and present the model evolution problem we shall consider, as well as its variational mixed formulations (with and without relaxation of the symmetry condition). In §2.2, we present the construction our lowest order elements in dimension 2 and how to obtain mass lumping. Section 2.3 is devoted to the generalisation to higher order elements and §2.4 to the generalisation to the 3D case.

The second part concerns the analysis of these new elements in the 2D case. This analysis involves two main difficulties. The first one is due to the fact that the classical Ladyzhenskaya-Babuska-Brezzi assumptions for the discrete problem (c.f [11, 10, 3]) are not satisfied (defect of coerciveness). That is because, compared to more classical elements, we enrich the approximate stress space but not the velocity one. The second difficulty in the analysis concerns the symmetry of the stress tensor. In a recent paper [7], we developed and analyzed the same type of element for a scalar model problem : the anisotropic wave equation. However, the theory developed there can not be directly applied to the elastodynamic problem.

The outline of the second part of the paper is as follows. Section 3 is concerned with the mixed approximation of an elliptic problem which is the stationary problem associated to the evolution problem of §2. This will be used to study an elliptic projection operator that is naturally involved in the analysis of the approximation of the evolution problem. We show in §3.2 why the analysis of the new element fits neither the classical theory nor the theory developped in [7]. We thus develop in §3.4 a novel abstract theory for obtaining error estimates. This theory is based on weaker inf-sup and coercivity conditions but requires stronger approximation properties. In §3.5.1, we show that our new elements fits within this framework and error estimates are given. To verify the (modified) inf-sup condition, we use a macro-element approach. §4 is devoted to relating the error estimates on the time domain solution to the error estimates obtained in the previous section on the elliptic problem. This part of the analysis is inspired by [16] and essentially relies on energy estimates.

2 Presentation of the new mixed finite elements

2.1 The model problem : the 2D elastic wave equation

In the following, we identify the space of 2×2 tensors with the space $\mathcal{L}(\mathbb{R}^2)$ of linear applications from $\mathbb{R}^2$ to $\mathbb{R}^2$, in which we define the linear form

(1) $$\mathbf{as}(\sigma) = \sigma_{12} - \sigma_{21}.$$

Let $\mathcal{L}^s(\mathbb{R}^2)$ be the subspace of symmetric tensors of $\mathcal{L}(\mathbb{R}^2)$, that is

(2) $$\mathcal{L}^s(\mathbb{R}^2) = \left\{\sigma \in \mathcal{L}(\mathbb{R}^2) \ / \ \mathbf{as}(\sigma) = 0\right\}.$$

The scalar product in $\mathcal{L}(\mathbb{R}^2)$ is defined by

(3) $$\sigma : \tau = \sigma_{ij}\tau_{ij}, \qquad \forall(\sigma, \tau) \in \mathcal{L}(\mathbb{R}^2),$$

and $|\sigma|$ is the associated norm. In the sequel, a tensor σ will be indefferently identified with an element of $\mathbb{R}^4$ or an element of $\mathbb{R}^2 \times \mathbb{R}^2$. In particular, to each σ in $\mathcal{L}(\mathbb{R}^2)$, we shall associate the two column vectors

(4) $$\sigma_1 = \begin{bmatrix} \sigma_{11} \\ \sigma_{12} \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} \sigma_{21} \\ \sigma_{22} \end{bmatrix}.$$

If Ω is an open subspace of $\mathbb{R}^2$, σ a tensor field in Ω ($\sigma \in \mathcal{D}'(\Omega; \mathcal{L}(\mathbb{R}^2))$), we set

$$\mathbf{div}\ \sigma = \begin{bmatrix} \mathrm{div}\ \sigma_1 \\ \\ \mathrm{div}\ \sigma_2 \end{bmatrix} = \begin{bmatrix} \dfrac{\partial \sigma_{11}}{\partial x_1} + \dfrac{\partial \sigma_{12}}{\partial x_2} \\ \dfrac{\partial \sigma_{21}}{\partial x_1} + \dfrac{\partial \sigma_{22}}{\partial x_2} \end{bmatrix}.$$

Finally, if $(u, v) \in L^2(\Omega)^d$ (d=1,2,3), $(u, v)_\Omega$ will denote the usual inner product of $L^2(\Omega)^d$

2.1.1 The elastodynamic problem

We consider now that Ω is filled by an elastic material. We denote by $u(x,t)$ the displacement field in Ω, at time t ($x \in \Omega, t > 0$). We associate to $u(x,t)$ the strain tensor $\varepsilon(u)$

(5) $$\varepsilon_{ij}(u) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right).$$

In the following, we consider ε as a linear operator from $\mathcal{D}'(\Omega; \mathbb{R}^2)$ to $\mathcal{D}'(\Omega; \mathcal{L}^s(\mathbb{R}^2))$. The displacement is governed by the following system

(6) $$\varrho \frac{\partial^2 u}{\partial t^2} - \mathbf{div}\ \sigma(u) = f,$$

where $\varrho = \varrho(x)$ denotes the density which verifies

(7) $$0 < \varrho_- \leq \varrho(x) \leq \varrho_+ < +\infty, \qquad \text{a.e. } x \in \Omega.$$

We should add to (6) the initial conditions

(8) $$u(t=0) = u_0 \ ; \ \frac{\partial u}{\partial t}(t=0) = u_1,$$

and a boundary condition on $\partial\Omega$

(9) $$u = 0 \text{ on } \partial\Omega.$$

The stress tensor $\sigma(u)$ is related to the deformation tensor by Hooke's law

(10) $$\sigma(u)(x,t) = C(x)\varepsilon(u)(x,t),$$

where for all x in Ω, $C(x)$ is a 4×4 positive tensor having the usual symmetry properties [2] which means that $C(x)$ is a positive symmetric definite operator in $\mathcal{L}^s(\mathbb{R}^2)$. We set $A(x) = C(x)^{-1}$ and assume that $x \to A(x)$ is measurable and satisfies the uniform inequalities (α and M are strictly positive real numbers)

(11) $$0 < \alpha\,|\sigma|^2 \leq A(x)\sigma : \sigma \leq M\,|\sigma|^2\,,\ \forall \sigma \in \mathcal{L}^s(\mathbb{R}^2),\ \text{a.e. } x \in \Omega.$$

The elastodynamic problem in Ω can be written as a first order hyperbolic system, the velocity-stress system with the unknowns

$$\begin{cases} v = \dfrac{\partial u}{\partial t} & : \quad \text{the velocity vector in } \Omega, \\ \sigma = \sigma(u) & : \quad \text{the stress tensor in } \Omega, \end{cases}$$

(12) $$\begin{cases} \varrho \dfrac{\partial v}{\partial t} - \mathbf{div}\ \sigma & = \quad f, \\ A \dfrac{\partial \sigma}{\partial t} - \varepsilon(v) & = \quad 0, \end{cases}$$

subject to the initial conditions

(13) $$v(0) = v_0 \equiv u_1 \ ; \ \sigma(0) = \sigma_0 \equiv \sigma(u_0),$$

and the boundary condition on $\partial\Omega$

(14) $$v = 0 \text{ on } \partial\Omega.$$

Remark 1 *The homogeneous Dirichlet condition on $\partial\Omega$ has just been considered for simplicity. This has nothing to do with a free boundary condition on the boundary of a crack interior to Ω that will be taken into account with the ficitious domain method as explained in the introduction.*

2.1.2 Variational formulation in $\underline{\underline{X}}^{sym}$

We introduce the Hilbert spaces

$$\underline{M} = (L^2(\Omega;\mathbb{R}^2))^2,\ \underline{\underline{H}} = (L^2(\Omega;\mathcal{L}(\mathbb{R}^2)))^4 \text{ and } \underline{\underline{X}} = \left\{\sigma \in \underline{\underline{H}} \,/\, \mathbf{div}\,\sigma \in \underline{M}\right\}.$$

We denote $\underline{\underline{X}}^{sym} \subset \underline{\underline{X}}$ the subspace of symmetric tensors in $\underline{\underline{X}}$

$$\underline{\underline{X}}^{sym} = \left\{\sigma \in \underline{\underline{X}} \,/\, \mathbf{as}(\sigma) = 0\right\}. \tag{15}$$

We can write now a mixed variational formulation of system (12, 14) in the following form

$$\left\{\begin{array}{l} \text{Find } (\sigma, v) : [0,T] \longmapsto \underline{\underline{X}}^{sym} \times \underline{M} \text{ such that} \\[2ex] \dfrac{d}{dt} a(\sigma(t),\tau) + b(v(t),\tau) = 0\ , \qquad \forall \tau \in \underline{\underline{X}}^{sym}, \\[2ex] \dfrac{d}{dt} c(v(t),w) - b(w,\sigma(t)) = (f,w)\ , \quad \forall w \in \underline{M}, \end{array}\right. \tag{16}$$

where

$$\left\{\begin{array}{ll} a(\sigma,\tau) = \displaystyle\int_\Omega A\sigma : \tau dx, & \forall (\sigma,\tau) \in \underline{\underline{H}} \times \underline{\underline{H}}, \\[2ex] c(v,w) = \displaystyle\int_\Omega \varrho v \cdot w dx, & \forall (v,w) \in \underline{M} \times \underline{M}, \\[2ex] b(w,\tau) = \displaystyle\int_\Omega \mathbf{div}\ \tau \cdot w dx, & \forall (w,\tau) \in \underline{\underline{X}} \times \underline{M}. \end{array}\right. \tag{17}$$

Remark 2 *In this section we are not interested in the regularity of (σ, v) with respect to time variable t. This is why we simply write $(\sigma, v) : [0,T] \longmapsto \underline{\underline{X}}^{sym} \times \underline{M}$.*

The bilinear form $a(\cdot,\cdot)$ (resp. $b(\cdot,\cdot)$) is continuous on $\underline{\underline{H}} \times \underline{\underline{H}}$ (resp. on $\underline{\underline{X}} \times \underline{M}$), thus we can define a linear continuous operator $\mathcal{A} : \underline{\underline{H}} \to \underline{\underline{H}}'$ by $\langle \mathcal{A}\sigma, \tau\rangle_{H' \times H} = a(\sigma,\tau)$ and $B : \underline{\underline{X}} \to \underline{M}'$ by $\langle B\tau, w\rangle_{M' \times M} = b(w,\tau)$ (H' holds for the dual space of H, and so on). They satisfy the following properties (see for instance [11])

(18)

(i) The continuous inf-sup condition

$$\exists\ \beta > 0 \,/\, \forall\ w \in \underline{M},\ \exists\ \tau \in \underline{\underline{X}} \,/\, b(w,\tau) \geq \beta \left\|w\right\|_M \left\|\tau\right\|_X .$$

(ii) The coercivity of the form $a(\cdot,\cdot)$ on $\mathrm{Ker}B$

$$\exists\ \alpha > 0 \,/\, \forall\ \sigma \in \mathrm{Ker}B,\ a(\sigma,\sigma) \geq \alpha \left\|\sigma\right\|_X^2 ,$$

with $\mathrm{Ker}B = \left\{\tau \in \underline{\underline{X}} \,/\, b(w,\tau) = 0,\ \forall\ w \in M\right\}$.

The mixed formulation (16) is the one we shall work with for the numerical approximation. It is crucial to work in the space $\underline{\underline{X}}^{sym}$ of symmetric tensors : the operator $-\varepsilon$ is not the adjoint of $\mathbf{div}$ if one works in all of $\underline{\underline{X}}$.

However, as we will see in the following, for the analysis of the discrete problem associated to (16) we need to introduce an equivalent problem in which the symmetry of the stress tensor is imposed in a weak way. This is the object of the next section.

2.1.3 A relaxed variational formulation in $\underline{\underline{X}}$

The content of this section is classical. We refer the reader to D. N. Arnold, F. Brezzi and J. Douglas [1], R. Stenberg [25] and M. E. Morley [20] for further details.

If we set $\gamma = \dfrac{1}{2}\,\mathbf{rot} v$ (with $\mathbf{rot} v = \dfrac{\partial v_2}{\partial x_1} - \dfrac{\partial v_1}{\partial x_2}$) , we get

$$\varepsilon(v) = \nabla v - \gamma\chi \quad \text{where} \quad \chi = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

and we can rewrite the elastodynamic problem in the following form

$$(19)\qquad \left\{\begin{array}{rcl} \varrho\dfrac{\partial v}{\partial t} - \mathbf{div}\,\sigma & = & f, \\ A\dfrac{\partial \sigma}{\partial t} - \nabla v + \gamma\chi & = & 0, \\ \mathbf{as}(\sigma) & = & 0. \end{array}\right.$$

Systems (19) and (12) are equivalent in the sense that (σ, v, γ) solves (19) if and only if $\gamma = 1/2\ \mathbf{rot}v$ and (σ, v) solves (12). We introduce now the variational formulation of problem (19)

$$(20)\qquad \left\{\begin{array}{lll} \text{Find } (\sigma, v, \gamma) : [0,T] \longmapsto \underline{\underline{X}} \times \underline{M} \times L \text{ such that} & & \\ \dfrac{d}{dt}a(\sigma(t),\tau) + b(v(t),\tau) + (\gamma, \mathbf{as}(\tau))_L & = 0, & \forall \tau \in \underline{\underline{X}}, \\ \dfrac{d}{dt}c(v(t),w) - b(w,\sigma(t)) & = (f,w), & \forall w \in \underline{M}, \\ \hfill (\eta, \mathbf{as}(\sigma))_L & = 0, & \forall \eta \in L, \end{array}\right.$$

where $L = L^2(\Omega)$. Problem (20) has a unique solution (σ, v, γ), namely the solution of (19). To simplify the notations, we introduce a bilinear form $d(\cdot,\cdot)$ continuous on $L \times \underline{\underline{H}}$ defined by

$$d(\eta, \tau) = (\eta, \mathbf{as}(\tau))_L, \quad \forall (\eta, \tau) \in L \times \underline{\underline{H}},$$

and we associate to $d(\cdot,\cdot)$ the linear continuous operator $D : \underline{\underline{H}} \to L'$ with $\langle D\tau, \eta\rangle_{L' \times L} = d(\eta, \tau)$. We then have the following properties (c.f [11])

$$(21)\qquad \left|\begin{array}{ll} (i) & \text{The continuous inf-sup condition} \\ & \exists\, \beta > 0 \,/\, \forall\, (w,\eta) \in \underline{M} \times L,\ \exists\, \tau \in \underline{\underline{X}}\,/ \\ & b(w,\tau) + d(\eta,\tau) \geq \beta\left(\|w\|_M + \|\eta\|_L\right)\|\tau\|_X\,. \\ (ii) & \text{The coercivity of the form } a(\cdot,\cdot) \text{ on } V = \mathrm{Ker}B \cap \mathrm{Ker}D \\ & \exists\, \alpha > 0 \,/\, \forall\, \sigma \in V,\ a(\sigma,\sigma) \geq \alpha\,\|\sigma\|_X^2\,. \end{array}\right.$$

Note that V is characterized by

$$V = \left\{\tau \in \underline{\underline{X}} \,/\, b(w,\tau) + d(\eta,\tau) = 0,\ \forall\,(w,\eta) \in \underline{M} \times L\right\},$$

2.2 Presentation of the finite element in the lowest order

In what follows, we shall use the following standard notation for spaces of polynomials of two variables. We denote P_k the space of polynomials of degree not greater than k, then P_{kl} is defined by

$$P_{kl} = \left\{ p(x,y) \;\mid\; p(x,y) = \sum_{i \leq k, j \leq l} a_{ij} x^i y^j \right\}.$$

and $Q_k = P_{k,k}$.

We suppose now that Ω is a union of rectangles in such a way that we can consider a regular mesh $(\mathcal{T}_h)$ with squares elements (K) of edge $h > 0$. To obtain our finite element spaces, we shall adopt a constructive approach which aims in particular at exploiting the geometry of the mesh. We look for approximation spaces of the form

$$(22)\qquad \begin{array}{l} \underline{M_h} = \{u_h \in \underline{M} \,/\, \forall K \in \mathcal{T}_h,\ u_h|_K \in \mathcal{P}\}, \\ \underline{\underline{X_h}} = \left\{\sigma_h \in \underline{\underline{X}} \,/\, \forall K \in \mathcal{T}_h,\ \sigma_h|_K \in \mathcal{Q}\right\}, \\ \underline{\underline{X_h}}^{sym} = \left\{\sigma_h \in \underline{\underline{X_h}};\, /\ \mathbf{as}(\sigma_h) = 0\right\}, \end{array}$$

where $\mathcal{P}$ and $\mathcal{Q}$ are finite dimensional spaces of C^∞ functions, for instance polynomials, to be determined. We can remark that $\underline{\underline{X_h}}^{sym}$ is seeked as a subspace of $\underline{\underline{X}}^{sym}$: we want to take into account the symmetry condition in the strong sense, contrary to the approach presented in section 2.1.3. Then the approximate problem associated to the mixed velocity-stress system for elastodynamics can be written in the following form

$$\text{(23)} \quad \left\{ \begin{array}{lll} \text{Find } (\sigma_h, v_h) : [0,T] \longmapsto \underline{\underline{X_h}}^{sym} \times \underline{M_h} \text{ such that} & & \\ & & \\ \dfrac{d}{dt} a(\sigma_h, \tau_h) + b(v_h, \tau_h) & = 0, & \forall \tau_h \in \underline{\underline{X_h}}^{sym}, \\ \dfrac{d}{dt} c(v_h, w_h) - b(w_h, \sigma_h) & = (f, w_h), & \forall w_h \in \underline{M_h}, \end{array} \right.$$

with initial conditions ($(\sigma_{0,h}, v_{0,h})$ being approximations of (σ_0, v_0))

$$\sigma_h(0) = \sigma_{0,h} \; ; \; v_h(0) = v_{0,h}.$$

We want to construct the lowest order spaces $\underline{\underline{X_h}}$ and $\underline{M_h}$. First, it is natural to approximate the velocity with piecewise constants

$$\text{(24)} \quad \underline{M_h} = \left\{ v_h \in \underline{M} \; / \; \forall K \in \mathcal{T}_h, \; v_h|_K \in (P_0)^2 \right\}.$$

A possible choice for $\underline{\underline{X_h}}$ is the lowest order Raviart Thomas element $RT_{[0]}$

$$X_h^{RT} = \left\{ \sigma_h \in \underline{\underline{X}} / \forall K \in \mathcal{T}_h, \; (\sigma_{h1}, \sigma_{h2})|_K \in (RT_{[0]})^2 \right\}, \quad \text{where } RT_{[0]} = P_{1,0} \times P_{0,1}.$$

However, this choice is not satisfactory, since the space $X_h^{RT} \cap \underline{\underline{X}}^{sym}$ is too small and thus cannot be considered as a good approximation space for $\underline{\underline{X}}^{sym}$: indeed, if σ_h is a symmetric tensor in X_h^{RT}, then σ_{12} is necessarily constant in Ω. To see this, note that σ_{12} is linear in x_2 and constant in x_1 and on the other hand σ_{21} is linear in x_1 and constant in x_2. Thus imposing $\sigma_{12} = \sigma_{21}$ implies that they should be both constant inside each cell. Finally using the continuity relations in $H(\mathrm{div})$ we can show that they should be constant everywhere. That is a kind of numerical locking.

Remark 3 *If one wants to choose σ_h in X_h^{RT}, one must impose, as it has been developed for the stationary problem ([1], [25], [20]), the symmetry condition in a weak way using the relaxed variational formulation (20). We then need introduce some approximation space for L, the space for the Lagrange multiplier*

$$\text{(25)} \quad L_h = \left\{ \eta_h \in L / \forall K \in \mathcal{T}_h, \; \eta_h|_K \in \hat{L} \right\},$$

with $\hat{L}$ a polynomial space on K. In this case the approximate problem is

$$\text{(26)} \quad \left\{ \begin{array}{lll} \textit{Find } (\sigma_h, v_h, \gamma_h) : [0,T] \longmapsto \underline{\underline{X_h}} \times \underline{M_h} \times L_h \textit{ such that} & & \\ & & \\ \dfrac{d}{dt} a(\sigma_h, \tau_h) + b(v_h, \tau_h) + (\gamma_h, \mathbf{as}(\tau_h))_L & = 0, & \forall \tau_h \in \underline{\underline{X_h}}, \\ \dfrac{d}{dt} c(v_h, w_h) - b(w_h, \sigma_h) & = (f, w_h), & \forall w_h \in \underline{M_h}, \\ (\eta_h, \mathbf{as}(\tau_h))_L & = 0, & \forall \eta_h \in L_h. \end{array} \right.$$

This formulation was analyzed by several authors ([1], [25], [20]) in the case of the stationary problem. We do not use it because we do not see how to do mass lumping, in particular because of the presence of the Lagrange multiplier.

We shall construct our space $\underline{\underline{X_h}}^{sym}$ as a subspace of $\underline{\underline{X}}^{sym}$, guided by the following theorem.

Theorem 1 *For all $\sigma_h \in \underline{\underline{X_h}}$, where $\underline{\underline{X_h}}$ is given by (22), we have*

$$\sigma_h \in \underline{\underline{X_h}}^{sym} \Leftrightarrow \left[\begin{array}{c} \begin{pmatrix} \sigma_{11} \\ \sigma_{22} \end{pmatrix} \in H(\mathrm{div}, \Omega) \\ \\ \sigma_{12} = \sigma_{21} \in H^1(\Omega) \end{array} \right]$$

Proof. We only prove the direct statement, the inverse being obvious. We first remind that for all $\sigma_h \in \underline{\underline{X_h}}$, we have $\sigma_1 = \begin{bmatrix} \sigma_{11} \\ \sigma_{12} \end{bmatrix} \in H(\text{div}\ ,\Omega)$ and $\sigma_2 = \begin{bmatrix} \sigma_{21} \\ \sigma_{22} \end{bmatrix} \in H(\text{div},\Omega)$. Because of the geometry of the mesh we deduce that

- σ_{11} and σ_{21} are continuous in the x_1 direction and discontinuous in the x_2 direction, that means, they can have a jump only through the vertical edges of the mesh,
- in the same way σ_{12} and σ_{22} can only have a jump through the horizontal edges of the mesh.

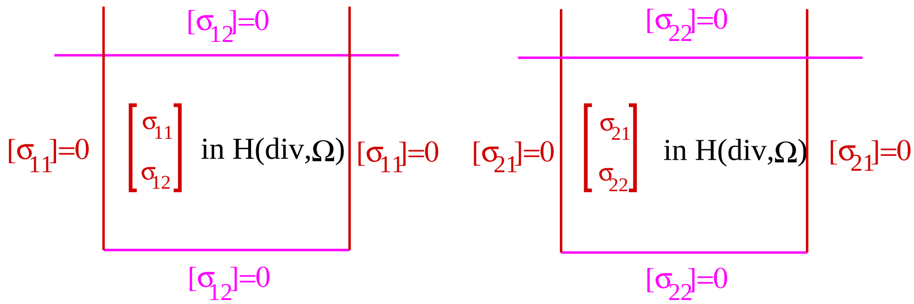


Figure 1: Continuity of $\sigma_h \in \underline{\underline{X_h}}$

If moreover $\sigma_h \in \underline{\underline{X_h}}^{sym}$, then $\sigma_{12} = \sigma_{21}$ is continuous across any edge of the mesh (see figure 2) and thus belongs to $H^1(\Omega)$.

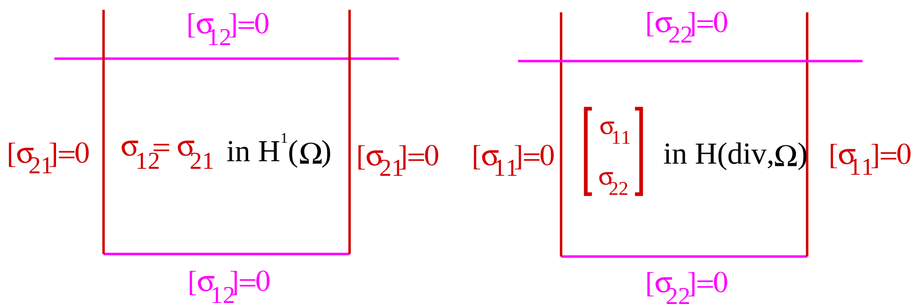


Figure 2: Continuity of $\sigma_h \in \underline{\underline{X_h}}^{sym}$

Moreover, as $(\sigma_{11},\sigma_{22})^t\,|_K \in H(\text{div},K)$ for any K and as σ_{11}(resp. σ_{22}) is continuous in x_1(resp. in x_2), we deduce that the vector $(\sigma_{11},\ \sigma_{22})^t$ belongs to $H(\text{div},\Omega)$. ■

Theorem 1 shows that σ_{12} must belong to an approximation space of H^1. Thus, in order to define the lowest order element, it is natural to choose

$$\sigma_{12} \in H^1(\Omega)/\forall K \in \mathcal{T}_h,\ \sigma_{12}\,|_K \in Q_1.$$

It remains now to define the approximation space of $H(\text{div},\Omega)$ in which the vector $(\sigma_{11},\sigma_{22})^t$ belongs. Once again, a natural choice is the lowest order Raviart-Thomas element $RT_{[0]}$. We denote

$$\widetilde{X}_h^{sym} = \left\{\sigma_h \in \underline{\underline{X}}/\forall K \in \mathcal{T}_h,\ \sigma_{12}\,|_K \in Q_1 \text{ and } (\sigma_{11},\sigma_{22})\,|_K \in RT_{[0]}\right\}.$$

However, we can easily show that this choice does not permit to get an explicit time discretization scheme. This was explained in [7] for the anisotropic wave equation. The reason is that the degrees of freedom for the stress tensor are associated whether to a vertex of an element K(for σ_{12}) or to an edge (for $(\sigma_{11},\sigma_{22})$) as we can see in figure 3.

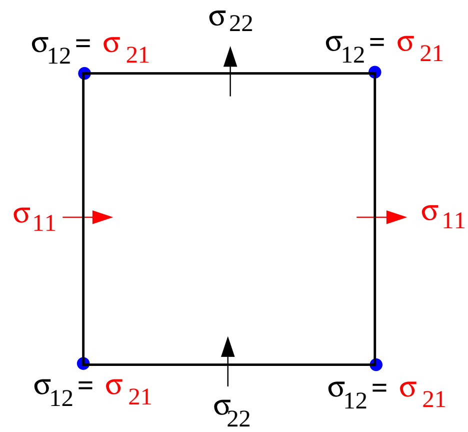


Figure 3: Degrees of freedom associated to the stress tensor for the space $\widetilde{X}_h^{sym}$

Indeed, in order to obtain an explicit time discretization scheme, we want to use a mass lumping technique for the approximation of the mass matrix associated to the bilinear form $a(\sigma_h, \tau_h)$ (the reader can verify that the matrix associated to $c(v_h, w_h)$ is already diagonal in the usual basis of $\underline{M_h}$). Thus we are led to approximate the mass matrix $a(\sigma_h, \tau_h)$ by

$$a_h(\sigma_h, \tau_h) = \sum_{K \in \mathcal{T}_h} I_K(A\sigma_h : \tau_h),$$

where I_K is some quadrature formula, to be determined. Unfortunately, we did not find any quadrature formula leading to a diagonal (or block diagonal) mass matrix (for more details see ([7]), ([5]). So, we have to choose another space for the approximation of $H(\mathrm{div}, \Omega)$. The idea is to regroup all the degrees of freedom at the nodes of the quadrature formula, namely the nodes of the mesh. Under this condition the adequate choice for $(\sigma_{11}, \sigma_{22})$ is the lowest order element of the second family of mixed finite elements proposed by Nédélec in [21], that is

$$\left\{q_h \in H(\mathrm{div}, \Omega) \text{ such that } q_h\,|_K \in (Q_1(K))^2 \quad \forall K \in \mathcal{T}_h\right\},$$

and thus our choice for the space $\underline{\underline{X_h}}^{sym}$ can be written as

$$\begin{aligned}\underline{\underline{X_h}}^{sym} = \quad & \left\{\sigma_{12} \in H^1(\Omega)/\sigma_{12}\,|_K,\ \forall K \in \mathcal{T}_h \quad \text{and}\right.\\ & \left.(\sigma_{11}, \sigma_{22}) \in H(\mathrm{div}, \Omega)/(\sigma_{11}, \sigma_{22})\,|_K \in (Q_1)^2,\ \forall K \in \mathcal{T}_h\right\}.\end{aligned}$$

The key point is that now the degrees of freedom of the stress tensor are all associated to the vertices of an element K as we can see in figures 4 and 5. In this case the approximation of $a(\sigma_h, \tau_h)$ using the quadrature formula

$$\int_k f dx \approx I_K(f) = \frac{h^2}{4} \sum_{M \text{vertices of } K} f(M), \qquad \forall f \in C^o(K), \tag{27}$$

on K leads to a block diagonal mass matrix. Each block is associated to a node of the mesh and its dimension is equal to the number of degrees of freedom at this point (that is 5, see figure 5).

Remark 4 *The space $\underline{\underline{X_h}}^{sym}$ can also be obtained as follows*

- *Choose both σ_{h1} and σ_{h2} (defined as in 4) in the second family of mixed finite elements of Nédélec [21]:*

$$\underline{\underline{X_h}} = \left\{\sigma_h \in \underline{\underline{X}}/(\sigma_{h1}, \sigma_{h2})\,|_K \in (Q_1(K)^2 \times Q_1(K)^2) \quad \forall K \in \mathcal{T}_h\right\},$$

- *Impose the symmetry in a strong way*

$$\underline{\underline{X_h}}^{sym} = \underline{\underline{X_h}} \cap \underline{\underline{X}}^{sym}.$$

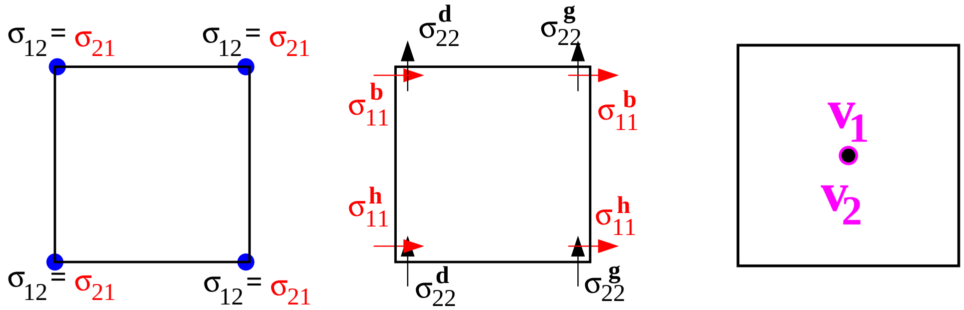


Figure 4: Degrees of freedom for the stress tensor (left and center) and the velocity (right)

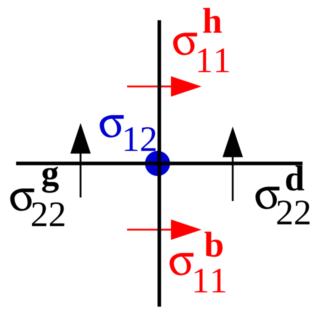


Figure 5: Degrees of freedom for the stress tensor associated to a node of the mesh

2.3 Extension to higher orders and mass lumping

The natural generalization of the lowest order element presented in the previous section, consists in taking (see remark 4)

$$
\left\{
\begin{array}{l}
\underline{\underline{X_h}} = \left\{ \tau_h \in X \,/\, \forall K \in \mathcal{T}_h,\ \tau_h|_K \in (Q_{k+1})^4 \right\} = X_h \times X_h, \\[2mm]
\underline{\underline{X_h}}^{sym} = \underline{\underline{X_h}} \cap \underline{\underline{X}}^{sym}, \\[2mm]
\underline{M_h} = \left\{ w_h \in \underline{M} \,/\ \ \forall K \in \mathcal{T}_h,\ w_h|_K \in (Q_k)^2 \right\} = M_h \times M_h.
\end{array}
\right.
\tag{28}
$$

This will be referred as the $Q_{k+1} - Q_k$ element. For simplicity, we focus here on the mixed finite elements corresponding to $k = 1, 2$, that we use in practice. We represent the degrees of freedom for these elements in Figures 6 and 7. We also indicate in these figures the number of degrees of freedom per node.

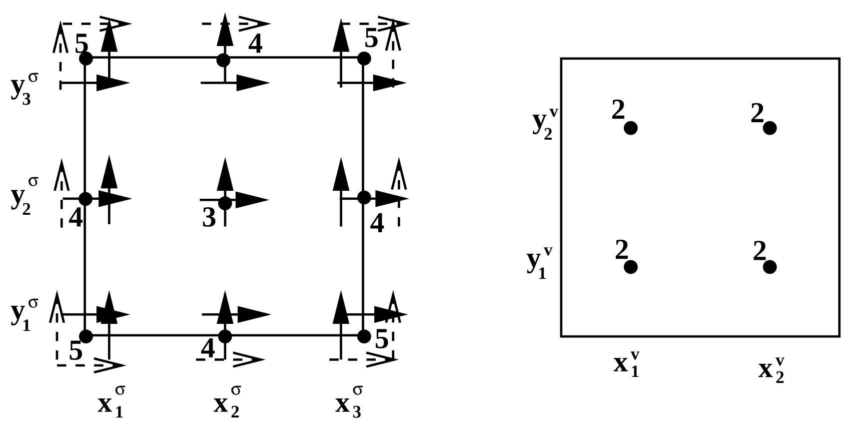


Figure 6: Degrees of freedom in the $Q_2 - Q_1$ element

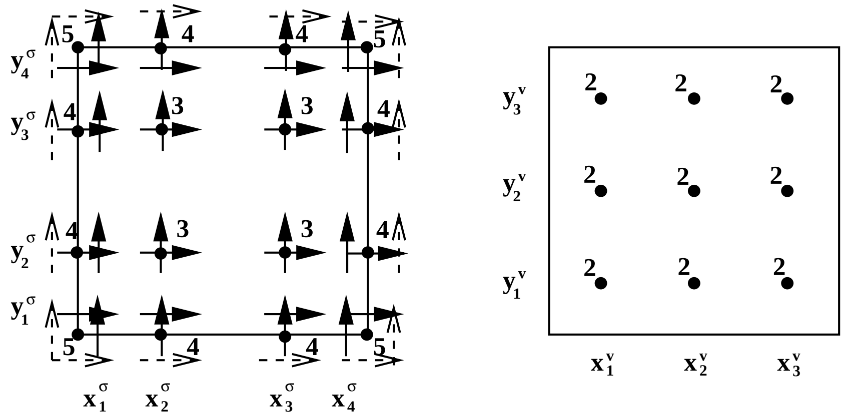


Figure 7: Degrees of freedom in the $Q_3 - Q_2$ element

The locations of the degrees of freedom correspond to tensor products of 1D quadrature points associated to Gauss-Lobatto (for σ_h) or Gauss-Legendre (for v_h) quadrature formulas. The exact location of the quadrature points and the associated weights are given by:

For $k = 1$ (Fig. 6)

- The Gauss-Lobatto points and weights:

$$\left\{\begin{array}{l} x_1^\sigma = y_1^\sigma = 0,\ x_2^\sigma = y_2^\sigma = \dfrac{1}{2},\ x_3^\sigma = y_3^\sigma = 1, \\ \omega_{11}^\sigma = \omega_{31}^\sigma = \omega_{33}^\sigma = \omega_{13}^\sigma = \dfrac{1}{36}, \\ \omega_{21}^\sigma = \omega_{32}^\sigma = \omega_{23}^\sigma = \omega_{12}^\sigma = \dfrac{1}{9}, \\ \omega_{22}^\sigma = \dfrac{4}{9}. \end{array}\right.$$

- The Gauss-Legendre points and weights:

$$\left\{\begin{array}{l} x_1^v = y_1^v = \dfrac{3-\sqrt{3}}{6},\ x_2^v = y_2^v = \dfrac{3+\sqrt{3}}{6}, \\ \omega_{11}^v = \omega_{21}^v = \omega_{22}^v = \omega_{12}^v = \dfrac{1}{4}. \end{array}\right.$$

For $k = 2$ (Fig. 7)

- The Gauss-Lobatto points and weights:

$$\left\{\begin{array}{l} x_1^\sigma = y_1^\sigma = 0,\ x_2^\sigma = y_2^\sigma = \dfrac{5-\sqrt{5}}{10},\ x_3^\sigma = y_3^\sigma = \dfrac{5+\sqrt{5}}{10},\ x_4^\sigma = y_4^\sigma = 1, \\ \omega_{11}^\sigma = \omega_{41}^\sigma = \omega_{44}^\sigma = \omega_{14}^\sigma = \dfrac{1}{144}, \\ \omega_{21}^\sigma = \omega_{31}^\sigma = \omega_{42}^\sigma = \omega_{43}^\sigma = \omega_{34}^\sigma = \omega_{24}^\sigma = \omega_{13}^\sigma = \omega_{12}^\sigma = \dfrac{5}{144}, \\ \omega_{22}^\sigma = \omega_{32}^\sigma = \omega_{33}^\sigma = \omega_{23}^\sigma = \dfrac{25}{144}. \end{array}\right.$$

- The Gauss-Legendre points and weights:

$$\begin{cases} x_1^v = y_1^v = \dfrac{5-\sqrt{15}}{10}, \; x_2^v = y_2^v = \dfrac{1}{2}, \; x_3^v = y_3^v = \dfrac{5+\sqrt{15}}{10}, \\ \omega_{11}^v = \omega_{31}^v = \omega_{33}^v = \omega_{13}^v = \dfrac{25}{324}, \\ \omega_{21}^v = \omega_{32}^v = \omega_{23}^v = \omega_{12}^v = \dfrac{10}{81}, \\ \omega_{22}^v = \dfrac{16}{81}. \end{cases}$$

Mass lumping can be achieved using the same technique as for the anisotropic wave equation [7]. That consists in approximating the integrals in the mass matrices $a(\sigma_h, \tau_h)$ and $c(v_h, w_h)$ by adequate quadrature formulas. More precisely

- we use the Gauss-Lobatto quadrature formulas to compute $a(\sigma_h, \tau_h)$ (our degrees of freedom for the stress tensors are associated to the Gauss-Lobatto points). The resulting matrix is now block diagonal. Each block is associated to one quadrature point and its dimension is equal to the number of degrees of freedom at this point: from 3×3 for the "interior" nodes to 5×5 for the vertices of the mesh.

- The matrix $c(v_h, w_h)$ does not really need to be mass lumped, it suffices to remark that $\underline{M_h}$ is discontinuous and thus we always obtain a block diagonal matrix. Each block is associated to an element of the mesh and its dimension is the number of degrees of freedom in the element($(k+1)^2$). However, for convenience, we also use a quadrature formula for the approximation of this matrix : the Gauss-Legendre quadrature formula which leads to a diagonal matrix.

Remark 5 *The generalization of the previous techniques to higher orders ($k \geq 3$) can be done without great difficulties, using higher order Gauss Lobatto and Gauss Legendre quadrature formulas [15].*

2.4 Extension to 3D

We shall only present in this section the lowest order 3D finite element. The extension to higher orders is similar to the 2D case but the presentation would be tedious.

Let Ω be a bounded domain of $\mathbb{R}^3$ and consider a regular mesh $(\mathcal{T}_h)$ of Ω with cubic elements (K) of edge $h > 0$. Applying the same approach as in the 2D case, we remark that imposing the symmetry as a strong condition implies the following :

- $(\sigma_{11},\ \sigma_{22},\ \sigma_{33}) \in H(\mathrm{div}, \Omega)$ since σ_{11} is continuous only in the x_1 direction, σ_{22} in the x_2 and σ_{33} in the x_3 direction,

- σ_{12}, σ_{13} and σ_{23} are continuous in two directions. More precisely, if i, j, k denotes a circular permutation of $1, 2, 3$

$$x_i \rightarrow \sigma_{jk}(\cdot, x_i) \in L^2_{x_i}(H^1_{x_j, x_k}).$$

We introduce the approximation spaces $\underline{\underline{X_h}} \subset \underline{\underline{X}}$, $\underline{\underline{X_h}}^{sym} \subset \underline{\underline{X}}^{sym}$ and $\underline{M_h} \subset \underline{M}$ defined by (22) with

$$\mathcal{Q} = (Q_1)^9, \qquad \mathcal{P} = (Q_0)^3.$$

We give in figure 8 a schematic view of this element, via the degrees of freedom.

Remark 6 *We can remark that the number of degrees of freedom associated to the stress tensor σ (18 per node of the mesh) is quite important even for the lowest order element. Thus the reader could naturally think that this element is quite expensive. However we can overcome this inconvenience by eliminating the unknown associated to σ. That is possible because of mass-lumping. Thus the only unknown remaining is the one associated to the velocity and that corresponds to 3 unknowns per element (in the case of the lowest order element).*

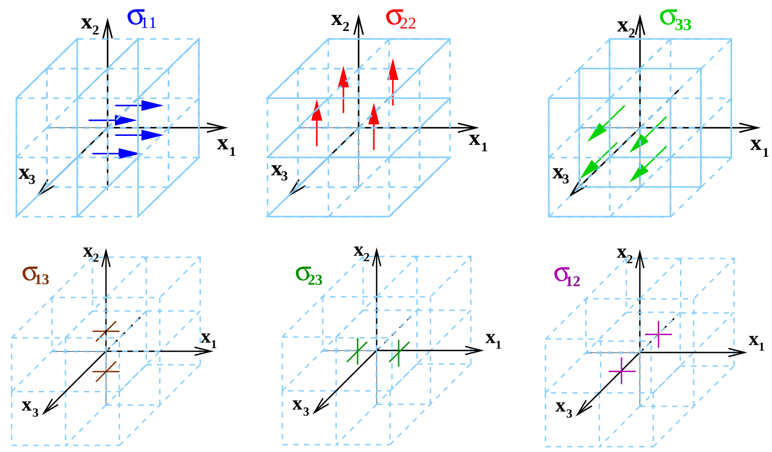


Figure 8: A schematic view of the lowest order element in 3D.

3 Analysis of the new mixed finite element for an elliptic problem : the static elasticity case

For the study of the time dependent problem, we shall follow the approach of [7] concerning the anisotropic wave equation. It is made of two main steps. One step consists in relating, thanks to energy estimates, error estimates for the evolution problem to the estimation of the difference between the exact solution and its elliptic projection (that has to be cleverly defined). This step is rather standard and delayed to section 4. The second step, which amounts to analyzing the elliptic projection error, is much more difficult. It directly follows from the analysis of the approximation of the stationary problem associated to the evolution problem (23). This is the object of the present section.

In the following, in order to simplify the presentation, we will consider that Ω is a square in such a way that we can consider a regular mesh $(\mathcal{T}_h)$ composed by $N \times N$ squares elements (K) of edge $h = 1/N$. However our results can be extended without any difficulty to any general structured mesh. Take then $\underline{\underline{X_h}}, \underline{\underline{X_h}}^{sym}$ and $\underline{M_h}$ the new family of mixed finite elements defined by

$$\left\{\begin{array}{l} \underline{\underline{X_h}} = \left\{\tau_h \in X \,/\, \forall K \in \mathcal{T}_h,\ \tau_h|_K \in (Q_{k+1})^4\right\}, \\[2mm] \underline{\underline{X_h}}^{sym} = \underline{\underline{X_h}} \cap \underline{\underline{X}}^{sym}, \\[2mm] \underline{M_h} = \left\{w_h \in \underline{M} /\ \ \forall K \in \mathcal{T}_h,\ w_h|_K \in (Q_k)^2\right\}. \end{array}\right.$$

3.1 The elliptic problem in $\underline{\underline{X}}^{sym}$ and its approximation

The stationary problem associated to the evolution problem (16) can be written

$$(29) \qquad \left\{\begin{array}{llll} \text{Find } (\sigma, v) \in \underline{\underline{X}}^{sym} \times \underline{M} \text{ such that} & & & \\[2mm] a(\sigma,\tau) + & b(v,\tau) & = 0\,, & \forall \tau \in \underline{\underline{X}}^{sym}, \\[2mm] & b(w,\sigma) & = -(f,w), & \forall w \in \underline{M}, \end{array}\right.$$

where $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ are defined by (17) and satisfy properties (18). As a consequence [11], there exists a unique solution (σ, v) in $\underline{\underline{X}}^{sym} \times \underline{M}$ of problem (29). The corresponding discrete problem is

$$(30) \qquad \left\{\begin{array}{llll} \text{Find } (\sigma_h, v_h) \in \underline{\underline{X_h}}^{sym} \times \underline{M_h} \text{ such that} & & & \\[2mm] a(\sigma_h,\tau_h) + & b(v_h,\tau_h) & = 0, & \forall \tau_h \in \underline{\underline{X_h}}^{sym}, \\[2mm] & b(w_h,\sigma_h) & = -(f,w_h), & \forall w \in \underline{M_h}. \end{array}\right.$$

3.2 The difficulties of the convergence analysis

According to the classical theory (c.f [11]), we know that (30) admits a unique solution (σ_h, v_h) in $\underline{\underline{X_h}}^{sym} \times \underline{M_h}$ with the convergence property

$$(31) \qquad (\sigma_h, v_h) \longrightarrow (\sigma, v) \quad \in \quad \underline{\underline{X}} \times \underline{M},$$

when the following assumptions are satisfied

(H0.0) $\forall f \in \text{Im } B$, $V_h(f) \neq \emptyset$, where we set

$$V_h(f) = \left\{ \tau_h \in \underline{\underline{X_h}}^{sym} \;/\; b(w_h, \tau_h) = -(f, w_h),\ \forall\ w_h \in \underline{M_h} \right\}$$

(H0.1) The uniform discrete inf-sup condition

There exists a constant $\beta > 0$ independent of h such that

$$\forall\ w_h \in \underline{M_h},\ \exists\ \tau_h \in \underline{\underline{X_h}}^{sym} \;/\; b(w_h, \tau_h) \geq \beta \left\| w_h \right\|_M \left\| \tau_h \right\|_X .$$

(H0.2) The coercivity of the form $a(\cdot,\cdot)$ on $V_h \stackrel{def}{=} V_h(0)$

There exists a constant $\alpha > 0$ independent of h such that

$$\forall\ \sigma_h \in V_h, \quad a(\sigma_h, \sigma_h) \geq \alpha \left\| \sigma_h \right\|_X^2 .$$

(H0.3) Approximation properties

$$\begin{cases} \lim\limits_{h \to 0} \inf\limits_{\tau_h \in X_h^{sym}} \left\| \sigma - \tau_h \right\|_X = 0, & \forall \sigma \in X^{sym}, \\ \lim\limits_{h \to 0} \inf\limits_{w_h \in M_h} \left\| v - w_h \right\|_M = 0, & \forall v \in M. \end{cases}$$

With our choice (28) for $\underline{\underline{X_h}}^{sym}$ and $\underline{M_h}$, it is easy to check that property (H0.2) is not satisfied. The proof is the same as for the scalar problem described in [7].

The abstract theory we developed in [7] (shortly described in [6]) involves a third Hilbert space $\underline{\underline{H}}$ such that $\underline{\underline{X}} \subset \underline{\underline{H}}$, with $\underline{\underline{X}}$ dense in $\underline{\underline{H}}$. One replaces conditions (H0.0) to (H0.3) by the following set of conditions

(H1.0) $\forall f \in \text{Im } B$, $V_h(f) \neq \emptyset$, where we set

$$V_h(f) = \left\{ \tau_h \in \underline{\underline{X_h}}^{sym} \;/\; b(w_h, \tau_h) = -(f, w_h),\ \forall w_h \in \underline{M_h} \right\}, \quad V_h = V_h(0).$$

(H1.1) Orthogonal decomposition of $\underline{\underline{X_h}}^{sym}$

$$\left| \begin{array}{l} \underline{\underline{X_h}}^{sym} = \underline{\underline{X_h}}^{s} \oplus \underline{\underline{X_h}}^{r} \ \ (\sigma_h = \sigma_h^s + \sigma_h^r) \ \ , \ \ \underline{\underline{X_h}}^{r} \subset V_h, \\ (\sigma_h^s, \sigma_h^r)_H = 0, \ \ \forall (\sigma_h^s, \sigma_h^r) \in \underline{\underline{X_h}}^{s} \times \underline{\underline{X_h}}^{r}. \end{array} \right.$$

(H1.2) "Strong" discrete uniform inf-sup condition

There exists a constant $\beta > 0$, independent of h, such that

$$\forall\ w_h \in \underline{M_h}, \quad \exists\ \tau_h^s \in \underline{\underline{X_h}}^{s} \quad / \quad b(w_h, \tau_h^s) \geq \beta \left\| w_h \right\|_{M/\text{Ker } B_h^t} \left\| \tau_h^s \right\|_X .$$

(H1.3) "Weak" coercivity

There exists a constant $\alpha > 0$, independent of h, such that

$$\forall\ \sigma_h \in V_h, \ \ a(\sigma_h, \sigma_h) \geq \alpha \left(\left\| \sigma_h^s \right\|_X^2 + \left| \sigma_h^r \right|_H^2 \right).$$

(H1.4) Approximation properties

$$\left\{\begin{array}{l} \lim\limits_{h\to 0} \inf\limits_{\tau_h^s \in \underline{\underline{X_h}}^s} \|\sigma - \tau_h^s\|_X = 0, \quad \forall \sigma \in \underline{\underline{X}}^{sym} \\ \lim\limits_{h\to 0} \inf\limits_{w_h \in \underline{M_h}} \|v - w_h\|_M = 0, \quad \forall v \in \underline{M}. \end{array}\right.$$

Remark 7 *The approximation properties (H1.4) are close to the classical ones. The only difference is that $\underline{\underline{X}}^{sym}$ must be approximated by $\underline{\underline{X_h}}^s$ and not only by $\underline{\underline{X_h}}^{sym}$.*

Under these assumptions [7], (30) admits a unique solution $(\sigma_h = \sigma_h^s + \sigma_h^r, v_h)$ in $\underline{\underline{X_h}}^{sym} \times \underline{M_h}$ with the convergence property

$$(32) \qquad \begin{array}{l} (\sigma_h^s, v_h) \longrightarrow (\sigma, v) \quad \text{in} \quad \underline{\underline{X}} \times \underline{M}, \\ \sigma_h^r \longrightarrow 0 \quad \text{in} \quad \underline{\underline{H}}, \end{array}$$

If we wanted to apply this theory to our problem, $\underline{\underline{H}}$ must be the space of L^2 tensor-valued functions. According to [7], the candidate for the decomposition (H1.1) is then

$$\underline{\underline{X_h}}^s = \underline{\underline{X_h}}^{RT} \cap \underline{\underline{X}}^{sym}.$$

where

$$\underline{\underline{X_h}}^{RT} = \{\sigma_h = (\sigma_{1,h}, \sigma_{2,h}) \in \underline{\underline{X}} \; / \; \forall K \in \mathcal{T}_h, (\sigma_{1,h}, \sigma_{2,h})|_K \in (P_{k+1,k} \times P_{k,k+1})^2\}.$$

With such a choice, conditions (H1.0) to (H1.3) are satisfied. The problem is that, as we have already explained in section 2.2, $\underline{\underline{X_h}}^s$ is too small to be an approximation space for $\underline{\underline{X}}^{sym}$. In other words, property (H1.4) is not satisfied. It is clear at this point that the main difficulty is due to our choice of imposing the symmetry of the stress tensor in a strong way. In order to overcome this problem, we will step back for a moment by considering an equivalent discrete problem under the form (20) where the symmetry condition has been relaxed.

3.3 A relaxed discrete elliptic problem in $\underline{\underline{X}}_h$

The stationary problem associated to the evolution problem (20) is the following

$$(33) \qquad \left\{\begin{array}{llll} \text{Find } (\sigma, v, \gamma) \in \underline{\underline{X}} \times \underline{M} \times L \text{ such that} & & & \\ a(\sigma,\tau) + \; b(v,\tau) + \; (\gamma, \mathbf{as}(\tau))_L & = 0, & \forall \tau \in \underline{\underline{X}}, \\ b(w,\sigma) & = -(f,w), & \forall w \in \underline{M}, \\ (\eta, \mathbf{as}(\sigma))_L & = 0, & \forall \eta \in L, \end{array}\right.$$

where $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ are defined by (17) and satisfy properties (21). We can rewrite problem (33) in the following form

$$(34) \qquad \left\{\begin{array}{lll} \text{Find } (\sigma, v, \gamma) \in \underline{\underline{X}} \times \underline{M} \times L \text{ such that} & & \\ a(\sigma,\tau) + \; \widetilde{b}(v,\gamma;\tau) = 0, & \forall \tau \in \underline{\underline{X}}, \\ \widetilde{b}(w,\eta;\sigma) = -(f,w), & \forall (w,\eta) \in \underline{M} \times L, \end{array}\right.$$

by setting

$$(35) \qquad \widetilde{b}(w,\eta;\tau) = b(w,\tau) + (\eta, \mathbf{as}(\tau))_L, \quad \forall (\tau, w, \eta) \in \underline{\underline{X}} \times \underline{M} \times L.$$

We know that problems (34) and (29) are equivalent. To get a discrete version of (34), we would like to find a finite dimensional subspace (to be determined) of L, denoted L_h, such that the approximate problem (30) is equivalent to the following one, where we work in $\underline{\underline{X_h}}$ instead of $\underline{\underline{X_h}}^{sym}$

$$(36)\qquad \begin{cases} \text{Find } (\sigma_h, v_h, \gamma_h) \in \underline{\underline{X_h}} \times \underline{M_h} \times L_h \text{ such that} \\ a(\sigma_h, \tau_h) + \quad b(v_h, \tau_h) + \quad (\gamma_h, \mathbf{as}(\tau_h))_L \quad = 0, \qquad \forall \tau_h \in \underline{\underline{X_h}}, \\ \qquad b(w_h, \sigma_h) \qquad\qquad = -(f, w_h), \quad \forall w_h \in \underline{M_h}, \\ \qquad\qquad (\eta_h, \mathbf{as}(\sigma_h))_L \quad = 0, \qquad \forall \eta_h \in L_h, \end{cases}$$

which we can rewrite in the following form

$$(37)\qquad \begin{cases} \text{Find } (\sigma_h, v_h, \gamma_h) \in \underline{\underline{X_h}} \times \underline{M_h} \times L_h \text{ such that} \\ a(\sigma_h, \tau_h) + \quad \widetilde{b}(v_h, \gamma_h; \tau_h) \quad = 0, \qquad \forall \tau_h \in \underline{\underline{X_h}}, \\ \qquad \widetilde{b}(w_h, \eta_h; \sigma_h) \quad = -(f, w_h), \quad \forall (w_h, \eta_h) \in \underline{M_h} \times L_h. \end{cases}$$

Assume for the moment that we can define L_h in this way, then we can obtain error estimates for problem (30) by studying the equivalent problem (37).

However, once again, neither the classical theory nor the abstract theory of [7] can be applied to the analysis of (37).

First, as far as the classical theory is concerned, as for problem (30), one can easily show that property (H0.2) (v_h being replaced (v_h, γ_h), $\underline{\underline{X_h}}^{sym}$ by $\underline{\underline{X_h}}$, M_h by $M_h \times L_h$ and $b(\cdot,\ \cdot)$ by $\widetilde{b}(\cdot,\ \cdot;\ \cdot)$ is not satisfied.

Concerning the abstract theory developed in [7], we have not succeeded to verify condition (H1.2)(we conjecture it is not true).

Thus, in order to obtain error estimates for problem (37), we introduce in the following section a new abstract theory, which consists in modifying hypotheses (H1.0)-(H1.4).

Remark 8 *Let us emphasize that the introduction of the equivalent problem (37) is necessary only for the theoretical analysis. However, we never solve numerically problem (37)(or more precisely its corresponding time dependent problem).*

3.4 An abstract result

We cannot avoid introducing a lot of notation in order to state our result. Let $\underline{M}$, $\underline{\underline{X}}$, $\underline{\underline{H}}$ and L be four Hilbert spaces with

$$(38)\qquad \underline{\underline{X}} \subset \underline{\underline{H}},\ \underline{\underline{X}} \text{ dense in } \underline{\underline{H}} \text{ and } |\cdot|_H \leq \|\cdot\|_X .$$

We shall denote by $\underline{M}'$, $\underline{\underline{X}}'$, $\underline{\underline{H}}'$ and L' the dual spaces. The reader can have in mind that, for our application, we shall have

$$\underline{\underline{H}} = (L^2(\Omega))^4,\ \underline{\underline{X}} = (H(\mathrm{div}, \Omega))^2,\ \underline{M} = (L^2(\Omega))^2 \text{ and } L = L^2(\Omega).$$

Let $a(\cdot,\cdot)$, $b(\cdot,\cdot)$ and $d(\cdot,\cdot)$ be three continuous bilinear forms respectively in $\underline{\underline{H}} \times \underline{\underline{H}}$, $\underline{M} \times \underline{\underline{X}}$ and $L \times \underline{\underline{H}}$. We define $\widetilde{b}(\cdot, \cdot\ ;\ \cdot)$ as

$$\widetilde{b}(w, \eta; \tau) = b(w, \tau) + d(\eta, \tau), \quad \forall (w, \eta, \tau) \in \underline{M} \times L \times \underline{\underline{X}}.$$

We shall see in the sequel that the bilinear forms $b(\cdot,\cdot)$ and $d(\cdot,\cdot)$ play a different role, essentially because their continuity with respect to the second variable τ involves two different topologies. From the continuity of the bilinear form $a(\cdot,\cdot)$, we know that there exists an operator $\mathcal{A}$ in $\mathcal{L}(H)$ such that

$$a(\sigma, \tau) = (\mathcal{A}\sigma, \tau)_H, \quad \forall\, \sigma, \tau\ \in\ \underline{\underline{H}} \times \underline{\underline{H}}.$$

We also define the operators $B\ : \underline{\underline{X}} \to \underline{M}'$ and $B^t : \underline{M} \to \underline{\underline{X}}'$ by

$$\langle B\tau, w\rangle = \left\langle \tau, B^t w\right\rangle = b(w,\tau), \quad \forall (\tau, w) \in \underline{\underline{X}} \times \underline{M},$$

and their kernels by

$$\text{Ker } B = \left\{\tau \in \underline{\underline{X}} \,/\, b(w,\tau) = 0,\ \forall w \in \underline{M}\right\},$$

$$\text{Ker } B^t = \left\{w \in \underline{M} \,/\, b(w,\tau) = 0,\ \forall \tau \in \underline{\underline{X}}\right\}.$$

In the same way, we define $D : \underline{\underline{H}} \to L'$ and $D^t : L \to \underline{\underline{H}}'$ by

$$\langle D\tau, \eta\rangle = \left\langle \tau, D^t\eta\right\rangle = d(\eta,\tau), \quad \forall (\tau,\eta) \in \underline{\underline{H}} \times L,$$

and their kernels

$$\text{Ker } D = \left\{\tau \in \underline{\underline{H}} \,/\, d(\eta,\tau) = 0,\ \forall \eta \in L\right\},$$

$$\text{Ker } D^t = \left\{\eta \in L \,/\, d(\eta,\tau) = 0,\ \forall \tau \in \underline{\underline{H}}\right\}.$$

Finally, we can define $\widetilde{B} : \underline{\underline{X}} \to \underline{M}' \times L'$ and $\widetilde{B}^t : \underline{M} \times L \to \underline{\underline{X}}'$ such that

$$\left\langle \widetilde{B}\tau; w, \eta\right\rangle = \left\langle \tau; \widetilde{B}^t(w,\eta)\right\rangle = \widetilde{b}(w,\eta;\tau), \quad \forall (\tau, w, \eta) \in \underline{\underline{X}} \times \underline{M} \times L,$$

we then define the kernel of $\widetilde{B}$ and $\widetilde{B}^t$ as follows

$$\text{Ker } \widetilde{B} \equiv V = \left\{\tau \in \underline{\underline{X}} \,/\, \widetilde{b}(w,\eta;\tau) = 0,\ \forall (w,\eta) \in \underline{M} \times L\right\},$$

$$\text{Ker } \widetilde{B}^t \equiv V^t = \left\{(w,\eta) \in \underline{M} \times L \,/\, \widetilde{b}(w,\eta;\tau) = 0,\ \forall \tau \in \underline{\underline{X}}\right\}.$$

In the following, we shall identify the quotient space $(\underline{M} \times L)/\text{Ker } \widetilde{B}^t$ with the orthogonal complement of Ker $\widetilde{B}^t$

$$\begin{aligned}(M \times L)/\text{Ker } \widetilde{B}^t \ &\equiv (\text{Ker } \widetilde{B}^t)^{\perp} \\ &\equiv \left\{(w,\eta) \in \underline{M} \times L \,/\, (v,w)_M + (\gamma,\eta)_L = 0, \forall\ (v,\gamma) \in \text{Ker } \widetilde{B}\right\},\end{aligned}$$

and we assume the following properties ($\|(w,\eta)\|_{M\times L} = \|w\|_M + \|\eta\|_L$.)

$$(39) \qquad \left|\begin{array}{ll} (i) & \exists\ c > 0 \,/\, \forall\ (w,\eta) \in \underline{M} \times L,\ \exists\ \tau \in \underline{\underline{X}} \,/ \\ & \qquad \widetilde{b}(w,\tau;\eta) \geq c\, \|(w,\eta)\|_{(M\times L)/\text{Ker } \widetilde{B}^t}\, \|\tau\|_X\,, \\ (ii) & \exists\ \alpha > 0 \,/\, \forall\ \tau \in V, \quad a(\tau,\tau) \geq \alpha\, \|\tau\|_X^2\,, \end{array}\right.$$

We are interested in the numerical approximation of the solution (σ, v, γ) of the following problem, with $f \in \underline{M}'$

$$(40) \qquad \left\{\begin{array}{llll} \text{Find } (\sigma,v,\gamma) \in \underline{\underline{X}} \times \underline{M} \times L \text{ such that} & & & \\ a(\sigma,\tau) + & \widetilde{b}(v,\gamma;\tau) & = 0, & \forall \tau \in \underline{\underline{X}}, \\ & \widetilde{b}(w,\eta;\sigma) & = -\langle f, w\rangle\,, & \forall (w,\eta) \in \underline{M} \times L, \end{array}\right.$$

Thanks to (39), we have the classical result (see [11])

Theorem 2 *For all $(f,0) \in Im\widetilde{B}$, problem (40) has a unique solution (σ,v,γ) in $\underline{\underline{X}} \times ((\underline{M} \times L)/\text{Ker } \widetilde{B}^t)$. Moreover*

$$\|(v,\gamma)\|_{(M\times L)/\text{Ker } \widetilde{B}^t} + \|\sigma\|_X \leq C\, \|f\|_{M'}\,.$$

We introduce now the same notation for the discrete problem. Suppose now that $\underline{\underline{X_h}} \subset \underline{\underline{X}}$, $\underline{M_h} \subset \underline{M}$ and $L_h \subset L$ are finite dimensional approximation spaces. We consider then the approximate problem

$$
(41) \qquad \left\{
\begin{array}{lll}
\text{Find } (\sigma_h, v_h, \gamma_h) \in \underline{\underline{X_h}} \times \underline{M_h} \times L_h \text{ such that} & & \\
a(\sigma_h, \tau_h) + \quad \widetilde{b}(v_h, \gamma_h; \tau_h) & = 0, & \forall \tau_h \in \underline{\underline{X_h}}, \\
\qquad\qquad \widetilde{b}(w_h, \eta_h; \sigma_h) & = -\left\langle f, w_h \right\rangle, & \forall (w_h, \eta_h) \in \underline{M_h} \times L_h.
\end{array}
\right.
$$

We can introduce, as in the case of the continuous problem, the operators $B_h \; : \underline{\underline{X_h}} \rightarrow \underline{M_h}$ and $B_h^t : \underline{M_h} \rightarrow \underline{\underline{X_h}}$ such that

$$
(B_h \tau_h, w_h)_M = \left(\tau_h, B_h^t w_h\right)_X = b(w_h, \tau_h), \quad \forall (\tau_h, w_h) \in \underline{\underline{X_h}} \times \underline{M_h},
$$

and their kernels

$$
\begin{aligned}
Z_h \equiv \; \text{Ker } B_h &= \left\{\tau_h \in \underline{\underline{X_h}} \,/\, b(w_h, \tau_h) = 0, \; \forall w_h \in \underline{M_h}\right\}, \\
\text{Ker } B_h^t &= \left\{w_h \in \underline{M_h} \,/\, b(w_h, \tau_h) = 0, \; \forall \tau_h \in \underline{\underline{X_h}}\right\}.
\end{aligned}
$$

In the same way, we define $D_h : \underline{\underline{H_h}} \rightarrow L_h$ and $D_h^t \; : L_h \rightarrow \underline{\underline{H_h}}$ such that

$$
(D_h \tau_h, \eta_h)_L = \left(\tau_h, D_h^t \eta_h\right)_H = d(\eta_h, \tau_h), \quad \forall (\tau_h, \eta_h) \in \underline{\underline{X_h}} \times L_h,
$$

and their kernels

$$
\begin{aligned}
\text{Ker } D_h &= \left\{\tau_h \in \underline{\underline{X_h}} \,/\, d(\eta_h, \tau_h) = 0, \; \forall \eta_h \in L_h\right\}, \\
\text{Ker } D_h^t &= \left\{\eta_h \in L_h \,/\, d(\eta_h, \tau_h) = 0, \; \forall \tau_h \in \underline{\underline{X_h}}\right\}.
\end{aligned}
$$

Finally we define $\widetilde{B}_h$ from $\underline{\underline{X_h}}$ to $\underline{M_h} \times L_h$ and $\widetilde{B}_h^t$ from $\underline{M_h} \times L_h$ to $\underline{\underline{X_h}}$ by

$$
\begin{aligned}
\left(\widetilde{B}_h \sigma_h; w_h, \eta_h\right)_{(M \times L)} &= \left(\sigma_h; \widetilde{B}_h^t(w_h, \eta_h)\right)_X \\
&= \widetilde{b}(w_h, \eta_h; \sigma_h), \quad \forall \sigma_h \in \underline{\underline{X_h}}, \; \forall (w_h, \eta_h) \in \underline{M_h} \times L_h.
\end{aligned}
$$

Finally we introduce

$$
\begin{aligned}
V_h \equiv \; \text{Ker} \widetilde{B}_h &= \left\{\sigma_h \in \underline{\underline{X_h}} \,/\, \widetilde{b}(w_h, \eta_h; \sigma_h) = 0, \; \forall (w_h, \eta_h) \in \underline{M_h} \times L_h\right\}, \\
\text{Ker} \widetilde{B}_h^t &= \left\{(w_h, \eta_h) \in \underline{M_h} \times L_h \,/\, \widetilde{b}(w_h, \eta_h; \sigma_h) = 0, \; \forall \sigma_h \in \underline{\underline{X_h}}\right\}.
\end{aligned}
$$

Our convergence theory will be based on the following hypotheses

(H0) $\forall (f, 0) \in \text{Im } \widetilde{B}, V_h(f) = \left\{\tau_h \in \underline{\underline{X_h}} \,/ \widetilde{b}(w_h, \eta_h; \tau_h) = -(f, w_h)\right\} \neq \emptyset.$

(H1) Orthogonal decomposition of X_h

$$
\left|
\begin{array}{l}
\underline{\underline{X_h}} = \underline{\underline{X_h}}^s \oplus \underline{\underline{X_h}}^r \quad (\tau_h = \tau_h^s + \tau_h^r) \quad , \quad \underline{\underline{X_h}}^r \subset Z_h, \\
(\tau_h^s, \tau_h^r)_H = 0, \; \forall \; (\tau_h^s, \tau_h^r) \in \underline{\underline{X_h}}^s \times \underline{\underline{X_h}}^r.
\end{array}
\right.
$$

(H2) "Weak" discrete uniform inf-sup condition

$$
\left|
\begin{array}{l}
\text{There exists a constant } \beta > 0, \text{ independent of } h, \text{ such that} \\
\forall \; (w_h, \eta_h) \in \underline{M_h} \times L_h, \; \exists \; \tau_h \in \underline{\underline{X_h}} \,/ \\
\widetilde{b}(w_h, \eta_h; \tau_h) \geq \beta \, \|(w_h, \eta_h)\|_{(M \times L)/\text{Ker } B_h^t} \, (\|\tau_h^s\|_X + |\tau_h^r|_H).
\end{array}
\right.
$$

(H3) "Weak" coercivity

There exists a constant $\alpha > 0$, independent of h, such that

$$\forall\ \tau_h \in V_h,\ a(\tau_h,\tau_h) \geq \alpha\left(\|\tau_h^s\|_X^2 + |\tau_h^r|_H^2\right).$$

(H4) "Strong" approximation properties

$$\begin{cases} \lim\limits_{h\to 0}\ \inf\limits_{\tau_h^s\in X_h^s} \|\sigma - \tau_h^s\|_X = 0, & \forall \sigma \in X, \\ \lim\limits_{h\to 0}\ \inf\limits_{w_h\in M_h} \|v - w_h\|_M = 0, & \forall u \in M, \\ \lim\limits_{h\to 0}\ \inf\limits_{\eta_h\in L_h} \|\gamma - \eta_h\|_L = 0, & \forall \gamma \in L. \end{cases}$$

(H4b) Additional approximation property in L

$$\lim_{h\to 0}\ \inf_{\eta^{(h)}\in\Lambda_r^{(h)}} \left\|\gamma - \eta^{(h)}\right\|_L = 0, \quad \forall \gamma \in L,$$

where

$$\Lambda_r^{(h)} = \left\{\eta^{(h)} \in L \,/\, d(\eta^{(h)},\tau_h^r) = 0,\ \forall \tau_h^r \in \underline{\underline{X_h}}^r\right\}.$$

Remark 9 *Once again, the approximation properties (H4) are close to the classical ones ($\underline{\underline{X}}$ must be approximated by $\underline{\underline{X_h}}^s$ and not only by $\underline{\underline{X_h}}$).*

On the other hand, hypothesis (H4b) is new, since the space $\Lambda_r^{(h)}$ is not included in L_h (it has infinite dimension). In practice, in order to verify (H4b), we will search for $L^{(h)}$ as a finite dimensional subspace of $\Lambda_r^{(h)}$ which satisfies the approximation property (H4b):

$$\lim_{h\to 0}\ \inf_{\eta^{(h)}\in L^{(h)}} \|\gamma - \eta^{(h)}\|_L = 0 \quad \forall \gamma \in L. \tag{42}$$

We then have

$$\lim_{h\to 0}\ \inf_{\eta^{(h)}\in\Lambda_r^{(h)}} \|\gamma - \eta^{(h)}\|_L \leq \lim_{h\to 0}\ \inf_{\eta^{(h)}\in L^{(h)}} \|\gamma - \eta^{(h)}\|_L = 0 \quad \forall \gamma \in L.$$

Remark 10 *Note that, since $\underline{\underline{X}}$ is dense in $\underline{\underline{H}}$, (H4) implies that:*

$$\lim_{h\to 0}\ \inf_{\tau_h^s\in X_h^s} |\sigma - \tau_h^s|_H = 0, \quad \forall \sigma \in H.$$

Remark 11 *It may be more convenient to characterize hypothesis (H0) by*

$$(H0) - (ii) \quad \text{Ker } \widetilde{B}_h^t = \text{Ker } \widetilde{B}^t \cap (\underline{M_h} \times L_h)$$

Theorem 3 *With the hypotheses (H0) to (H4b), problem (41) admits a unique solution*

$$(\sigma_h = \sigma_h^s + \sigma_h^r, v_h, \gamma_h) \in \underline{\underline{X_h}} \times ((\underline{M_h} \times L_h)/\text{Ker } \widetilde{B}_h^t),$$

and we have the following convergence result

- $(\sigma_h^s, v_h, \gamma_h) \quad \to (\sigma, v, \gamma) \quad$ *in* $\underline{\underline{X}} \times \underline{M} \times L$,
- $\sigma_h^r \quad \to 0 \quad$ *in* $\underline{\underline{H}}$.

More precisely, we obtain the error estimates

$$\begin{aligned} &|\sigma_h^r|_H + \|\sigma - \sigma_h^s\|_X + \|(v - v_h, \gamma - \gamma_h)\|_{(M\times L)/\text{Ker } \widetilde{B}_h^t} \leq \\ &\leq C\left\{\inf_{\tau_h^s\in X_h^s} \|\sigma - \tau_h^s\|_X + \inf_{w_h\in M_h} \|v - w_h\|_M + \inf_{\eta_h\in L_h} \|\gamma - \eta_h\|_L\right. \\ &\qquad \left. + \inf_{z_h^s\in X_h^s} |\mathcal{A}\sigma - z_h^s|_H + \inf_{\eta^{(h)}\in\Lambda_r^{(h)}} \|\gamma - \eta^{(h)}\|_L\right\}. \end{aligned} \tag{43}$$

The convergence follows from the error estimate (43), thanks to (H4) and remark 10.

Proof.

(i) Existence and uniqueness

First note that hypothesis (H3) implies a non-uniform discrete coercivity on the kernel Ker $\widetilde{B}_h$, i.e.,

$$\exists \alpha_h > 0, \ \forall \sigma_h \in V_h \ \ , a(\sigma_h, \sigma_h) \geq \alpha_h \, \|\sigma_h\|_X^2 \, . \tag{44}$$

That is because in finite dimension all the norms are equivalent. Thus using (44) and (H0) we obtain existence and uniqueness of the solution $(\sigma_h, v_h, \gamma_h)$ of the discrete problem (41) in $\underline{\underline{X_h}} \times ((\underline{M_h} \times L_h)/\text{Ker } \widetilde{B}_h^t)$.

(ii) Error estimates on $\sigma - \sigma_h$

In its principle, our proof follows the same lines than the one of the classical theory. We shall use the triangular inequality on the identity $\sigma - \sigma_h = \sigma - \tau_h + \tau_h - \sigma_h$ where τ_h is an arbitrary element of $V_h(f)$ and use the discrete coercivity of a (assumption (H2)) in order to estimate $\tau_h - \sigma_h$. The new difficulty is that we have to deal with the very specific norm in $\underline{\underline{X_h}}$,

$$\tau_h \rightarrow (\|\tau_h^s\|_X^2 + |\tau_h^r|_H^2)^{\frac{1}{2}},$$

appearing in the coercivity inequality (H3) and the inf-sup condition (H2). Therefore, we shall have systematically to get rid of terms, such as terms of the form $b(\cdot, \tau_h^r)$, whose estimate would make appear $\|\tau_h^r\|_X$. This is essentially what makes our proof much longer than the usual one.

From the second equation of (41), σ_h belongs to $V_h(f)$. Thus, for any τ_h in $V_h(f)$

$$\sigma_h - \tau_h \in V_h.$$

We can write

$$a(\sigma_h - \tau_h, \sigma_h - \tau_h) = a(\sigma - \tau_h, \sigma_h - \tau_h) + a(\sigma_h - \sigma, \sigma_h - \tau_h). \tag{45}$$

Taking the difference between the first equation of the continuous (40) and the discrete (41) problem we get

$$a(\sigma_h - \sigma, \sigma_h - \tau_h) = \widetilde{b}(v - w_h, \gamma - \eta_h; \sigma_h - \tau_h), \ \forall (w_h, \eta_h) \in \underline{M_h} \times L_h. \tag{46}$$

Using (H1) in (46), we can write, with obvious notations

$$\begin{aligned} a(\sigma_h - \sigma, \sigma_h - \tau_h) \ &= b(v - w_h, \sigma_h^s - \tau_h^s) \\ &+ b(v, \sigma_h^r - \tau_h^r) + d(\gamma - \eta_h, \sigma_h - \tau_h). \end{aligned} \tag{47}$$

We have used the fact that $b(w_h, \sigma_h^r - \tau_h^r) = 0$, since $X_h^r \subset Z_h$. Now, since we only have a "weak" coercivity, we want to keep only terms leading to H-norm on the X_h^r part, thus we want to eliminate $b(u, \sigma_h^r - \tau_h^r)$ (which would give $\|\sigma_h^r - \tau_h^r\|_X$ from the continuity of b). By using then the first equation of the continuous problem (40), we obtain

$$\left| \begin{aligned} a(\sigma_h - \sigma, \sigma_h - \tau_h) \ &= b(v - w_h, \sigma_h^s - \tau_h^s) - a(\sigma, \sigma_h^r - \tau_h^r) \\ &- d(\gamma, \sigma_h^r - \tau_h^r) + d(\gamma - \eta_h, \sigma_h - \tau_h) \\ &= b(v - w_h, \sigma_h^s - \tau_h^s) - (\mathcal{A}\sigma, \sigma_h^r - \tau_h^r)_H \\ &- d(\gamma, \sigma_h^r - \tau_h^r) + d(\gamma - \eta_h, \sigma_h - \tau_h). \end{aligned} \right. \tag{48}$$

Thus, using the orthogonality of X_h^r and X_h^s (H1) and hypothesis (H4b), we get for any $(z_h^s, \eta^{(h)}) \in \underline{\underline{X_h}}^s \times \Lambda_r^{(h)}$

$$\begin{aligned} a(\sigma_h - \sigma, \sigma_h - \tau_h) \ &= b(v - w_h, \sigma_h^s - \tau_h^s) - (\mathcal{A}\sigma - z_h^s, \sigma_h^r - \tau_h^r)_H \\ &- d(\gamma - \eta^{(h)}, \sigma_h^r - \tau_h^r) + d(\gamma - \eta_h, \sigma_h - \tau_h), \end{aligned} \tag{49}$$

From (45) through (48) and (49), we obtain

$$(50)\quad \begin{aligned} a(\sigma_h-\tau_h,\sigma_h-\tau_h) &= a(\sigma-\tau_h,\sigma_h-\tau_h)+b(v-w_h,\sigma_h^s-\tau_h^s)\\ &-(\mathcal{A}\sigma-z_h^s,\sigma_h^r-\tau_h^r)_H\\ &-d(\gamma-\eta^{(h)},\sigma_h^r-\tau_h^r)+d(\gamma-\eta_h,\sigma_h-\tau_h). \end{aligned}$$

Furthermore from (H3), (50) leads to

$$\left|\begin{aligned} \alpha(\|\sigma_h^s-\tau_h^s\|_X^2+|\sigma_h^r-\tau_h^r|_H^2) &\le \|a\|\,|\sigma-\tau_h|_H\,|\sigma_h-\tau_h|_H\\ &+\|b\|\,\|v-w_h\|_M\,\|\sigma_h^s-\tau_h^s\|_X\\ &+|\mathcal{A}\sigma-z_h^s|_H\,|\sigma_h^r-\tau_h^r|_H\\ &+\|d\|\,\left\|\gamma-\eta^{(h)}\right\|_L|\sigma_h^r-\tau_h^r|_H\\ &+\|d\|\,\|\gamma-\eta_h\|_L\,|\sigma_h-\tau_h|_H\,. \end{aligned}\right.$$

We deduce the existence of a constant $C>0$ depending on $\|a\|$, $\|b\|$, $\|d\|$ and α such that, for any $(\tau_h,w_h,\eta_h,z_h^s,\eta^{(h)})\in V_h(f)\times\underline{M_h}\times L_h\times\underline{\underline{X_h}}^s\times\Lambda_r^{(h)}$ $(|\sigma-\tau_h|_H\le\|\sigma_h^s-\tau_h^s\|_X+|\sigma_h^r-\tau_h^r|_H)$ that is to say

$$(51)\quad \begin{aligned} \|\sigma_h^s-\tau_h^s\|_X+|\sigma_h^r-\tau_h^r|_H &\le C\Big(\;|\sigma-\tau_h^s|_H+|\tau_h^r|_H+\|v-w_h\|_M+\|\gamma-\eta_h\|_L\\ &\quad+\;|\mathcal{A}\sigma-z_h^s|_H+\|\gamma-\eta^{(h)}\|_L\;\Big)\,. \end{aligned}$$

We then use the triangle inequality to get, for any $\tau_h\in V_h(f)$

$$\begin{aligned} \|\sigma-\sigma_h^s\|_X+|\sigma_h^r|_H &= \|\sigma-\tau_h^s+\tau_h^s-\sigma_h^s\|_X+|\sigma_h^r-\tau_h^r+\tau_h^r|_H\\ &\le\|\sigma-\tau_h^s\|_X+\|\tau_h^s-\sigma_h^s\|_X+|\sigma_h^r-\tau_h^r|_H+|\tau_h^r|_H\,, \end{aligned}$$

which gives, using (51)

$$\left|\begin{aligned} \|\sigma-\sigma_h^s\|_X+|\sigma_h^r|_H &\le(1+C)\left(\inf_{\tau_h\in V_h(f)}(\|\sigma-\tau_h^s\|_X+|\tau_h^r|_H)\right)\\ &+C\left(\inf_{w_h\in M_h}\|v-w_h\|_M+\inf_{\eta_h\in L_h}\|\gamma-\eta_h\|_L\right.\\ &\left.+\inf_{z_h^s\in\underline{\underline{X_h}}^s}|\mathcal{A}\sigma-z_h^s|_H+\inf_{\eta^{(h)}\in\Lambda_r^{(h)}}\|\gamma-\eta^{(h)}\|_L\right). \end{aligned}\right.$$

To conclude, let us recall that the inf-sup condition (H2) implies (cf. [11], Proposition 2.5)

$$\inf_{\tau_h\in V_h(f)}(\|\sigma-\tau_h^s\|_X+|\tau_h^r|_H)\le c_1\inf_{\tau_h\in\underline{\underline{X_h}}}(\|\sigma-\tau_h^s\|_X+|\tau_h^r|_H)\,,$$

in particular

$$\inf_{\tau_h\in V_h(f)}(\|\sigma-\tau_h^s\|_X+|\tau_h^r|_H)\le c_1\inf_{\tau_h^s\in\underline{\underline{X_h}}^s}\|\sigma-\tau_h^s\|_X\,.$$

(iii) Error estimates on $v-v_h$ and $\gamma-\gamma_h$

Once again, we shall use the triangle inequality

$$(52)\quad \begin{aligned} \|(v-v_h,\gamma-\gamma_h)\|_{(M\times L)/\mathrm{Ker}\ \widetilde{B}_h^t}\le\;& \|(v_h-w_h,\gamma_h-\eta_h)\|_{(M\times L)/\mathrm{Ker}\ \widetilde{B}_h^t}+\\ &\|(v-w_h,\gamma-\eta_h)\|_{(M\times L)/\mathrm{Ker}\ \widetilde{B}_h^t}\,, \end{aligned}$$

where (w_h, η_h) is arbitrary in $\underline{M_h} \times L_h$. To estimate $\|(v_h - w_h, \gamma_h - \eta_h)\|_{(M\times L)/\mathrm{Ker}\ \widetilde{B}_h^t}$, we use the discrete inf-sup condition (H2) which expresses that, for any $(w_h, \eta_h) \in \underline{M_h} \times L_h$

$$\|(v_h - w_h, \gamma_h - \eta_h)\|_{(M\times L)/\mathrm{Ker}\ \widetilde{B}_h^t} \leq \frac{1}{\beta} \sup_{\tau_h \in \underline{\underline{X_h}}} \frac{\widetilde{b}(v_h - w_h, \gamma_h - \eta_h; \tau_h)}{\|\tau_h^s\|_X + |\tau_h^r|_H}. \tag{53}$$

Using the decomposition of τ_h in $\tau_h^s + \tau_h^r$ and the fact that $X_h^r \subset Z_h$, we get the following expression of $\widetilde{b}(v_h - w_h, \gamma_h - \eta_h; \tau_h)$

$$\widetilde{b}(v_h - w_h, \gamma_h - \eta_h; \tau_h) = b(v_h - w_h, \tau_h^s) + d(\gamma_h - \eta_h, \tau_h^s) + d(\gamma_h - \eta_h, \tau_h^r) \tag{54}$$

Now, let us subtract the first equation of (41) from the first equation of (40). We get

$$a(\sigma - \sigma_h, \tau_h) + \widetilde{b}(v - v_h, \gamma - \gamma_h; \tau_h) = 0, \quad \forall \tau_h \in \underline{\underline{X_h}}.$$

In particular, for any $\tau_h^s \in \underline{\underline{X_h}}^s$, we obtain

$$a(\sigma - \sigma_h, \tau_h^s) + b(v - v_h, \tau_h^s) + d(\gamma - \gamma_h, \tau_h^s) = 0, \quad \forall \tau_h^s \in \underline{\underline{X_h}}^s,$$

which, introducing an arbitrary (w_h, η_h) in $\underline{M_h} \times L_h$, can be written

$$a(\sigma - \sigma_h, \tau_h^s) + b(v - w_h, \tau_h^s) + b(w_h - v_h, \tau_h^s) + d(\gamma - \eta_h, \tau_h^s) + d(\eta_h - \gamma_h, \tau_h^s) = 0,$$

that we shall use in (54) as

$$b(v_h - w_h, \tau_h^s) + d(\gamma_h - \eta_h, \tau_h^s) = a(\sigma - \sigma_h, \tau_h^s) + b(v - w_h, \tau_h^s) + d(\gamma - \eta_h, \tau_h^s),$$

to obtain

$$\begin{aligned} \widetilde{b}(v_h - w_h, \gamma_h - \eta_h; \tau_h) &= a(\sigma - \sigma_h, \tau_h^s) + b(v - w_h, \tau_h^s) \\ &\quad + d(\gamma - \eta_h, \tau_h^s) + d(\gamma_h - \eta_h, \tau_h^r). \end{aligned} \tag{55}$$

Since we have already estimated $\sigma - \sigma_h$ in (i), and since $b(v - w_h, \tau_h^s)$ and $d(\gamma - \eta_h, \tau_h^s)$ can be estimated by $C\,\|\tau_h^s\|_X\,(\|v - w_h\|_M + \|\gamma - \eta_h\|_L)$, only the term $d(\gamma_h - \eta_h, \tau_h^r)$ needs to be treated. It can be rewritten as follows, thanks to (H4b)

$$\begin{aligned} d(\gamma_h - \eta_h, \tau_h^r) &= d(\gamma_h - \gamma, \tau_h^r) + d(\gamma - \eta_h, \tau_h^r) \\ &= d(\gamma_h, \tau_h^r) + d(\eta^{(h)} - \gamma, \tau_h^r) + d(\gamma - \eta_h, \tau_h^r), \quad \forall \eta^{(h)} \in \Lambda_r^{(h)}. \end{aligned} \tag{56}$$

If we take then $\tau_h = \tau_h^r$ in the first equation of the discrete problem (41), we get

$$a(\sigma_h, \tau_h^r) + d(\gamma_h, \tau_h^r) = 0,$$

or equivalently

$$d(\gamma_h, \tau_h^r) = -a(\sigma_h, \tau_h^r) = -(\mathcal{A}\sigma_h, \tau_h^r) = (z_h^s - \mathcal{A}\sigma_h, \tau_h^r), \quad \forall z_h^s \in \underline{\underline{X_h}}^s. \tag{57}$$

From (55) through (56) and (57), we obtain

$$\begin{aligned} \widetilde{b}(v_h - w_h, \gamma_h - \eta_h; \tau_h) &= a(\sigma - \sigma_h, \tau_h^s) + b(v - w_h, \tau_h^s) + d(\gamma - \eta_h, \tau_h^s + \tau_h^r) \\ &\quad + (z_h^s - \mathcal{A}\sigma_h, \tau_h^r) + d(\eta^{(h)} - \gamma, \tau_h^r) \\ &\leq C\{\, |\sigma - \sigma_h|_H + \|v - w_h\|_M + \|\gamma - \eta_h\|_L \\ &\quad + |z_h^s - \mathcal{A}\sigma_h|_H + \|\eta^{(h)} - \gamma\|_L\}\,\{\|\tau_h^s\|_X + |\tau_h^r|_H\}, \end{aligned} \tag{58}$$

where we shall use

$$
(59)\qquad \begin{aligned} |z_h^s - \mathcal{A}\sigma_h|_H &= |z_h^s - \mathcal{A}\sigma + \mathcal{A}\sigma - \mathcal{A}\sigma_h|_H \\ &\leq |z_h^s - \mathcal{A}\sigma|_H + \|a\| |\sigma - \sigma_h|_H . \end{aligned}
$$

Since w_h, η_h, z_h^s and $\eta^{(h)}$ are arbitrary, using (58) and (59) in (52) and (53), we finally obtain

$$
\begin{aligned} \|(v - v_h, \gamma - \gamma_h)\|_{M\times L/\mathrm{Ker}\ \widetilde{B}_h^t} &\leq C' \{ |\sigma - \sigma_h^s|_H + |\sigma_h^r|_H \\ &+ \inf_{w_h \in M_h} \|v - w_h\|_M + \inf_{\eta_h\in L_h} \|\gamma - \eta_h\|_M \\ &+ \inf_{z_h^s \in \underline{\underline{X_h}}^s} |z_h^s - \mathcal{A}\sigma|_H + \inf_{\eta^{(h)}\in \Lambda_r^{(h)}} \|\eta^{(h)} - \gamma\|_L \Big\}. \end{aligned}
$$

that permits to conclude. ■

Remark 12 *If* Ker $\widetilde{B}_h^t = 0$, *we obtain error estimates for the usual norm* $\|(v - v_h, \gamma - \gamma_h)\|_{M\times L}$. *That is what will happen in our case.*

Remark 13 *Theorem 3 can be easily generalized to the following problem*

$$
\left\{ \begin{aligned} &\textit{Find } (\sigma, v, \gamma) \in \underline{\underline{X}} \times \underline{M} \times L \textit{ such that} \\ &a(\sigma,\tau) + \ \widetilde{b}(v,\gamma;\tau) \ = (g,\tau)_H , \quad \forall \tau \in \underline{\underline{X}}, \\ &\qquad\qquad \widetilde{b}(w,\eta;\sigma) \ = -\langle f, w\rangle , \quad \forall (w,\eta) \in \underline{M} \times L, \end{aligned} \right.
$$

with $g \in \underline{\underline{H}}$. *Error estimates for this problem are slightly different from (43). More precisely we obtain*

$$
(60)\qquad \begin{aligned} |\sigma_h^r|_H + \|\sigma - \sigma_h^s\|_X + \|(v - v_h, \gamma-\gamma_h)\|_{(M\times L)/\mathrm{Ker}\ \widetilde{B}_h^t} &\leq C \Big\{ \inf_{\tau_h^s \in X_h^s} \|\sigma - \tau_h^s\|_X \\ + \inf_{w_h\in M_h} \|v - w_h\|_M + \inf_{\eta_h \in L_h} \|\gamma - \eta_h\|_L & \\ + \inf_{z_h^s \in X_h^s} |(g - \mathcal{A}\sigma) - z_h^s|_H + \inf_{\eta^{(h)} \in \Lambda_r^{(h)}} \|\gamma - \eta^{(h)}\|_L \Big\}. & \end{aligned}
$$

3.5 Application to the elastic problem

3.5.1 Statement of the main results

We first recall the functional spaces for the continuous elastic problem

$$
\underline{\underline{H}} = (L^2(\Omega))^4,\ \underline{\underline{X}} = (H(\mathrm{div}\ ,\ \Omega))^2,\ \underline{M} = (L^2(\Omega))^2 \text{ and } L = L^2(\Omega).
$$

The operators B and D are defined by

$$
(61)\qquad B\tau = \mathrm{div}\ \tau, \quad \forall\ \tau \in \underline{\underline{X}}, \quad D\tau = \mathrm{as}(\tau) \quad \forall\ \tau \in \underline{\underline{H}},
$$

and the operator $\mathcal{A}$ is given by

$$
(62)\qquad \mathcal{A}\tau(x) = A(x)\tau(x).
$$

Consider $\underline{\underline{X_h}}$, $\underline{M_h}$ the new family of mixed finite elements defined by (28). We then define L_h as

$$
(63)\qquad L_h = D(\underline{\underline{X_h}}) = \mathrm{as}(\underline{\underline{X_h}}).
$$

Note that, in this case, D_h is nothing but the restriction of D to $\underline{\underline{X_h}}$ and that

$$
(64)\qquad L_h = \mathrm{Im}\ D_h, \quad \underline{\underline{X_h}}^{sym} = \mathrm{Ker}\ D_h.
$$

Our first result is the equivalence between problems (30) and (37).

Theorem 4 *If L_h is defined by (63), problem (37) admits a unique solution $(\sigma_h, v_h, \gamma_h) \in \underline{\underline{X_h}} \times \underline{M_h} \times L_h$ where (σ_h, v_h) is the unique solution of problem (30).*

The main result of this section is the following convergence theorem, that we restrict to the cases $k = 0, 1$ and 2 for reasons that we explain in remark 14.

Theorem 5 *Assume $k \leq 2$ and let $(\sigma_h, v_h, \gamma_h) \in \underline{\underline{X_h}} \times \underline{M_h} \times L_h$ be the solution of (37) and $(\sigma, v, \gamma) \in \underline{\underline{X}} \times \underline{M} \times L$ be the solution of (34), then σ_h can be decomposed as $\sigma_h = \sigma_h^s + \sigma_h^r$ such that we have*

- $(\sigma_h^s, v_h, \gamma_h) \rightarrow (\sigma, v, \gamma) \quad \in (H(\mathrm{div}\ , \ \Omega))^2 \times (L^2(\Omega))^2 \times L^2(\Omega)$,
- $\sigma_h^r \rightarrow 0 \quad \in (L^2(\Omega))^4$.

Furthermore if we assume that the solution (σ, v, γ) of (34) is such that $(\sigma, v, \gamma) \in (H^m(\Omega))^4 \times (H^m(\Omega))^2 \times H^m(\Omega)$ and that $(\mathcal{A}\sigma, \mathrm{div}\ \sigma \in (H^m(\Omega))^4 \times (H^m(\Omega))^2$ for $m = k+1$, then we get the error estimate ($|\cdot|_{H^m}$ denoting the usual semi-norm in H^m)

$$
\begin{aligned}
(65)\qquad & \|\sigma - \sigma_h^s\|_{H(\mathrm{div})} + |\sigma_h^r|_{L^2} + \|v - v_h\|_{L^2} + \|\gamma - \gamma_h\|_{L^2} \leq \\
& \qquad \leq \ C\ h^m\ (|\sigma|_{H^m} + |\mathrm{div}(\sigma)|_{H^m} + |v|_{H^m} + |\mathcal{A}\sigma|_{H^m}).
\end{aligned}
$$

In order to prove Theorem 5, we will use the abstract theory presented in the previous section to problem (34) and its approximation (37). The basic steps of the proof are the following. First, we will justify the choice of L_h defined by (63). Then, we will characterize this space and finally we will check hypothesis (H0)-(H4) for the new family of finite elements. Finally, we obtain the error estimates (65) by using the usual interpolation results (c.f [24]) in $\underline{\underline{X_h}}^s, \underline{M_h}, L_h$ (see also §3.5.3-d and Lemma 3).

Remark 14 *In fact we shall see in §3.5.4 that the proof of the convergence result of Theorem 5 for any k can be reduced to check that a rectangular matrix $\hat{A}$, whose dimension increases with k, has a full rank. It appears difficult to check this properly by hand for a general k. We have thus written an algorithm which is able to form the matrix $\hat{A}$ for each k and then compute its rank. To prove that Theorem 5 is valid we have implemented this algorithm with MAPLE for $k = 0, 1$ and 2. However, for a given $k \geq 3$, it would be sufficient to use again this algorithm to conclude that Theorem 5 is still true if we find that matrix $\hat{A}$ has a full rank.*

3.5.2 Equivalence with the relaxed problem : Characterization of the space L_h

The proof of theorems 4 and 5 will follow from the next lemma. This will justify our choice for L_h if we are able to check the assumptions of this lemma, which will be the object of the two next sections.

Lemma 1 *Assume that the spaces $\underline{\underline{X_h}}, \underline{M_h}, L_h$ defined by (28) and (63) verify assumptions (H0) to (H4b), then theorems 4 and 5 hold.*

Proof.
(i) **Existence and uniqueness for problem (37)**
First note that assumptions (H0) to (H4b) ensure the existence and uniqueness of the solution of (37) thanks to theorem 2.
(ii) **Existence and uniqueness for problem (30)**
Next we remark that $L_h = D(\underline{\underline{X_h}})$ implies

$$
(66)\qquad V_h(f) \quad \equiv \left\{ \tau_h \in \underline{\underline{X_h}}^{sym} \ / \ b(w_h, \tau_h) = -(f, w_h), \ \forall w_h \in \underline{M_h} \right\}.
$$

Indeed, $\widetilde{b}(w_h, \eta_h; \tau_h) = -(f, w_h)$ for any (w_h, η_h) implies in particular (choose $w_h = 0$) that $d(\eta_h, \tau_h) = (D\tau_h, \eta_h) = 0$ for any η_h in L_h. Then take $\eta_h = D\tau_h \in L_h$, we get $D\tau_h = 0$, that is to say $\tau_h \in \underline{\underline{X_h}}^{sym}$. Then, it suffices to remark that $\widetilde{b}(w_h, \eta_h; \tau_h) = b(w_h, \tau_h)$ when τ_h belongs to $\underline{\underline{X_h}}^{sym}$, to deduce that

$$
V_h(f) \subset \left\{ \tau_h \in \underline{\underline{X_h}}^{sym} \ / \ b(w_h, \tau_h) = -(f, w_h), \ \forall w_h \in \underline{M_h} \right\}.
$$

The converse inclusion is straightforward. Then hypotheses (H0), (H1) and (H3) imply (using the equivalence of norms in a finite dimensional space) the following discrete coercivity result

$$\exists \alpha_h > 0,\ \forall \sigma_h \in V_h(0),\ \ a(\sigma_h, \sigma_h) \geq \alpha_h \, \|\sigma_h\|_X^2 . \tag{67}$$

Using the classical theory of mixed variational problems hypothesis (H0) with (66) and the discrete coercivity condition (67) ensure the existence and uniqueness of the solution (σ_h, v_h) of the discrete problem (30) in $\underline{\underline{X_h}}^{sym} \times \underline{M_h}$ (Ker $B_h^t = 0$).

(iii) **Equivalence between problems (37) and (30)**

Finally, let $(\sigma_h, v_h, \gamma_h) \in \underline{\underline{X_h}} \times \underline{M_h} \times L_h$ be the solution of (37), clearly σ_h belongs to $V_h(f)$ which implies, cf. (66), that $\sigma_h \in \underline{\underline{X_h}}^{sym}$. Then, taking $\tau_h \in \underline{\underline{X_h}}^{sym}$ in the first equation of problem (37) shows that (σ_h, v_h) satisfies problem (30). ■

We next characterize the space L. We introduce the following spaces

- The space of Q_k discontinuous functions

$$Q_{h,k}^d = \left\{ q_h \in L^2(\Omega) \,/\, \forall K \in \mathcal{T}_h,\ q_h|_K \in Q_k \right\}.$$

- The space of Q_k continuous functions

$$Q_{h,k}^c = \left\{ q_h \in H^1(\Omega) \,/\, \forall K \in \mathcal{T}_h,\ q_h|_K \in Q_k \right\}.$$

We define then $\widetilde{L}_h \subset L$ as follows

Definition: *$\widetilde{L}_h$ is the subspace of functions in $Q_{h,k+1}^d$ which satisfy an additional relation at each interior vertex of the mesh $\mathcal{T}_h$. More precisely, the four values $q_k, 1 \leq k \leq 4$ of the function $q_h \in \widetilde{L}_h$ associated to each interior vertex $M_{i,j}$ of the mesh $\mathcal{T}_h$ (see Figure 9) must satisfy*

$$q_1 + q_3 = q_2 + q_4. \tag{68}$$

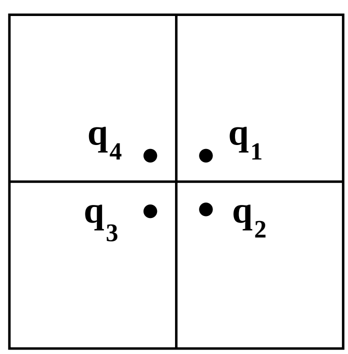


Figure 9: Degrees of freedom associated to an interior vertex of the mesh

Lemma 2 *The space $L_h = D(\underline{\underline{X_h}})$ is a subspace of $\widetilde{L}_h$.*

Proof. By definition of L_h:

$$\forall \eta_h \in L_h, \exists \tau_h \in \underline{\underline{X_h}} \text{ such that } \eta_h = \mathrm{as}(\tau_h) = \tau_h^{21} - \tau_h^{12}.$$

Using then the definition of $\underline{\underline{X_h}}$ we deduce that

$$\forall K \in \mathcal{T}_h,\ \eta_h|_K \in Q_{k+1}.$$

To conclude, it remains to prove that η_h also satisfy the local relation (68) at each interior vertex of the mesh. Indeed, consider M_j an interior vertex of the mesh as shown in figure 10. By definition of as(.), we have

$$\eta_{K_1}\big|_{M_j} = \tau^{21}_{K_1}\big|_{M_j} - \tau^{12}_{K_1}\big|_{M_j} \ ,$$

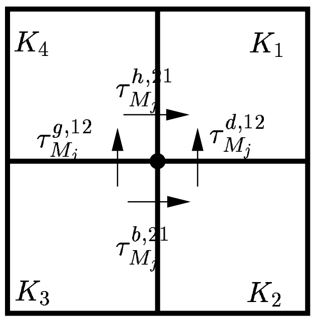
$$\eta_{K_2}\big|_{M_j} = \tau^{21}_{K_2}\big|_{M_j} - \tau^{12}_{K_2}\big|_{M_j} \ ,$$

$$\eta_{K_3}\big|_{M_j} = \tau^{21}_{K_3}\big|_{M_j} - \tau^{12}_{K_3}\big|_{M_j} \ ,$$

$$\eta_{K_4}\big|_{M_j} = \tau^{21}_{K_4}\big|_{M_j} - \tau^{12}_{K_4}\big|_{M_j} \ ,$$

where we denote by $\eta_{K_i}\big|_{M_j}$ the value of $\eta_h\,|_{K_i}$ at point M_j (and the same notation for τ_h^{12} and τ_h^{21}).

Using now the continuity relations in $\underline{\underline{X_h}}$ we get



$$\tau^{21}_{K_1}\big|_{M_j} = \tau^{21}_{K_4}\big|_{M_j} \equiv \tau^{h,21}_{M_j},$$

$$\tau^{21}_{K_2}\big|_{M_j} = \tau^{21}_{K_3}\big|_{M_j} \equiv \tau^{b,21}_{M_j},$$

$$\tau^{12}_{K_1}\big|_{M_j} = \tau^{12}_{K_2}\big|_{M_j} \equiv \tau^{d,12}_{M_j},$$

$$\tau^{12}_{K_3}\big|_{M_j} = \tau^{12}_{K_4}\big|_{M_j} \equiv \tau^{g,12}_{M_j},$$

Figure 10: Continuity relations.

so that we can write

$$\eta_{K_1}\big|_{M_j} = \tau^{h,21}_{M_j} - \tau^{d,12}_{M_j},$$

$$\eta_{K_2}\big|_{M_j} = \tau^{b,21}_{M_j} - \tau^{d,12}_{M_j},$$

$$\eta_{K_3}\big|_{M_j} = \tau^{b,21}_{M_j} - \tau^{g,12}_{M_j},$$

$$\eta_{K_4}\big|_{M_j} = \tau^{h,21}_{M_j} - \tau^{g,12}_{M_j}.$$

One then easily checks that

$$\eta_{K_1}\big|_{M_j} + \eta_{K_3}\big|_{M_j} = \eta_{K_2}\big|_{M_j} + \eta_{K_4}\big|_{M_j} \ . \qquad \blacksquare$$

Our next result characterizes entirely L_h.

Theorem 6 *The dimension of L_h is equal to the dimension of the space $\widetilde{L}_h$. Therefore we have*

$$L_h = \widetilde{L}_h.$$

Proof. Let us first compute the dimension of the space $\widetilde{L}_h$.

$$\begin{aligned} \textbf{Dim}\ \widetilde{L}_h &= \textbf{Dim}\ Q^d_{h,k+1} - \text{ Number of interior vertices of the mesh} \\ &= N^2\ (k+2)^2 - (N-1)^2 \qquad (69) \\ &= N^2(k^2+4k+3) + 2N - 1. \end{aligned}$$

On the other hand, we remark that, since $L_h = D(\underline{\underline{X_h}}) \equiv \text{Im}\ D_h$ and $\underline{\underline{X_h}}^{sym} = \text{Ker}\ D_h$

$$\textbf{Dim}\ L_h = \textbf{Dim}\ \underline{\underline{X_h}} - \textbf{Dim}\ \underline{\underline{X_h}}^{sym}. \qquad (70)$$

Considering $\underline{\underline{X_h}}$ as a subspace of $\left(Q^d_{h,k+1}\right)^4$, we have

$$\mathbf{Dim}\ \underline{\underline{X_h}} = \mathbf{Dim}\ Q^d_{h,k+1} \times 4 - \ \text{Num. of continuity relations.} \tag{71}$$

Since τ_h^{11}, τ_h^{21} (resp. τ_h^{12}, τ_h^{22}) are continuous across each vertical (resp. horizontal) edge, the number of continuity relations in $\underline{\underline{X_h}}$ is given by

$$\begin{aligned}\text{Num. of cont. relat.} \ &= \ \text{Num. of edges} \ \times \ \text{Num. of deg. of freed. per edge} \ \times 2 \\ &= 2N(N-1) \times (k+2) \times 2.\end{aligned}$$

Therefore, (71) gives

$$\mathbf{Dim}\ \underline{\underline{X_h}} = 4N^2(k^2+3k+2) + 4N(k+2). \tag{72}$$

In order to compute the dimension of $\underline{\underline{X_h}}^{sym}$ we will rewrite it as a direct sum of two subspaces corresponding to the decomposition of a tensor as the sum of its diagonal part and of its extra-diagonal part

$$\underline{\underline{X_h}}^{sym} = \underline{\underline{X_h}}^{sym,d} + \underline{\underline{X_h}}^{sym,e}, \tag{73}$$

where

$$\begin{aligned}&\underline{\underline{X_h}}^{sym,d} \text{ is isomorphic to} \left\{(\tau_h^{11}, \tau_h^{22}) \in H(\text{div}),\ \forall K \in \mathcal{T}_h,\ (\tau_h^{11}, \tau_h^{22})\,|_K \in (Q_{k+1})^2\right\}, \\ &\underline{\underline{X_h}}^{sym,e} \text{ is isomorphic to} \left\{\tau_h^{12} = \tau_h^{21} \in H^1 \ /\ \forall K \in \mathcal{T}_h,\ \tau_h^{12}\,|_K \in Q_{k+1}\right\}.\end{aligned}$$

Therefore, we have

$$\mathbf{Dim}\ \underline{\underline{X_h}}^{sym,e} = \mathbf{Dim}\ Q^c_{h,k+1} = (N+1)^2 + 2N(N+1)k + N^2k^2, \tag{74}$$

while, as for the the computation of $\mathbf{Dim}\underline{\underline{X_h}}$,

$$\begin{aligned}\mathbf{Dim}\ \underline{\underline{X_h}}^{sym,d} \ &= \mathbf{Dim}\ Q^d_{h,k+1} \times 2 - \ \text{Num of continuity relations} \\ &= 2N^2(k+2)^2 - 2N(N-1) \times (k+2) \\ &= 2N^2(k^2+3k+2) + 2N(k+2) \quad (= \frac{1}{2}\mathbf{Dim}\underline{\underline{X_h}}),\end{aligned} \tag{75}$$

$$\begin{aligned}\mathbf{Dim}\ \underline{\underline{X_h}}^{sym} \ &= \mathbf{Dim}\ \underline{\underline{X_h}}^{sym,d} + \mathbf{Dim}\ \underline{\underline{X_h}}^{sym,e} \\ &= N^2(3k^2+8k+5) + N(4k+6) + 1.\end{aligned} \tag{76}$$

From (70) through (72) and (76), we obtain

$$\begin{aligned}\mathbf{Dim}L_h \ &= 4N^2(k^2+3k+2) + 4N(k+2) - N^2(3k^2+8k+5) - N(4k+6) - 1 \\ &= N^2(k^2+4k+3) + 2N - 1.\end{aligned} \tag{77}$$

(69) and (77) show that $\widetilde{L}_h$ and L_h have the same dimension. ■

In the sequel, we shall set $NL_h = \mathbf{Dim}\ L_h$ and shall denote by

$$\{q_j\ ,\ 1 \leq j \leq NL_h\} \tag{78}$$

the basis of L_h associated to the following set of degrees of freedom (here we distinguish the "vertices" of $\mathcal{T}_h$ from the "nodes" which correspond to the refined mesh obtained by dividing each square K into k^2 equal squares):

- Degrees of freedom associated to the boundary:

  - The value of the function at each vertex of Ω, node type 1, (there are 4),
  - The value of the function at each node of the boundary which is interior to an edge, node type 2 (there are $4kN$),
  - The two values of the function at each point of the boundary which is a vertex of the mesh, but not one of the vertices of Ω, node type 3 (there are $4(N-1) \times 2$).
- Degrees of freedom associated to the interior:
  - The value of the function at each node which is interior to an element, node type 4 (there are k^2N^2),
  - The two values of the function at each node located on an interior edge, interior to this edge, node type 5 (there are $2kN(N-1) \times 2$),
  - For each interior vertex of the mesh, node type 6 (there are $(N-1)^2 \times 3$), the three quantities: $(q_1+q_2+q_3+q_4)/4, (q_1-q_2-q_3+q_4)/4, (q_1+q_2-q_3-q_4)/4$.

We present in figure 11 the differnt types of nodes for L_h.

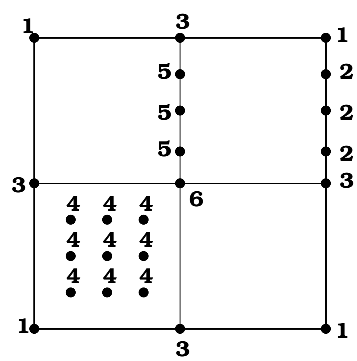


Figure 11: The six types of nodes for L_h.

The reader will easily verify that the support of each basis function is made of one element K, except for the degrees of freedom associated to the interior vertices, for which the support of the basis function is made of four elements.

It remains, in order to apply lemma 1, to check assumptions (H0) to (H4b) for the spaces $\underline{\underline{X_h}}, \underline{M_h}, L_h$. Section 3.5.3 is devoted to the proof of (H1), (H0), (H3), (H4) and (H4b), in this order. The proof of (H2) is much more complicated and delayed to section 3.5.4

3.5.3 Proof of the main theorem : Verification of the conditions (H1), (H0), (H3), (H4) and (H4b)

a - Hypothesis (H1) It appears as a simple generalisation of the results presented in [7]. We introduced in [7] the spaces X_h defined by

(79) $$X_h = \{q_h \in X = H(\mathrm{div};\Omega) \,/\, \forall K \in \mathcal{T}_h,\ q_h|_K \in X_k = Q_{k+1} \times Q_{k+1}\},$$

and we have shown that X_h admits the following orthogonal (in L^2) decomposition

$$X_h = X_h^s \oplus X_h^r,$$

$$X_h^s = \left\{p_h \in H(\mathrm{div};\Omega) \,/\, \forall K \in \mathcal{T}_h,\ p_h|_K \in RT_{[k]}\right\},$$

$$X_h^r = \{p_h \in H(\mathrm{div};\Omega) \,/\, \forall K \in \mathcal{T}_h,\ p_h|_K \in \Psi_k\},$$

where Ψ_k is defined as the orthogonal complement in X_k of RT_k (for the inner product $L^2(K)$)

(80) $$\Psi_k(K) = \left\{\psi \in \hat{X} \,/\, \int_K \psi\,\phi\,dx = 0,\ \forall \phi \in RT_k\right\}.$$

Thus, as

$$\underline{\underline{X_h}} = X_h \times X_h,$$

it is straightforward that $\underline{\underline{X_h}}$ admits the orthogonal decomposition

$$\underline{\underline{X_h}} = \underline{\underline{X_h}}^s \oplus \underline{\underline{X_h}}^r, \tag{81}$$

with

$$\underline{\underline{X_h}}^s = X_h^s \times X_h^s \text{ and } \underline{\underline{X_h}}^r = X_h^r \times X_h^r. \tag{82}$$

Moreover, one can prove easily (see [7]) that

$$\underline{\underline{X_h}}^r \subset Z_h.$$

This is in fact a corollary of the following strongest property

$$\forall K \in \mathcal{T}_h,\ \forall w \in Q_k,\ \forall p_h \in X_h^r,\ (\mathrm{div} p_h, w)_K = 0. \tag{83}$$

b - Hypothesis (H0) From remark 11, we know that (H0) is equivalent to

$$\mathrm{Ker}\ \widetilde{B}_h^t = 0.$$

Therefore, we have to prove that for (w_h, η_h) in $\underline{M_h} \times L_h$ which satisfy

$$\widetilde{b}(w_h, \eta_h; \tau_h) \equiv b(w_h, \tau_h) + d(\eta_h, \tau_h) = 0,\ \forall \tau_h \in \underline{\underline{X_h}}, \tag{84}$$

we have

$$w_h = 0 \text{ and } \eta_h = 0.$$

Indeed, take first in (84) $\tau_h = \tau_h^d$, a diagonal tensor of $\underline{\underline{X_h}}$, we then get

$$\widetilde{b}(w_h, \eta_h; \tau_h^d) \equiv b(w_h, \tau_h^d) = 0,\ \forall \tau_h^d \in \underline{\underline{X_h}}. \tag{85}$$

It is easy now to check that (85) implies $w_h = 0$. That is a direct consequence of the well known properties of the Raviart-Thomas element, given that the space $\underline{\underline{X_h}}$ contains the Raviart-Thomas space $(\underline{\underline{X_h}}^s)$. To conclude it remains now to prove that

$$(\mathrm{as}(\tau_h), \eta_h) = 0,\ \ \forall \tau_h \in \underline{\underline{X_h}} \quad \Rightarrow \eta_h = 0.$$

Which is true, because from the definition of L_h we have

$$L_h = D(\underline{\underline{X_h}}) \left(\equiv \mathrm{as}(\underline{\underline{X_h}})\right).$$

c - Hypothesis (H3) Checking hypothesis (H3) is also a simple generalisation of results presented in [7]. From (11), we get

$$a(\sigma_h, \sigma_h) \geq \alpha\, |\sigma_h|_H^2 = \alpha \left(|\sigma_h^s|_H^2 + |\sigma_h^r|_H^2\right).$$

Remark now that $\sigma_h \in V_h$ implies

$$\begin{cases} (a) \quad (\mathrm{as}(\sigma_h), \eta_h) = 0, \quad \forall \eta_h \in L_h, \\ (b) \quad (\mathrm{div}(\sigma_h), w_h) = 0, \quad \forall w_h \in \underline{M_h}. \end{cases}$$

Moreover

$$\begin{cases} (i) \quad \mathrm{div}\sigma_h^s \in \underline{M_h}, \\ (ii) \quad (\mathrm{div}\sigma_h^s, w_h) = 0, \quad \forall w_h \in \underline{M_h}, \end{cases}$$

(i) is a well known property of the Raviart Thomas space and (ii) is a straightforward consequence of (b) and $\underline{\underline{X_h}}^r \subset Z_h$. Thanks to (i) we can take $w_h = \mathrm{div}\sigma_h^s$ in (ii), which gives

$$\mathrm{div}\sigma_h^s = 0.$$

Therefore $\|\sigma_h^s\|_X^2 = |\sigma_h^s|_H^2$ and consequently

$$\forall \sigma_h \in V_h, \quad a(\sigma_h, \sigma_h) \geq \alpha \left(\|\sigma_h^s\|_X^2 + |\sigma_h^r|_H^2\right).$$

d - Hypotheses (H4) - (H4b) The approximations properties in $\underline{\underline{X_h}}^s$ and $\underline{M_h}$ are a generalisation of the well known properties of spaces X_h^s and M_h defined by (79) (cf. [23], [24], [11]). Moreover, as L_h contains the Q_{k+1} continuous functions so we have

$$\lim_{h\to 0} \inf_{\eta_h \in L_h} \|\gamma - \eta_h\|_L = 0 \quad \forall \gamma \in L.$$

using the well-known properties of Q_{k+1} continuous finite elements. Concerning hypothesis (H4b) (cf. Remark 7) we will search for a finite dimensional subspace $L^{(h)}$ of $\Lambda_r^{(h)}$ which satisfies the approximation property (42).

Lemma 3 *The space $L^{(h)} = Q_{h,k}^d$ is included in $\Lambda_r^{(h)}$. and satisfies*

(86) $$\lim_{h\to 0} \inf_{\eta^{(h)} \in L^{(h)}} \|\gamma - \eta^{(h)}\|_L = 0, \quad \forall \gamma \in L.$$

Proof. Property (86) is well known. Thus, it only remains to prove that $L^{(h)} = Q_{h,k}^d$ is also a subspace of

$$\begin{aligned} \Lambda_r^{(h)} &= \left\{ \eta^{(h)} \in L \,/\, d(\eta^{(h)}, \tau_h^r) = 0, \ \forall \tau_h^r \in \underline{\underline{X_h}}^r \right\} \\ &= \left\{ \eta^{(h)} \in L \,/\, (\mathrm{as}(\tau_h^r), \eta^{(h)}) = 0, \ \forall \tau_h^r \in \underline{\underline{X_h}}^r \right\}. \end{aligned}$$

For this, it suffices to prove that

(87) $$\forall K \in \mathcal{T}_h, \quad (\mathrm{as}(\tau_h^r), \eta^{(h)})_{L^2(K)} = 0, \ \forall (\tau_h^r, \eta^{(h)}) \in \underline{\underline{X_h}}^r \times L^{(h)}.$$

This follows from the construction of the space $\underline{\underline{X_h}}^r$. Let us simply recall some basic properties of this space. For simplicity and without any loss of generality, we can work with the reference element $K = \hat{K} = [0,1] \times [0,1]$. The restriction $\hat{\Psi}$ to $\hat{K}$ of any element $\hat{\Psi}_h$ of $\underline{\underline{X_h}}^r$ is of the form (cf. [7])

$$\hat{\Psi} = \left\{ \begin{pmatrix} \hat{\psi}_{11} & \hat{\psi}_{12} \\ \hat{\psi}_{21} & \hat{\psi}_{22} \end{pmatrix} = \begin{pmatrix} p_1(x_1)\sigma_k(x_2) & p_2(x_2)\sigma_k(x_1) \\ p_3(x_1)\sigma_k(x_2) & p_4(x_2)\sigma_k(x_1) \end{pmatrix}, \ p_{i, 1 \le i \le 4} \in P_{k+1} \right\},$$

where the function σ_k is characterized, up to a multiplicative constant, by

$$\begin{cases} \displaystyle\int_0^1 \sigma_k(x) p(x) dx = 0, \ \forall p \in P_k, \\ \sigma_k \in P_{k+1}, \ \sigma_k \not\equiv 0 \,. \end{cases}$$

Thus, as any function $q(x_1, x_2) \in Q_k$ can be written as a linear combination of terms of the form $q_1(x_1)q_2(x_2)$ with $(q_1(x_1), q_2(x_2)) \in P_k(x_1) \times P_k(x_2)$, one easily checks that

$$\int_{\hat{K}} \sigma_k(x_2) q_1(x_1) q_2(x_2) dx_1 dx_2 = \int_{\hat{K}} \sigma_k(x_1) q(x_1) q(x_2) dx_1 dx_2 = 0.$$

Therefore, we have

$$\forall (i,j), i = 1,2, \ j = 1,2, \quad \left(\hat{\psi}_{ij}, q\right)_{L^2(\hat{K})} = 0, \ \ \forall \, q \in Q_k(\hat{K}).$$

This implies in particular (87). ■

3.5.4 Proof of the main theorem : Verification of the condition (H2)

Finally we are going to verify the discrete inf-sup condition(hypothesis (H2)). To do so, we will split it into two parts, by showing the following two lemmas.

Lemma 4 *For all $v_h \in \underline{M_h}$, there exists $\tau_h^1 = \tau_h^{1,s} \in \underline{\underline{X_h}}^s$ such that*

$$\begin{aligned} &(i) \quad (\mathrm{div}\tau_h^1, w_h) = (v_h, w_h), \ \ \forall w_h \in \underline{M_h}, \\ &(ii) \quad (as(\tau_h^1), \mu_h) = 0, \ \ \forall \mu_h \in L_h, \\ &(iii) \quad \|\tau_h^{1,s}\|_X + |\tau_h^{1,r}|_H = \|\tau_h^{1,s}\|_X \le c_1 \|v_h\|_M, \end{aligned}$$

with c_1 a positive constant independent of h.

Lemma 5 *For all $\eta_h \in L_h$ there exists a $\tau_h^2 \in \underline{\underline{X_h}}$ such that*

$$\begin{array}{ll} (i) & (\mathrm{div}\tau_h^2, w_h) = 0, \ \ \forall w_h \in \underline{M_h}, \\ (ii) & (as(\tau_h^2), \eta_h) \geq c_2 \, \|\eta_h\|_L^2 \, , \\ (iii) & \|\tau_h^{2,s}\|_X + |\tau_h^{2,r}|_H \leq c_3 \|\eta_h\|_L, \end{array}$$

with c_2, c_3 positive constants independent of h.

Before proving these lemmas let us first prove that they imply the following theorem which is equivalent to hypothesis (H2).

Theorem 7 *For all $(v_h, \eta_h) \in M_h \times L_h$ there exists a $\tau_h \in \underline{\underline{X_h}}$ such that*

$$(\mathrm{div}\tau_h, v_h) + (as(\tau_h), \eta_h) \geq C \left(\|v_h\|_M + \|\eta_h\|_L\right) \left(\|\tau_h^s\|_X + |\tau_h^r|_H\right),$$

with C a positive constant independent of h.

Proof. For all $(v_h, \eta_h) \in M_h \times L_h$, we take $\tau_h = \tau_h^1 + \tau_h^2 \in \underline{\underline{X_h}}$, with τ_h^1 (resp. τ_h^2) verifying Lemma 4 (resp. Lemma 5). Then we get

$$\begin{aligned} (\mathrm{div}\tau_h, v_h) + (\mathrm{as}(\tau_h), \eta_h) \ &= (\mathrm{div}\tau_h^1, v_h) + (\mathrm{as}(\tau_h^2), \eta_h) \quad \text{(Lemma 4-(ii) and 5-(i))} \\ &\geq \|v_h\|_M^2 + c_2 \, \|\eta_h\|_L^2 \qquad \text{(Lemma 4-(i) and 5-(ii))} \\ &\geq c \left(\|v_h\|_M + \|\eta_h\|_L\right)^2 \\ &\geq C \left(\|v_h\|_M + \|\eta_h\|_L\right) \left(\|\tau_h^s\|_X + |\tau_h^r|_H\right) \quad \text{(Lemma 4 and 5,(iii))} \end{aligned}$$

■

To prove Lemma 4, we will follow the usual technique for proving the discrete inf-sup condition (see [11]). More precisely in Lemma 6, we first obtain for the continuous problem a result analogous to the one of Lemma 4. Then, by using the well known properties of the Raviart Thomas mixed finite element space, we are able to conclude.

Lemma 6 *For all $v \in \underline{M}$ there exists a $\tau \in \underline{\underline{X}}$ such that*

$$\begin{array}{ll} (i) & \tau \text{ is diagonal (thus } as(\tau) = 0), \\ (ii) & \mathrm{div}\tau = v, \\ (iii) & \|\tau\|_X \leq C_1 \|v\|_M. \end{array}$$

Proof of Lemma 6. In the case of a rectangular domain Ω it is easy to prove that for all $v = (v_1, v_2) \in (L^2(\Omega))^2 (= \underline{M})$, there exist a $\tau = (\tau_1, \tau_2) \in H(\mathrm{div}, \Omega) (= \underline{\underline{X}})$ with

$$\tau_1 = \begin{pmatrix} \tau_{11} \\ 0 \end{pmatrix}, \ \ \tau_2 = \begin{pmatrix} 0 \\ \tau_{22} \end{pmatrix},$$

where τ_1, τ_2 verify

$$\mathrm{div}\tau_1 = \frac{\partial \tau_{11}}{\partial x_1} = v_1 \text{ and } \mathrm{div}\tau_2 = \frac{\partial \tau_{22}}{\partial x_2} = v_2.$$

Indeed, we define $\widetilde{v}_1$(resp. $\widetilde{v}_2$) the extension by zero of v_1(resp. v_2) at the exterior of Ω. We can define then τ_{11} (resp. τ_{22}) as a primitive of $\widetilde{v}_1$ (resp. $\widetilde{v}_2$)

$$\tau_{11}(x_1, x_2) = \int_0^{x_1} \widetilde{v}_1(s, x_2) ds, \ \ \tau_{22}(x_1, x_2) = \int_0^{x_1} \widetilde{v}_2(x_1, s) ds.$$

It is clear then, that $\tau = (\tau_1, \tau_2)$ satisfy (i), (ii) and (iii) of Lemma 6. ■

Proof of Lemma 4. Take v_h any element of $\underline{M_h}$. By Lemma 6, we can construct $\tau^h \in \underline{\underline{X}}$ such that

(88)
$$
\begin{aligned}
&(a1) \quad \tau^h \text{ is diagonal } \Rightarrow \mathrm{as}(\tau^h) = 0,\\
&(a2) \quad \mathrm{div}\tau^h = v_h,\\
&(a3) \quad \|\tau^h\|_X \le C_1 \|v_h\|_M.
\end{aligned}
$$

Let Π_h^s be the usual interpolation operator on the Raviart Thomas space $\underline{\underline{X_h}}^s$. It is well known (see [11]) that

(89)
$$
\begin{aligned}
&(b1) \quad (\mathrm{div}(\tau - \Pi_h^s \tau), w_h) = 0, \ \forall (\tau, w_h) \in \underline{\underline{X}} \times \underline{M_h},\\
&(b2) \quad \|\Pi_h^s \tau\|_X \le C_2 \|\tau\|_X, \ \forall \tau \in \underline{\underline{X}}.
\end{aligned}
$$

Moreover, because of the particular mesh we work with, we have

(90)
$$\tau \text{ is diagonal } \Rightarrow \Pi_h^s \tau \text{ is diagonal too.}$$

Let us take now $\tau_h^1 = \tau_h^{1,s} = \Pi_h^s \tau^h$ and check that it satisfies properties (i), (ii) and (iii) of Lemma 4. Indeed,

(i) is a consequence of (88)-(a2) and (89)-(b1).

(ii) is a consequence of (88)-(a1) and (90).

(iii) is a consequence of (88)-(a3) and (89)-(b2). ■

A macroelement partitionning. To prove now the second part of the inf-sup condition that is Lemma 5 we will use a macroelement technique (c.f [26]). This technique will permit us to obtain a global stability estimate by simply adding together analogous local estimates. We need to introduce some notation.

We shall define a reference macroelement $\widehat{M}$ as a finite connected union of 4 disjoint unit squares.

$$
\begin{aligned}
&\widehat{M} = \bigcup_{j=1,\dots,4} \widehat{K}_j \text{ where } \widehat{K}_j, \ j = 1, \dots, 4 \text{ are unit squares,}\\
&\widehat{K}_j = T_j(\widehat{K}), \ \widehat{K} = [0,1]^2 \text{ and } T_j \text{ is a translation, } T_j(\widehat{x}) = \widehat{x} + t_j, \ t_j \in \{-1, 0\}^2
\end{aligned}
$$

A current macroelement M_e will be associated to each vertex S_e of the mesh $\mathcal{T}_h$ and defined as

$$M_e = \bigcup_{j=1,\dots,4} K_j^e \text{ whith } K_j^e = F_e(\widehat{K}_j) = F_e \circ T_j(\widehat{K}),$$

where F_e is the affine transformation:

$$F_e(\widehat{x}) = S_e + h\,\widehat{x}.$$

We have obviously:

(91)
$$\Omega \subset \bigcup_{e=1,\dots,N^2} M_e,$$

and the finite overlaping property

(92) Each element $K \in \mathcal{T}_h$ is included in at most 4 macroelements,

We define the following spaces of functions on the reference macroelement $\widehat{M}$, that are local spaces associated to $\underline{\underline{X_h}}$ and $\underline{M_h}$

(93)
$$
\begin{aligned}
&\underline{\underline{\widehat{X}}} = \{\widehat{\tau} \in H_0(\mathrm{div}, \widehat{M}) \ / \ \forall\, \widehat{K} \in \widehat{M}, \widehat{\tau}|_{\widehat{K}} \in Q_{k+1}^4\},\\
&\underline{\widehat{M}} = \{\widehat{w} \in L^2(\widehat{M})^2 \ / \ \forall\, \widehat{K} \in \widehat{M}, \widehat{w}|_{\widehat{K}} \in Q_k^2\},
\end{aligned}
$$

where we recall that

$$H_0(\mathrm{div}, \widehat{M}) = \{\widehat{\tau} \in H(\mathrm{div}, \widehat{M}) \ / \ \widehat{\tau} \cdot n|_{\partial \widehat{M}} = 0\}.$$

Of course, as a local equivalent of the decomposition (81), (82), $\underline{\underline{\widehat{X}}}$ can be decomposed as

(94)
$$\left|\begin{array}{l} \underline{\underline{\widehat{X}}} = \underline{\underline{\widehat{X}}}^s + \underline{\underline{\widehat{X}}}^r, \quad \underline{\underline{\widehat{X}}}^s = \{\widehat{\tau} \in \underline{\underline{\widehat{X}}} \ / \ \forall \, \widehat{K} \in \widehat{M}, \widehat{\tau}|_{\widehat{K}} \in RT_k^2\}, \\ \underline{\underline{\widehat{X}}}^r \text{ is the orthogonal complement in } \underline{\underline{\widehat{X}}} \text{ of } \underline{\underline{\widehat{X}}}^s \text{ for the scalar product of } L^2(\widehat{M})^4, \end{array}\right.$$

where we recall that $\underline{\underline{\widehat{X}}}^r$ has the fundamental property (cf (83))

(95)
$$\forall \, \widehat{K} \in \widehat{M}, \ \forall \, \widehat{w} \in Q_k^2, \ \forall \, \widehat{\tau}^r \in \underline{\underline{\widehat{X}}}^r, \quad (\mathrm{div}\widehat{\tau}^r, \widehat{w})_{\widehat{K}} = 0.$$

We then define the local space corresponding to L_h

(96)
$$\widehat{L} \ = \mathrm{as} \ (\underline{\underline{\widehat{X}}}).$$

Using the same arguments as in the proof of Lemma 6, one characterizes the space $\widehat{L}$ as

$$\left|\begin{array}{l} \widehat{L} \ = \{\widehat{\eta} \in L^2(\widehat{M}) \ / \ \forall \, \widehat{K} \in \widehat{M}, \widehat{\eta}|_{\widehat{K}} \in Q_{k+1} \text{ and } \widehat{\eta} \text{ satisfies } (i), (ii), (iii)\}. \\ \text{(i)} \quad \widehat{\eta} \text{ satisfies the continuity condition (REF) at each interior node } \widehat{M} \ . \\ \text{(ii)} \quad \widehat{\eta} \text{ is continuous on } \partial\widehat{M}. \\ \text{(iii)} \ \widehat{\eta} = 0 \text{ at each of the four vertices of } \widehat{M}. \end{array}\right.$$

The difference with the characterization of L_h comes from the additional conditions associated to the nodes of the boundary. These conditions are the consequence of the boundary condition imposed in $\underline{\underline{\widehat{X}}}$. Note that the dimension of $\widehat{L}$ is $NL = 4(k^2 + 4k + 3) - 9$. In the sequel, we shall note $\{\widehat{\mu}_i, 1 \leq i \leq NL\}$ the basis of $\widehat{L}$ associated to the following degrees of freedom:

- the value of the function at each boundary node which is not a vertex of $\widehat{M}$ (there are $8(k+1) - 4$),
- the value of the function at each node which is interior to an element (there are $4k^2$),
- the two values of the function at each node located to an iterior edge, interior to this edge (there are $8k$),
- for the interior vertex, the three quantities $(q_1 + q_2 + q_3 + q_4)/4$, $(q_1 - q_2 + q_3 - q_4)/4$, $(q_1 - q_2 - q_3 + q_4)/4$, where (q_1, q_2, q_3, q_4) have been defined in figure (9) (there are 3).

We then define the same type of spaces for each macroelement

(97)
$$\begin{array}{l} \underline{\underline{X}}_e = \{\tau_e = \widehat{\tau} \circ F_e^{-1} \ , \ \widehat{\tau} \in \underline{\underline{\widehat{X}}}\}, \\ \underline{M}_e = \{w_e = \widehat{w} \circ F_e^{-1} \ , \ \widehat{w} \in \underline{\widehat{M}}\}, \\ L_e = \{\eta_e = \widehat{\eta} \circ F_e^{-1} \ , \ \widehat{\eta} \in \widehat{L}\}. \end{array}$$

Obviously, to the orthogonal decomposition (94) of $\underline{\underline{\widehat{X}}}$, corresponds the following orthogonal decomposition of $\underline{\underline{X}}_e$

$$\left|\begin{array}{l} \underline{\underline{X}}_e = \underline{\underline{X}}_e^s + \underline{\underline{X}}_e^r, \quad \underline{\underline{X}}_e^s = \{\tau_e \in \underline{\underline{\widehat{X}}}_e \ / \ \forall \, \widehat{K}_e \subset \widehat{M}_e, \tau_e|_{K_e} \in RT_k^2\} = \{\widehat{\tau} \circ F_e^{-1}, \ \widehat{\tau} \in \underline{\underline{\widehat{X}}}^s\} \\ \underline{\underline{X}}_e^r \text{ is the orthogonal complement in } \underline{\underline{\widehat{X}}}_e \text{ of } \underline{\underline{\widehat{X}}}_e^s \text{ for the scalar product of } L^2(\widehat{M}_e)^4. \end{array}\right.$$

As a consequence of (95), we have

(98)
$$\forall \, K_e \in M_e, \ \forall \, w_e \in Q_k^2, \ \forall \, \tau_e^r \in \underline{\underline{X}}_e^r, \quad (\mathrm{div}\tau_e^r, w_e)_{K_e} = 0.$$

We recall here that, thanks to a well-known property of the space RT_k:

(99) The divergence operator is sujective from $\underline{\underline{X}}_e^s$ onto $\underline{M}_e$.

Also, notice that if $\mu_i^e = \widehat{\mu}_i \circ F_e^{-1}$, then $\{\mu_i^e, 1 \leq i \leq NL\}$ is a basis of L_e. In the sequel, we shall use the following definition

Definition 1 *If u_e is a function defined in M_e, $\widetilde{u}_e$ represents the restriction to Ω of its extension by 0.*

The reader will check easily that, by construction, these spaces have the following property

(100)
$$\begin{aligned}
&(\tau_e, w_e, \eta_e) \in \underline{\underline{X}}_e \times \underline{M}_e \times L_e \Longrightarrow (\widetilde{\tau}_e, \widetilde{w}_e, \widetilde{\eta}_e) \in \underline{\underline{X_h}} \times \underline{M_h} \times L_h, \\
&(\tau_e^s, \tau_e^r) \in \underline{\underline{X}}_e^s \times \underline{\underline{X}}_e^r \Longrightarrow (\widetilde{\tau}_e^s, \widetilde{\tau}_e^r) \in \underline{\underline{X_h}}^s \times \underline{\underline{X_h}}^r.
\end{aligned}$$

From now on, to avoid two much notation, we shall use the same notation $\|.\|_X$ for the $H(\text{div})$ norm in $\underline{\underline{X}}, \widehat{\underline{\underline{X}}}$ and $\underline{\underline{X}}_e$ and will use the same convention for the notation $\|.\|_M, \|.\|_L$ or $|.|_H$.

We shall use the following property of L_h

Lemma 7 *Any $\eta_h \in L_h$ can be decomposed as*

(101)
$$\eta_h = \sum_e \widetilde{\eta}_e, \quad \eta_e \in L_e.$$

where each η_e satisfies (C is a constant independent on h and η_e):

(102)
$$\|\eta_e\|_L \leq C \ \|\eta_h|_{M_e}\|_L$$

which implies that

(103)
$$\sum_{e \in \mathcal{E}} \|\eta_e\|_L^2 \leq C \ \|\eta_h\|_L^2$$

Proof: (i) We first prove (101). Using the basis (78), any η_h in L_h can be written as

$$\eta_h = \sum_{j=1}^{NL_h} \eta_j \ q_j.$$

A consequence of the properties of the macroelement partitioning $\mathcal{M}_h$ is that, for each q_j, there exists at least one macroelement M_e (we choose one of them denoted $e = E(j)$) and a basis function μ_i^e ($i = I(e,j)$) of L_e such that

$$q_j = \widetilde{\mu}_i^e.$$

Moreover, for convenience, we shall note $\zeta_i^e = \eta_j$. Thus, η_h in L_h can be rewritten as

$$\eta_h = \sum_{j=1}^{NL_h} \zeta_i^e \ \cdot \ \widetilde{\mu}_i^e, \quad \text{where} \quad e = E(j), \quad i = I(e,j)$$

Introducing the set of indices $\mathcal{I}_1^e = \{I(e,j), \ \forall j \text{ such that } \ E(j) = e\}$, this can be rearranged as

$$\eta_h = \sum_e \sum_{i \in \mathcal{I}_1^e} \zeta_i^e \ \ \widetilde{\mu}_i^e,$$

which gives (101) if we set

(104)
$$\eta_e = \sum_{i \in \mathcal{I}_1^e} \zeta_i^e \ \ \mu_i^e.$$

(ii) Inequality (102) is due to the fact that (see (104)) η_e is only defined with degrees of freedom of η_h which are local to the macroelement M_e and that the number of these degrees of freedom is bounded independently on h.

(iii) To prove (103), we simply write using the finite overlaping property (92)

$$\sum_{e\in\mathcal{E}} \|\eta_h|_{M_e}\|_L^2 = \sum_{e\in\mathcal{E}}\sum_{j=1,4} \|\eta_h|_{K_j^e}\|_L^2 \leq 4 \sum_{K\in\mathcal{T}_h} \|\eta_h|_K\|_L^2 = 4\|\eta_h\|_L^2. \qquad \blacksquare$$

The proof of Lemma 5. It is based on a similar local property in the reference macroelement:

$$(\widehat{P_1}) \quad \begin{cases} \forall \hat{\eta} \in \widehat{L}, \quad \exists \hat{\tau} = \hat{\tau}^s + \hat{\tau}^r \in \underline{\underline{\widehat{X}}} = \underline{\underline{\widehat{X}}}^s + \underline{\underline{\widehat{X}}}^r \quad \text{such that} \\ (i) \qquad (\text{div }\hat{\tau}, \hat{w})_{\hat{K}} = 0, \ \ \forall \hat{w} \in \underline{\widehat{M}}, \ \forall \widehat{K} \in \widehat{M}, \\ (ii) \qquad \text{as}(\hat{\tau}) = \hat{\eta}, \ \ \forall \hat{\mu} \in \widehat{L}, \\ (iii) \qquad \|\hat{\tau}^s\|_X + |\hat{\tau}^r|_H \leq C\, \|\hat{\eta}\|_L, \end{cases}$$

Lemma 8 *Suppose that $(\widehat{P_1})$ is satisfied, then Lemma 5 is true.*

Proof: According to lemma 7, $\eta_h \in L_h$ can be written as

$$\eta_h = \sum_e \widetilde{\eta}_e, \quad \eta_e \in L_e.$$

The local relations (i) and (ii) imply that, for each η_e, there exists $\tau_e \in \underline{\underline{X}}_{Me}$ (if $\eta_e = \widehat{\eta}\circ F_e^{-1}$, we take $\tau_e = \widehat{\tau}\circ F_e^{-1}$, where $\widehat{\tau}$ satisfies assumptions of $(\widehat{P_1})$), such that, if $K_e = F_e(\hat{K})$,

$$(a) \quad \forall w_e \quad (= \widehat{w}\circ F_e^{-1}) \in \underline{M}_e, \quad (\text{div }\tau_e, w_e)_{K_e} = h\,(\text{div }\hat{\tau}, \hat{w})_{\hat{K}} = 0,$$

$$(b) \quad \text{as}(\tau_e) = \eta_e \quad (\text{as}(\widetilde{\tau}_e) = \widetilde{\eta}_e).$$

If $\tau_e = \tau_e^r + \tau_e^s$, as the operator divergence is surjective from $\underline{\underline{X}}_e^s$ onto $\underline{M}_e$, we can take $w_e = \text{div }\tau_e^s$ in (a), which shows that $\text{div}\tau_e^s = 0$. Therefore, using (iii), we obtain

$$(c) \quad \|\tau_e^s\|_X + |\tau_e^r|_H = |\tau_e^s|_H + |\tau_e^r|_H = h\,(|\hat{\tau}^s|_H + |\hat{\tau}^r|_H) \leq c\,h\,\|\hat{\eta}\|_L \ \leq C\,\|\eta_e\|_L.$$

We define then τ_h^2 as

$$\tau_h^2 = \sum_e \widetilde{\tau}_e.$$

First, we remark that, using (a) and (b)

$$(\text{div }\tau_h^2, w_h) = \sum_{K_e} (\text{div }\widetilde{\tau}_e, w_h) = \sum_e (\text{div }\tau_e, w_h|_{M_e \cap\,\Omega})_{K_e} = 0, \ \ \forall w_h \in M_h,$$

$$\text{as}(\tau_h^2) = \sum_e \text{as}(\widetilde{\tau}_e) = \sum_e \widetilde{\eta}_e = \eta_h.$$

On the other hand, one has $\tau_h^2 = \tau_h^{2,s} + \tau_h^{2,r}, (\tau_h^{2,s}, \tau_h^{2,r}) \in \underline{\underline{X_h}}^s \times \underline{\underline{X_h}}^r$, where $\tau_h^{2,s} = \sum_e \widetilde{\tau_e^s}$ and $\tau_h^{2,r} = \sum_e \widetilde{\tau_e^r}$. Then, using (c) and (103)

$$(iii) \ \ \|\tau_h^{2,s}\|_X + |\tau_h^{2,r}|_H \leq \sum_e (\|\tau_e^s\|_X + |\tau_e^r|_H) \leq C \sum_e \|\eta_e\|_L \leq C'\,\|\eta_h\|_L. \qquad \blacksquare$$

To conclude, it remains to prove that property $(\widehat{P_1})$ holds. We shall use the rotational operator

$$\varphi \in H^1 \to \text{rot }\varphi = (-\frac{\partial\varphi}{\partial x_2}, \frac{\partial\varphi}{\partial x_1})^t \in (L^2)^2,$$

which satisfies

$$\text{div}\ (\text{rot}\ \varphi) = 0.$$

The idea is to look for $\hat{\tau} = \hat{\tau}^s + \hat{\tau}^r$ in a subspace $\mathcal{N} = \mathcal{N}^s \oplus \mathcal{N}^r$ of $\underline{\underline{\widehat{X}}}$ ($\mathcal{N}^s \subset \underline{\underline{\widehat{X}}}^s$ and $\mathcal{N}^r \subset \underline{\underline{\widehat{X}}}^r$) such that property (i) of Lemma 8 is automatically satisfied. More precisely, we search $\hat{\tau}$ in the form $\hat{\tau} = \hat{\Phi}(\hat{\eta})$ where $\hat{\Phi}$ is a linear map from $\widehat{L}$ into $\mathcal{N}$. Then we also have the property (iii) satisfied, since $\widehat{L}$ has a finite dimension. Thus, it suffices to define properly $\hat{\Phi}$ in order to satisfy (ii).

We first construct $\mathcal{N} = \mathcal{N}^s \oplus \mathcal{N}^r$. Let us recall that (i) will be satisfied by $\hat{\tau} = \hat{\tau}^s + \hat{\tau}^r$ if and only if $\mathbf{div}\hat{\tau}^s = 0$. Moreover, condition (ii) is independent of $\hat{\tau}_{11}$ and $\hat{\tau}_{22}$. This justifies to define

$$(105)\qquad \left|\begin{array}{ll} \mathcal{N}^s = & \left\{\hat{\tau} = (\hat{\tau}_1, \hat{\tau}_2) = (\text{rot}\ \varphi_1, \text{rot}\ \varphi_2),\ \forall\ (\varphi_1, \varphi_2) \in \widehat{V} \times \widehat{V}\right\}, \\ & \text{where } \widehat{V} = \left\{\varphi \in H^1_0(\widehat{M})\ /\ \varphi|_{\widehat{K}} \in Q_{k+1},\ \forall \widehat{K} \in \widehat{M}\right\}, \\ \mathcal{N}^r = & \left\{\hat{\tau}^r \in \underline{\underline{X}}^r\ /\ \hat{\tau}^r \text{ is of the form } \begin{pmatrix} 0 & \hat{\psi}_{12} \\ \hat{\psi}_{21} & 0 \end{pmatrix}\right\}. \end{array}\right.$$

We remark that $\varphi \in \widehat{V} \to \text{rot}\ \varphi$ is injective, so that

$$(106)\qquad \mathbf{Dim}\,\mathcal{N}^s = N_s = 2\mathbf{Dim}\ \widehat{V} = 8k^2 + 8k + 2.$$

On the other hand, using the characterization of $\underline{\underline{X}}^r$ given in [7], we compute that

$$\mathbf{Dim}\,\mathcal{N}^r = N_r = 8k + 4$$

Thus, the number of degrees of freedom defining $\hat{\tau}$ is

$$N_f = N_r + N_s = 8k^2 + 16k + 6.$$

On the other hand, (ii) is nothing but a linear system of N_{eq} equations (constraints), that $\hat{\tau}$ should satisfy. A simple computation gives

$$N_{eq} = \mathbf{Dim}\ \widehat{L} = 4k^2 + 16k + 7.$$

We notice that $N_{eq} \leq N_f$ if and only if

$$k \geq 1.$$

To write in practice the linear equations corresponding to (ii), we consider a basis $\{\hat{\tau}^s_l, 1 \leq l \leq N_s\}$ of $\mathcal{N}^s$, a basis $\{\hat{\tau}^r_m, 1 \leq m \leq N_r\}$ of $\mathcal{N}^r$ and a basis $\{\hat{\eta}_q, 1 \leq q \leq N_{eq}\}$ of $\widehat{L}$. We then write

$$\hat{\tau} = \sum_{l=1}^{N_s} x^s_l \hat{\tau}^s_l + \sum_{m=1}^{N_s} x^r_m \hat{\tau}^r_m$$

$$\hat{\eta} = \sum_{q=1}^{N_{eq}} b_q \hat{\eta}_q$$

Introducing $\hat{x}$ the vector of $\mathbb{R}^{N_f}$ whose the N_s first components are the x^s_l's and the N_r others are the x^r_l's, and $\hat{b}$ the vector of $\mathbb{R}^{N_{eq}}$ with components b_q, then (ii) can be rewritten

$$(107)\qquad \hat{A}\hat{x} = \hat{b}$$

where $\hat{A}$ is a $N_f \times N_{eq}$ matrix that can be computed explicitly with the three bases $\{\hat{\tau}^s_l\}$, $\{\hat{\tau}^r_m\}$ and $\{w_q\}$.

Let us assume that we are able to prove that $\hat{A}$ has full rank, i.e. that it is surjective. Then, we know that (107) has at least one solution $\hat{x}$, which is not unique when $N_f > N_{eq}$. We can take for instance the solution of minimal norm in $\mathbb{R}^{N_f}$, using the pseudo-inverse of $\hat{A}$

$$\hat{x} = \hat{A}^*\ (\hat{A}\hat{A}^*)^{-1}\hat{b}$$

We then define the map $\hat{\Phi}$ by

$$\hat{\tau} = \hat{\Phi}(\hat{\eta}) \Longleftrightarrow \hat{x} = \hat{A}^* \, (\hat{A}\hat{A}^*)^{-1}\hat{b} \tag{108}$$

which concludes the proof.

Proof for $k = 1, 2$. It remains to check that $\hat{A}$ is surjective. It seems difficult to give a general abstract proof for any k. Therefore, a separate proof must be done for each value ok k. In practice, for $k = 1, 2$, we form the matrix $\hat{A}$ and compute its rank using MAPLE. The results are the following:

- For $k = 1$, $n = 2$, $\hat{A}$ is (27×30) and $\text{rank}(\hat{A}) = 27$ ($\widehat{P_1}$ holds).
- For $k = 2$, $n = 2$, $\hat{A}$ is (57×70) and $\text{rank}(\hat{A}) = 57$ ($\widehat{P_1}$ holds).

According to lemma 8, we have proven the following lemma:

Lemma 9 *Lemma 5, and thus assumption (H2), is satisfied for k=1 and k=2.*

Proof for $k = 0$. On the other hand, we have to modify the proof for the lowest order element. For $k = 0$ the matrix $\hat{A}$ is 7×6 and $\text{rank}(\hat{A}) = 6$. Let $\hat{L}^\star$ be the subspace of $\hat{L}$ (of dimension 6), which is the image of $\mathcal{N}$ by the application $\hat{\Phi}$. It is clear that property $(\widehat{P_1})$ is replaced by

$$(\widehat{P_1}^\star) \quad \begin{cases} \forall \hat{\eta} \in \widehat{L}^\star, \quad \exists \hat{\tau} = \hat{\tau}^s + \hat{\tau}^r \in \underline{\underline{\widehat{X}}} = \underline{\underline{\widehat{X}}}^s + \underline{\underline{\widehat{X}}}^r \quad \text{such that} \\ (i) \qquad (\text{div}\, \hat{\tau}, \hat{w})_{\hat{K}} = 0, \ \ \forall \hat{w} \in \underline{\widehat{M}}, \ \forall \widehat{K} \in \widehat{M}, \\ (ii) \qquad \text{as}(\hat{\tau}) = \hat{\eta}, \\ (iii) \quad \|\hat{\tau}^s\|_X + |\hat{\tau}^r|_H \le C \, \|\hat{\eta}\|_L, \end{cases}$$

It is not difficult to characterize $\hat{L}^\star$. One finds (we omit the calculations)

$$\hat{L}^\star = \left\{ \hat{\eta} \in \hat{L}, / \sum_{j=1}^{4} \sum_{i=1}^{4} \hat{\eta}|_{K_j}(S_j^i) = 0 \right\} \tag{109}$$

This space is generated by the 6 basis functions $\{\widehat{\mu}_1^d, \widehat{\mu}_2^d\} \cup \{\widehat{\mu}_1^c, \widehat{\mu}_2^c, \widehat{\mu}_3^c, \widehat{\mu}_4^c\}$, where the functions $\widehat{\mu}_i^d, i = 1, 2$ are discontinuous and represented in figure 12, where we give on each element the four values of the function at the four vertices of the element (by convention, this value is omitted if it is equal to 0).

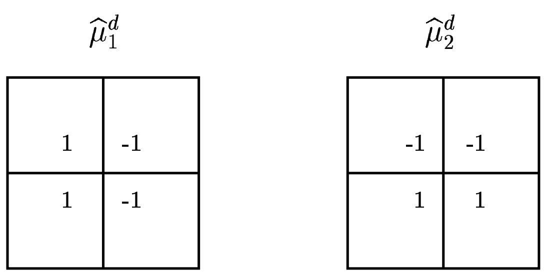


Figure 12: The basis functions $\widehat{\mu}_1^d$ and μ_2^d.

The four functions $\widehat{\mu}_i^c, i = 1, 4$ are continuous and represented in Fig. 13-left.

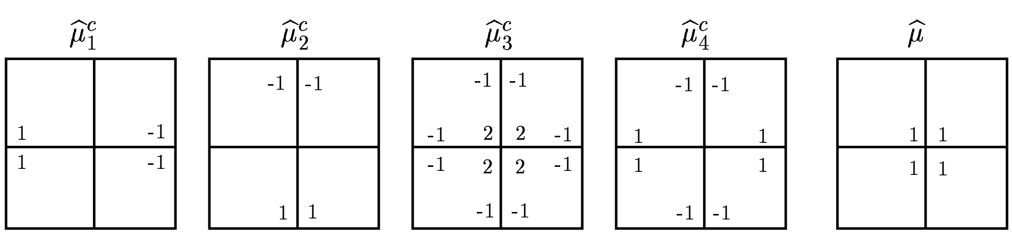


Figure 13: The basis functions $\widehat{\mu}_j^c$, $j = 1, .., 4$ (left) and $\widehat{\mu}$ (right).

We complete the space $\widehat{L}$ by adding the usual continuous Q_1 function $\widehat{\mu}$ associated to the center of the macroelement, see Fig. 13-right.

From $\hat{L}^\star$ we define for each macroelement

$$L_e^\star = \{\eta_e = \widehat{\eta} \circ F_e^{-1} \ , \ \widehat{\eta} \in \widehat{L}^\star\}.$$

and we introduce the space

$$L_h^\star = \{\eta_h = \sum_e \widetilde{\eta}_e, \eta_e \in L_e^\star\}$$

Using the same arguments as the one used for proving lemmas 7 and 8, we prove the following result: for all $\eta_h \in L_h^\star$ there exists a $\tau_h^2 \in \underline{\underline{X_h}}$ such that

$$\begin{aligned}
&(i) \qquad (\text{div } \tau_h^2, w_h) = 0, \ \ \forall w_h \in \underline{M_h},\\
&(ii) \qquad (\text{as}(\tau_h^2), \eta_h) \geq C_2 \, \|\eta_h\|_L^2 \, ,\\
&(iii) \qquad \|\tau_h^{2,s}\|_X + |\tau_h^{2,r}|_H \leq C_3 \|\eta_h\|_L,
\end{aligned}$$

with C_2, C_3 positive constants independent of h. This is nothing but Lemma 5 if we are able to prove that $L_h^\star = L_h$. Since

$$L_h = \{\eta_h = \sum_e \widetilde{\eta}_e, \eta_e \in L_e\}$$

it remains to prove that any $\widetilde{\eta}_e$ is in $L_h^\star$. As any $\eta_e \in L_e$ can be written as

$$\eta_e = \eta_e^\star + \alpha_e \mu_e \quad \text{with} \quad \eta_e^\star \in L_e^\star \text{ and } \mu_e = \widehat{\mu} \circ F_e^{-1}$$

it is sufficient to prove that $\widetilde{\mu}_e \in L_h^\star$. As we will see, this is true thanks to the overlaping between macroelements.

For the proof it seems more elegant to consider the mesh $\mathcal{T}_h$ as a subset of the infinite uniform mesh of $\mathbb{R}^2$ and to identify the index e to the couple (i,j), if the point S_e denotes the vertex of coordinates (ih, jh). In what follows, we shall work with the functions of $L_h^\star$ associated to this infinite mesh, that we shall represent in the figures by their four values at each vertex. Let S_e be one of the vertices of $\mathcal{T}_h$, we denote by $(\mu_k^d)^e = \widehat{\mu}_k^d \circ F_e^{-1}$ for $k = 1,2$ and $(\mu_k^c)^e = \widehat{\mu}_k^c \circ F_e^{-1}$ for $k = 1,4$.

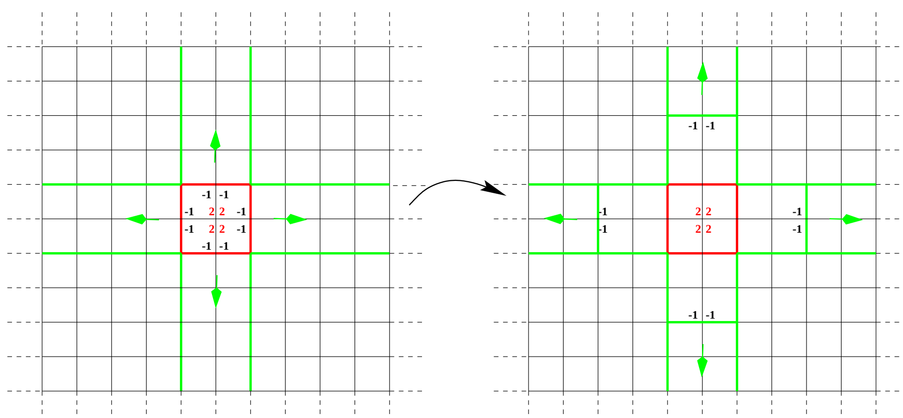


Figure 14: The functions $(\widetilde{\mu}_3^c)^e$ on the left and $\widetilde{\eta}_e^1$ on the right.

Let $t_1 = (1,0)$ and $t_2 = (0,1)$, we represent in Figure 14 the function:

$$\begin{aligned}
\widetilde{\eta}_e^1 \; &= (\widetilde{\mu}_3^c)^e + (\widetilde{\mu}_1^d)^{(e+t_1)} - (\widetilde{\mu}_1^d)^{(e-t_1)} + (\widetilde{\mu}_2^d)^{(e+t_2)} - (\widetilde{\mu}_2^d)^{(e-t_2)}\\
&+ (\widetilde{\mu}_1^c)^{(e+2t_1)} - (\widetilde{\mu}_1^c)^{(e-2t_1)} + (\widetilde{\mu}_2^c)^{(e+2t_2)} - (\widetilde{\mu}_2^c)^{(e-2t_2)}
\end{aligned}$$

By induction, it is clear that the restriction to Ω of the function of $L_h^\star$

$$(\widetilde{\mu}_3^c)^e \quad + \sum_{k=1}^{M} \Big((\widetilde{\mu}_1^d)^{(e+kt_1)} - (\widetilde{\mu}_1^d)^{(e-kt_1)} + (\widetilde{\mu}_2^d)^{(e+kt_2)} - (\widetilde{\mu}_2^d)^{(e-kt_2)}$$
$$+ (\widetilde{\mu}_1^c)^{(e+2kt_1)} - (\widetilde{\mu}_1^c)^{(e-2kt_1)} + (\widetilde{\mu}_2^c)^{(e+2kt_2)} - (\widetilde{\mu}_2^c)^{(e-2kt_2)} \Big)$$

coincides for M large enough with the function $2\widetilde{\mu}_e$. ∎

4 Error estimates for the evolution problem

We will show in this section, how we can use the abstract theory described in section 3.4 in order to obtain error estimates for the evolution problem (16), (13) (14). We follow the same technique as in [7], that is we first define an elliptic operator and then we relate error estimates for the evolution problem to an elliptic projection error thanks to energy estimates.

We consider first the variational formulation of the continuous elastodynamic problem in $\underline{\underline{X}} \times \underline{M} \times L$, that is

$$(110) \qquad \begin{cases} \text{Find } (\sigma, v, \gamma) : [0,T] \to \underline{\underline{X}} \times \underline{M} \times L \text{ such that} \\ \dfrac{d}{dt} a(\sigma(t), \tau) + \widetilde{b}(v(t), \gamma(t); \tau) = 0, & \forall \tau \in \underline{\underline{X}}, \\ \dfrac{d}{dt} c(v(t), w) - \widetilde{b}(w, \eta; \sigma(t)) = (f, w), & \forall (w, \eta) \in \underline{M} \times L, \end{cases}$$

with the initial conditions (13). Take now $\underline{\underline{X_h}}$, $\underline{\underline{X_h}}^{sym}$, $\underline{M_h}$ the new family of mixed finite elements (28), L_h being defined by (63) and consider the discrete problem

$$(111) \qquad \begin{cases} \text{Find } (\sigma_h, v_h, \gamma_h) : [0,T] \to \underline{\underline{X_h}} \times \underline{M_h} \times L_h \text{ such that} \\ \dfrac{d}{dt} a(\sigma_h(t), \tau_h) + \widetilde{b}(v_h(t), \gamma_h(t); \tau_h) = 0, & \forall \tau_h \in \underline{\underline{X_h}}, \\ \dfrac{d}{dt} c(v_h(t), w_h) - \widetilde{b}(w_h, \eta_h; \sigma_h(t)) = (f, w_h), & \forall (w_h, \eta_h) \in \underline{M_h} \times L_h, \end{cases}$$

with the initial conditions

$$(112) \qquad \sigma_h(t=0) = \sigma_{h,0}, \; v_h(t=0) = v_{h,0}.$$

From the classical theory of ODE's, we get the following result

Theorem 8 *If $f \in C^0(0,T; \underline{M_h})$, then problem (111) admits a unique solution $(\sigma_h, v_h, \gamma_h)$ in $C^1(0,T; \underline{\underline{X_h}}) \times C^1(0,T; \underline{M_h}) \times C^0(0,T; L_h)$.*

Our aim in this section is to study the error between the solution (σ, v, γ) of the continuous problem (110) and the solution $(\sigma_h, v_h, \gamma_h)$ of the approximation problem (111).

Remark 15 *As we have already mentioned, numerically we do not solve problem (111) but its equivalent (23) with the strong symmetry condition. Although we have constructed the new family of mixed finite elements in order to be able to do mass lumping, we analyze here the error for the discrete problem without mass lumping. Of course, when doing mass lumping, one should add to this error the quadrature error due to the numerical integration (see [27], [4]).*

4.1 The elliptic projection error

Following [13, 16] we introduce the elliptic operator defined in the following way

$$(113)\qquad \begin{cases} \text{Find } \Pi_h(\sigma,v,\gamma) = (\widehat{\sigma}_h, \widehat{v}_h, \widehat{\gamma}_h) \in \underline{\underline{X_h}} \times \underline{M_h} \times L_h \text{ such that} \\ a(\sigma - \widehat{\sigma}_h, \tau_h) + \widetilde{b}(v - \widehat{v}_h, \gamma - \widehat{\gamma}_h; \tau_h) = 0, \quad \forall \tau_h \in \underline{\underline{X_h}}, \\ \widetilde{b}(w_h, \eta_h; \sigma - \widehat{\sigma}_h) = 0, \quad \forall (w_h, \eta_h) \in \underline{M_h} \times L_h, \end{cases}$$

where (σ, v, γ) is an element of $\underline{\underline{X}} \times \underline{M} \times L$. We introduce now the following notations

$$\mathcal{C}^{m,r} = C^m(0,T;\underline{\underline{H}}) \cap C^r(0,T;\underline{\underline{X}}),$$

$$(114)\qquad \begin{cases} |||\sigma - \widehat{\sigma}_h||| = \|\sigma - \widehat{\sigma}_h{}^s\|_X + |\widehat{\sigma}_h{}^r|_H, \\ \|\,[(\sigma,v,\gamma) - \Pi_h(\sigma,v,\gamma)]\,\| = |||\sigma - \widehat{\sigma}_h||| + \|v - \widehat{v}_h\|_M + \|\gamma - \widehat{\gamma}_h\|_L. \end{cases}$$

We can show then the following theorem

Lemma 10 *Let $(\sigma,v,\gamma)(t)$ be the solution of problem (110) and assume $(\sigma,v,\gamma)(t) \in C^{1,0} \times C^1(0,T;\underline{M}) \times C^0(0,T;L)$, then*

(i) For all $\forall t \in [0,T]$, problem (113) admits a unique solution

$$\Pi_h(\sigma,v,\gamma)(t) = (\widehat{\sigma}_h{}^s + \widehat{\sigma}_h{}^r, \widehat{v}_h, \widehat{\gamma}_h)\,(t) \in \underline{\underline{X_h}} \times \underline{M_h} \times L_h.$$

Moreover we have the following error estimates

$$\|\,[(\sigma,v,\gamma) - \Pi_h(\sigma,v,\gamma)]\,(t)\| \le C\ \mathcal{E}_h(\sigma, \partial_t\sigma, v, \gamma)(t),$$

with

$$\begin{aligned} \mathcal{E}_h(\sigma,\partial_t\sigma,v,\gamma)(t) &= \Big\{ \inf_{\tau_h^s \in X_h^s} \|\sigma(t) - \tau_h^s\|_X + \inf_{w_h \in M_h} \|v(t) - w_h\|_M \\ &+ \inf_{\eta_h \in L_h} \|\gamma(t) - \eta_h\|_L + \inf_{z_h^s \in X_h^s} |\mathcal{A}(\partial_t\sigma)(t) - z_h^s|_H \\ &+ \inf_{\eta^{(h)} \in \Lambda_r^{(h)}} \left\|\gamma(t) - \eta^{(h)}\right\|_L \Big\}, \end{aligned}$$

which shows in particular that $|||\sigma - \widehat{\sigma}_h|||$, $\|v - \widehat{v}_h\|_M$ and $\|\gamma - \widehat{\gamma}_h\|_L$ tend to zero uniformly in time ($t \in [0,T]$).

(ii) In the same way, if $(\sigma,v,\gamma)(t) \in C^{k+1,k} \times C^{k+1}(0,T;\underline{M}) \times C^k(0,T;L)$ we have

$$(115)\qquad \left\|\left[(\partial_t^k\sigma, \partial_t^k v, \partial_t^k\gamma) - \Pi_h\left(\partial_t^k\sigma, \partial_t^k v, \partial_t^k\gamma\right)\right](t)\right\| \le C\ \mathcal{E}_h\left(\partial_t^k\sigma, \partial_t^{k+1}\sigma, \partial_t^k v, \partial_t^k\gamma\right)(t),$$

where we used the notation $\partial_t^k g = \dfrac{\partial^k g}{\partial t^k}$.

Proof. The proof is a simple application of the abstract theorem 3. It suffices to rewrite problem (110) as an elliptic problem in the following way

$$(116)\qquad \begin{cases} a(\sigma(t),\tau) + \widetilde{b}(v(t),\gamma(t);\tau) = a(\sigma(t),\tau) - \dfrac{d}{dt}a(\sigma(t),\tau), \quad \forall \tau \in \underline{\underline{X}}, \\ \widetilde{b}(w,\eta;\sigma(t)) = -(f,w) + \dfrac{d}{dt}c(v(t),w), \quad \forall (w,\eta) \in \underline{M} \times L. \end{cases}$$

Then we can remark that problem (113) is an approximation of problem (116). Indeed we can rewrite problem (113) as follows

$$
(117)\qquad \left\{
\begin{array}{lll}
\multicolumn{3}{l}{\text{Find } \Pi_h(\sigma,v,\gamma)(t) = (\widehat{\sigma}_h, \widehat{v}_h, \widehat{\gamma}_h)(t) \in \underline{\underline{X_h}} \times \underline{M_h} \times L_h \text{ such that}} \\[2mm]
a(\widehat{\sigma}_h, \tau_h) + & \widetilde{b}(\widehat{v}_h, \widehat{\gamma}_h; \tau_h) & = a(\sigma(t), \tau_h) - \dfrac{d}{dt} a(\sigma(t), \tau_h), \quad \forall \tau_h \in \underline{\underline{X_h}}, \\[2mm]
& \widetilde{b}(w_h, \eta_h; \widehat{\sigma}_h) & = -(f, w_h) + \dfrac{d}{dt} c(v(t), w_h), \quad \forall (w_h, \eta_h) \in \underline{M_h} \times L_h.
\end{array}
\right.
$$

In order to apply the abstract Theorem 3 or more precisely its generalization (see Remark 13) we need to check that the term

$$a(\sigma(t), \tau_h) - \frac{d}{dt} a(\sigma(t), \tau_h),$$

can be written as a H-scalar product. Indeed, we have

$$a(\sigma(t), \tau_h) - \frac{d}{dt} a(\sigma(t), \tau_h) = (\mathcal{A}(\sigma(t) - \partial_t \sigma(t)), \tau_h)_H.$$

We obtain the expression of $\mathcal{E}_h$ by replacing g by $\mathcal{A}(\sigma(t) - \partial_t \sigma(t))$ in (60) (see Remark 13). In the same way if (σ, v, γ) is sufficiently regular in time, we can differentiate problems (116) and (117) with respect to t and then apply theorem (3) and get (ii). ■

4.2 From error estimates for the elliptic problem to error estimates for the evolution problem

We will prove here the following theorem

Theorem 9 *Let (σ, v, γ) be the solution of (110) and $(\sigma_h, v_h, \gamma_h)$ the solution of the approximate problem (111) with the following initial conditions*

$$(118)\qquad (\sigma_{0,h}, v_{0,h}) = \Pi_h(\sigma_0 \;,\; v_0).$$

If $(\sigma, v, \gamma) \in C^{3,2} \times C^3(0,T; \underline{M}) \times C^2(0,T; L)$ and $(\sigma_h, v_h, \gamma_h) \in C^2(0,T; \underline{\underline{X_h}}) \times C^2(0,T; \underline{M_h} \times C^1(0,T; L_h))$, we have the following convergence result $\forall t \in [0,T]$,

$$\|(\sigma - \sigma_h^s)(t)\|_X \to 0 \;;\; |\sigma_h^r(t)|_H \to 0 \;;\; \|(v - v_h)(t)\|_M \to 0 \;;\; \|(\gamma - \gamma_h)(t)\|_L \to 0.$$

More precisely, we obtain the error estimates

$$
\begin{aligned}
\|(\sigma - \sigma_h^s)(t)\|_X + \|\sigma_h^r(t)\|_H \leq\; & C\,\{\mathcal{E}_h(\sigma, \partial_t\sigma, v, \gamma)(t) + \mathcal{E}_h(\partial_t\sigma, \partial_t^2\sigma, \partial_t v, \partial_t\gamma)(t) \\
& + \int_0^t \big(\mathcal{E}_h(\sigma, \partial_t\sigma, v, \gamma)(s) + \mathcal{E}_h(\partial_t\sigma, \partial_t^2\sigma, \partial_t v, \partial_t\gamma)(s) \\
& + \mathcal{E}_h(\partial_t^2\sigma, \partial_t^3\sigma, \partial_t^2 v, \partial_t^2\gamma)(s)\big)\; ds\}, \\
\|((v - v_h)(t), (\gamma - \gamma_h)(t))\|_{M\times L} \leq\; & C\,\{\mathcal{E}_h(\sigma, \partial_t\sigma, v, \gamma)(t) + \mathcal{E}_h(\partial_t\sigma, \partial_t^2\sigma, \partial_t v, \partial_t\gamma)(t) \\
& + \int_0^t \big(\mathcal{E}_h(\partial_t\sigma, \partial_t^2\sigma, \partial_t v, \partial_t\gamma)(s) \\
& + \mathcal{E}_h(\partial_t^2\sigma, \partial_t^3\sigma, \partial_t^2 v, \partial_t^2\gamma)(s)\big)\; ds\}.
\end{aligned}
$$

Furthermore if we assume that the solution $(\sigma, v, \gamma) \in C^3(0,T; H^{k+1}(div, \Omega)) \times C^3(0,T; H^{k+1}(\Omega)) \times C^2(0,T; H^{k+1}(\Omega)$ we obtain using the usual interpolation results (c.f [24])

$$(119)\qquad \|(\sigma - \sigma_h^s)(t)\|_X + \|(\sigma_h^r)(t)\|_H + \|((v - v_h)(t), (\gamma - \gamma_h)(t))\|_{M\times L} \leq C(t) h^k,$$

with $C(t) = O(\|\sigma\|_{C^3(0,t;H^{k+1}(div,\Omega))} + \|\mathcal{A}\partial_t\sigma\|_{C^2(0,t;H^{k+1}(\Omega))} + \|v\|_{C^3(0,t;H^{k+1}(\Omega))} + +\|\gamma\|_{C^3(0,t;H^{k+1}(\Omega))})$.

In order to prove theorem 9 we need the following lemma

Lemma 11 *Let (σ, v, γ) be the solution of (110) and $(\sigma_h, v_h, \gamma_h)$ the solution of the approximate problem (111) with the initial conditions (118). Let $\Pi_h(\sigma, v, \gamma) = (\widehat{\sigma}_h{}^s + \widehat{\sigma}_h{}^r, \widehat{v}_h, \widehat{\gamma}_h)$ the elliptic projection defined by (113). We set $\varepsilon_h = \sigma - \widehat{\sigma}_h$, $\delta_h = v - \widehat{v}_h$.*

- *If $(\sigma, v, \gamma) \in \mathcal{C}^{2,1} \times C^2(0,T;\underline{M}) \times C^1(0,T;L)$, there exists a constant C_1, independent of h such that $\forall t \in [0,T]$*

$$
|(\widehat{\sigma}_h{}^s - \sigma_h^s)(t)|_H + |(\widehat{\sigma}_h{}^r - p_h^r)(t)|_H + \|(\widehat{v}_h - v_h)(t)\|_M \leq C_1 \int_0^t \left(|\partial_t \varepsilon_h|_H(s) + |\varepsilon_h(s)|_H + \|\partial_t \delta_h(s)\|_M\right)\, ds. \tag{120}
$$

- *Moreover if $(\sigma, v, \gamma) \in \mathcal{C}^{3,2} \times C^3(0,T;\underline{M}) \times C^2(0,T;L)$ and $(\sigma_h, v_h, \gamma_h) \in C^2(0,T;\underline{\underline{X_h}}) \times C^2(0,T;Mh \times C^1(0,T;L_h))$, there exist constants C_2, C_3, independent of h such that $\forall t \in [0,T]$*

$$
\begin{aligned}
&(i) \quad \|((\widehat{v}_h - v_h)(t), (\widehat{\gamma}_h - \gamma_h)(t))\|_{M\times L} \equiv \|(\widehat{\gamma}_h - \gamma_h))(t)\|_L + \|(\widehat{v}_h - v_h)(t)\|_L \\
&\leq C_2 \left\{ |\partial_t \varepsilon_h(t)|_H + |\varepsilon_h(t)|_H + \int_0^t \left(|\partial_t^2 \varepsilon_h(s)|_H + |\partial_t \varepsilon_h(s)|_H + \|\partial_t^2 \delta_h(s)\|_M\right)\, ds \right\}, \\
&(ii) \quad \|(\widehat{\sigma}_h{}^s - \sigma_h^s)(t)\|_X \leq C_3 \left\{ \|\partial_t \delta_h(t)\|_M + \int_0^t \left(|\partial_t^2 \varepsilon_h(s)|_H \right.\right. \\
&\left.\left. + |\partial_t \varepsilon_h(s)|_H + |\varepsilon_h(s)|_H + \|\partial_t^2 \delta_h(s)\|_M + \|\partial_t \delta_h(s)\|_M\right)\, ds \right\}.
\end{aligned} \tag{121}
$$

Proof of Lemma 11.

- Estimation (120) : we begin by rewriting equations (110) with the test functions $\tau = \tau_h \in \underline{\underline{X_h}} \subset X$, $w = w_h \in \underline{M_h} \subset M$ and $\eta = \eta_h \in L_h \subset L$ and we subtract it from (111)

$$
\left\{
\begin{array}{ll}
\dfrac{d}{dt} a(\sigma - \sigma_h, \tau_h) + \widetilde{b}(v - v_h, \gamma - \gamma_h; \tau_h) & = 0, \quad \forall \tau_h \in \underline{\underline{X_h}}, \\
\dfrac{d}{dt} c(v - v_h, w_h) - \widetilde{b}(w_h, \eta_h; \sigma - \sigma_h) & = 0, \quad \forall w_h \in \underline{M_h}, \\
(\sigma - \sigma_h)(0) = \sigma_0 - \sigma_{0,h} \;\; ; \;\; (v - v_h)(0) = v_0 - v_{0,h}. &
\end{array}
\right.
$$

Introducing the elliptic projection $\Pi_h(\sigma, v, \gamma) = (\widehat{\sigma}_h, \widehat{v}_h, \widehat{\gamma}_h)$, we split the error between the approximate solution and the exact solution into two parts

$$
\left\{
\begin{array}{rl}
\|(\sigma - \sigma_h)(t)\|_X = & \|(\sigma - \widehat{\sigma}_h)(t)\|_X + \|(\widehat{\sigma}_h - \sigma_h)(t)\|_X, \\
\|(v - v_h)(t)\|_M = & \|(v - \widehat{v}_h)(t)\|_M + \|(\widehat{v}_h - v_h)(t)\|_M, \\
\|(\gamma - \gamma_h)(t)\|_L = & \|(\gamma - \widehat{\gamma}_h)(t)\|_L + \|(\widehat{\gamma}_h - \gamma_h)(t)\|_L,
\end{array}
\right. \tag{122}
$$

and we choose as approximate initial conditions the elliptic projection of the exact initial condition, (118), so that at time $t = 0$ we have

$$
(\widehat{\sigma}_h - \sigma_h)(0) = 0 \;\; ; \;\; \widehat{v}_h - v_h(0) = 0.
$$

Using now the error decomposition (122) we obtain

$$
\left\{
\begin{array}{ll}
a(\partial_t(\widehat{\sigma}_h - \sigma_h), \tau_h) + \widetilde{b}(\widehat{v}_h - v_h, \widehat{\gamma}_h - \gamma_h; \tau_h) & = -a(\partial_t(\sigma - \widehat{\sigma}_h), \tau_h) \\
& \quad - \widetilde{b}(v - \widehat{v}_h, \gamma - \widehat{\gamma}_h; \tau_h), \quad \forall \tau_h \in \underline{\underline{X_h}}, \\
c(\partial_t(\widehat{v}_h - v_h), w_h) - \widetilde{b}(w_h, \eta_h; \widehat{\sigma}_h - \sigma_h) & = -c(\partial_t(v - \widehat{v}_h), w_h) \\
& \quad + \widetilde{b}(w_h, \eta_h; \sigma - \widehat{\sigma}_h), \quad \forall (w_h, \eta_h) \in \underline{M_h} \times L_h.
\end{array}
\right. \tag{123}
$$

From (123) and (113) we get

$$
(124) \qquad \left\{
\begin{array}{ll}
a(\partial_t(\widehat{\sigma}_h - \sigma_h), \tau_h) + \widetilde{b}(\widehat{v}_h - v_h, \widehat{\gamma}_h - \gamma_h; \tau_h) & = -a(\partial_t(\sigma - \widehat{\sigma}_h), \tau_h) \\
 & \quad + a(\sigma - \widehat{\sigma}_h, \tau_h), \quad \forall \tau_h \in \underline{\underline{X_h}}, \\
c(\partial_t(\widehat{v}_h - v_h), w_h) - \widetilde{b}(w_h, \eta_h; \widehat{\sigma}_h - \sigma_h) & = -c(\partial_t(v - \widehat{v}_h), w_h), \\
 & \qquad \forall (w_h, \eta_h) \in \underline{M_h} \times L_h.
\end{array}
\right.
$$

Furthermore by taking $\tau_h = \widehat{\sigma}_h - \sigma_h, \; w_h = \widehat{v}_h - v_h, \; \eta_h = \widehat{\gamma}_h - \gamma_h$ in (123) and by adding the two equations, we get

$$
(125) \qquad
\begin{array}{rl}
a(\partial_t(\widehat{\sigma}_h - \sigma_h), \widehat{\sigma}_h - \sigma_h) + c(\partial_t(\widehat{v}_h - v_h), \widehat{v}_h - v_h) = & -a(\partial_t(\sigma - \widehat{\sigma}_h), \widehat{\sigma}_h - \sigma_h) \\
 & + a(\sigma - \widehat{\sigma}_h, \widehat{\sigma}_h - \sigma_h) \\
 & - c(\partial_t(v - \widehat{v}_h), \widehat{v}_h - v_h).
\end{array}
$$

Next set $E_h = \dfrac{1}{2}\left(a(\widehat{\sigma}_h - \sigma_h, \widehat{\sigma}_h - \sigma_h) + c(\widehat{v}_h - v_h, \widehat{v}_h - v_h)\right)$, we have

$$
(126) \qquad \frac{dE_h}{dt} = a(\partial_t(\widehat{\sigma}_h - \sigma_h), \widehat{\sigma}_h - \sigma_h) + c(\partial_t(\widehat{v}_h - v_h), \widehat{v}_h - v_h),
$$

Moreover $\exists C > 0$, independent of h such that

$$
(127) \qquad E_h^{1/2}(t) \geq C \left(|(\widehat{\sigma}_h - \sigma_h)(t)|_H^2 + \|(\widehat{v}_h - v_h)(t)\|_M^2 \right)^{1/2}.
$$

From (126) and (125) we obtain

$$
\begin{array}{rl}
2E_h^{1/2}(t) \dfrac{dE_h^{1/2}}{dt}(t) \leq & c\left(|\partial_t(\sigma - \widehat{\sigma}_h)(t)|_H + |(\sigma - \widehat{\sigma}_h)(t)|_H + \|\partial_t(v - \widehat{v}_h)(t)\|_M \right) \\
 & \left(|(\widehat{\sigma}_h - \sigma_h)(t)|_H^2 + \|(\widehat{v}_h - v_h)(t)\|_M^2 \right)^{1/2}.
\end{array}
$$

Using then (127) it comes

$$
(128) \qquad
\begin{array}{rl}
\dfrac{dE_h^{1/2}}{dt}(t) & \leq C' \left\{ |\partial_t(\sigma - \widehat{\sigma}_h)(t)|_H + |(\sigma - \widehat{\sigma}_h)(t)|_H + \|\partial_t(v - \widehat{v}_h)(t)\|_M \right\} \\
 & \leq C' \left\{ |\partial_t \varepsilon_h(t)|_H + |\varepsilon_h(t)|_H + \|\partial_t \delta_h(t)\|_M \right\},
\end{array}
$$

and from the choice (118) for the initial conditions

$$
E_h(0) = 0.
$$

It is then easy to see that (128) leads to (120).

• Estimation (121)-(i) : to get this error estimate, we will use the inf-sup condition, written in the following way

$$
\forall (w_h, \eta_h) \in \underline{M_h} \times L_h, \exists c \text{ independent of } h, \text{ such that}
$$

$$
\|(w_h, \eta_h)\|_{M \times L} \leq \frac{1}{c} \sup_{\tau_h \in \underline{\underline{X_h}}} \frac{\widetilde{b}(w_h, \eta_h; \tau_h)}{\|\tau_h^s\|_X + |\tau_h^r|_H}.
$$

We apply now the inf-sup condition on $w_h = \widehat{v}_h - v_h$ and $\eta_h = \widehat{\gamma}_h - \gamma_h$

$$
(129) \qquad \|(\widehat{v}_h - v_h, \widehat{\gamma}_h - \gamma_h)\|_{M \times L} \leq \frac{1}{c} \sup_{\tau_h \in \underline{\underline{X_h}}} \frac{\widetilde{b}(\widehat{v}_h - v_h, \widehat{\gamma}_h - \gamma_h; \tau_h)}{\|\tau_h^s\|_X + |\tau_h^r|_H}.
$$

From the first equation of system (124) we have

$$\widetilde{b}(\widehat{v}_h - v_h, \widehat{\gamma}_h - \gamma_h; \tau_h) = -a(\partial_t(\sigma - \widehat{\sigma}_h), \tau_h) + a(\sigma - \widehat{\sigma}_h, \tau_h) - a(\partial_t(\widehat{\sigma}_h - \sigma_h), \tau_h), \quad \forall \tau_h \in \underline{\underline{X_h}}.$$

Substituting that in (129) and using then the continuity of the bilinear form $a(\cdot,\cdot)$ we obtain

$$(130)\qquad \begin{aligned} \|((\widehat{v}_h - v_h)(t), (\widehat{\gamma}_h - \gamma_h)(t))\|_{M\times L} \quad &\leq \frac{\|a\|}{c}\,(|\partial_t \varepsilon_h(t)|_H \\ &+|\varepsilon_h(t)|_H + |\partial_t(\widehat{\sigma}_h - \sigma_h)(t)|_H)\,. \end{aligned}$$

Till now, we only have used the C^2 regularity of the solution. In order to bound $|\partial_t(\widehat{\sigma}_h - \sigma_h)|_H$, we need C^3. Indeed, we want to apply (120) replacing σ_h by $\partial_t \sigma_h$, $\widehat{\sigma}_h$ by $\partial_t \widehat{\sigma}_h$ and so on... More precisely, we have

$$(131)\qquad |\partial_t(\widehat{\sigma}_h - \sigma_h)(t)|_H \leq C \int_0^t \left(|\partial_t^2 \varepsilon_h(s)|_H + |\partial_t \varepsilon_h(s)|_H + \|\partial_t^2 \delta_h(s)\|_M\right)\, ds.$$

Finally, combining (130), (131) we get

$$\begin{aligned} \|((\widehat{v}_h - v_h)(t), (\widehat{\gamma}_h - \gamma_h)(t))\|_{M\times L} &\leq C_2 \{|\partial_t \varepsilon_h(t)|_H \\ +|\varepsilon_h(t)|_H \int_0^t &\left(|\partial_t^2 \varepsilon_h(s)|_H + |\partial_t \varepsilon_h(s)|_H + \|\partial_t^2 \delta_h(s)\|_M\right)\, ds \Big\}. \end{aligned}$$

- Estimation (121)-(ii) : to get the estimate on the $\underline{\underline{X}}$ norm, we will use the inverse inf-sup condition, that is

$$\forall \tau_h \in \underline{\underline{X_h}}, \exists c \text{ independent of } h, \quad \text{such that :}$$

$$|\tau_h|_{X/\mathrm{Ker}\ \widetilde{B}_h} \leq \frac{1}{c} \sup_{(w_h,\eta_h)\in \underline{M_h}\times L_h} \frac{\widetilde{b}(w_h, \eta_h; \tau_h)}{\|(w_h, \eta_h)\|_{M\times L}},$$

where we use the notation

$$|\tau_h|_X = \|\tau_h^s\|_X + |\tau_h^r|_H.$$

Let us first remark that for $\tau_h = \tau_h^s \in \underline{\underline{X_h}}^s$ we have

$$|\tau_h^s|_{X/\mathrm{Ker}\ \widetilde{B}_h} \equiv \|\tau_h^s\|_{X/\mathrm{Ker}\ \widetilde{B}_h}.$$

Indeed,

$$|\tau_h^s|_{X/\mathrm{Ker}\ \widetilde{B}_h} = \inf_{q_h \in \mathrm{Ker}\ \widetilde{B}_h} (\|\tau_h^s + q_h^s\|_X + |q_h^r|_H) = \inf_{q_h \in \mathrm{Ker}\ \widetilde{B}_h} (\|\tau_h^s + q_h^s\|_X) = \|\tau_h^s\|_{X/\mathrm{Ker}\ \widetilde{B}_h}.$$

So that, for any $\tau_h^s \in \underline{\underline{X_h}}^s \subset \underline{\underline{X_h}}$, the inverse inf-sup condition gives

$$(132)\qquad \|\tau_h^s\|_{X/\mathrm{Ker}\ \widetilde{B}_h} \leq \frac{1}{c} \sup_{(w_h,\eta_h)\in \underline{M_h}\times L_h} \frac{\widetilde{b}(w_h, \eta_h; \tau_h^s)}{\|(w_h, \eta_h)\|_{M\times L}}.$$

We recall now that for all $\tau_h^s \in \underline{\underline{X_h}}^s$ we have $\tau_h^s = \tau_1^s + \tau_2^s$ with $\tau_1^s \in \mathrm{Ker}\ B_h$ and $\tau_2^s \in (\mathrm{Ker}\ B_h)^\perp$ so

$$\|\tau_h^s\|_X^2 = |\tau_1^s|_H^2 + \|\tau_2^s\|_X^2,$$

(using that $\|\tau_1^s\|_X = \|\tau_1^s\|_H \ \ \forall \tau_1^s \in \mathrm{Ker}\ B_h$). We set $\tau_h^s = \widehat{\sigma}_h^s - \sigma_h^s$, the term $|\tau_1^s|_H$ is already estimated from the inequality (120), therefore in order to get the second inequality (121)-(ii) we only need to estimate $\|\tau_2^s\|_X$. To do so, we will use (132). We also know that $\|\tau_2^s\|_X = \|\tau_2^s\|_{X/\mathrm{Ker}\ \widetilde{B}_h}$. Thus by taking $\tau_h^s = \tau_2^s$ in (132) we get

$$\|\tau_2^s\|_X \leq \frac{1}{c} \sup_{(w_h,\eta_h)\in \underline{M_h}\times L_h} \frac{\widetilde{b}(w_h, \eta_h; \tau_2^s)}{\|(w_h, \eta_h)\|_{M\times L}},$$

using now the second equation of (124) we obtain

$$\|\tau_2^s\|_X \le C' \left\{\|\partial_t(v-\widehat{v}_h)\|_M + \|\partial_t(\widehat{v}_h - v_h)\|_M\right\}. \tag{133}$$

To estimate $\|\partial_t(\widehat{v}_h - v_h)\|_M$, we replace as previously v_h by $\partial_t v_h$, $\widehat{v}_h$ by $\partial_t \widehat{v}_h$ and so on... in (120). We obtain then

$$|\partial_t(\widehat{v}_h - v_h)(t)|_M \le C\int_0^t \left(|\partial_t^2 \varepsilon_h(s)|_H + |\partial_t \varepsilon_h(s)|_H + \|\partial_t^2 \delta_h(s)\|_M\right)\, ds. \tag{134}$$

Finally, combining (133), (134) we get

$$\|\tau_2^s\|_X \le C_3 \left\{\|\partial_t \delta_h\|_M + \int_0^t \left(|\partial_t^2 \varepsilon_h(s)|_H + |\partial_t \varepsilon_h(s)|_H + \|\partial_t^2 \delta_h(s)\|_M\right)\, ds\right\}.$$

and the proof is achieved. ■

Proof of Theorem 9. We combine results given in lemma 10 and in lemma 11.

• Error estimates on $\|\sigma - \sigma_h^s\|_X(t) + |\sigma_h^r|_H(t)$

We have

$$\begin{aligned}\|(\sigma - \sigma_h^s)(t)\|_X + \|\sigma_h^r(t)\|_H &= \|(\sigma - \widehat{\sigma}_h^s + \widehat{\sigma}_h^s - \sigma_h^s)(t)\|_X + \|(\sigma_h^r - \widehat{\sigma}_h^r + \widehat{\sigma}_h^r)(t)\|_H \\ &\le \underbrace{\|(\sigma - \widehat{\sigma}_h^s)(t)\|_X + |\widehat{\sigma}_h^r(t)|_H}_{\text{from lemma 10-(i)}} + \underbrace{\|(\widehat{\sigma}_h^s - \sigma_h^s)(t)\|_X}_{\text{from (121)-(ii)}} \\ &\quad + \underbrace{|(\widehat{\sigma}_h^r - \sigma_h^r)(t)|_H}_{\text{from (120)}}.\end{aligned} \tag{135}$$

The first term is estimated using (i) of Lemma 10 and we obtain

$$\|(\sigma - \widehat{\sigma}_h^s)(t)\|_X + |\widehat{\sigma}_h^r)(t)|_H \le \mathcal{E}_h(\sigma, \partial_t\sigma, v, \gamma)(t).$$

The second term is estimated using (121)-(ii), this requires to estimate

$$|\partial_t \varepsilon_h|_H,\ |\partial_t^2 \varepsilon_h|_H,\ \|\partial_t \delta_h\|_M \text{ and } \|\partial_t^2 \delta_h\|_M.$$

For this we use (ii) of Lemma 10 for $k = 1$ and 2, we get

$$\begin{aligned}|\partial_t \varepsilon_h(s)|_H + \|\partial_t \delta_h(s)\|_M &\le \mathcal{E}_h(\partial_t\sigma, \partial_t^2\sigma, \partial_t v, \partial_t\gamma)(s), \\ |\partial_t^2 \varepsilon_h(s)|_H + \|\partial_t^2 \delta_h(s)\|_M &\le \mathcal{E}_h(\partial_t^2\sigma, \partial_t^3\sigma, \partial_t^2 v, \partial_t^2\gamma)(s).\end{aligned} \tag{136}$$

Finally for the third term we use (120). In this case the terms $|\partial_t \varepsilon_h|_H$, $\|\partial_t \delta_h\|_M$ appear and to estimate them we use Lemma 10 for $k = 1$ or more precisely (136).

• Error estimates on $\|(v - v_h, \gamma - \gamma_h)\|_{M\times L}(t)$

In the same way we have

$$\begin{aligned}\|((v - v_h)(t), (\gamma - \gamma_h)(t))\|_{M\times L} &= \|(v - \widehat{v}_h + \widehat{v}_h - v_h)(t)\|_M + \|(\gamma - \widehat{\gamma}_h + \widehat{\gamma}_h - \gamma_h)(t)\|_L \\ &\le \underbrace{\|(v - \widehat{v}_h)(t)\|_M + \|(\gamma - \widehat{\gamma}_h)(t)\|_L}_{\text{from lemma 10-(i)}} + \underbrace{\|(\widehat{v}_h - v_h)(t)\|_M + \|(\widehat{\gamma}_h - \gamma_h)(t)\|_L}_{\text{from 121-(i)}}.\end{aligned} \tag{137}$$

The first term is again estimated from (i) of Lemma 10. For the second term we use now (121)-(i), in which the terms $|\partial_t \varepsilon_h|_H$, $|\partial_t^2 \varepsilon_h|_H$, $\|\partial_t \delta_h\|_M$ and $\|\partial_t^2 \delta_h\|_M$ appear. Finally we obtain the error estimates (119) by using the usual interpolation results (c.f [24]) in $\underline{\underline{X_h}}^s, \underline{M_h}, L_h$ (see also Remark 3.5.3 and Lemma 3). ■

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Unit´e de recherche INRIA Lorraine, Technopôle de Nancy-Brabois, Campus scientifique,
615 rue du Jardin Botanique, BP 101, 54600 VILLERS LÈS NANCY
Unit´e de recherche INRIA Rennes, Irisa, Campus universitaire de Beaulieu, 35042 RENNES Cedex
Unit´e de recherche INRIA Rhône-Alpes, 655, avenue de l'Europe, 38330 MONTBONNOT ST MARTIN
Unit´e de recherche INRIA Rocquencourt, Domaine de Voluceau, Rocquencourt, BP 105, 78153 LE CHESNAY Cedex
Unit´e de recherche INRIA Sophia-Antipolis, 2004 route des Lucioles, BP 93, 06902 SOPHIA-ANTIPOLIS Cedex

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