# Pomset Logic as a Calculus of Directed Cographs

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**Abstract:** We give an abstract formulation of proof-structures and nets for pomset logic i.e. linear logic enriched with a non commutative and self-dual connective. A proof-structure is described as a directed R&B-cograph, that is an edge bicoloured graph: one colour is a perfect matching and the other a directed cograph — directed cographs are a simple generalization of cographs and series-parallel orders. The proof-nets or correct proof-structures are the directed R&B-cographs such that every alternate elementary circuit contains a chord.

This representation is even more compact than usual descriptions: the algebraic properties of the connectives, associativity and commutativity, are interpreted by equality, as well as the presence or not of final disjunctions (final *par*'s). But the main advantage is that any directed R&B-cograph, without any further specification, is a proof-structure.

We then study a step by step invertible transformation from proof-structures with links to directed R&B-cographs; this transformation and its inverse preserve correctness. Next we study the impact of the graph rewriting which axiomatizes the inclusion of directed cographs (found with D. Bechet and Ph. de Groote) on the correctness of directed R&B-cographs, and we show that all rewriting rules but one preserve the correctness. This yields cut-elimination (strongly normalizing and confluent) and suggests a complete sequent calculus for Pomset logic.

These results also apply to linear logic enriched with the *mix* rule, since Pomset logic is a conservative extension of it.

**Key-words:** Proof theory; linear logic. Graph theory

*(Résumé : tsvp)*

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# La logique ordonnée vue comme un calcul de cographes orientés

**Résumé :** Nous donnons une définition abstraite des réseaux de démonstration de la logique ordonnée c'est-à-dire de la logique linéaire enrichie par un connecteur non commutatif et auto-dual. Un préréseau est décrit par un graphe R&B orienté, c'est-à-dire par un graphe aux arêtes bicolores: une couleur définit un couplage parfait, et l'autre un cographe orienté — une généralisation des ordres séries-parallèles et des cographes. Les réseaux sont les préréseaux satisfaisant le critère de correction suivant: tout circuit élémentaire alternant contient une corde.

Les (pré)réseaux ainsi définis sont encore plus compacts que d'habitude: les propriétés algébriques des connecteurs, commutativité et associativité, sont interprétées par l'égalité, ainsi que la présence ou l'absence des disjonctions finales (les *par* finaux). Mais l'avantage principal est que tout cographe R&B orienté, sans aucune spécification supplémentaire est un préréseau.

Nous étudions alors une transformation graduelle et inversible des (pré)réseaux usuels avec liens en graphes R&B orientés; cette transformation et son inverse préservent la correction. Nous étudions ensuite l'effet de la réécriture de graphes qui axiomatise l'inclusion des cographes orientés (trouvée avec D. Bechet et Ph. de Groote) sur la correction des graphes R&B orientés, et montrons qu'elle est préservée par toutes les règles de réécriture sauf une. Ceci suffit à établir l'élimination des coupures (confluente et fortement normalisante) et suggère également un calcul des séquents complet pour la logique ordonnée.

Ces résultats valent aussi pour la logique linéaire multiplicative enrichie par la règle *mix*, puisque la logique ordonnée en est une extension conservative.

**Mots-clé :** Théorie de la démonstration; logique linéaire. Théorie des graphes.

## Introduction

Pomset logic is naturally issued from coherent semantics, where besides the two usual multiplicative commutative connectives ($\wp$, "*par*" and $\otimes$ "*times*") there is a third multiplicative connective $<$ called "*before*". This connective is associative, non-commutative, and self-dual, and lies in between the conjunction $\otimes$ and the disjunction $\wp$ with respect to linear implication. After some investigations it turns out that one can define a proof-net calculus extending multiplicative linear logic plus *mix* to this new connective, which enjoys cut-elimination (strongly normalizing and confluent) and has a simple denotational semantics. [9, 11]

On an applicative ground, with A. Lecomte we explored the grammatical use of this calculus: we defined a categorial grammar in which partial proof-nets are associated with words (rather than plain formulas). The richness of this calculus which handles partial orders rather than linear orders enables to model sophisticated linguistic phenomena — using such a classical calculus is possible since it does not induce a cyclic exchange rule. [4, 5, 6]

A. Asperti [1] and A. Guglielmi [3] have studied the possible use of this calculus for concurrency, and the non-commutative connective "*before*" corresponds to sequential composition, which is a natural primitive for concurrency.

Nevertheless there is quite a big problem with this calculus, namely the absence of a complete sequent calculus corresponding to the proof-net calculus. Sound sequent calculi have been proposed but up to now we do not know whether all proof-nets correspond to sequent calculus proofs. This has a more technical drawback: in the absence of such a result, proving a usually simple property may be difficult, as there is no inductive definition for proof-nets. For instance, truth value semantics would be easier to find if there were rules.

In this paper we provide a new description of proof-structures and nets for pomset logic, motivated by the afore-mentioned problem. The graphs representing proof-structures and nets have both directed and undirected edges. Their vertices are the propositional variables of the conclusion sequent (and of the cut-formulas, to be precise). Theses graphs are endowed with a perfect matching B that is a set of pairwise non adjacent undirected edges; the edges not in the perfect matching define a directed cograph, and they can be directed or undirected. From a logical viewpoint and the B-edges of the perfect matching are the axioms, and the other edges, called R-edges define a binary relation on the vertices (propositional variables) which totally encodes the structure of the conclusion sequent, that we may assume to be reduced to a single formula $F$, since it makes no difference. The R-edges encodes the conclusion as follows:

- There is an undirected R-edge between $p$ and $q$ whenever $F = G[A(p) \otimes B(q)]$ — i.e. when $p$ and $q$ meet on a *times* in the subformula tree of $F$.
- There is a directed R-edge from $p$ to $q$ whenever $F = G[A(p) < B(q)]$ — i.e. when $p$ and $q$ meet on a *before* in the subformula tree of $F$.
- There is no R-edge at all $p$ and $q$ whenever $F = G[A(p) \wp B(q)]$ — i.e. when $p$ and $q$ meet on a *par* in the subformula tree of $F$.

We christened such relations/graphs “directed cographs” because they generalize in the simplest way series-parallel orders and cographs. The main advantage of this description as opposed to other presentations of proof-nets is that any directed R&B-cograph does correspond to a proof-structure without any further restriction like on the bricks it is made of, the degree of the vertices etc.

The criterion for recognizing proof-nets among proof-structures is that every circuit using alternatively B and R edges should contain a chord, where a chord is an edge (directed or undirected) between two vertices in the circuit such that neither the edge itself or its opposite lies in the circuit.

Section 1 recalls the definition of pomset proof-structures and nets with links, section 2 introduces the new description, directed R&B-cographs, and section 3 shows the equivalence of the two descriptions. We then study, in section 4, a step by step transformation which turns a proof-structure with links into a directed R&B-cograph. At an intermediate step of this process, a proof-structure consists in a proof-structure with links plus a directed cograph on its conclusions. This transformation and its inverse are shown to preserve correctness, that is the absence of chordless Æ-circuit. Next, in section 5, we show that the rewriting system of [2] which axiomatizes the inclusion of (directed) cographs preserves the correctness of directed R&B-cographs —except one rewriting rule concerning the compositions of directed cographs corresponding to $\wp$ and $\otimes$. From these results we obtain another look at cut-elimination for pomset logic, which is described in section 6; it is shown to be a strongly normalizing and confluent process; directed R&B-cographs and the rewriting rules suggest some extensions of the use of the cut-rule and the reduction procedure which are briefly sketched.

Notice that we could have proceeded the other way round, that is to deduce that the rewriting rules preserve the correctness from the transformation step by step, cut-elimination and the fact that rewriting rules correspond to provable linear implications. Our choice yields a simpler presentation and allows some generalization of cuts and cut-elimination.

The most promising result in here is the rewriting system for pomset logic. Indeed all correct directed R&B-cographs are included into the complete directed R&B-cograph with

$2n$ vertices — the B-edges being $n$ axioms. So the cograph of the complete directed R&B-cograph rewrites to any correct R&B-cograph with $n$ axioms. Assume that this also holds for when the rewriting rule which does not preserve correctness is excluded. Then we have a rewriting system which derives exactly all the proofs, so we can describe the logic as a Hilbert system, from which it is not too difficult to extract a complete sequent calculus. Indeed this holds for MLL and MLL+mix [14, 13]. We are willing to do the same for Pomset logic, i.e. to prove that the rewriting rules which preserve correctness are enough to derive all the correct directed R&B-cographs. This would provide an inductive definition of correct directed R&B-cographs from which it is easy to extract a complete sequent calculus.

# 1 Standard description of Pomset logic proof-nets: proof-structures and nets with links

We recall here the standard description of Pomset proof-nets, that we restrict to series-parallel partial orders.

## 1.1 Graphs, directed R&B-graphs

In this paragraph we present the graph-theoretical notions that we use. They are standard (see e.g. the first pages of [7]) except that we consider graph with both directed and undirected edges, which is unusual for dealing with matchings. Beware that we will make a distinction between edge (undirected) and arc (directed edge).

A **graph** (or undirected graph) consists in a set of **vertices** $V$ and a *multiset* of **edges** or unordered pairs of distinct vertices $\{x, y\}$ — possibly $\{x, y\}_i$ when there are several $\{x, y\}$ edges. A *set* $B$ of edges in an undirected graph $\mathcal{G}$ is called a **matching** if no two edges of $B$ are adjacent — $\{x, y\}_i, \{u, v\}_j \in B$ entails $\{x, y\} = \{u, v\}$ and $i = j$ or $\{x, y\} \cap \{u, v\} = \emptyset$. The matching $B$ is said to be a **perfect matching** whenever each vertex is incident to exactly one edge of $B$.

A directed graph or **digraph** $\mathcal{G} = (V, E)$ consists in a set of **vertices** $V$ and a multiset of **arcs** (or directed edges) $E \subset V^2$ whose end vertices are always distinct — $\forall x \in V\ (x, x) \notin E$. We often simply denote an arc $(x, y)$ by $xy$. Given a digraph, the **underlying undirected graph** is defined as follows: if we have $p$ arcs $xy$ and $q$ arcs $yx$ then there will be $\max(p, q)$ edges $\{x, y\}$ in the underlying undirected graph: an arc $xy$ and an arc $yx$ result in a single undirected edge $\{x, y\}$, and an arc $xy$ with no corresponding arc $yx$ results in an undirected edge $\{x, y\}$. A **matching** in a directed graph is a *set* of invertible arcs $B$ such that $(x, y) \in B$ entails $(y, x) \in B$ and such that $(x, y), (u, v) \in B$ entails $\{x, y\} = \{u, v\}$ or $\{x, y\} \cap \{u, v\} = \emptyset$. So it is a set of invertible arcs which, when mapped to the underlying undirected graph yields a matching in the usual sense. A matching in a directed graph is said to be a **perfect matching** whenever each vertex is incident to exactly two opposite arcs of the matching.

A **path** of length $n$ in a graph (resp. digraph) is an alternate sequence of vertices and edges (resp. arcs), from which we often leave out the vertices:

$$x_0\ \{x_0,x_1\}\ x_1 \cdots x_{n-1}\ \{x_{n-1},x_n\}\ x_n \quad \text{(resp. } x_0\ \ x_0x_1\ \ x_1 \cdots x_{n-1}\ \ x_{n-1}x_n\ \ x_n\text{)}$$

It is said to be **elementary** whenever no two vertices are equal but, possibly, the first and last. In this latter case the path is said to be a **cycle** (resp. **circuit**).

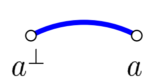
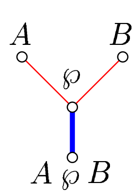
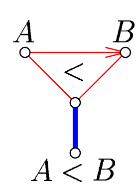
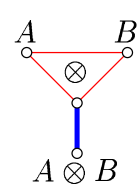
Given a (di)graph $\mathcal{G}$ and a matching $B$, a path $\mathcal{P}$ is said to be **alternating** if the edges (arcs) of $\mathcal{P}$ are alternately in $B$ and not in $B$. Given a (di)graph and a matching, an alternating elementary path will be written an **Æ-path**. An alternating elementary circuit *of even length* is called an **Æ-cycle** ( **Æ-circuit**).

An **R&B-(di)graph** $\mathcal{G} = (V; B, R)$ is a (di)graph $\underline{\mathcal{G}} = (V; B \uplus R)$ endowed with a perfect matching $B$ such that the (di)graph $(V; R)$ contains no multiple edge (arc). Since a matching does not contain multiple arcs as well, the maximum number of arcs we can have between two vertices $x$ and $y$ is a B-edge $\{x,y\}$ that is a pair of B-arcs $(x,y)$ and $(y,x)$ and the R-arcs $(x,y)$ and $(y,x)$ too. The (di)graph $\underline{\mathcal{G}}$ is said to be the underlying (di)graph of $\mathcal{G}$.

Clearly R&B-(di)graphs can be pictured as arcs-bicoloured digraph, the two colours being $B$ (Bold, Blue) for the perfect matching and $R$ (Regular, Red) for the other arcs.

## 1.2 Links, proof-structures with links

In [11] Pomset logic is described as a standard proof-net calculus with links. The links are defined as bicoloured graphs as follows:

| Name | *axiom*-link | *par*-link | *before*-link | *times*-link |
|---|---|---|---|---|
| Premises | none | $A$ and $B$ | $A$ and $B$ | $A$ and $B$ |
| R&B-graph |  |  |  |  |
| Conclusions | $a$ and $a^\perp$ | $A \wp B$ | $A < B$ | $A \otimes B$ |

A proof-structure with links is an R&B-graph made out of links such that:

– each formula vertex is the conclusion of exactly one link

– each formula vertex is the premise of at most one link

The formula vertices which are not the premise of any link are said to be the conclusions of the proof-structure with links.

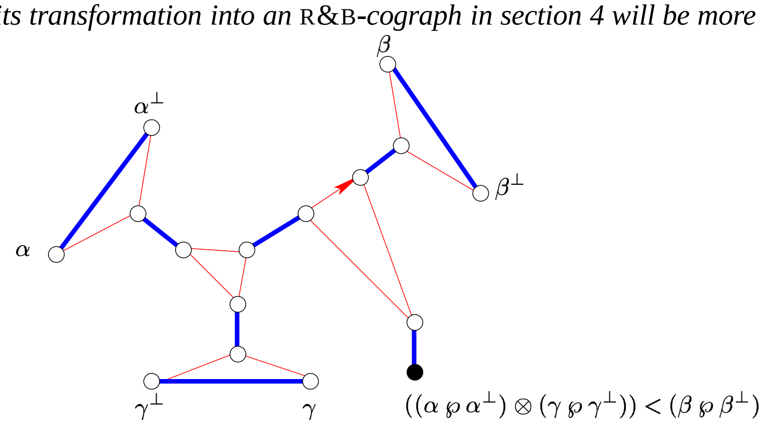
Actually there is a slight difference with [11]. There is no order on conclusions, so we are implicitly restricting ourselves to series-parallel partial orders. Indeed, as argued in [11] series-parallel orders maybe faithfully encoded by the connectives *before* and *par*, and conversely a conclusion which is a combination of a multi-set $\mathcal{F}$ of formulas by means of *before* and *par* corresponds to a series-parallel partial order on the conclusions in $\mathcal{F}$: unsurprisingly *before* corresponds to series-composition, and *par* to parallel composition.

### 1.3 The correctness criterion for pomset logic proof-nets with links

This criterion is the simplest extension of the one for MLL+mix:

**Correctness criterion 1** *A proof-structure is said to be a proof-net or to be correct whenever it does not contain any Æ-circuit.*

**Example 1** *Here is an example of a correct proof-net with a single conclusion. It looks unusual because we did not respect the habit of writing conclusion below premises. This way its transformation into an* R&B-*cograph in section 4 will be more visible.*




It may seem weird to state a correctness criterion without offering a sound and complete sequent calculus which would faithfully correspond to it. Let us explain the motivations for this correctness criterion:

- When restricted to the commutative connectives $\wp$ and $\otimes$ it is exactly the criterion for MLL+mix.
- This criterion is preserved under cut-elimination.
- This criterion is equivalent to the interpretability of proofs in coherence spaces [10] (extending [12] which only handles MLL+mix without before).
- It is the most general criterion enjoying these properties.
- The proofs of the sequent calculus proposed in [11] maps to proof-nets or correct proof-structures.

# 2 New description of pomset logic proof nets: directed R&B-cographs

## 2.1 Directed cographs

We consider only binary relations, that is simple digraphs, i.e. digraphs without multiple arcs. Directed cographs generalize series-parallel partial orders and cographs, in the simplest manner. A good reference on these classes of graphs and relations is [8].

Given a relation $R \subset E^2$ its directed part $R^{\uparrow}$ and undirected part $R^{\updownarrow}$ are defined by: $R^{\uparrow} = \{(x,y) \in R | (y,x) \notin R\}$ and $R^{\updownarrow} = \{(x,y) \in R | (y,x) \in R\}$. Clearly: $R = R^{\uparrow} \uplus R^{\updownarrow}$.

Given two relations on disjoint sets $R \subset E^2$ and $S \subset F^2$ with $E \cap F = \emptyset$, let us define directed series composition, symmetric series composition and parallel composition as the following relations on $E \uplus F$:

- $R \mathbin{\widehat{\wp}} S = R \uplus S$ — **parallel composition**
- $R \mathbin{\widehat{<}} S = R \uplus S \uplus (E \times F)$ — **series composition**
- $R \mathbin{\widehat{\otimes}} S = R \uplus S \uplus (E \times F) \uplus (F \times E)$ — **symmetric (series) composition**

Let $\mathsf{1}$ be the class of empty relations on singletons.

The smallest class of relations containing $\mathsf{1}$ and closed under parallel and series compositions is the class of **series-parallel partial orders**. A series-parallel order $R$ is characterized by being an $N$-free order: $R$ is antireflexive and transitive (thus antisymmetric) and $(x,y)(z,y)(z,t) \in R \Rightarrow \{(y,t),(z,x),(x,t)\} \cap R \neq \emptyset$.

The smallest class of relations containing $\mathsf{1}$ and closed under parallel and symmetric series compositions is known as the class of series-parallel graphs or **cographs**. A cograph is characterized as being a $P_4$ free graph, i.e. $R$ is antireflexive and symmetric, and

$(x,y)(z,y)(x,t) \in R \Rightarrow \{(y,t),(z,x),(z,t)\} \cap R \neq \emptyset$. [This looks exactly as being $N$-free but here the relation is symmetric, so $(u,v) \in R \Leftrightarrow (v,u) \in R$].

What about the smallest class of relations containing $\mathtt{1}$ , and closed under parallel, series, and symmetric-series compositions? Let us call them **directed cographs**. In [2] we characterized them as follows:

- $R$ is antireflexive — $\forall x \in E\ (x,x) \notin R$.
- $R^{\uparrow}$, the directed part of $R$ is N-free
- $R^{\updownarrow}$, undirected part of $R$ is $P_4$ free
- $R$ is **weakly transitive**, i.e. $(x,y) \in R \wedge (y,z) \in R^{\uparrow} \Rightarrow (x,z) \in R$ and $(x,y) \in R^{\uparrow} \wedge (y,z) \in R \Rightarrow (x,z) \in R$

In other words:

- $R$ is antireflexive
- the directed part of $R$ is a series-parallel order
- the undirected part of $R$ is a cograph
- $R$ is weakly transitive

Obviously, a directed cograph $R$ can be written as a linear term over the vertices, i.e. as a term in which each vertex appears exactly once — the operations being the three compositions: series, symmetric-series and parallel compositions. Such a term is called a **coterm** for $R$ and is unique up to the algebraic properties of the operations: commutativity of symmetric-series and parallel compositions "$\widehat{\otimes}$" and "$\widehat{\wp}$" and associativity of all compositions "$\widehat{\otimes}$" , "$\widehat{<}$" and "$\widehat{\wp}$". Two vertices $x$ and $y$ are said to be **twins** whenever:

$$\forall z \neq x,y \quad \left| \begin{array}{lcl} (x,z) \in R & \Leftrightarrow & (y,z) \in R \\ (z,x) \in R & \Leftrightarrow & (z,y) \in R \end{array} \right.$$

When $x$ and $y$ are twins then:

- if $R|_{x,y} = \{(x,y),(y,x)\}$ then $R$ admits a coterm $T[x \mathbin{\widehat{\otimes}} y]$,
- if $R|_{x,y} = \{(x,y)\}$ then $R$ admits a coterm $T[x \mathbin{\widehat{<}} y]$,

- if $R|_{x,y} = \{(y,x))\}$ then $R$ admits a coterm $T[y \mathbin{\widehat{<}} x]$,

- if $R|_{x,y} = \emptyset$ then $R$ admits a coterm $T[x \mathbin{\widehat{\wp}} y]$

Conversely, each time $R$ admits a coterm $T[x \mathbin{\widehat{\bullet}} y]$ with $\bullet$ being either $\widehat{\wp}$ or $\widehat{<}$ or $\widehat{\otimes}$ then the vertices $x$ and $y$ are twins.

**Remark 2** *Observe that from the universal properties characterizing directed cographs, the restriction of a directed cograph to a subset of its vertices is also a directed cograph.*

## 2.2 Proof-structures as directed R&B-cographs

Extending what we did for MLL and MLL+mix in [14, 13] we will describe pomset proof-nets as directed R&B-cographs, i.e. as R&B-graphs whose R-arcs are a directed cograph.

The R-arcs will represent the structure of the sequent, and the B-edges the axioms. It should be observed that the algebraic properties of the connectives (associativity of all, commutativity of all but *before*) are interpreted by the equality of the directed cographs. Similarly it makes no difference between a proof of conclusion $\vdash A, B$ and a proof of conclusion $\vdash A \wp B$. But the main advantage of description with respect to the one with links is that any directed R&B-cograph describes a proof-structure, since the only restriction concerns the names of the vertices: the end vertices of B-edges are asked to have dual names, i.e. $x$ and $x^\perp$. A part from this, there is no other structural property like degree of vertices, or the blocks (links) which the proof-structures are made of: *any directed* R&B-*cograph is a proof-structure*.

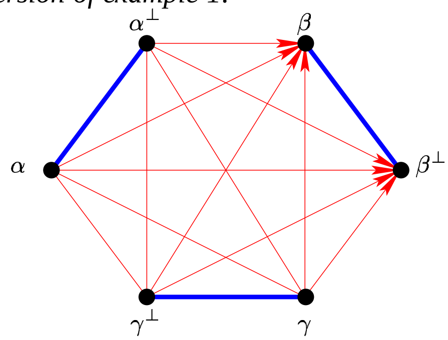
## 2.3 The correctness criterion for directed R&B-cographs

Let us recall that a **chord** of a path or circuit $\gamma$ is an arc $xy$ between two vertices of the circuit such that neither $xy$ nor $yx$ lies in the circuit.

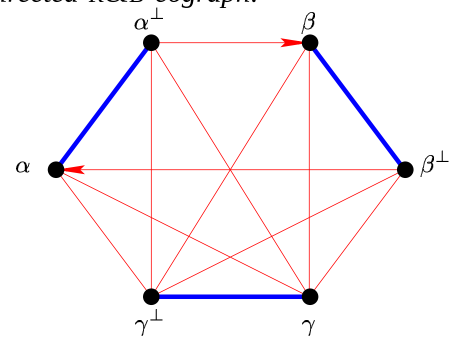
The correctness criterion is stated as follows:

**Correctness criterion 2** *A directed* R&B-*cograph is said to be a proof-net or to be correct whenever any* Æ-*circuit contains a chord.*

**Example 3** *For instance, here is a correct directed* R&B*-cograph, the abstract version of example 1:*




**Example 4** *And here is an incorrect directed* R&B*-cograph:*




**Remark 5** *It should be observed that instead of the criterion we gave in section 1 we could have given this one: indeed, because of the shape of the links, no* Æ*-circuit of a proof-structure with links can contain a chord. So to ask that each* Æ*-circuit contains a chord is the same as asking that there is no* Æ*-circuit at all.*

## 3 Equivalence of the two descriptions

We shall consider the following map.

$$\boxed{\textit{proof-structures with links}} \xrightarrow{\rho} \boxed{\textit{directed } \mathrm{R\&B}\textit{-cographs}}$$

The definition of this map is completely straightforward. Let $\Pi$ be a proof-structure with links having the conclusions $A_1, \ldots, A_n$ and the axioms $\{x_i, x_i^\perp\}$ for $i \in I$; then $\rho(\Pi)$ is defined by:

- the vertices of $\rho(\Pi)$ are $\{x_i, x_i^\perp | i \in I\}$,
- the B-edges of $\rho(\Pi)$ are $\{x_i x_i^\perp\}$
- and the directed cograph of $\rho(\Pi)$ is $\widehat{A_1} \mathbin{\widehat{\wp}} \cdots \mathbin{\widehat{\wp}} \widehat{A_n}$ where $\widehat{\cdots}$ consists in over-lining each connective with a hat-symbol.

Observe that the associativity of *times*, *before* and *par* is interpreted by equality, as well as the commutativity of *times* and *par*; the presence or absence of final *par* is also interpreted by equality: a proof-structure with conclusions $A_1, \ldots, A_n$ is interpreted as the proof proof-structure with the single conclusion $A_1 \wp \cdots \wp A_n$ obtained from it by adding several *par* links — no matter in which order since associativity is interpreted by equality.

**Remark 6** *Assume $\rho(\Pi) = \rho(\Pi')$; then it is easy to see that $\Pi$ and $\Pi'$ may only differ up to associativity of the connectives and to the presence or not of final $\wp$ links; consequently $\Pi$ is correct if and only if $\Pi'$ is correct.*

The aim of this section is to prove the following, which results from lemmas 16 and 17 proven below.

**Theorem 7** *A proof-structure $\Pi$ with links is correct (contains no Æ-circuit) if and only if the associated directed R&B-cograph $\rho(\Pi)$ is correct (contains no chordless Æ-circuit).*

As always, a proof-structure is completely determined by its conclusions and the axioms — viewing *cuts* as *times*. Axioms simply are an indexing of the propositional variables such that each index appears exactly twice, the two occurrences being dual propositional variables. We are to consider the restriction of a proof-structure to a family of axioms, by restricting the conclusions to the occurrences of variables in the family of axioms, and the indexing to the occurrences of variables in the axioms. Clearly, the result is a proof-structure as well. Notice that this is totally independent of the actual representation of the proof-structures, so it makes sense for both the representations we are dealing with.

More precisely, given a formula $F$, let us define its restriction to a family $f$ of variable occurrences as follows. Replace each occurrence of a propositional variable not in $f$ by $\star$ and then compute the restriction of $F$ to the family of variable occurrences by simplification as if $\star$ was the unit of all multiplicative connectives. The restriction of a sequent to a family of variable occurrences is defined in the same way, and if one of the formulas of the sequent reduces to $\star$, it is simply erased. Notice that when the family of variable occurrences is not empty, the resulting sequent is a usual sequent of usual formulas, i.e. contains no $\star$. If we restrict a proof-structure, that is a sequent with an indexing, to a family of variable occurrences corresponding to a family of axioms, we obtain a proof-structure as well: the indexing of the restriction still satisfies that each index appears on exactly two dual propositional variables.

Observe that restricting a directed R&B-cograph to a family of axioms (that is to a subset of the B-edges) simply consists in taking the full subgraph induced by the vertices corresponding to the propositional variables of the chosen axioms.

The three following remarks on restriction are straightforward:

**Remark 8** *Let $\Pi$ be a proof structure with links, and let us denote by $(\cdots)^\circ$ the operation consisting to restrict it to a given family of axioms, then $\rho(\Pi^\circ) = (\rho(\Pi))^\circ$.*

**Remark 9** *A proof-structure with links contains no Æ-circuits if and only if all its restrictions also do.*

**Remark 10** *A directed* R&B*-cograph contains no chordless Æ-circuit if and only if all its restrictions also do.*

Given an Æ-circuit $\gamma$ in a proof-structure with links, the **essential** links of $\gamma$ are the *times* and *before* links such that an R-arc premises belongs to $\gamma$. Let us say a link $\ell$ is **below** a link $\ell'$ whenever the conclusion of $\ell'$ is a subformula of the conclusion of $\ell$ — so $\ell$ is below $\ell'$ in the subformula tree of a conclusion of the proof-structure. Let us say that $\gamma$ is **independent** whenever two essential links of $\gamma$ never are one below the other, that is to say the conclusion of one link never is a subformula of the conclusion of the other. A proof-structure with links is said to be a **basic Æ-circuit** whenever it consists in

- axioms $\{x_i, x_i^\perp\}$ for $1 \leq i \leq n$
- *before* and *times* links

  $$C_1 = (x_0 *^0 x_1) \quad C_2 = (x_1 *^1 x_2) \cdots C_{n-1} = (x_{n-1} *^{n-1} x_n) \quad C_n = (x_n *^n x_0)$$

  where $*^i$ is either $\otimes$ or $<$
- *par* links which are all *below* the $*^i$ links.

**Remark 11** *If a proof-structure with links* $\Pi$ *is a basic Æ-circuit,* $\rho(\Pi)$ *consists in Æ-circuit* $\gamma$ *plus possibly some* R*-arcs whose opposite arcs are in* $\gamma$ *(when* $*^i$ *is* $\otimes$*), so* $\rho(\Pi)$ *contains a chordless Æ-circuit.*

**Proposition 12** *If an Æ-circuit of a proof-structure with links* $\Pi$ *is not independent, then* $\Pi$ *contains an Æ-circuit with less essential links.*

**Proof:** Assume that an Æ-circuit $\gamma$ is not independent, and let us consider two essential links $\ell$ and $\ell'$ of $\gamma$, with $\ell$ below $\ell'$. Assume $\gamma$ contains the R-arc $AB$ in $\ell$ and the R-arc $A'B'$ in $\ell'$ — the undirected R-edge of a $\otimes$ link is a pair of R-arcs. If $\ell'$ is above $A$, replacing the part of $\gamma$ between $A'$ and $B$ by $A' \cdots A \to B$ we have an Æ-circuit with one essential link less. If $\ell'$ is above $B$, replacing the part of $\gamma$ between $A$ and $B'$ by $A \to B \cdots B'$ we have an Æ-circuit with one essential link less. $\square$

**Proposition 13** *In a proof-structure with links* $\Pi$ *if two essential links of an independent Æ-circuit meets on a* before *or* times *link, then* $\Pi$ *contains an Æ-circuit with less essential links.*

**Proof:** Let $\gamma$ be an independent Æ-circuit. Consider two conclusions $C_i$ and $C_j$ which meets on a *before* or a *times* link, which is not one of the $C_k$, because $\gamma$ is independent. If $C_i$ and $C_j$ meet on $F[C_i] < G[C_j]$ (resp. $G[C_j] < F[C_i]$), then replacing the $C_k$ of $\gamma$ between $C_j$ and $C_i$ (resp. $C_i$ and $C_j$) including $C_i$ and $C_j$, with $F[C_i] < G[C_j]$ (resp. $G[C_j] < F[C_i]$), one obtains an Æ-circuit with at least one essential link less. If $C_i$ and $C_j$ meet on $F[C_i] \otimes G[C_j]$ then both previous solutions for decreasing the number of essential links work. □

**Proposition 14** *A proof-structure with links contains an Æ-circuit, if and only if one of its restriction is a basic Æ-circuit.*

**Proof:** Let us consider an Æ-circuit $\gamma$ with a minimal number of essential links. Then it is independent, by proposition 12 and its essential links only meet on *par* links by proposition 13. Thus when we restrict the proof-structure to the axioms of this Æ-circuit, we obtain a proof-structure with only the essential links of $\gamma$ (because $\gamma$ is independent), and where the essential links only meets on *par* links: thus $\gamma$ is a basic Æ-circuit.

Conversely, if one of the restrictions of $\Pi$ is a basic Æ-circuit, then one of its restriction contains an Æ-circuit, and by remark 9 $\Pi$ contains an Æ-circuit too. To view the Æ-circuit in $\Pi$ let us say its essential links are the *before* and *times* links of the basic Æ-circuit, and its axioms are the same. □

Because directed cographs are preserved under restriction (remark 2) we have:

**Proposition 15** *A directed* R&B-*cograph contains no chordless Æ-circuit if and only if all its restrictions also do.*

**Lemma 16** *If a proof-structure with links* $\Pi$ *contains an Æ-circuit, then* $\rho(\Pi)$ *contains a chordless Æ-circuit.*

**Proof:** If $\Pi$ contains an Æ-circuit, then, by proposition 14 one of its restriction, say $\Pi^\circ$, is a basic Æ-circuit. Because restrictions and $\rho$ commute (remark 8) we have $(\rho(\Pi))^\circ = \rho(\Pi^\circ)$. As $\rho(\Pi^\circ)$ contains a chordless Æ-circuit (remark 11), the restriction $(\rho(\Pi))^\circ$ of $\rho(\Pi)$ contains a chordless Æ-circuit, and thus $\rho(\Pi)$ contains a chordless Æ-circuit by proposition 15. □

**Lemma 17** *If a directed* R&B-*cograph* $\mathcal{G}$ *contains a chordless Æ-circuit then one (or all) of the proof-structure with links* $\Pi$ *such that* $\rho(\Pi) = \mathcal{G}$ *contain(s) an Æ-circuit.*

**Proof:** Let us take consider the restriction $\mathcal{G}^\circ$ of $\mathcal{G} = \rho(\Pi)$ to the axioms in this chordless Æ-circuit. Then $(\rho(\Pi))^\circ = \rho(\Pi^\circ)$ is a basic Æ-circuit, and in this case $\Pi^\circ$ is a basic Æ-circuit. Thus, by proposition 14, $\Pi$ contains an Æ-circuit. □

# 4 Folding and unfolding proof-structures

An alternative proof of theorem 7 consists in expanding little by little the links of a proof-structure with links into a directed cograph of R-edges between conclusions, and showing that both this transformation and its inverse preserve the absence of chordless Æ-circuit.

In order to do so, let us introduce corelated proof-structures. A corelated proof-structure consists in a proof-structure with links together with a directed cograph on its conclusions. The directed cograph $R$ between the conclusions is simply represented by putting an R-arc from a conclusion $X$ to a conclusion $Y$ whenever $(X, Y) \in R$. A corelated proof-structure is said to be correct whenever it contains no chordless Æ-circuit. Two extreme cases of corelated proof-structures are:

- proof-structures with links: a corelated proof-structure with the empty cograph on the conclusions.

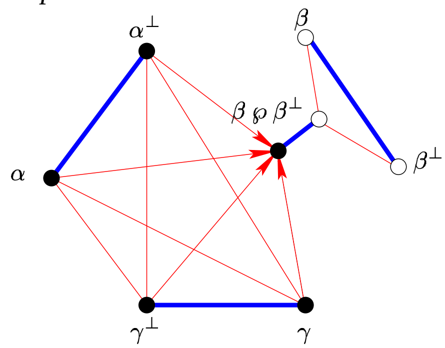
- directed R&B-cographs: the proof-structure with links has only axiom links, i.e. is a family of axioms; all the links are encoded by the R-cograph on the conclusions.

The correctness criterion for these corelated proof-structure is that they contain no chordless Æ-circuit. Indeed, as observed in remark 5 a proof-structure with links contains no Æ-circuit if and only if it contains no chordless Æ-circuit.

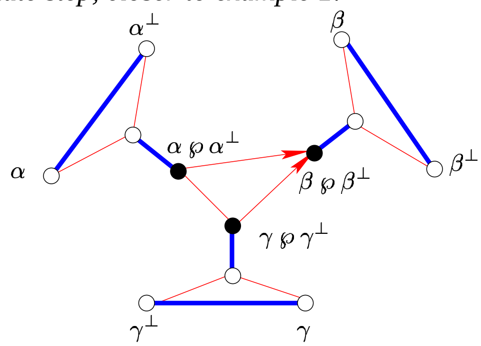
It should be observed that to recover the formulas from the plain R&B-graph we should either know which B-edges are axioms, or which vertices are conclusions, otherwise there might by several readings of the plain directed R&B-cograph. In order to avoid these ambiguities, we mark the conclusions as black vertices.

Let us define the transformation and its inverse. For sake of brevity we use the following compact notation and convention. A proof-structure with links is simply denoted by a sequent where propositional variables are indexed, in such a way that each index appears exactly twice, once on a propositional variable and once on its dual. Directed cographs are denoted by a coterm the variable of which are the conclusions. The expression $R[x]$ means a coterm with a $x$ among its variables, and the expression $R[U]$ means $R[U/x]$ — $U$ and $R$ are assumed to have no common variables. The symbol $\bullet$ range over $\otimes, <, \wp$, and so $\widehat{\bullet}$ ranges over $\widehat{\otimes}, \widehat{<}, \widehat{\wp}$ accordingly: $(\widehat{\bullet}, \bullet) \in \{(\widehat{\otimes}, \otimes), (\widehat{<}, <), (\widehat{\wp}, \wp)\}$

**Example 18** *Here is an intermediate step between the directed* R&B*-cograph of example 3 and the proof-structure with links of example 1:*




**Example 19** *And here is another intermediate step, closer to example 1:*




Let $\mathcal{C}$ be a corelated proof-structure, and let $X$ and $Y$ be twin conclusions w.r.t $R$, that is to say two conclusions which have exactly the same R-antecedent and images by the relation $R$. So $\mathcal{C}$ may be written as $(X, Y, A_1, \ldots, A_n; R[X\widehat{\bullet}Y])$. Folding $X$ and $Y$ — they ought to be twins for doing so — yields the corelated proof-structure $(X \bullet Y, A_1, \ldots, A_n; R[X \bullet Y])$. Notice that $X \bullet Y$ is a conclusion and a single variable with respect to the coterm $R$. Also notice that a directed cograph always has at least a pair of twins: thus, a folding two conclusions is always possible, unless there is a single conclusion. As soon as $R$ is empty it is a plain proof-structure with links and only $(\widehat{\wp}, \wp)$ folding can be performed.

Let $\mathcal{C}$ be a corelated proof-structure, and let $X \bullet Y$ be one of its compound conclusions. So $\mathcal{C}$ may be written as $(X \bullet Y, A_1, \ldots, A_n; R[X \bullet Y])$. Unfolding $X \bullet Y$ yields the corelated proof-structure $(X, Y, A_1, \ldots, A_n; R[X\widehat{\bullet}Y])$. Notice that $X\widehat{\bullet}Y$ are two twins conclusions of the unfolding and two variables with respect to the coterm $R$. Unfolding may always be performed unless all conclusions are propositional variables: in this case the corelated proof-structure simply is a directed R&B-cograph.

Clearly unfolding a folded conclusion yields back the same corelated proof-structure and linking the two twin conclusions of an unfolding yields back the same corelated proof-structure.

**Theorem 20** *Let $\Pi_\bullet$ and $\Pi_{\widehat{\bullet}}$ be the following corelated proof-structures:*

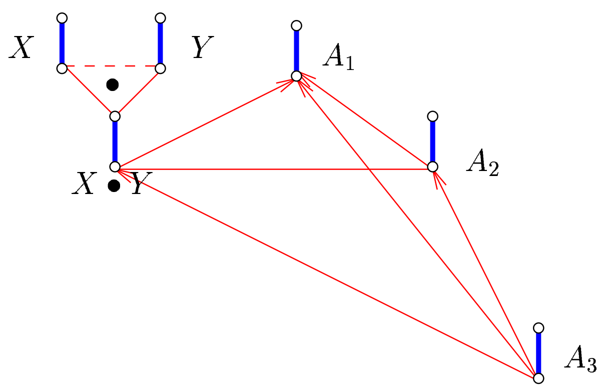
$$\left|\begin{array}{lllll} \Pi_\bullet & = ( & X \bullet Y & , A_1, \ldots, A_n & ; & R[X \bullet Y] & ) \\ \Pi_{\widehat{\bullet}} & = ( & X, Y & , A_1, \ldots, A_n & ; & R[X \widehat{\bullet} Y] & ) \end{array}\right.$$

*The following properties are equivalent:*

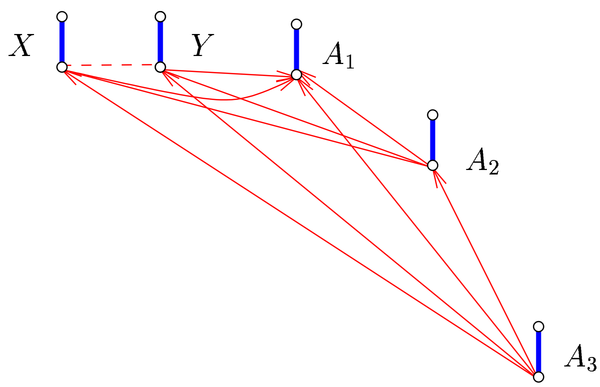
1. $\Pi_\bullet$ *contains a chordless Æ-circuit*
2. $\Pi_{\widehat{\bullet}}$ *contains a chordless Æ-circuit.*

*Consequently, folding and unfolding preserve correctness, that is the absence of chordless Æ-circuit.*

$\boxed{\Pi_\bullet}$




$\boxed{\Pi_{\widehat{\bullet}}}$



**Proof:** Let $\mathcal{A}$ be the R-arcs of $\Pi_{\hat{\bullet}}$ with one end vertex in $\{X, Y\}$ and the other in $\{A_1, \ldots, A_n\}$. Clearly, the two following directed R&B-cographs are equal:

- $\Pi_{\hat{\bullet}}$ minus the R-arcs in $\mathcal{A}$
- $\Pi_{\bullet}$ minus the B-edge incident to the $X \bullet Y$ conclusion, the pair of R-edges (or the four R-arcs) of the link $\bullet$ leading to the premises, and the R-edges incident to $X \bullet Y$.

$\boxed{2 \Rightarrow 1}$ Let $\gamma$ be a chordless Æ-circuit in $\Pi_{\hat{\bullet}}$. Observe that $\gamma$ contains at most one R-arc of $\mathcal{A}$. Indeed two such R-arcs are either adjacent (and $\gamma$ would not be elementary) or there is an R-arc adjacent to both (since $X$ and $Y$ are twins) and the Æ-circuit would not be chordless. Also observe that if $\gamma$ contains the R-arc $XY$ (if any) $\gamma$ does not contain any R-arc in $\mathcal{A}$ since $XY$ is incident to any R-arc of $\mathcal{A}$.

- If $\gamma$ does not contain any arc of $\mathcal{A}$ then $\gamma$ is an Æ-circuit in $\Pi_{\bullet}$ as well, and none of the R-arcs in $\Pi_{\bullet}$ not in $\Pi_{\hat{\bullet}}$ may be a chord of $\gamma$ in $\Pi_{\bullet}$: they are all incident to $X \bullet Y$ which is not a vertex of $\Pi_{\bullet}$.
- If $\gamma$ contains an arc of $\mathcal{A}$, say $XA_i$ first observe that $\gamma$ does not pass through $Y$: $YA_i$ would be a chord. Let us replace this R-arc $XA_i$ by the following sequence: the R-arc $X\bullet$ the B-edge $\bullet(X \bullet Y)$ and the R-arc $(X \bullet Y)A_i$. We thus obtain an Æ-circuit $\gamma'$ in $\Pi_{\bullet}$. But none of the R-arcs in $\Pi_{\bullet}$ not in $\Pi_{\hat{\bullet}}$ may be a chord of $\gamma'$ in $\Pi_{\bullet}$. Indeed the R-arc $Y\bullet$ is excluded since $Y$ is neither a vertex of $\gamma$, nor of $\gamma'$, and if some R-arc $(X \bullet Y)A_j$ was a chord in $\Pi_{\bullet}$ then $XA_j$ would be a chord of $\gamma$ in $\Pi_{\hat{\bullet}}$.

$\boxed{1 \Rightarrow 2}$ Let $\gamma$ be a chordless Æ-circuit of $\Pi_{\bullet}$.

If $\gamma$ does neither pass through $X$ nor through $Y$, then $\gamma$ does not contain the B-edge incident to $X \bullet Y$; therefore $\gamma$ is itself an Æ-circuit of $\Pi_{\hat{\bullet}}$, and none of the arcs in $\mathcal{A}$ may be a chord of $\gamma$ in $\Pi_{\hat{\bullet}}$. In this case $\gamma$ is itself a chordless Æ-path of $\Pi_{\hat{\bullet}}$.

So we can assume that $\gamma$ passes through $X$ or through $Y$, that is to say we are in one of the two following cases:

1. $\gamma$ passes through $X$ and $Y$ — in this case $\gamma$ contains the R-arc $XY$ (or $YX$ but it does not matter) and does not pass through $X \bullet Y$.
2. $\gamma$ passes through $X$ and not through $Y$ (or the converse but it does not matter) in this case it passes through $(X \bullet Y)$.

Let us exhibit a chordless Æ-circuit in $\Pi_{\hat{\bullet}}$ in both cases:

1. Assume that $\gamma$ contains the R-arc $XY$ and does not pass through $(X \bullet Y)$.

   Let us show the following: *if $\gamma$ contains the R-arc $XY$ and passes through an $A_i$ such that $(X \bullet Y)A_i$ or $A_i(X \bullet Y)$ is an R-arc of $\Pi_\bullet$ then there exists another chordless Æ-circuit which does not pass through the R-arc $XY$.* Let us consider the first such $A_i$ after $Y$ along $\gamma$. If there is an R-arc $A_i(X \bullet Y)$ then there is a chordless Æ-circuit $Y$(part of $\gamma$)$A_i(X \bullet Y) \bullet Y$. If there is an R-arc $(X \bullet Y)A_i$ but not the opposite $A_i(X \bullet Y)$ then, letting $A_k$ be the vertex immediately after $A_i$ along $\gamma$, because of the weak transitivity of $R$, there is an R-arc $(X \bullet Y)A_k$. So we have an Æ-circuit $X \bullet (X \bullet Y)A_k$(part of $\gamma$)$X$. It is fairly possible that this Æ-circuit contains a chord $(X \bullet Y)A_l$. Let $A_l$ be the first $A_i$ after $X$ along $\gamma$. The chord is necessarily $(X \bullet Y)A_l$: otherwise there would be, because of weak transitivity a chord $A_l A_i$ in $\gamma$. So we have an Æ-circuit $X \bullet (X \bullet Y)A_l$(part of $\gamma$)$X$ which is, this time, chordless.

   By the previous paragraph we can assume that $\gamma$ does not pass through an $A_i$ with $(X \bullet Y)A_i$ or $A_i(X \bullet Y)$ in $R$. The Æ-circuit $\gamma$ of $\Pi_\bullet$ is an Æ-circuit of $\Pi_{\hat{\bullet}}$ as well. The R-arcs of $\Pi_{\hat{\bullet}}$ which could be chords of $\gamma$ are all incident with some $A_i$ such that $(X \bullet Y)A_i$ or $A_i(X \bullet Y)$ in $R$ in $\Pi_\bullet$. But $\gamma$ does not pass through any of the $A_i$'s so the R-arcs of $\Pi_{\hat{\bullet}}$ cannot be chords of $\gamma$.

2. Assume that $\gamma$ passes through $(X \bullet Y)$, in a sequence $X \bullet \in R$, $\bullet(X \bullet Y) \in B$ $(X \bullet Y)A_i \in R$ — thus $\gamma$ does not pass through $Y$. Simply replacing this sequence by the R-arc $XA_i$ of $\Pi_{\hat{\bullet}}$ we obtain an Æ-circuit $\gamma'$ in $\Pi_{\hat{\bullet}}$. The Æ-circuit $\gamma'$ is chordless: indeed, if it contains a chord, this chord necessarily it is an R-arc $XA_i$ (resp. $A_iX$) but then $(X \bullet Y)A_i$ (resp. $A_i(X \bullet Y)$) would be a chord of $\gamma$ in $\Pi_\bullet$.

□

# 5 Directed-cograph inclusion and directed R&B-cographs

In [2] we provided a complete rewriting system modulo the algebraic properties of parallel, series, symmetric-series compositions which axiomatizes the inclusion of directed cographs. As usual for rewriting systems, these rules apply to subterms: it $U$ rewrites to $U'$ then $R[U]$ rewrites to $R[U']$. This rewriting system is defined for directed cographs modulo associativity that is to say one should add the invertible rewriting rules corresponding to the associativity of parallel, series and symmetric-series compositions and the rewriting rules corresponding to the commutativity of parallel and symmetric compositions.

The expanded version of this rewriting system is a bit lengthy (11 rules), but they admit a short description, as all possible instantiations of the single rule:

$$(R\nabla S)\blacktriangle(R'\nabla S') \rightarrow (R\blacktriangle R')\nabla(S\blacktriangle S')$$

The symbol $\blacktriangle$ stands for a connective which is stronger than $\nabla$ that is to say $(\blacktriangle, \nabla) \in \{(\widehat{\otimes}, \widehat{<}), (\widehat{\otimes}, \widehat{\wp}), (\widehat{<}, \widehat{\wp})\}$. This yields three such rules, but then one has to consider the possibility for the coterms $R, R', S, S'$ to be the common unit of all these operation, namely the empty directed cograph. So the rule gets instantiated to the following ones; we use two symbols, $\rightarrow$ and $-\!\bullet$ because we are to study the subsystem consisting only in the rules with a $-\!\bullet$.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $(\otimes\wp 4)$ | $(X \mathbin{\widehat{\wp}} Y)$ | $\widehat{\otimes}$ | $(U \mathbin{\widehat{\wp}} V)$ | $\longrightarrow$ | $(X \mathbin{\widehat{\otimes}} U)$ | $\widehat{\wp}$ | $(Y \mathbin{\widehat{\otimes}} V)$ |
| $(\otimes\wp 3)$ | $(X \mathbin{\widehat{\wp}} Y)$ | $\widehat{\otimes}$ | $U$ | $-\!\bullet$ | $(X \mathbin{\widehat{\otimes}} U)$ | $\widehat{\wp}$ | $Y$ |
| $(\otimes\wp 2)$ | $Y$ | $\widehat{\otimes}$ | $U$ | $-\!\bullet$ | $U$ | $\widehat{\wp}$ | $Y$ |
| $(\otimes{<}4)$ | $(X \mathbin{\widehat{<}} Y)$ | $\widehat{\otimes}$ | $(U \mathbin{\widehat{<}} V)$ | $-\!\bullet$ | $(X \mathbin{\widehat{\otimes}} U)$ | $\widehat{<}$ | $(Y \mathbin{\widehat{\otimes}} V)$ |
| $(\otimes{<}l3)$ | $(X \mathbin{\widehat{<}} Y)$ | $\widehat{\otimes}$ | $U$ | $-\!\bullet$ | $(X \mathbin{\widehat{\otimes}} U)$ | $\widehat{<}$ | $Y$ |
| $(\otimes{<}r3)$ | $Y$ | $\widehat{\otimes}$ | $(U \mathbin{\widehat{<}} V)$ | $-\!\bullet$ | $U$ | $\widehat{<}$ | $(Y \mathbin{\widehat{\otimes}} V)$ |
| $(\otimes{<}2)$ | $Y$ | $\widehat{\otimes}$ | $U$ | $-\!\bullet$ | $U$ | $\widehat{<}$ | $Y$ |
| $({<}\wp 4)$ | $(X \mathbin{\widehat{\wp}} Y)$ | $\widehat{<}$ | $(U \mathbin{\widehat{\wp}} V)$ | $-\!\bullet$ | $(X \mathbin{\widehat{<}} U)$ | $\widehat{\wp}$ | $(Y \mathbin{\widehat{<}} V)$ |
| $({<}\wp l3)$ | $(X \mathbin{\widehat{\wp}} Y)$ | $\widehat{<}$ | $U$ | $-\!\bullet$ | $(X \mathbin{\widehat{<}} U)$ | $\widehat{\wp}$ | $Y$ |
| $({<}\wp r3)$ | $Y$ | $\widehat{<}$ | $(U \mathbin{\widehat{\wp}} V)$ | $-\!\bullet$ | $U$ | $\widehat{\wp}$ | $(Y \mathbin{\widehat{<}} V)$ |
| $({<}\wp 2)$ | $Y$ | $\widehat{<}$ | $U$ | $-\!\bullet$ | $U$ | $\widehat{\wp}$ | $Y$ |

This system is a bit redundant: $(\otimes\wp 2)$ may be derived from $(\otimes{<}2)$ and $({<}\wp 2)$.

It is easily seen that $(\otimes\wp 4)$ does not preserve the absence of chordless Æ-circuit. That is why we excluded this rule from $-\!\bullet$. The directed R&B-cograph $\mathcal{H}$ is correct, but rewrites by $(\otimes\wp 4)$ into $\mathcal{H}'$ which is not correct:

$$\begin{array}{lllllll}
\mathcal{H} & = ( & \{a, a^\perp, b, b^\perp\} & ; & \{aa^\perp, bb^\perp\} & , & (a \mathbin{\widehat{\wp}} a^\perp) \quad \widehat{\otimes} \quad (b \mathbin{\widehat{\wp}} b^\perp) \quad ) \\
\mathcal{H}' & = ( & \{a, a^\perp, b, b^\perp\} & ; & \{aa^\perp, bb^\perp\} & , & (a \mathbin{\widehat{\otimes}} b) \quad \widehat{\wp} \quad (b^\perp \mathbin{\widehat{\otimes}} a^\perp) \quad )
\end{array}$$

**Theorem 21** *Let $\mathcal{G} = (V; B, R)$ be a directed* R&B*-cograph without chordless Æ-circuit. Assume $R -\!\bullet R'$. Then $\mathcal{G}' = (V; B, R')$ is a directed* R&B*-cograph without chordless Æ-circuit. (In other words, the rewriting system "$-\!\bullet$" preserves the correctness of directed* R&B*-cographs.)*

**Proof:** Clearly it is enough to show this for a single rewriting step, that we assume to be $Q -\!\bullet Q'$ where $Q$ is a coterm made out of the coterms $Q_i$ with $1 \leq i \leq k$ with $k = 2, 3, 4$ according to the rewriting rule. Thus $R$ may be written as $S[Q]$ and $R'$ as $S[Q']$.

We use the transformation of section 4 to establish this result. Let us construct by a sequence of folding a corelated proof-structure $\Pi$ with two kinds of conclusions: on one hand the $Q_i$ for $i \in \{1, k\}$ and, on the other hand, the (occurrences of) propositional variables which do not appear in any of the $Q_i$.

By theorem 20 $\Pi$ is free from chordless Æ-circuit. The directed cograph on the conclusions of $\Pi$ when restricted to the $Q_i$ precisely is the R-subgraph $q$ corresponding to $Q$ the left hand side of the rewriting rule. If we replace this R-subgraph $q$ by the R-graph $q' \subset q$ corresponding to $Q'$ we obtain a corelated proof-structure $\Pi'$ which unfolds to $\mathcal{G}'$. Because of theorem 20, to prove that $\mathcal{G}'$ contains no chordless Æ-circuit it is sufficient to prove that *if* $\Pi$ *contains no chordless Æ-circuit, then* $\Pi'$ *as well contains no chordless Æ-circuit*. Because $q' \subset q$, an Æ-circuit of $\Pi'$ is an Æ-circuit in $\Pi'$.

We assume that there exists a chordless Æ-circuit $\gamma'$ in $\Pi'$, and let us show that there is a chordless Æ-circuit in $\Pi$ as well.

If $\gamma'$ does not pass through any of the $Q_i$, then $\gamma'$ has no chord in $\Pi$ as well. Indeed all the suppressed R-arcs have their end vertices among the $Q_i$.

Let us establish the following:

**(*)** If $\gamma'$ contains an R-arc $Q_iA$ or $AQ_i$ with $A$ not among the $Q_i$ then $\gamma'$ does not pass through any other $Q_j$.
Otherwise $\gamma'$ would contain a chord in $\Pi'$, namely $AQ_j$. Indeed, if $\exists i\ Q_iA \in R \vee AQ_i \in R$ then $\forall i\ Q_iA \in R \vee AQ_i \in R$; in other words the $Q_i$ are all equivalent regarding their relation with other vertices, because $R'$ is $S[Q']$.

**(**)** If $\gamma'$ passes through $Q_i$ and does not contain any of the R-arcs of $q'$ then $\gamma'$ is a chordless Æ-circuit of $\Pi$ as well.
Indeed, if $\gamma'$ contains a chord in $\Pi$, it means that $\gamma'$ passes through $Q_i$ and at least through another $Q_j$: R-adjacent vertices in $\Pi$ but not in $\Pi'$ can only be $Q_i$'s. As $\gamma'$ does not contain any R-arcs of $q'$, $\gamma'$ contains an arc $Q_iA$ or $AQ_i$ and passes through $Q_j$, thus conflicting with (*).

So we can assume that $\gamma'$ contains some of the R-arcs in $q$.

$\gamma'$ **contains a single R-arc of** $q'$ say $Q_1Q_2$. Then $\gamma$ does not pass through any other $Q_i$ (if $k \geq 2$): otherwise, being elementary it would necessarily contain an R-arc $Q_3A$ or $AQ_3$ with $A \neq Q_1, Q_2, Q_3$, thus conflicting with (*). But the R-arcs of $q$ but not of $q'$ which may be chords of $\gamma'$ in $\Pi$ are either $Q_2Q_1$ (which cannot be a chord of $\gamma'$) or are incident to $Q_3$ or $Q_4$ (which cannot be chord of $\gamma'$ because $Q_3$ and $Q_4$ are not vertices of $\gamma'$). Therefore $\gamma'$ is a chordless Æ-circuit of $\Pi$.

$\gamma'$ **contains two R-arcs of** $q'$ say $Q_1Q_2$ and $Q_3Q_4$ — observe that being elementary, $Q_1Q_2$ and $Q_3Q_4$ must have no common end vertex, and so $k = 4$.
Let $\gamma' = \gamma^1\ Q_1Q_2\ \gamma^2\ Q_3Q_4$.

($<\wp 4$) Consider the Æ-circuit of $\Pi$ $\delta = \gamma^1\ Q_1Q_4$. It is chordless because if there would exists an R-arc $Q_1A$ it would be a chord of $\gamma'$ in $\Pi'$ as well.

($\otimes{<}4$) None of the R-arcs in $q$ but not in $q'$ may be a chord of $\gamma'$ in $\Pi$ without being a chord in $\Pi'$: their opposite are in $\Pi'$.

□

# 6 Cut-elimination as particular case of cograph inclusion

Up to now we did not pay any special attention to cuts. Indeed, as in MLL(+mix), they can simply be viewed as a final *times* between two dual formulas (or directed cographs). Conclusions are cographs, and duality of cographs is defined according to de Morgan laws:

$$\begin{array}{rcl}
(x)^\perp & = & x^\perp \\
(x^\perp)^\perp & = & x \\
(\phi\ \widehat{\wp}\ \phi')^\perp & = & (\phi)^\perp\ \widehat{\otimes}\ (\phi')^\perp \\
(\phi\ \widehat{<}\ \phi')^\perp & = & (\phi)^\perp\ \widehat{<}\ (\phi')^\perp \\
(\phi\ \widehat{\otimes}\ \phi')^\perp & = & (\phi)^\perp\ \widehat{\wp}\ (\phi')^\perp
\end{array}$$

Hence a cut is a *times* $\phi \otimes \phi^\perp$ between two dual conclusions $\phi$ and $\phi^\perp$. In [9, 11] we allow not only *par* between cuts and other conclusions but any partial order, so in particular series-parallel orders, that is to say we allow cuts and other conclusions to be related by *par*'s and *before*'s — while in usual MLL and MLL+mix if cuts have no relation to other conclusions and cuts,or in terms of R&B-cographs they are related by *par*'s. For directed R&B-cographs, if we wish to have no relation at all between cuts and other cuts and conclusions we ask that $R = S\ \widehat{\wp}\ (\phi\ \widehat{\otimes}\ \phi^\perp)$ or if we only want to have a series-parallel order between conclusions and cuts we ask that $R = S[X_1, \ldots, X_n, \phi\ \widehat{\otimes}\ \phi^\perp]$ where $S$ is a series parallel order. Here we will consider both kinds of cuts and cut-elimination, and it is easily seen that the reduction step we give preserve that there is no relation at all between conclusions and cuts, or that it is a series-parallel order.

**Theorem 22** *Let* $C = \phi\,\widehat{\otimes}\,\phi^\perp$ *be a cut in* $\mathcal{G} = (V; B, R[\phi\,\widehat{\otimes}\,\phi^\perp]\,)$ *with*

1. $R = S\,\widehat{\wp}\,(\phi\,\widehat{\otimes}\,\phi^\perp)$ *(corresponding to the case without any relation between cuts and conclusions)*

2. *or* $S[X_1, \ldots, X_n, \phi \mathbin{\widehat{\otimes}} \phi^\perp]$ *where* $S$ *is a series-parallel order (corresponding to the case where there is a series-parallel order on cuts and conclusions)*

*The result of one step of cut-elimination performed on* $\phi \mathbin{\widehat{\otimes}} \phi^\perp$ *is obtained as follows:*

**axiom** *if* $\phi = x$ *and* $\phi^\perp = x^\perp$ *i.e.* $\phi$ *and* $\phi^\perp$ *consist in dual vertices, then in both cases we define the reduction step by:*

$$\begin{array}{llllll} \mathcal{G} & = ( \quad V \uplus \{x, x^\perp\} & ; & B \uplus \{\{\underline{x}^\perp, x\}, \{x^\perp \underline{x}\}\} & , & R[x \mathbin{\widehat{\otimes}} x^\perp] \quad ) \\ & \textit{reduces to} & & & & \\ \mathcal{G}' & = ( \qquad V & ; & B \uplus \{\{\underline{x}^\perp, \underline{x}\}\} & , & R|_V \quad ) \end{array}$$

*The vertex* $\underline{x}$ *(resp.* $\underline{x}^\perp$*) stands for the unique vertex of* $\mathcal{G}$ *which is* B*-adjacent to* $x^\perp$ *(resp.* $x$*), labeled* $x$ *(resp.* $x^\perp$*). We wrote* $\{a, b\}$ *instead of* $ab, ba$ *which is consistent because* B*-edges are undirected.*

**"*times*" versus "*par*"** *if* $\phi = \phi_1 \mathbin{\widehat{\otimes}} \phi_2$ *then in both case we define the reduction step by:*

$$\begin{array}{ll} \mathcal{G} & = ( \quad V \quad ; \quad B \quad , \quad R[(\phi_1 \mathbin{\widehat{\otimes}} \phi_2) \mathbin{\widehat{\otimes}} (\phi_1^\perp \mathbin{\widehat{\wp}} \phi_2^\perp)] \quad ) \\ & \textit{reduces to} \\ \mathcal{G}' & = ( \quad V \quad ; \quad B \quad , \quad R[(\phi_1 \mathbin{\widehat{\otimes}} \phi_1^\perp) \mathbin{\widehat{\wp}} (\phi_2^\perp \mathbin{\widehat{\otimes}} \phi_2^\perp)] \quad ) \end{array}$$

**"*before*" versus "*before*"** *if* $\phi = \phi_1 \mathbin{\widehat{<}} \phi_2$ *then we define the reduction step according to whether we allow a series-parallel order between conclusions and cuts:*

1. *If we do not allow any relation between conclusions and cuts, i.e. if* $R = S \mathbin{\widehat{\wp}} (\phi \otimes \phi^\perp)$ *then the reduction step is defined by:*

$$\begin{array}{ll} \mathcal{G} & = ( \quad V \quad ; \quad B \quad , \quad R[(\phi_1 \mathbin{\widehat{<}} \phi_2) \mathbin{\widehat{\otimes}} (\phi_1^\perp \mathbin{\widehat{<}} \phi_2^\perp)] \quad ) \\ & \textit{reduces to} \\ \mathcal{G}' & = ( \quad V \quad ; \quad B \quad , \quad R[(\phi_1 \mathbin{\widehat{\otimes}} \phi_1^\perp) \mathbin{\widehat{\wp}} (\phi_2^\perp \mathbin{\widehat{\otimes}} \phi_2^\perp)] \quad ) \end{array}$$

2. *If we allow a series-parallel order between conclusions and cuts, i.e. if* $R = S[X_1, \ldots, X_n, \phi \mathbin{\widehat{\otimes}} \phi^\perp]$ *where* $S$ *is a series-parallel order, then the reduction step is defined by:*

$$\begin{array}{ll} \mathcal{G} & = ( \quad V \quad ; \quad B \quad , \quad R[(\phi_1 \mathbin{\widehat{<}} \phi_2) \mathbin{\widehat{\otimes}} (\phi_1^\perp \mathbin{\widehat{<}} \phi_2^\perp)] \quad ) \\ & \textit{reduces to} \\ \mathcal{G}' & = ( \quad V \quad ; \quad B \quad , \quad R[(\phi_1 \mathbin{\widehat{\otimes}} \phi_1^\perp) \mathbin{\widehat{<}} (\phi_2^\perp \mathbin{\widehat{\otimes}} \phi_2^\perp)] \quad ) \end{array}$$

*Each of these cut-elimination steps preserves the absence of chordless Æ-circuit. These cut-elimination steps defines a strongly normalizing and confluent rewriting system.*

**Proof:**

**axiom** If, as in section 4 we fold the $x, x^\perp$ which are twins, we obtain a corelated proof-structure $\Pi$ with conclusions $V \uplus \{x \otimes x^\perp\}$ and the cograph $R[x \otimes x^\perp]$ between conclusions and cuts. By theorem 20 $\Pi$ does not contain any chordless Æ-circuit. Then $\mathcal{G}'$ is obtained from $\Pi$ by

1. (unnecessary when $R = (x \otimes x^\perp) \widehat{\wp} S$) Suppressing the B-edge incident to $x \otimes x^\perp$ as well as its adjacent R-arcs, and any full subgraph of a directed R&B-cograph without any chordless Æ-circuit does neither contain a chordless Æ-circuit.
2. Replacing the sequence $\underline{x}^\perp x \in B$, $xx^\perp \in R$, $x^\perp \underline{x} \in B$, with $\underline{xx}^\perp \in B$; this does not create any new Æ-circuit, and the suppressed R-arc $xx^\perp$ can not be a chord of an Æ-circuit: neither $x$ nor $x^\perp$ is R-adjacent to any other vertex, so the result i.e. $\mathcal{G}'$ contains no chordless Æ-circuit.

**"*times*" versus "*par*"** observe that

$$\begin{array}{rcl} R[(\phi_1 \mathbin{\widehat{\otimes}} \phi_2) \mathbin{\widehat{\otimes}} (\phi_1^\perp \mathbin{\widehat{\wp}} \phi_2^\perp)] & = & R[\phi_2 \mathbin{\widehat{\otimes}} (\phi_1 \mathbin{\widehat{\otimes}} (\phi_1^\perp \mathbin{\widehat{\wp}} \phi_2^\perp))] \\ & \multimap & R[\phi_2 \mathbin{\widehat{\otimes}} ((\phi_1 \mathbin{\widehat{\otimes}} \phi_1^\perp) \mathbin{\widehat{\wp}} \phi_2^\perp)] \\ & \multimap & R[(\phi_1 \mathbin{\widehat{\otimes}} \phi_1^\perp) \mathbin{\widehat{\wp}} (\phi_2 \mathbin{\widehat{\otimes}} \phi_2^\perp)] \end{array}$$

by two applications of $(\otimes\wp 3)$, so by theorem 21 if $\mathcal{G}$ does not contain any chordless Æ-circuit, neither does $\mathcal{G}'$.

**"*before*" versus "*before*"** observe that

$$\begin{array}{rcl} R[(\phi_1 \mathbin{\widehat{<}} \phi_2) \mathbin{\widehat{\otimes}} (\phi_1^\perp \mathbin{\widehat{<}} \phi_2^\perp)] & \multimap & R[(\phi_1 \mathbin{\widehat{\otimes}} \phi_1^\perp) \mathbin{\widehat{<}} (\phi_2 \mathbin{\widehat{\otimes}} \phi_2^\perp)] \\ & \multimap & R[(\phi_1 \mathbin{\widehat{\otimes}} \phi_1^\perp) \mathbin{\widehat{\wp}} (\phi_2 \mathbin{\widehat{\otimes}} \phi_2^\perp)] \end{array}$$

by $(\otimes{<}4)$, and $({<}\wp 2)$ so by theorem 21 if $\mathcal{G}$ does not contain any chordless Æ-circuit, neither does $\mathcal{G}'$ in both cases. Indeed, in the first case, when no relation between cuts and conclusion is allowed, $\mathcal{G}'$ is the directed R&B-cograph obtained after the two reductions, while in the second case, where we allow a series-parallel partial order between conclusions and cuts, $\mathcal{G}'$ is the directed R&B-cograph obtained after the first rewriting $(\otimes{<}4)$.

Strong normalization is obvious: at each reduction step the number of R-arcs decreases. Confluence follows from the fact that the configurations which may be reduced always are disjoint. □

This suggests a natural extension of cut elimination: why not allowing a general directed cographs between conclusions and cuts, i.e. to allow *times* relations between conclusions and cuts? In other words, why not allowing a cut to be any subformula (sub directed-cograph) of the shape $\phi \otimes \phi^{\perp}$ ? Thus we could even have a cut as the premise of some other cut. Indeed the rewriting corresponding to cut-elimination may apply anywhere in a proof.

Nevertheless, with respect to the conclusions of the proof-net, we definitely not want that a proof of something reduces to a proof of something else — for instance to have a denotational semantics preserved under cut-elimination. The trick is to consider, for computing the sequent proved by the proof-net, that cuts are a unit for all multiplicative connectives, say $\star$; on the semantic side $\star$ corresponds to the coherence space with a single token: $(\{*\}, \emptyset)$, which is the interpretation of $\exists a\ (a \otimes a^{\perp})$. Assigning this value $\star$ to the cuts, the conclusion would not be modified during the cut-elimination process, even in the presence of internal cuts.

This may deserve to be developed in a future research, if there is a use or an intuition for it.

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