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***Properties of untimed Routed Petri Nets***

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———— THÈME 1 ————

 ***Rapport  
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## Properties of untimed Routed Petri Nets

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Thème 1 — Réseaux et systèmes  
Projet TRIO

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**Abstract:** In this paper, we study the properties of routing functions in non-timed Petri nets with respect to the nets' semantics. We consider two types of routing (*origin dependent* and *origin independent*) which are not mutually representable, and describe the net semantics induced. For unequivocal routing, the process behavior – definable only in the origin dependent case – is shown to be unique up to isomorphism. We show that every routed net is weakly routing place bisimilar to a Free Choice net. This fact, in turn, facilitates the subsequent study of fairness properties, which are characterized by the local *equitability* of the routing functions.

**Key-words:** Petri nets, Semantics, Routing, Fairness

(Résumé : *tsvp*)

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## Propriétés des Réseaux de Petri routés non temporisés

**Résumé :** Dans cet article, nous étudions les propriétés des fonctions de routage dans un réseau de Petri non temporisé par rapport à la sémantique du réseau. Nous regardons deux types de routage (*dépendent* et *indépendent de l'origine*) qui ne peuvent pas être représentés mutuellement, et décrivons le comportement induit du réseau. Dans le cas de routage univoque, l'unicité sauf isomorphisme du processus – définissable uniquement dans le cas dépendent de l'origine – sera démontrée. Nous démontrons que tout réseau routé est faiblement bisimilaire à un réseau Free Choice. Ce fait facilitera à son tour l'étude des propriétés de fairness; celles-là seront caractérisées par l'*équité* locale des fonctions de routage.

**Mots-clé :** Réseaux de Petri, Sémantique, Routage, Fairness

## 1 Introduction

While there are several different semantics for Petri nets and also different possible firing rules and strategies, the “fate” of tokens is usually the same: once a token is created on a place  $p$ , it can be removed by *any* post-transition  $q$  of  $p$ , provided  $q$  is enabled under the firing rule, and selected by the strategy to actually fire<sup>1</sup>.

This picture changes completely in *routed* Petri nets; here, the tokens are assigned a set of possible post-transitions depending on the number of tokens previously arrived.

This concept of routing – already familiar in, e.g., queueing networks – has been extended to Petri nets – and used to investigate liveness properties and performance – by several authors, see for example [3], [4], [5], [10].

In routed nets, tokens arriving on a place  $p$  are *routed towards* some post-transition(s) of  $p$  in the sense that they contribute to possible enablings only of those transitions and are not available for the others. The firing itself proceeds as usual. Routing thus refines the usual firing semantics of Petri nets where all tokens are available to all post-transitions.

We show here that every routed Petri nets is weakly bisimilar to a FC one; this facilitates the study of infinite behaviour properties, as we will show by ensuring fairness in terms of properties of the routing functions.

## 2 Petri nets and Routing Functions

### 2.1 Notations

For the following, let  $M$  be a set. The set of words over  $M$  is denoted by  $M^*$ , the set of infinite sequences over  $M$  by  $M^\omega$ ; set  $M^\omega := M^* \cup M^\infty$ . — A **multi-set** over  $M$  is a mapping  $\chi : M \rightarrow \mathbb{N}_0$ . For  $x \in M$ ,  $\chi(x)$  is the **multiplicity** of  $x$  in  $M$ . If  $\chi(x) > 0$ , we write  $x \in \chi$ . We further denote the set given by  $\chi$  as  $set(\chi) := \{x \in M : x \in \chi\}$ . The set of multi-sets over a set  $M$  is denoted by  $\mathcal{MULT}(M)$ . Addition and subtraction of multi-sets are defined as

$$\begin{aligned} (\chi_1 + \chi_2)(x) &:= \chi_1(x) + \chi_2(x) \\ (\chi_1 - \chi_2)(x) &:= \max(0, \chi_1(x) - \chi_2(x)); \end{aligned}$$

for  $n \in \mathbb{N}$ , set  $n\chi = \sum_{i=1}^n \chi$ . For a product set  $\mathcal{X} = \bigotimes_{i=1}^n \mathcal{X}_i$ ,  $pr_i$  denotes the projection mapping onto the  $i$ th component, i.e.  $pr_i(x_1, \dots, x_n) = x_i$ .

A binary relation over  $\mathcal{X}$  is a set  $\mathfrak{R} \subseteq \mathcal{X} \times \mathcal{X}$ ; the identity relation of  $\mathcal{X}$  is

$$id_{\mathcal{X}} := \{(x, x) : x \in \mathcal{X}\}.$$

For  $x \in \mathcal{X}$ , we write

$$\mathfrak{R}x := \{y \in \mathcal{X} : (y, x) \in \mathfrak{R}\} \quad \text{and} \quad x\mathfrak{R} := \{y \in \mathcal{X} : (x, y) \in \mathfrak{R}\};$$

---

<sup>1</sup>the same applies for coloured nets if nets representing a single colour are considered

By extension, we also set – for  $\theta$  a multi-set over  $\mathcal{X}$  –

$$\mathfrak{R}\theta := \bigcup_{x \in \theta} \mathfrak{R}x, \quad \text{and} \quad \theta\mathfrak{R} := \bigcup_{x \in \theta} x\mathfrak{R}.$$

## 2.2 Definitions

A **Petri net** is a tuple  $\mathcal{N} = (\mathcal{P}, \mathcal{Q}, pre, post, \mathcal{M}_0)$ , where  $\mathcal{P}$  is a countable set of places,  $\mathcal{Q}$  a countable set of transitions with  $\mathcal{P} \cap \mathcal{Q} = \emptyset$ , and

$$\begin{aligned} pre &: \mathcal{Q} \rightarrow \mathcal{MULT}(\mathcal{P}) \\ post &: \mathcal{Q} \rightarrow \mathcal{MULT}(\mathcal{P}) \end{aligned}$$

the mappings assigning to each transition  $q$  its (multi-) sets  $pre^q/post^q$  of input /slash output places, respectively. We write  $\mathcal{P}(\mathcal{N})$  and  $\mathcal{Q}(\mathcal{N})$  if necessary to avoid confusion, and set  $\mathcal{X}(\mathcal{N}) := \mathcal{P}(\mathcal{N}) \cup \mathcal{Q}(\mathcal{N})$ . The numbers  $pre^q(p)$  and  $post^q$  are the *arc weights* from  $p$  to  $q$  and from  $q$  to  $p$ , respectively.

Any  $\mathcal{M} \in \mathcal{MULT}(\mathcal{P})$  is called a **marking** of  $\mathcal{N}$ ; the marking  $\mathcal{M}_0$  is the **initial marking**, the set of markings of  $\mathcal{N}$  is denoted  $\mathcal{MARK}(\mathcal{N}) = \mathcal{MULT}(\mathcal{P})$ . The set of **arcs** of  $\mathcal{N}$  is given by  $pre$  and  $post$  according to

$$\begin{aligned} F &:= \{(p, q) \in \mathcal{P} \times \mathcal{Q} : p \in pre^q\} \\ &\cup \{(q, p) \in \mathcal{Q} \times \mathcal{P} : p \in post^q\}. \end{aligned}$$

Set  $G := [F \cup F^{-1}]$ ; obviously,  $Gx = Fx \cup xF$ . Further, set  $\bigcirc := G \cup id_{\mathcal{X}}$ .

The multisets over  $\mathcal{Q}$  are called **steps**. All nets considered here are assumed to be locally finite, i.e.

$$\forall x \in \mathcal{P} \cup \mathcal{Q} : |\bigcirc x| \in \mathbb{N}.$$

If, for all  $p \in \mathcal{P}$  and all  $q \in \mathcal{Q}$ ,  $|pre^q(p)| \leq 1$ ,  $\mathcal{N}$  is called **forward ordinary**; if also  $|post^q(p)| \leq 1$ ,  $\mathcal{N}$  is called **ordinary**.

The firing of a transition  $q$  puts exactly  $pre^q(p)$  tokens on every post-place  $p$  of  $q$ .

Finally, we introduce net the net topology; for this, we need the notion of **boundary** of a set  $E \subseteq \mathcal{X}(\mathcal{N})$ . Set

$$\begin{aligned} \partial^{in}(E) &:= \{x \in E : \exists y \in \mathcal{X}(\mathcal{N}) - E : F(y, x)\} \\ \partial^{out}(E) &:= \{x \in E : \exists y \in \mathcal{X}(\mathcal{N}) - E : F(x, y)\} \\ \partial(E) &:= \{x \in E : \exists y \in \mathcal{X}(\mathcal{N}) - E : G(x, y)\} \\ &= \partial^{in}(E) \cup \partial^{out}(E). \end{aligned}$$

The net topology on  $\mathcal{N}$  is defined as follows:  $E$  is **open** iff  $\partial(E) \subseteq \mathcal{P}(\mathcal{N})$  and closed iff  $\partial(E) \subseteq \mathcal{Q}(\mathcal{N})$ . A

**Definition 1** Let  $\mathcal{N}_1 = (\mathcal{P}_1, \mathcal{Q}_1, pre_1, post_1, \mathcal{M}_0^1)$  and  $\mathcal{N}_2 = (\mathcal{P}_2, \mathcal{Q}_2, pre_2, post_2, \mathcal{M}_0^2)$  be nets. A mapping

$$\phi : \mathcal{X}(\mathcal{N}_1) \rightarrow \mathcal{X}(\mathcal{N}_2)$$

is a **morphism** iff it is continuous in the net topology. A surjective morphism is an **epimorphism**, an injective epimorphism is an **isomorphism**.

### 2.3 Routing

We now introduce policies for the routing of tokens. A token will be routed towards a transition depending on its *arrival number*, that is, number  $n$  indicates that the token in question is the  $n$ th to arrive on the place in question. If several tokens arrive simultaneously on  $p$  from some  $q$  – i.e. if  $|post^q(p)| > 1$ , and / or if  $q$  performs several firings “*auto-concurrently*” –, we assume that these tokens arrive at  $p$  one after the other (although separated by merely logical instead of physical time intervals); if two (or more) transitions  $q_i$  with  $p \in \bigcap_i post^q$  fire simultaneously, the arriving tokens are also assumed to be ordered and thus obtain their arrival numbers.

Such a number is not provided for the tokens in the initial marking. In order to define routing policies consistently, we therefore modify the original net  $\mathcal{N} = (\mathcal{P}, \mathcal{Q}, pre, post, \mathcal{M}_0)$  in the following way: add a new transition  $q_{start}$  to  $\mathcal{Q}$  and a new place  $p_{start}$  to  $\mathcal{P}$  such that

1.  $pre^q = 1_{\{p_{start}\}}$  and  $pF_{start} = q_{start}$ ;
2.  $post^{q_{start}} = \mathcal{M}_0$
3. for each place  $p$ ,  $pre^{q_{start}}(p) = \mathcal{M}_0(p)$ , and
4. replace  $\mathcal{M}_0$  by the marking  $1_{\{p_{start}\}}$ .

The transition  $q_{new}$  had to be introduced for consistency, since for each token in the initial marking, a corresponding number and – for o.d. routing – a transition of origin (i.e. “ $n$ th token from  $q$ ”) is needed to rout the token: the above construction ensures that  $q_{start}$  fires exactly once, emptying  $p_{start}$  (both  $q_{start}$  and  $p_{start}$  can be ignored afterwards) and creating  $\mathcal{M}_0$  on the original net, with each token assigned  $q_{start}$  as its transition of origin.

**Definition 2** Let  $\mathcal{N} = (\mathcal{P}, \mathcal{Q}, pre, post, \mathcal{M}_0)$  be a net.

1. **origin independent policy (o.i.p.)** for a place  $p$  is a mapping  $\nu_p : \mathbb{N} \rightarrow 2^{p^F}$  such that for all  $p \in \mathcal{P}$  and  $n$ ,
  - (a)  $\nu_p(n) \subseteq p^\bullet$  and
  - (b)  $(p^\bullet \neq \emptyset) \Rightarrow (\nu_p(n) \neq \emptyset)$ .
2. **origin dependent policy (o.d.p.)** for a place  $p$  is a mapping  $\rho_p : Fp \times \mathbb{N} \rightarrow 2^{p^F}$  such that for all  $p \in \mathcal{P}$ ,  $q \in Fp$  and  $n \in \mathbb{N}$ ,



- (a)  $\rho_p(q, n) \subseteq p^\bullet$  and  
 (b)  $(p^\bullet \neq \emptyset) \Rightarrow (\rho_p(q, n) \neq \emptyset)$ .

For  $n \in \mathbb{N}$ , an o.i.p.  $\nu$  assigns to the  $n$ th token arriving on  $p$  the set  $\nu^p(n)$  of those transitions in  $pF$  to which it is available (i.e. may contribute to enabling); for  $n \in \mathbb{N}$  and  $q_1 \in Fp$ , an o.d.p. assigns the set of those transitions in  $pF$  to which it is visible (i.e. may contribute to enabling).

**Definition 3** For  $p \in \mathcal{P}$ ,  $q_1 \in Fp$ ,  $q, q_2 \in pF$  and  $m \in \mathbb{N}$ , the o.i. / o.d. cumulative routing functions are given by

$$\Psi_{p,q}(m) := \sum_{l=1}^m 1_{\{q \in \nu^p(l)\}}. \quad (1)$$

$$\Phi_{q_1,p,q_2}(m) := \sum_{l=1}^m 1_{\{q_2 \in \rho^p(q_1, l)\}}. \quad (2)$$

respectively.

Note that  $\Psi$  and  $\nu$  are closely linked;  $\Psi$  is uniquely determined given  $\nu$  according to (1), and for given  $\Psi$ ,  $p$  and  $n \in \mathbb{N}$ , it holds that

$$\nu_p(n+1) = \{q \in p^\bullet : \Psi_{p,q}(n+1) - \Psi_{p,q}(n) = 1\}.$$

Similarly,  $\Phi$  and  $\rho$  determine one another uniquely in the origin dependent case.

o.i. routings can in general not be represented by o.d. routings (on the same net) and vice versa; see the discussion of Figure 1 below.

## 2.4 The spaces of routing policies

We define a refinement relation between routing policies:

**Definition 4** Let  $\mathcal{N} = (\mathcal{P}, \mathcal{Q}, pre, post, \mathcal{M}_0)$  be a Petri net with the above notation.

1. An o.i.p.  $\nu_1$  is said to **refine** the o.i.p.  $\nu_2$ , denote by  $\nu_1 \sqsubseteq_i \nu_2$ , iff

$$\forall p \in \mathcal{P} \forall n \in \mathbb{N} : \nu_1(n) \subseteq \nu_2(n) \quad .$$

2. An o.d.p.  $\rho_1$  is said to **refine** the o.d.p.  $\rho_2$ , denote by  $\rho_1 \sqsubseteq_d \rho_2$ , iff

$$\forall p \in \mathcal{P} \forall q \in Fp \forall n \in \mathbb{N} : \rho_1(q, n) \subseteq \rho_2(q, n) \quad .$$

The unique maximal elements  $\nu_0$  of  $(\mathcal{OIP}, \sqsubseteq_i)$  and  $\rho_0$  of  $(\mathcal{ODP}, \sqsubseteq_d)$  are called *null routings*; in fact, they give exactly the behavior of the underlying non-routed Petri net. The

meet and join of two routings are given, for  $p \in \mathcal{P}$ ,  $\bar{q} \in Fp$ ,  $q \in pF$  and  $n \in \mathbb{N}$ , by pointwise union/intersection, i.e.

$$\begin{aligned} (\nu_1 \vee \nu_2)^p(n) &:= \nu_1^p(n) \cup \nu_2^p(n) \\ (\nu_1 \wedge \nu_2)^p(n) &:= \nu_1^p(n) \cap \nu_2^p(n) \\ (\rho_1 \vee \rho_2)^p(\bar{q}, n) &:= \rho_1^p(\bar{q}, n) \cup \rho_2^p(\bar{q}, n) \\ (\rho_1 \wedge \rho_2)^p(\bar{q}, n) &:= \rho_1^p(\bar{q}, n) \cap \rho_2^p(\bar{q}, n); \end{aligned}$$

we have:

**Lemma 1** ( $\mathcal{OIP}, \sqsubseteq_i$ ) and ( $\mathcal{ODP}, \sqsubseteq_d$ ) are semilattices.

**Proof:** It suffices to give explicitly  $\bigvee S_i$  and  $\bigvee S_d$  for  $S \subseteq \mathcal{OIP}$  or  $S \subseteq \mathcal{ODP}$  non-empty, which we do here:

$$\begin{aligned} \left(\bigvee S_i\right)^p(n) &= \bigcup_{\nu \in S_i} \nu^p(n) \\ \left(\bigvee S_d\right)^p(n) &= \bigcup_{\rho \in S_d} \rho^p(\bar{q}, n). \end{aligned}$$

for  $p \in \mathcal{P}$ ,  $\bar{q} \in Fp$ ,  $q \in pF$  and  $n \in \mathbb{N}$ . □

Of course,  $\bigvee \mathcal{OIP} = \nu_0$  and  $\bigvee \mathcal{ODP} = \rho_0$ . The *minimal* elements in both spaces are those routings that select, for each token, at most one transition (none at all, of course, iff the place in question has no post-transitions); we call these routings *unequivocal*. For  $\nu_1, \nu_2$  (or  $\rho_1, \rho_2$ ) two different unequivocal routings, the meet does not exist since empty routing sets are not allowed; therefore, neither ( $\mathcal{OIP}, \sqsubseteq_i$ ) nor ( $\mathcal{ODP}, \sqsubseteq_d$ ) are lattices.

## 3 Semantics

### 3.1 Enabling and Firing

In the standard PN semantics, a transition  $q$  is *enabled* in a marking  $\mathcal{M}$  iff

$$\mathcal{M} \geq pre^q$$

With routing, the enabling conditions become more restrictive; to state them, let us first define **token instances**:

**Definition 5** Let  $\mathcal{N} = (\mathcal{P}, \mathcal{Q}, pre, post, \mathcal{M}_0)$  be a Petri net.

1. For o.i. routing, **o.i. token instances** are the elements of the set

$$\mathcal{ TOK}_i := \mathbb{N} \times \mathcal{P}.$$

An **o.i. marking** is a mapping  $\mu_i : \mathcal{P} \rightarrow 2^{\mathcal{ TOK}_i}$  such that for all  $p \in \mathcal{P}$ :

- (a)  $\mu_i(p) \subseteq (\mathbb{N} \times \{p\})$
- (b)  $|\mu_i(p)| \in \mathbb{N}$ .

2. For a net with o.d.routing, **o.d. token instances** are the elements of the set

$$\mathcal{TOK}_d := \{(q, n, p) \in \mathcal{Q} \times \mathbb{N} \times \mathcal{P} : q \in Fp\}.$$

An **o.d. marking** is a mapping  $\mu_d : \mathcal{P} \rightarrow 2^{\mathcal{TOK}_d}$  such that for all  $p \in \mathcal{P}$ :

- (a)  $\mu_d(p) \subseteq (Fp \times \mathbb{N} \times \{p\})$
- (b)  $|\mu_d(p)| \in \mathbb{N}$ .

We now define the enabling and firing of a step for both types of routing. Of course, this includes the case of firing single transitions.

**Definition 6** Let  $\theta \in \mathcal{MUL}\mathcal{T}(\mathcal{Q})$  be a step of  $\mathcal{N} = (\mathcal{P}, \mathcal{Q}, pre, post, \mathcal{M}_0)$ .

1. *o.i. routing:*

- (a) Let  $\nu$  be an o.i. routing. An o.i. marking  $\mu_i$ ,  $\mu_i(p) = \{n_1(p), \dots, n_l(p)\} \times \{p\}$  with  $n_1(p) < \dots < n_l(p)$ , **enables**  $\theta$ , denoted  $\mu_i \xrightarrow{\theta} \nu$ , iff for all pre-places  $p$  of  $\theta$  the total number of tokens presently on  $p$  that are routed towards the transitions of  $\theta$  is greater than or equal to the sum of the corresponding arc weights:

$$\forall q \in \theta \forall p \in F\theta : |\Psi^{-1}(\{q\}) \cap \mu_i(p)| \geq \sum_{q \in \theta} \theta(q) pre^q(p).$$

$\theta$  **transforms**  $\mu_i$  into  $\bar{\mu}_i$ , denoted  $\mu_i \xrightarrow{\theta} \bar{\mu}_i$ , iff

- i.  $\mu_i \xrightarrow{\theta} \nu$ ,
- ii.  $\forall \bar{p} \in (\mathcal{P} - F\theta F) : \mu_i(\bar{p}) = \bar{\mu}_i(\bar{p})$ , and
- iii. with  $k_\theta^+(p) := \sum_{q \in \theta \cap Fp} |pre^q(p)|$ , it holds for all  $p$  that

$$\begin{aligned} |(\bar{\mu}_i - \mu_i)(p)| &= \sum_{q \in \theta \cap pF} \theta(q) |pre^q(p)| \\ (\bar{\mu}_i - \mu_i)(p) &\subseteq \bigcup_{q \in \theta \cap pF} (\nu^p)^{-1} \{q\} \\ (\bar{\mu}_i - \mu_i) &= \{n_{l+1}(p), \dots, n_{l+k_\theta^+(p)}(p)\} \times \{p\} \end{aligned}$$

- (b) Let  $\bar{\nu}$  be any o.i. routing. Then  $\mu_i \xrightarrow{\theta} \bar{\nu}$  iff  $\mu_i \xrightarrow{\theta} \nu$  for some  $\nu \sqsubseteq_i \bar{\nu}$ .

- 2. (a) Let  $\rho$  be an o.d. routing. An o.d. marking  $\mu_d$ ,  $\mu_d(p) = \{(\bar{q}, n_1(\bar{q}, p)), \dots, (\bar{q}, n_{l(\bar{q})}(\bar{q}, p)) : \bar{q} \in Fp\}$  with  $n_1(\bar{q}, p) < \dots < n_{l(\bar{q})}(\bar{q}, p)$  for all  $\bar{q} \in Fp$ , **enables**  $\theta$ , denoted  $\mu_d \xrightarrow{\theta} \rho$ , iff for all pre-places  $p$  of  $\theta$  the number of tokens presently on  $p$  that

are routed towards some  $q \in \theta$  is greater than or equal to the corresponding arc weight, regardless of their transition of origin:

$$\forall q \in \theta \forall p \in F\theta : |pr_2(\Phi^{-1}(\{q\})) \cap \mu_d(p)| > \sum_{q \in \theta} \theta(q) pre^q(p).$$

$\theta$  transforms  $\mu_d$  into  $\bar{\mu}_d$ , denoted  $\mu_d \xrightarrow{\theta} \bar{\mu}_d$ , iff

- i.  $\mu_d \xrightarrow{\theta} \bar{\mu}_d$ ,
- ii.  $\forall \bar{p} \in (\mathcal{P} - F\theta F) : \mu_d(\bar{p}) = \bar{\mu}_d(\bar{p})$ , and
- iii. for all  $p \in \mathcal{P}$  and  $\bar{q} \in Fq$ :

$$\begin{aligned} |(\mu_2 - \mu_1)(p)| &= \sum_{q \in \theta \cap pF} \theta(q) |pre^q(p)| \\ (\mu_2 - \mu_1)(p) &\subseteq \bigcup_{q \in \theta \cap pF} pr_2\left((\rho^p)^{-1}\{q\}\right) \\ (\mu_2 - \mu_1) &= \{n_{l(\bar{q})+1}(\bar{q}, p), \dots, n_{l(\bar{q})+k_{\bar{q}}^+(p)}(\bar{q}, p)\} \times \{p\} \end{aligned}$$

(b) Let  $\bar{\rho}$  be any o.d. routing. Then  $\mu_d \xrightarrow{\theta} \bar{\mu}_d$  iff  $\mu_d \xrightarrow{\theta} \bar{\mu}_d$  for some  $\rho \sqsubseteq_d \bar{\rho}$ .

### 3.2 Step Sequences

Any routed sequence gives rise to a firing sequence:

**Lemma 1** For any routed sequence

$$\begin{aligned} &\mu_0 \xrightarrow{\theta_0} \nu \mu_1 \dots \mu_n \xrightarrow{\theta_n} \nu \mu_{n+1} \\ \text{or } &\mu_0 \xrightarrow{\theta_0} \rho \mu_1 \dots \mu_n \xrightarrow{\theta_n} \rho \mu_{n+1} \quad , \end{aligned}$$

there exists a firing sequence  $\mathcal{M}_0 \xrightarrow{q_0} \mathcal{M}_1 \dots \mathcal{M}_n \xrightarrow{q_n} \mathcal{M}_{n+1}$  such that  $\mathcal{M}_k(p) = |\mu_k(p)|$  for all for  $p \in \mathcal{P}$  and  $k \in \{0, \dots, n\}$ .

This is a property valid for both o.i. and o.d. routing.

**Proof:** Follows from Definitions 1b and 2b since ordinary enabling is enabling under null-routing, and the null-routing is refined by any routing.  $\square$

Conversely, given a firing sequence  $\sigma$ , it is always possible to find a corresponding routed sequence; it suffices to choose the null routing. However, more is true:

**Theorem 1** Let  $\mathcal{M}_0 \xrightarrow{q_0} \mathcal{M}_1 \dots \mathcal{M}_n \xrightarrow{q_n} \mathcal{M}_{n+1} \dots$  be a firing sequence. Then there exists

1. an unequivocal o.i. routing  $\nu$  for  $\mathcal{N}$  yielding a routed sequence

$$\mu_0 \xrightarrow{q_0} \nu \mu_1 \dots \mu_n \xrightarrow{q_n} \nu \mu_{n+1} \dots$$

that satisfies  $\mathcal{M}_k(p) = |\mu_k(p)|$  ( $k \in \mathbb{N}$ ) for all  $p \in \mathcal{P}$ , and

2. an unequivocal o.d. routing  $\rho$  for  $\mathcal{N}$  yielding a routed sequence

$$\mu_0 \xrightarrow{\rho, q_0} \mu_1 \dots \mu_n \xrightarrow{\rho, q_n} \mu_{n+1} \dots$$

that satisfies  $\mathcal{M}_k(p) = |\mu_k(p)|$  ( $k \in \mathbb{N}$ ) for all  $p \in \mathcal{P}$ .

**Proof:** 1: Define  $\mu_0$  according to the above construction using a start transition. We will inductively construct  $\nu$ , starting with the null routing  $\nu_0$  and subsequently restricting the routing in each step of the construction, for those tokens needed in that step. Now assume we have constructed a sequence up to  $\mu_n$  corresponding to  $\mathcal{M}$ . For  $\mathcal{M}_n \xrightarrow{q} \mathcal{M}_{n+1}$  and  $p \in Fq$ , let  $\mu_n = \{(n_1, p), \dots\}$  such that for  $i < j$ ,  $n_i < n_j$ . Set

$$\forall : \nu^p(n_k) := \{q\}, k = 1, \dots, pre^q(p).$$

Generate  $\mathcal{M}_{n+1}$  according to Definition 1a; repeat. — 2: Proceeding as above, define  $\mu_0$  according to the above construction using a start transition. Now assume we have constructed a sequence up to  $\mu_n$  corresponding to  $\mathcal{M}$ . For  $\mathcal{M}_n \xrightarrow{q} \mathcal{M}_{n+1}$  and  $p \in Fq$ , let  $Fp = \{q_1, \dots, q_k\}$  and  $\mu_n = \{(q_1, n_1(q_1), p), \dots, (q_1, n_{u(q_1)}(q_1)), \dots, (q_k)\}$  such that for  $\bar{q} \in Fp$  and  $1 \leq i < j \leq u(\bar{q})$ ,  $n_i(\bar{q}) < n_j(\bar{q})$ . Take any collection of numbers  $m_1, \dots, m_k \in \mathbb{N}_0$  such that

1.  $\forall i \in \{1, \dots, k\}: m_i \leq u(q_i)$
2.  $\sum_{j=1}^k m_j = pre^q(p)$ ,

which exists by assumption. Generate  $\mathcal{M}_{n+1}$  according to Definition 2; repeat.  $\square$

Repeating the above argument for steps, one obtains the same result for step sequences:

**Lemma 2** For any step sequence  $\mathcal{M}_0 \xrightarrow{\theta_0} \mathcal{M}_1 \dots \mathcal{M}_n \xrightarrow{\theta_n} \mathcal{M}_{n+1}$  there exists unequivocal o.i. routing  $\nu$  and an unequivocal o.d. routing  $\rho$  for  $\mathcal{N}$  yielding routed sequences

$$\begin{aligned} \mu_0 &\xrightarrow{\nu, \theta_0} \mu_1 \dots \mu_n \xrightarrow{\nu, \theta_n} \mu_{n+1} \\ \mu_0 &\xrightarrow{\rho, \theta_0} \mu_1 \dots \mu_n \xrightarrow{\rho, \theta_n} \mu_{n+1} \end{aligned}$$

satisfying  $\mathcal{M}_k(p) := |\mu_k(p)|$  ( $k = 0, \dots, n$ ) for all  $p \in \mathcal{P}$ , and vice versa.

For unequivocal routing, more can be said about the set of firing sequences: the order in which concurrently enabled transitions are fired is irrelevant.

**Lemma 3** Let  $\mathcal{N} = (\mathcal{P}, \mathcal{Q}, pre, post, \mathcal{M}_0)$  be a net with unequivocal routings  $\nu$  (o.i.) and  $\rho$  (o.d.). Then:

1. If  $\mu_0 \xrightarrow{\nu, q_0} \mu_1 \dots \mu_{n-1} \xrightarrow{\nu, q_{n-1}} \mu_n \xrightarrow{\nu, q_n} \mu_{n+1}$  and  $\mu_0 \xrightarrow{\rho, q_n}$ , then there exist o.i. markings  $\bar{\mu}_i$ ,  $i \in \{0, \dots, n-1\}$ , such that

$$\mu_0 \xrightarrow{\nu, q_n} \bar{\mu}_0 \xrightarrow{\rho, q_0} \dots \bar{\mu}_{n-1} \xrightarrow{\nu, \theta_0} \mu_{n+1} \quad .$$

2. If  $\mu_0 \xrightarrow{q_0}_\rho \mu_1 \dots \mu_{n-1} \xrightarrow{q_{n-1}}_\rho \mu_n \xrightarrow{\theta_n}_\rho \mu_{n+1}$  and  $\mu_0 \xrightarrow{\theta_n}_\rho$ , then there exist o.i. markings  $\bar{\mu}_i, i \in \{0, \dots, n-1\}$ , such that

$$\mu_0 \xrightarrow{\theta_n}_\rho \bar{\mu}_0 \xrightarrow{\theta_0}_\rho \dots \bar{\mu}_{n-1} \xrightarrow{\theta_0}_\rho \mu_{n+1} \quad .$$

**Proof:** In both cases, the claim follows by induction since no two transitions can ever be in conflict over the same token.  $\square$

If no confusion can arise, we will drop the subscript indicating the routing.

The *step sequence graph*  $\mathcal{ST}(\mathcal{N})$  is obtained by concatenation: for  $w = uv$  set  $(\mathcal{M}_1 \xrightarrow{w} \mathcal{M}_2)$  iff there exists  $\mathcal{M}_3$  such that  $\mathcal{M}_1 \xrightarrow{u} \mathcal{M}_3 \xrightarrow{v} \mathcal{M}_2$ .

By allowing only single transition steps, one obtains the *graph*  $\mathcal{SI}(\mathcal{N})$ .

### 3.3 Independent Transitions

Two transitions  $\bar{q}$  and  $q$  are considered independent if the existence of a firing sequence  $v_1 \bar{q} q v_2$  implies the existence of  $v_1 q \bar{q} v_2$ , i.e. if  $\bar{q}$  and  $q$  can be fired in an arbitrary order. The equivalence classes of firing sequences induced by independence are the well-known *traces*. This semantic independence holds, in particular, if the following structural independence relation holds:  $\bigcirc \bar{q} \cap \bigcirc q = \emptyset$ . Under routing, for two such transitions, there is no routing decision for any one place that affects both  $\bar{q}$  and  $q$ . However, more can be said about semantic independence under routing: if for all  $p \in Fq \cap F\bar{q}$ , no token on  $p$  is ever routed to *both* transitions, and if

$$Fq \cap \bar{q}F = F\bar{q} \cap qF = \emptyset \quad ,$$

then, as we shall see, transitions  $q$  and  $\bar{q}$  are independent. We therefore define a routing - dependent independence relation on  $\mathcal{Q}$  as follows:

**Definition 7** Let  $\mathcal{N} = (\mathcal{P}, \mathcal{Q}, pre, post, \mathcal{M}_0)$  be a Petri net.

1. For an o.i. routing  $\nu$ , define the relation  $\mathcal{I}(\nu) \subset \mathcal{Q}^2$  by

$$q \mathcal{I}_\nu \bar{q} \quad : \iff \quad (Fq \cap \bar{q}F = F\bar{q} \cap qF = \emptyset \\ \vee \left[ Fq \cap F\bar{q} \neq \emptyset \wedge \forall p \in Fq \cap F\bar{q} : (\nu^p)^{-1}(q) \cap (\nu^p)^{-1}(\bar{q}) = \emptyset \right])$$

2. For an o.d. routing  $\rho$ , define the relation  $\mathcal{I}(\rho) \subset \mathcal{Q}^2$  by

$$q \mathcal{I}_\rho \bar{q} \quad : \iff \quad (Fq \cap \bar{q}F = F\bar{q} \cap qF = \emptyset \\ \vee \left[ Fq \cap F\bar{q} \neq \emptyset \wedge \forall p \in Fq \cap F\bar{q} \forall \tilde{q} \in Fp : (\rho^p(\tilde{q}, \bullet))^{-1}(q) \cap (\rho^p(\tilde{q}, \bullet))^{-1}(\bar{q}) = \emptyset \right])$$

The independence relations reflect the ordering of the routings:

**Lemma 4** Let  $\mathcal{N} = (\mathcal{P}, \mathcal{Q}, \text{pre}, \text{post}, \mathcal{M}_0)$  be a Petri net.

1. For o.i. routings  $\nu_1$  and  $\nu_2$  such that  $\nu_1 \sqsubseteq_i \nu_2$ , it holds that  $\mathcal{I}_{\nu_1} \supseteq \mathcal{I}_{\nu_2}$ .
2. For o.d. routings  $\rho_1$  and  $\rho_2$  such that  $\rho_1 \sqsubseteq_d \rho_2$ , it holds that  $\mathcal{I}_{\rho_1} \supseteq \mathcal{I}_{\rho_2}$ .
3. The unique minimal element of both  $\{\mathcal{I}_\nu : \nu \in \text{OIP}(\mathcal{N})\}$  and  $\{\mathcal{I}_\rho : \rho \in \text{ODP}(\mathcal{N})\}$  is

$$q_1 \mathcal{I} q_2 \Leftrightarrow [(q_1 F) \cap (F q_2) = \emptyset \wedge (q_2 F) \cap (F q_1) = \emptyset];$$

$\mathcal{I}$  corresponds to the null routing.

4. Any unequivocal routing has  $\mathcal{Q} \times \mathcal{Q}$  as its corresponding independence relation.

**Lemma 5** Let  $\mathcal{N} = (\mathcal{P}, \mathcal{Q}, \text{pre}, \text{post}, \mathcal{M}_0)$  be a Petri net.

1. Let  $\nu$  be an o.i. routing for  $\mathcal{N}$ ,  $M := \{q' : \mathcal{I}_\nu(q, q')\}$  and, for  $w \in M^*$ , assume  $\mu_0 \xrightarrow{q}_\nu$  and  $\mu_0 \xrightarrow{w}_\nu$ . Then there exist  $\mu, \mu' \in \text{ORM}(\mathcal{N})$  such that

$$\mu_0 \xrightarrow{q}_\nu \mu \xrightarrow{w}_\nu \quad \text{and} \quad \mu_0 \xrightarrow{w}_\nu \mu' \xrightarrow{q}_\nu.$$

2. Let  $\rho$  be an o.d. routing for  $\mathcal{N}$ ,  $M := \{q' : \mathcal{I}_\rho(q, q')\}$  and, for  $w \in M^*$ , assume  $\mu_0 \xrightarrow{q}_\rho$  and  $\mu_0 \xrightarrow{w}_\rho$ . Then there exist  $\mu, \mu' \in \text{ORM}(\mathcal{N})$  such that

$$\mu_0 \xrightarrow{q}_\rho \mu \xrightarrow{w}_\rho \quad \text{and} \quad \mu_0 \xrightarrow{w}_\rho \mu' \xrightarrow{q}_\rho.$$

Thus, under routing, two transitions may be interchangeable even if they share a pre- or post-place; they are always independent if no pre-place of one transition is a post-place of the other.

**Proof:** The independence follows in both cases by induction since  $q$  and  $q' \in M$  can not be in conflict over any token instance.  $\square$

Routing defined independences lie halfway between purely structural independences – as in [13] – and dynamical ones as studied, e.g., in [14].

Note that, for null routing, the *dependence* relation obtained above is the symmetric heart of the  $D$  underlying Diekert's partial traces ([12], [13]).

### 3.4 Processes

We now turn to *process semantics* in the sense of, e.g., [1] and [7].

**Definition 8** An ordinary net  $\mathcal{N} = (\mathcal{P}, \mathcal{Q}, \text{pre}, \text{post}, \mathcal{M}_0)$  is a **causal net** iff

1.  $F$  is acyclic, i.e.  $<_{\mathcal{N}} := F^+$  is a partial order,

2.  $<_{\mathcal{N}}$  is well-founded, i.e. contains no infinite strictly decreasing sequence,
3.  $\forall p : \mathcal{M}_0(p) \leq 1$ .

The *incomparability relation*  $co$  on a causal net holds between any two nodes  $a$  and  $b$  such that neither  $a = b$  nor  $a <_{\mathcal{N}} b$  nor  $b <_{\mathcal{N}} a$ . Any maximal clique  $c$  in the undirected graph of  $co$  is called a *cut*; if, in addition,  $c \in \mathcal{P}$ ,  $c$  is a **P-cut**. We denote the set of all  $<_{\mathcal{N}}$ -minimal nodes by  $c_0$ ;  $c_0$  is a P-cut iff no  $pre^q$  is empty.

**Definition 9** Given a net  $\mathcal{N} = (\mathcal{P}, \mathcal{Q}, pre, post, \mathcal{M}_0)$  and a causal net  $\mathcal{C}$ , a mapping  $f : \mathcal{P}_{\mathcal{C}} \cup \mathcal{Q}_{\mathcal{C}} \rightarrow \mathcal{P}_{\mathcal{N}} \cup \mathcal{Q}_{\mathcal{N}}$  is an **embedding** of  $\mathcal{C}$  into  $\mathcal{N}$  iff:

1.  $f$  preserves the type of nodes:  
 $f(\mathcal{P}_{\mathcal{C}}) \subset \mathcal{P}_{\mathcal{N}}$  and  $f(\mathcal{Q}_{\mathcal{C}}) \subset \mathcal{Q}_{\mathcal{N}}$ ;
2.  $f$  reflects the (weighted)  $F$ -neighbourhood of each transition:  $\forall q \in \mathcal{Q}_{\mathcal{C}}$ :  
 $\forall p \in F_{\mathcal{N}}f(q)F_{\mathcal{N}} : |f^{-1}(\{p\})| = |pre^q(p)|$
3.  $c_0$  corresponds, via  $f$ , to the initial marking:  
 $\forall p_{\mathcal{N}} \in \mathcal{P}_{\mathcal{N}} : \mathcal{M}_0(p_{\mathcal{N}}) = |c_0 \cap f^{-1}(p_{\mathcal{N}})|$

The pair  $\pi = (\mathcal{C}, f)$  is then a **process** of  $\mathcal{N}$ ; the set of processes of  $\mathcal{N}$  is denoted  $\Pi(\mathcal{N})$ .

If  $\mathcal{C}$  is line-finite, i. e. there exists a finite bound for the length of  $<_{\mathcal{C}}$ -chains, the set  $c_1$  consisting of the  $<_{\mathcal{C}}$ -maximal elements of  $\mathcal{C}$  yields, via  $f$ , a marking  $\mathcal{M}_1$  of  $\mathcal{N}$  (compare [6]); we then call  $\pi$  a **process move** of  $\mathcal{N}$  and write  $\mathcal{M}_0 \xrightarrow{\pi} \mathcal{M}_1$ . If there exists some process move  $\pi$  such that  $\mathcal{M} \xrightarrow{\pi} \mathcal{M}'$ , we write  $\mathcal{M} \xrightarrow{*} \mathcal{M}'$ ; note that  $\xrightarrow{*}$  is a transitive relation.

Now, we extend Definition 9 to both o.i. and o.d. routed nets. It turns out that – even in the unequivocal case – the o.i. routings, in general, allow an alternative of several processes while there is only one maximal process compatible with a given unequivocal o.d. routing. To illustrate this fact, consider Figure 1; it shows a net and its maximal processes. Under an o.d. routing  $\rho$ , there will always be at least one such process compatible with  $\rho$ . However, consider the o.i. routing given by

$$\nu^c(n) = \begin{cases} \{\gamma\} & : n \text{ odd} \\ \{\delta\} & : n \text{ even} \end{cases}$$

Obviously, only *I* and *III* are candidates for being compatible with  $\nu$ ; but which? The choice obviously depends on the order in which the two transitions  $\alpha$  and  $\beta$  fire; since they are concurrent, this cannot be decided in advance and is a time-dependent property.

By contrast, any o.d. routing  $\rho^c$  is compatible in an obvious way with appropriate processes of Figure 1; the choice of process(es) will, obviously, only depend on  $\rho^c(\bullet, 2)$  and  $\rho^c(\bullet, 2)$ .

**Theorem 2** 1. For any process move  $\mathcal{M}_0 \xrightarrow{\pi} \mathcal{M}_1$ , there exists an unequivocal o.d. routing  $\rho$  and an unequivocal o.i. routing for  $\mathcal{N}$  that yield a process move  $\mu_0 \xrightarrow{\pi} \mu_1$  (with the same  $\pi$ ) such that, for all for  $p \in \mathcal{P}$ ,  $m_k(p) := |\mu_k(p)|$  ( $k = 0, 1$ ).



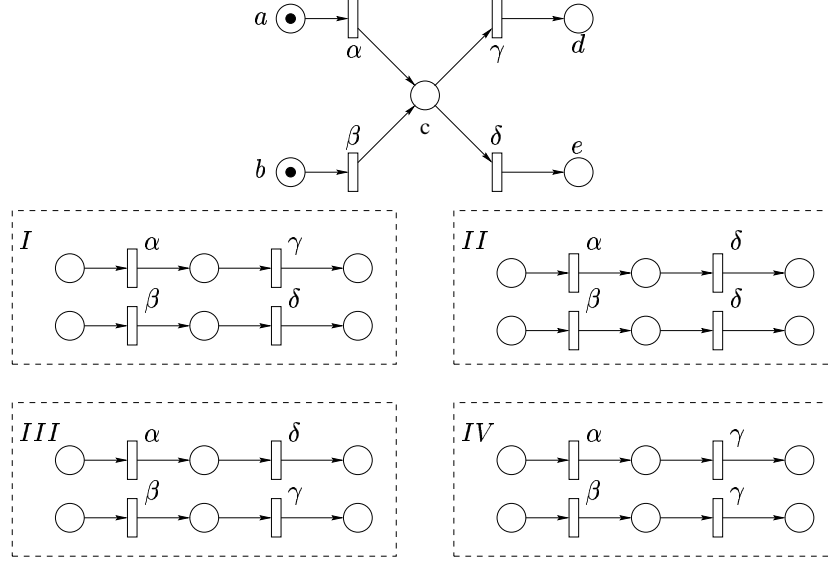


Figure 1: Comparison of process semantics for o.i. and o.d. routing

2. For any unequivocal o.d. routing  $\rho$  there exists a process move  $\mathcal{M}_0 \xrightarrow{\pi} \mathcal{M}_1$  such that, for all for  $p \in \mathcal{P}$ ,  $m_k(p) := |\mu_k(p)|$  ( $k = 0, 1$ ).

**Proof:**

1. Let  $\mathcal{M}_0 \xrightarrow{\pi} \mathcal{M}_1$  be given and let  $\mu_d^0$  be the initial o.d. marking obtained in the usual way. Now let  $\bar{\kappa}$  be the set of transitions  $\bar{t}$  of  $\mathcal{C}$  such that  $F_{\mathcal{C}}\bar{t} \subset c_0$ ; then  $\kappa(q) := |f^{-1}(q)|$  defines a step enabled in  $\mathcal{M}_0$ . Define  $\nu$  and  $\rho$  inductively as above; replace  $\mu_0$  by the o.d. marking  $\mu_d^1$  obtained when firing  $\kappa$ , replace  $c_0$  by  $(c_0 - F\bar{\kappa}) \cup \bar{\kappa}F$ . This procedure terminates since  $\mathcal{C}$  is line-finite ([7]).
2. Given a routed sequence  $\mu_0 \xrightarrow{q_0} \mu_1 \xrightarrow{q_1} \dots$ , create a process  $(\mathcal{C}, f)$  inductively:
  - $\mathcal{C}_0$  consists only of a set  $c_0$  of places that represent 1-1 the tokens in the initial marking  $\mu_d^0$ , i.e.  $\mu_d^0 f(c_0)$ .
  - If  $\mathcal{C}_n$  has been constructed, add a transition  $\bar{q}_n$  such that

$$\begin{aligned} f(F\bar{q}_n) &= \mu_d^n - \mu_d^{n+1} \\ f(\bar{q}_n F_n) &= \mu_d^{n+1} - \mu_d^n. \end{aligned}$$

- Set  $c_{n+1} := (c_n - F\bar{q}_n) \cup \bar{q}_n F_n$ .

□

**Remark 1** If  $Fq \neq \emptyset$  and  $qF \neq \emptyset$  for all  $q \in Q$ , simulating the firing of a transition  $t$ , as described in step 2 of this construction, transforms  $P$ -cuts into  $P$ -cuts (comp. e.g. [7]). Therefore, for any reachable marking  $\mathcal{M}$ , there exists a  $P$ -cut  $c_{\mathcal{M}}$  representing  $\mathcal{M}$  in the sense that

$$\forall p \in \mathcal{P} : \mathcal{M}(p) = |c \cap f^{-1}(p)|.$$

An immediate consequence of the definitions is, with the above notation:

**Lemma 6** If  $\nu_1 \sqsubseteq_i \nu_2$  and  $\rho_1 \sqsubseteq_d \rho_2$ , then

$$\begin{aligned} \mu_0 \xrightarrow{\pi}_{\nu_1} \mu_1 &\Rightarrow \mu_0 \xrightarrow{\pi}_{\nu_2} \mu_1 \\ \mu_0 \xrightarrow{\pi}_{\rho_1} \mu_1 &\Rightarrow \mu_0 \xrightarrow{\theta}_{\rho_2} \mu_2 \end{aligned}$$

Lemma 3 showed the well-known interchangeability of concurrently enabled transitions for firing sequences; on the level of processes, we have uniqueness up to isomorphism:

**Theorem 3** Let  $\rho$  be an o.d. routing for  $\mathcal{N}$ , and let  $\pi_1 = (\mathcal{C}_1, f_1)$  and  $\pi_2 = (\mathcal{C}_2, f_2)$  be two processes of  $\mathcal{N}$  for  $\rho$ . Then there exists an isomorphism  $\phi$  from  $\mathcal{C}_1$  to  $\mathcal{C}_2$  such that  $f_1 \circ \phi = f_2$  and  $f_2 \circ \phi = f_1$ .

**Proof:**

Every token  $\tau$  is represented by unique places  $p_1(\tau)$  and  $p_2(\tau)$ ; under unequivocal o.d. routing, the neighbourhoods  $p_1(\tau)G_{\mathcal{C}_1}$  and  $p_2(\tau)G_{\mathcal{C}_2}$  must be isomorphic. Hence the claim follows inductively.  $\square$

## 4 Weakly bisimilar Versions

A net may be bisimilar to a smaller one obtained by a net epimorphism when a *corresponding* routing is applied.

Let us start with an informal discussion of the two nets in figure 2. They are obviously not (trace-, step- or process-) bisimilar under the usual firing rule since, after firing  $q_3$ ,  $q_1$  remains firable, while firing  $q'_3$  and  $q'_4$  kills  $q'_1$ . Yet: taking corresponding unequivocal routings in both nets on the places  $p_1$  and  $p_2 / p'_1$  and  $p'_2$ , respectively, leads to “equivalent” behaviour. For this, note that if, e.g., the token on  $p_1$  is routed towards  $\{q_2\}$ ,  $q_1$  cannot fire even if the token on  $p_2$  is routed towards  $\{q_3\}$  (and, therefore,  $q_3$  fires). The net will then keep the token on  $p_1$  forever. — This is a case in which, as mentioned above, a firing sequence maximal w.r.t. one routing is not maximal w.r.t. another and, consequently, not maximal in the usual sense. It also becomes clear from this example that routing is more restrictive than priorities, for priorities can only disable an enabled transition by firing a conflicting one, while routing may forever hide tokens on a place  $p$  from a post-transition  $q \in pF$  that would otherwise be enabled.

Although the above discussion assumed o.i. routing, nothing has to be changed for o.d. routing.

Let us formalise the concepts.

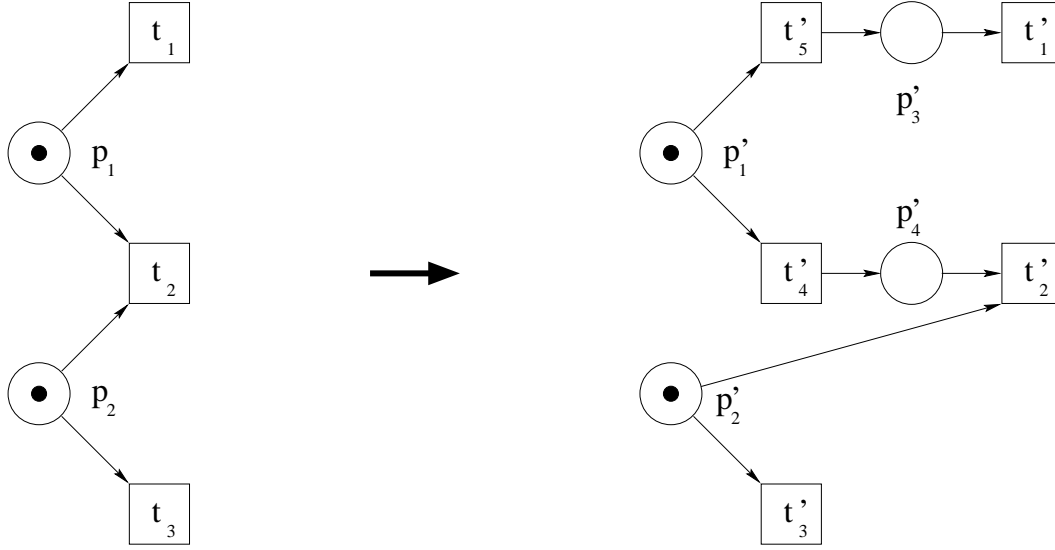


Figure 2: Routing bisimilar transformation into an FC-Net

#### 4.1 Weak Bisimulation

**Definition 10** A **labelled graph** is a tuple  $\mathcal{L} = (\mathcal{G}, \Theta, l)$ , where  $\mathcal{G} = (\mathcal{V}(\mathcal{G}), \mathcal{E}(\mathcal{G}))$  is a (directed) graph with node set  $\mathcal{V}$  and edge set  $\mathcal{E} \subset \mathcal{V}^2$ ,  $\mathcal{ALPH}$  a nonempty alphabet of label literals and  $l: \mathcal{E} \rightarrow (\mathcal{ALPH})^*$  a labelling function.  $\mathcal{L}$  is called *transitive* if, for vertices  $v_1, v_2, v_3 \in \mathcal{V}(\mathcal{G})$  such that  $e = (v_1, v_2) \in \mathcal{E}(\mathcal{G})$ ,  $e' = (v_2, v_3) \in \mathcal{E}(\mathcal{G})$ , and  $l(e) = u$ ,  $l(e') = v$

We have already defined two different labelled graphs associated to a net  $\mathcal{N} = (\mathcal{P}, \mathcal{Q}, pre, post, \mathcal{M}_0)$  and capturing its dynamics (the vertex set being the set of markings). We extend them by adding the *silent move* symbol  $\tau$

1. the *sequence move graph* of  $\mathcal{N}$ , denoted  $SI(\mathcal{N})$ , with  $\mathcal{ALPH} = (\mathcal{Q} \cup \{\tau\})^*$ ,
2. the *step sequence move graph*  $ST(\mathcal{N})$  with  $\mathcal{ALPH} = (\mathcal{MULT}(\mathcal{Q}) \cup \{\tau\})^*$ , and
3. the *process move graph*  $\mathcal{PR}(\mathcal{N})$  with  $\mathcal{ALPH} = (\Pi(\mathcal{N}) \cup \{\tau\})^*$ .

Note that the transitivity of  $SI(\mathcal{N})$  and  $\mathcal{PR}(\mathcal{N})$  is preserved by the insertion.

The weak bisimulations we will consider here are given as liftings of relations on the level of net nodes; since greater generality will not be needed, we restrict our definitions to that case. Let  $\mathcal{N}_1 = (\mathcal{P}_1, \mathcal{Q}_1, pre_1, post_1, \mathcal{M}_0^1)$  and  $\mathcal{N}_2 = (\mathcal{P}_2, \mathcal{Q}_2, pre_2, post_2, \mathcal{M}_0^2)$  be two

nets (with silent transitions) and  $\mathfrak{R} = (\mathfrak{R}_Q, \mathfrak{R}_P)$  a relation such that

$$\begin{aligned}\mathfrak{R}_Q &\subseteq (Q_1 \cup \{\lambda, \tau\}) \times (Q_2 \cup \{\lambda, \tau\}) \\ \mathfrak{R}_P &\subseteq (P_1) \times (P_2);\end{aligned}$$

denote by  $\tilde{\mathfrak{R}}$  the lifting of  $\mathfrak{R}$  to multisets given by

$$\begin{aligned}\tilde{\mathfrak{R}}_Q(\theta_1, \theta_2) &\iff \exists \{(q_1^1, q_2^1), \dots, (q_1^n, q_2^n)\} \in \text{MULT}(\mathfrak{R}_Q) : \\ &\quad \theta_1 = \{(q_1^1, \dots, q_1^n)\} \wedge \theta_2 = \{(q_2^1, \dots, q_2^n)\} \\ \tilde{\mathfrak{R}}_P(\theta_1, \theta_2) &\iff \exists \{(p_1^1, p_2^1), \dots, (p_1^n, p_2^n)\} \in \text{MULT}(\mathfrak{R}_P) : \\ &\quad \theta_1 = \{(p_1^1, \dots, p_1^n)\} \wedge \theta_2 = \{(p_2^1, \dots, p_2^n)\}.\end{aligned}$$

Then extend  $\tilde{\mathfrak{R}}$  to sequences:  $\tilde{\mathfrak{R}}(w_1, w_2)$  iff either  $\tilde{\mathfrak{R}}(w_1, w_2)$ , or  $w_1 = u_1 v_1$ ,  $w_2 = u_2 v_2$ , and  $\tilde{\mathfrak{R}}(u_1, u_2)$  and  $\tilde{\mathfrak{R}}(v_1, v_2)$ .

Moreover, we need a lifting of  $\mathfrak{R}$  to  $\Pi$ . For this, let  $\pi_1 = (\mathcal{C}_1, f_1) \in \Pi(\mathcal{N}_1)$  and  $\pi_2 = (\mathcal{C}_2, f_2) \in \Pi(\mathcal{N}_2)$ . Set  $\tilde{\mathfrak{R}}(\pi_1, \pi_2)$  iff there exists a net isomorphism

$$f : \mathcal{X}(\mathcal{N}_1) \rightarrow \mathcal{X}(\mathcal{N}_2)$$

such that  $f_2 \circ f = \mathfrak{R} \circ f_1$  and  $f_1 \circ f^{-1} = \rho^{-1} \circ f_2$ . It follows that  $\pi_1 = \pi_1' \pi_1''$ ,  $\pi_2 = \pi_2' \pi_2''$  and  $\tilde{\mathfrak{R}}(\pi_1', \pi_2')$ ,  $\tilde{\mathfrak{R}}(\pi_1'', \pi_2'')$  imply  $\tilde{\mathfrak{R}}(\pi_1, \pi_2)$ .

In the following, let  $\mathcal{N}_1 = (\mathcal{P}_1, \mathcal{Q}_1, pre_1, post_1, \mathcal{M}_0^1)$  and  $\mathcal{N}_2 = (\mathcal{P}_2, \mathcal{Q}_2, pre_2, post_2, \mathcal{M}_0^2)$  be fixed, and let  $\nu_1$  and  $\rho_1$  be an o.i. / o.d. routing for  $\mathcal{N}_1$  a fixed relation  $\mathfrak{R}$  and its liftings  $\tilde{\mathfrak{R}}^{SI} \subseteq \mathcal{V}(SI(\mathcal{N}_1)) \times \mathcal{V}(SI(\mathcal{N}_2))$ ,  $\tilde{\mathfrak{R}} \subseteq \mathcal{V}(ST(\mathcal{N}_1)) \times \mathcal{V}(ST(\mathcal{N}_2))$ , and  $\tilde{\mathfrak{R}} \subseteq \mathcal{V}(PR(\mathcal{N}_1)) \times \mathcal{V}(PR(\mathcal{N}_2))$ .

Let  $\nu_2$  and  $\rho_2$  be compatible under  $\mathfrak{R}$  with  $\nu_1$  and  $\rho_1$ , respectively, i.e., for all  $n \in \mathbb{N}$ ,  $p_1 \in \mathcal{P}_1$   $p_2 \in \mathcal{P}_2$  it holds that

$$\begin{aligned}\nu_1^{p_1}(n) &= \bigcup_{p_2 \in \mathfrak{R} p_1} \mathfrak{R} \nu_2^{p_2}(n) \\ \nu_2^{p_2}(n) &= \bigcup_{p_1 \in p_2 \mathfrak{R}} \mathfrak{R} \nu_1^{p_1}(n)\end{aligned}$$

or, respectively,

$$\begin{aligned}\forall q_1 \in F_1 p_1 : \rho_1^{p_1}(q_1, n) &= \bigcup_{p_2 \in \mathfrak{R} p_1} \bigcup_{q_2 \in \mathfrak{R} q_1} \mathfrak{R} \nu_2^{p_2}(q_2, n) \\ \forall q_2 \in F_2 p_2 : \rho_2^{p_2}(q_2, n) &= \bigcup_{p_1 \in p_2 \mathfrak{R}} \bigcup_{q_1 \in q_2 \mathfrak{R}} \nu_1^{p_1}(q_1, n) \mathfrak{R}.\end{aligned}$$

**Definition 11** Let  $\xi \in \{\nu, \rho\}$ .

1.  $\mathfrak{R}$  is a weak (trace)  $\xi_1 - \xi_2$ -bisimulation iff

(a) for all  $q_1, \mathcal{M}_1, \mathcal{M}'_1$  and  $\mathcal{M}_2$  such that

$$\bar{\mathfrak{R}}(\mathcal{M}_1, \mathcal{M}_2) \wedge \mathcal{M}_1 \xrightarrow{q_1}_{\xi_1} \mathcal{M}'_1 \quad ,$$

there exist  $q_2$  and  $\mathcal{M}'_2$  such that

$$\mathfrak{R}(q_1, q_2) \wedge \mathfrak{R}(\mathcal{M}'_1, \mathcal{M}'_2) \wedge \mathcal{M}_2 \xrightarrow{q_2}_{\xi_2} \mathcal{M}'_2 \quad .$$

(b)

2.  $\mathfrak{R}$  is a **weak step**  $\xi_1 - \xi_2$ -**bisimulation** iff for all  $\theta_1, \mathcal{M}_1, \mathcal{M}'_1$  and  $\mathcal{M}_2$  such that

$$\mathfrak{R}(\mathcal{M}_1, \mathcal{M}_2) \wedge \mathcal{M}_1 \xrightarrow{\theta_1}_{\xi_1} \mathcal{M}'_1 \quad ,$$

there exist  $\theta_2$  and  $\mathcal{M}'_2$  such that

$$\bar{\mathfrak{R}}(\theta_1, \theta_2) \wedge \bar{\mathfrak{R}}(\mathcal{M}'_1, \mathcal{M}'_2) \wedge \mathcal{M}_2 \xrightarrow{\theta_2}_{\xi_2} \mathcal{M}'_2 \quad .$$

3.  $\mathfrak{R}$  is a **weak process**  $\xi_1 - \xi_2$ -**bisimulation** iff for all  $\pi_1 \in \Pi(\mathcal{N}_1)$ ,  $\mathcal{M}_1, \mathcal{M}'_1$  and  $\mathcal{M}_2$  such that

$$\mathfrak{R}(\mathcal{M}_1, \mathcal{M}_2) \wedge \mathcal{M}_1 \xrightarrow{\theta_1}_{\xi_1} \mathcal{M}'_1 \quad ,$$

there exist  $\pi_2 \in \Pi(\mathcal{N}_2)$  and  $\mathcal{M}'_2$  such that

$$\tilde{\mathfrak{R}}(\pi_1, \pi_2) \wedge \bar{\mathfrak{R}}(\mathcal{M}'_1, \mathcal{M}'_2) \wedge \mathcal{M}_2 \xrightarrow{\pi_2}_{\xi_2} \mathcal{M}'_2 \quad .$$

We will now show that each net has a weakly bisimilar GFC version obtained as shown in Figure 2; in fact, together with arc weight removal as shown in Figure 6, one obtains weakly bisimilar FC versions, which allows to use the results of FCN theory ([11]) and thus facilitates the further investigations here (comp. [4]).

## 4.2 Free Choice nets and Clusters

It will turn out that special operations inducing epimorphisms can be used to transform a net into a weakly bisimilar Free Choice net; we will define these operations in such a way as to make the resulting net unique up to isomorphism.

A given net is not n-free iff it contains subnets of the kind shown in Figure 3; we call them non-trivial *clusters*:

**Definition 12** Let  $\mathcal{N} = (\mathcal{P}, \mathcal{Q}, pre, post, \mathcal{M}_0)$  be a net and  $p \in \mathcal{P}$ . The **cluster set**  $\mathcal{CL}(p)$  of  $p$  is the smallest set  $M_p \subseteq \mathcal{P} \cup \mathcal{Q}$  such that

1.  $p \in M_p$

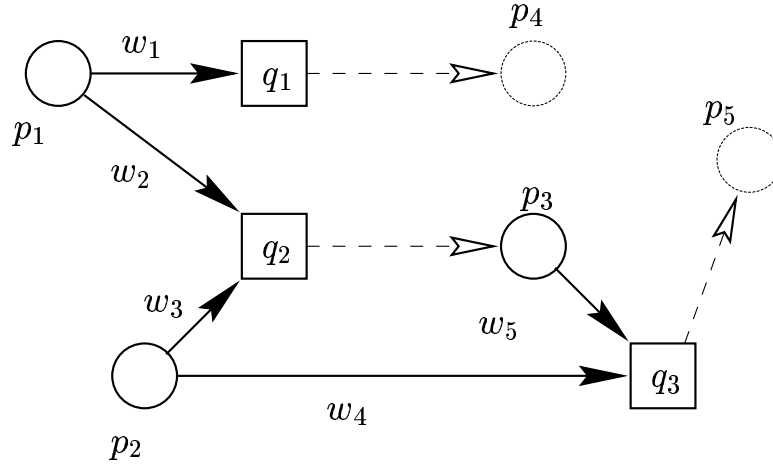


Figure 3: A cluster

$$2. \forall \bar{p} \in \mathcal{P} \cap M_p: pF \subset M_{\bar{p}}$$

$$3. \forall \bar{q} \in \mathcal{Q} \cap M_p: F\bar{q} \subset M_p$$

The subnet spanned by  $M_p$  is the **cluster (net)** of  $p$  in  $\mathcal{N}$ , denoted  $\mathcal{CN}(p)$ .

Clusters have some obvious properties:

**Lemma 7** Let  $\mathcal{N} = (\mathcal{P}, \mathcal{Q}, pre, post, \mathcal{M}_0)$  be a net.

$$1. \forall p, \bar{p} \in \mathcal{P}: \mathcal{CL}(p) \cap \mathcal{CL}(\bar{p}) \neq \emptyset \Rightarrow \mathcal{CL}(p) = \mathcal{CL}(\bar{p}).$$

$$2. \mathcal{N} \text{ is } n\text{-free iff } \forall p \in \mathcal{P}: \mathcal{CL}(p) \cap \mathcal{P} = \{p\}.$$

3. If  $\mathcal{N}$  is  $n$ -free, it is GFC iff:

$$\forall p \in \mathcal{P} \forall q \in pF: pre^q(p) > 1 \Rightarrow \mathcal{CL}(p) = \{p, q\}.$$

Let  $\mathcal{N}'$  be the net obtained from  $\mathcal{N}$  by replacing any arc from a place  $p$  to a transition  $q$  that is inside a non-trivial cluster  $\mathcal{CN}(p)$ , i.e.  $|\mathcal{CL}(p) \cap \mathcal{Q}| > 2$  and with weight greater than 1 by an extra place and transition as in Figure 4. We call  $\mathcal{N}'$  the **inflation** of  $\mathcal{N}$  and, conversely,  $\mathcal{N}$  the **contraction** of  $\mathcal{N}'$ .

**Definition 1** Let  $\mathcal{N} = (\mathcal{P}, \mathcal{Q}, pre, post)$  be an unmarked forward ordinary net with o.i. routing  $\nu$  or o.d. routing  $\rho$  and  $q, \bar{q} \in \mathcal{Q}$  such that  $qF \subset F\bar{q}$ . Then the net  $\bar{\mathcal{N}} = (\bar{\mathcal{P}}, \bar{\mathcal{Q}}, p\bar{r}e, p\bar{o}st)$  together with o.i. routing  $\bar{\nu}$  or o.d. routing  $\bar{\rho}$  satisfying

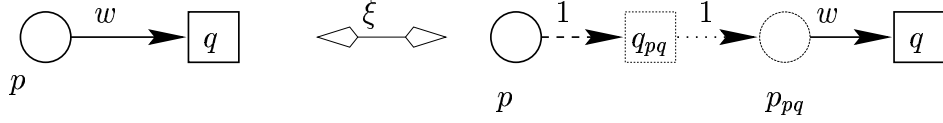


Figure 4: Contraction / Inflation

1.  $\bar{\mathcal{P}} = \mathcal{P} - qF$
2.  $\bar{\mathcal{Q}} = (\bar{\mathcal{Q}} - \{q, \bar{q}\}) \cup \{\tilde{q}\}$ ,
3.  $p\bar{r}e^q(p) = pre^q(p)$  for all  $p \in \bar{\mathcal{P}}$  and  $q \in \bar{\mathcal{Q}} \cap \bar{\mathcal{Q}}$ ,
4.  $p\bar{r}e^{\tilde{q}}(p) = pre^{\tilde{q}}(p)$  for  $p \in F\bar{q}$ ,
5.  $p\bar{r}e^{\tilde{q}}(p) = pre^q(p)$  for  $p \in Fq$ , and
6.  $\bar{\nu}$  given by
  - (a)  $\bar{\nu}^p(n) = (\nu^p(n) - \{q\}) \cup \{\tilde{q}\}$  for  $p \in Fq$  and  $n \in (\nu^p)^{-1}(\{q\})$ ,
  - (b)  $\bar{\nu}^p(n) = \nu^p(n)$  otherwise,
 or  $\bar{\rho}$  given – for  $q_{in} \in Fp$  – by
  - (a)  $\bar{\rho}^p(q_{in}, n) = (\rho^p(q_{in}, n) - \{q\}) \cup \{\tilde{q}\}$  for  $p \in Fq$  and  $n \in (\rho^p(q_{in}, \bullet))^{-1}$ ,
  - (b)  $\bar{\rho}^p(q_{in}, n) = \rho^p(q_{in}, n)$  otherwise.

is called the **q-contraction** of  $\mathcal{N}$ . Any  $\bar{\mathcal{N}}$  such that there exists a sequence  $\mathcal{N} = \mathcal{N}_1, \dots, \mathcal{N}_n = \bar{\mathcal{N}}$  where  $j = i + 1$  implies that  $\mathcal{N}_j$  is obtained from  $\mathcal{N}_i$  by a simultaneous application of q-contractions on  $\mathcal{N}_i$  such that no two operations affect the same node of  $\mathcal{N}_i$ ,  $\bar{\mathcal{N}}$  is called a **contraction** of  $\mathcal{N}$ .

The contraction just defined induces inclusion mappings  $\iota_{\mathcal{P}} : \mathcal{P} \rightarrow \bar{\mathcal{P}}$  and  $\iota_{\mathcal{Q}} : \mathcal{Q} \rightarrow \bar{\mathcal{Q}}$ , and thus the relation  $\mathfrak{R} \subseteq \mathcal{X}(\mathcal{N}) \times \mathcal{X}(\bar{\mathcal{N}})$ , where

$$\mathfrak{R} = \iota_{\mathcal{P}} \cup \{(\bar{p}, p)\} \cup \iota_{\mathcal{Q}} \cup \{(\bar{q}, \tau)\},$$

which, as we will see, is a weak bisimulation.

**Lemma 8** *Let  $\mathcal{N} = (\mathcal{P}, \mathcal{Q}, pre, post, \mathcal{M}_0)$ . Then*

1.  $\mathcal{N}'$  is unique up to isomorphism, and
2.  $\mathcal{N}'$  is GFC

**Proof:** Both properties follow from Lemma 7. □

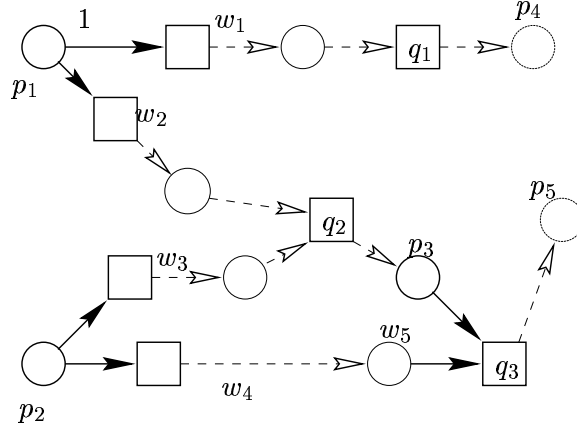


Figure 5: GFC version of the net from Figure 3. All unlabelled arc weights are 1.

**Theorem 4** Let  $\mathcal{N}_1 = (\mathcal{P}_1, \mathcal{Q}_1, pre_1, post_1, \mathcal{M}_0^1)$  be a net and  $\nu$  ( $\rho$ ) an o.i. (o.d.) routing. Let  $\mathcal{N}_2 = (\mathcal{P}_2, \mathcal{Q}_2, pre_2, post_2, \mathcal{M}_0^2)$  be the GFC version of  $\mathcal{N}_1$  with corresponding routing  $\bar{\nu}$  ( $\bar{\rho}$ ). Then

1.  $\mathcal{N}_2 = (\mathcal{P}_2, \mathcal{Q}_2, pre_2, post_2, \mathcal{M}_0^2)$  is weakly trace- and step-  $\xi - \bar{\xi}$ -bisimilar to  $\mathcal{N}_1$ ,  $\xi \in \{\nu, \rho\}$ , and
2.  $\mathcal{N}_2 = (\mathcal{P}_2, \mathcal{Q}_2, pre_2, post_2, \mathcal{M}_0^2)$  is weakly process  $\rho - \bar{\rho}$ -bisimilar to  $\mathcal{N}_1$ .

**Proof:** Both follow by composition of sequences and processes.  $\square$

For GFC nets, Figure 6 shows a way to create a (as we shall see, weakly bisimilar) ordinary – and thus FC - version.

If  $\mathcal{N} = (\mathcal{P}, \mathcal{Q}, pre, post, \mathcal{M}_0)$  is GFC and  $q \in \mathcal{Q}$ ,  $p \in Fq$  such that  $pre^q(p) =: w > 1$ , then  $pF = \{q\}$ . Hence the following construction has no effect on other transitions:

Let  $\mathcal{N}_1 = (\mathcal{P}_1, \mathcal{Q}_1, pre_1, post_1, \mathcal{M}_0^1)$  be a GFC net with o.i. routing  $\nu_1$  and  $\rho_2$ , and  $q \in \mathcal{Q}$ ,  $p \in Fq$  such that  $pre^q(p) =: w > 1$ . Then define  $\mathcal{N}_2 = (\mathcal{P}_2, \mathcal{Q}_2, pre_2, post_2, \mathcal{M}_0^2)$  such that

$$\begin{aligned} \mathcal{P}_2 &:= \mathcal{P}_1 \cup \{p_1, \dots, p_w\} & \text{and} & & \mathcal{Q}_2 &:= \mathcal{Q}_1 \cup \{q_1, \dots, q_w\}, \\ pre_2^q(p) &:= 0, \\ \forall i = 1, \dots, w : \\ pre_2^{q_i}(p_i) &:= 1 \\ \wedge post_2^{q_i}(\tilde{p}) &:= 1_{\{\tilde{p} \in \{p_1, \dots, p_w\}\}} \\ \wedge pre_2^{q_i}(\tilde{p}) &:= 1_{\{\tilde{p} = p\}} \end{aligned}$$

and routing  $\nu_2$  given by



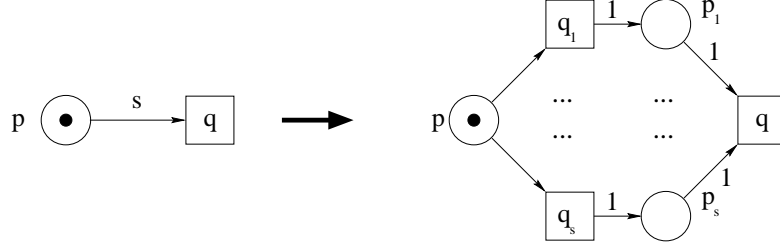


Figure 6: Locally removing arc weights

1.  $\nu_2^{\bar{p}}(n) = \{q_{(n \bmod w)+1}\}$
2.  $\forall \bar{p} \neq p : \nu_2^{\bar{p}} = \nu_1^{\bar{p}}$
3.  $\forall i = 1, \dots, w : \nu^{p_i} \equiv \{q\}$ ;

accordingly for  $\rho_2$ . Then  $\mathcal{N}_2$  is the **FC version** of  $\mathcal{N}_1$ ; by extension, if  $\mathcal{N}_1$  is the GFC version of  $\mathcal{N}$ ,  $\mathcal{N}_2$  is the FC version of  $\mathcal{N}$ .

Thus we have:

**Theorem 5** *Let  $\mathcal{N}_1 = (\mathcal{P}_1, \mathcal{Q}_1, pre_1, post_1, \mathcal{M}_0^1)$  be a routed net; then  $\mathcal{N}$  has a uniquely defined FC version  $\mathcal{N}_1 = (\mathcal{P}_1, \mathcal{Q}_1, pre_1, post_1, \mathcal{M}_0^1)$ , and*

1.  $\mathcal{N}_1$  is weakly trace- and step-  $\xi - \bar{\xi}$ -bisimilar to  $\mathcal{N}$ ,  $\xi \in \{\nu, \rho\}$ , and
2.  $\mathcal{N}_1$  is weakly process  $\rho - \bar{\rho}$ -bisimilar to  $\mathcal{N}$ .

Many analyses for routed nets may thus be carried out under the FC assumption without loss of generality; we will demonstrate this in the following section.

## 5 Equitability and fairness

Let  $\mathcal{N} = (\mathcal{P}, \mathcal{Q}, pre, post, \mathcal{M}_0)$  be a net and  $\sigma = (q_n)_{n \in \mathbb{N}_0} \in \mathcal{Q}^\omega$  an infinite firing sequence for  $\mathcal{N}$ , i.e.

$$\mathcal{M}_0 \xrightarrow{q_0} \mathcal{M}_1 \xrightarrow{q_1} \mathcal{M}_2 \xrightarrow{q_2} \mathcal{M}_3 \dots$$

Following the taxonomies of Francez [15] and Kwiatkowska [17], we define:

**Definition 13** *Let  $\mathcal{N} = (\mathcal{P}, \mathcal{Q}, pre, post, \mathcal{M}_0)$  be a net and  $\sigma = (q_n)_{n \in \mathbb{N}_0} \in MULT(\mathcal{Q})$  be a firing sequence for  $\mathcal{N}$ . Let  $\bar{\mathcal{Q}} \subset \mathcal{Q}$ . We say that  $\sigma$  is*

1. **weakly fair** (for  $\bar{\mathcal{Q}}$ ) *iff, for all  $q \in \bar{\mathcal{Q}}$  and all  $n \in \mathbb{N}_0$ , the following holds: If there exists an  $l \in \mathbb{N}$  such that  $\mathcal{M}_k \xrightarrow{t}$  for all  $k \in \{n, \dots, n+l\}$ , then  $|\sigma_n|_q > 0$ .*

2. **strongly fair** (for  $\bar{\mathcal{Q}}$ ) iff, for all  $q \in \bar{\mathcal{Q}}$ , the set  $\mathcal{EM}_q := \{k \in \mathbb{N}_0 : \mathcal{M}_k \xrightarrow{t}\}$  satisfies

$$|\mathcal{EM}_q| = \infty \Rightarrow |\sigma|_q = \infty.$$

Obviously, strong fairness implies weak fairness, and that implication cannot be reversed. There are, however, more subtle fairness issues not captured by the above definitions. Consider, for example, figure 7 and the firing sequences

$$\begin{aligned} \sigma &:= \alpha\beta q_1 \gamma \alpha \beta q_1 \dots \\ \tau &:= \delta q_2 \alpha \beta q_1 \gamma \varepsilon \delta q_2 \alpha \beta q_1 \gamma \varepsilon \dots \end{aligned}$$

Both  $\sigma$  and  $\tau$  are strongly fair; however, they treat, intuitively,  $t$  in an unfair way since their actions are “coordinated” in such a way that  $t$  never has a chance of firing. A behaviour that satisfies all preconditions for  $t$  perhaps infinitely often *individually* but only finitely many times *simultaneously* does not give  $q$  any opportunity to make use of the tokens passing it. We therefore introduce the following notion:

**Definition 14** Let  $\mathcal{N} = (\mathcal{P}, \mathcal{Q}, pre, post, \mathcal{M}_0)$  be a net and  $\sigma = (q_n)_{n \in \mathbb{N}_0} \in MULT(\mathcal{Q})$  be a firing sequence for  $\mathcal{N}$  with associated marking sequence  $(\mathcal{M}_n)_{n \in \mathbb{N}_0}$ . Set

$$\mathcal{WM}_p := \{k \in \mathbb{N}_0 : \mathcal{M}_k(q) \geq pre^q(p)\}.$$

Then we say that  $\sigma$  gives **opportunity** to  $q \in \mathcal{Q}$  iff for each  $p \in Fq$ ,

$$\forall p \in Fq : |\mathcal{WM}_p| = \infty \Rightarrow \left| \bigcap_{p \in Fq} \mathcal{WM}_p \right| = \infty.$$

Since opportunity for  $q$  does not imply anything about the actual firing of  $q$ , opportunity is independent of fairness. However, opportunity plus strong fairness captures exactly the intuitive meaning of fairness, namely, that the net should not “conspire against”  $q$ .

Now, consider an unequivocal o.i. routing  $\nu$  for  $\mathcal{N}$ . Since  $\nu$  works on the individual places rather than transitions or enabling conditions, there is no direct way of implementing fairness in it.

However, the following property (used for FC nets in [4] and [5]), which is instrumental in the study of liveness properties in general, ensures both opportunity and fairness for all  $t$ :

**Definition 15** An unequivocal o.i.p.  $\nu$  (resp. its routing function  $\Psi$ ) is **equitable** if the routing decisions satisfy

$$\forall p \in \mathcal{P} \forall q \in Fp : \lim_{n \rightarrow \infty} 1_{\nu_p(n)=\{q\}} = \infty.$$

**Remark 2** We define equitability only for o.i. routing here; there is no property for o.d. ensuring an equitable distribution over  $pF$  of the tokens arriving on  $p$ . Equitability in the above sense is obviously guaranteed if (not iff)

$$\forall p \in \mathcal{P} \forall q_1 \in Fp \forall q_2 \in pF : \lim_{n \rightarrow \infty} \Phi(q_1, n, q_2) = \infty$$

These routings – which we call strongly equitable – therefore satisfy all results derived below. So does an entire class of o.d. routings, which correspond dynamically to an equitable o.i. routing; for this class, however, a characterization in structural terms is not possible.

It is obvious that any unequivocal routing consistent with a strongly fair firing sequence is, a fortiori, equitable; the following Theorem shows that even more is true:

**Theorem 6** *Let  $\mathcal{N} = (\mathcal{P}, \mathcal{Q}, pre, post, \mathcal{M}_0)$  be a net with unequivocal routing  $\nu$   $\Psi = (\mu_0, \Psi)$  and  $\sigma$  a firing sequence under  $\nu$ . If  $\nu$  is equitable then*

1.  $\sigma$  gives opportunity to all  $q \in \mathcal{Q}$ , and
2.  $\sigma$  is strongly fair iff it is weakly fair.

**Proof:**

*Part 1:* Let  $q \in \mathcal{Q}$  and suppose that for all  $p \in Fq$ ,  $|\mathcal{MARK}_p| = \infty$ . This implies that an infinite number of token instances arrives on each  $p \in Fq$ ; by equitability, an infinite number of token instances on each  $p \in Fq$  is routed towards  $q$  and can – since  $\nu$  is unequivocal – not be removed from their places except by  $q$  itself; hence  $\sigma$  gives opportunity to  $q$ .

*Part 2:* We only have to show that weak fairness is sufficient for strong fairness under equitability. Suppose, therefore, that  $\sigma$  is weakly fair and that for some  $q \in \mathcal{Q}$ ,  $|\mathcal{MARK}_q| = \infty$ . By reasoning as in the proof of Part 1,  $\sigma$  gives opportunity to  $q$  which is therefore infinitely often enabled under  $\Psi$ . Since  $\nu$  is unequivocal,  $q$  can only be disabled by its own firing, therefore  $\sigma(q) = \infty$ .

□

Now we turn towards the problem of determining the number of firings for a transition, given a fixed unequivocal o.i. routing  $\nu$ . In general, there exist at least two different firing sequences  $\sigma_1$  and  $\sigma_2$  compatible with  $\nu$ ; to see this, assume in figure 7 that, for all  $m \in \mathbb{N}_0$ ,

$$\nu_{p_a}(m) = \{q_1\} \quad \text{and} \quad \nu_{p_b}(m) = \{q_2\}$$

such that, in principle,  $\alpha$  can fire infinitely often; however, there exists an infinite sequence of firings of  $\delta$ ,  $q_2$ , and  $\varepsilon$  alternately, in which  $\alpha$  never fires, and for each  $k \in \mathbb{N}_0$ , one can find an infinite sequence  $\sigma^k$  such that  $\sigma^k(\alpha) = k$ . The total number of firings for  $\alpha$  is thus undefined. Note, however, that no  $\sigma^k$  is weakly fair; in fact, the following Lemma shows that such extreme differences are impossible under weak fairness.

**Lemma 9** *Let  $\mathcal{N} = (\mathcal{P}, \mathcal{Q}, pre, post, \mathcal{M}_0)$  be a finite net with unequivocal and equitable routing  $\nu$  and  $\sigma_1$  and  $\sigma_2$  two weakly fair firing sequences generated according to  $\nu$ . Then for all  $q \in \mathcal{Q}$ ,*

$$|\sigma_1|_q = \infty \quad \Leftrightarrow \quad |\sigma_2|_q = \infty.$$

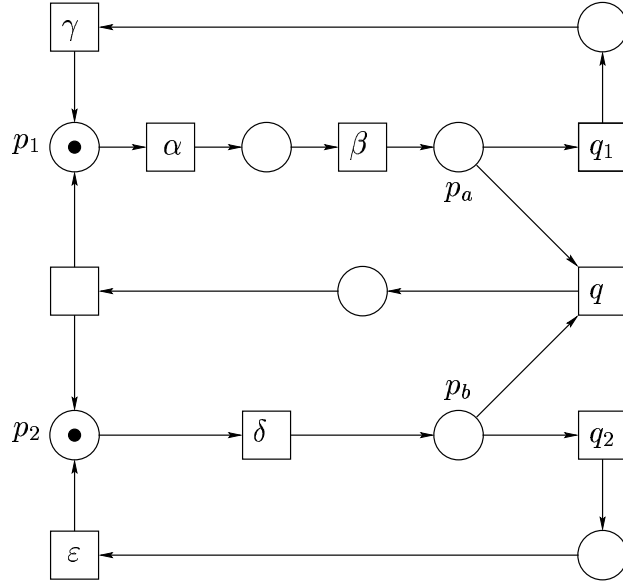


Figure 7: On fairness and equitability

We will write  $|\Psi|_q = \infty$  iff  $|\sigma|_q = \infty$  for some weakly fair sequence  $\sigma$  under  $\Psi$ .

**Proof:** Assume  $\mathcal{N}$  to be Free Choice and that  $qF \neq \emptyset$  for all  $q \in \mathcal{Q}$  (obviously, adding an output place  $p_q$  with empty post-domain for every transition  $t$  does not change the firing behaviour). Let  $\sigma = q_0 q_1 q_2 \dots$  and  $\sigma' = q'_0 q'_1 q'_2 \dots$ , where

$$\mu_0 \xrightarrow{q_0} \mu_1 \xrightarrow{q_1} \mu_2 \xrightarrow{q_2} \dots \quad \text{and} \quad \mu_0 \xrightarrow{q'_0} \mu'_1 \xrightarrow{q'_1} \mu'_2 \xrightarrow{q'_2} \dots;$$

further, for  $i \in \{1, 2\}$  set  $\mathcal{Q}_{starve} := \{q \in \mathcal{Q} : |\sigma|_q \in \mathbb{N}_0\}$  and

$$\mathcal{P}_{starve} := \{p \in \mathcal{P} : \text{in } \sigma, \text{ only a finite number of tokens arrive at } p\},$$

and accordingly  $\mathcal{Q}'_{starve}$  and  $\mathcal{P}'_{starve}$  for  $\sigma'$ . We observe first that  $\mathcal{P}_{starve}$  satisfies

$$F\mathcal{P}_{starve} = \mathcal{Q}_{starve} \quad \text{and} \quad \mathcal{P}F_{starve} = \mathcal{Q}_{starve} \quad (3)$$

and similarly for  $\mathcal{P}'_{starve}$  and  $\mathcal{Q}'_{starve}$  as well<sup>2</sup>. Let  $q \in \mathcal{Q}_{starve}$ . Suppose there exists  $p \in Fq - \mathcal{P}_{starve}$ ; by (3), there must therefore also exist  $p' \in Fq \cap \mathcal{P}'_{starve}$ . Since  $\mathcal{N}$  is Free Choice,  $pF = \{q\}$ . Therefore, after some finite number  $K$  of firings of  $q$ , any token entering  $p$  remains in  $p$ . Let  $\tilde{\mathcal{N}}$  be the net obtained from  $\mathcal{N}$  by removing any such  $p$ ; then

<sup>2</sup>that  $\mathcal{P}_{starve}$  ( $\mathcal{P}'_{starve}$ ) is therefore both a syphon and a trap.

$\bar{\mathcal{N}}$  and  $\mathcal{N}$  are weakly bisimilar, and both  $\sigma$  and  $\sigma'$  are firable in  $\bar{\mathcal{N}}$  under  $\nu$  restricted to  $\bar{\mathcal{P}} \subset \mathcal{P}$ . Hence we may assume that  $\mathcal{P}_{starve} \cup \mathcal{Q}_{starve}$  is disconnected from the rest of  $\mathcal{N}$ . Now replace  $\sigma$  and  $\sigma'$  by their restrictions  $\bar{\sigma}$  and  $\bar{\sigma}'$  to  $q \in \mathcal{Q}_{starve}$ . By assumption,  $\bar{\sigma}$  is finite and  $\bar{\sigma}'$  infinite, and, by reasoning as above, for all  $q \in \mathcal{Q}_{starve}$ ,  $|\bar{\sigma}'|_q \geq 1$  and  $|\bar{\sigma}|_q \geq 1$ . By assumption,  $\mu_0 \xrightarrow{q'_0}$ , and we just showed that there is a smallest index  $k$  such that  $q'_0 = q_k$ . Since the tokens used by  $t'_0$  are present in  $\mu_0$  and can not be removed by any move in  $\bar{\sigma}$  before  $q_k$ , we can replace  $\bar{\sigma}$  by the sequence obtained by moving  $q_k$  in the 0-th position. But then it suffices to compare the suffixes  $\bar{\sigma}_1$  and  $\bar{\sigma}'_1$  after deleting the 0-th move; by assumption, repeating this argument leads to an ordered marking which is dead in one sequence but not in the other under the same routing; since that contradicts itself, we are done.  $\square$

## 6 Concluding remarks

Token routing can be viewed as a refinement for the semantics of Petri nets. By investigating routing functions, new looks upon system properties are possible. The refinement provided by routing allows to restrict oneself, without loss, to Free Choice systems – which, obviously, must be used with care.

Little can be said in general about the properties of routings that are neither unequivocal nor trivial; their potential for modelling will be worth studying.

One would hope to obtain a result to the effect that the correspondence between routings and semantics reflected some inclusion ordering of the semantics (like, e.g., subgraph relations between reachability graphs) by ordering of the routings. This however, is not the case; there are incomparable routings with comparable reachability graphs. In figure 8, let  $\nu_1$  and  $\nu_2$  be such that

$$\begin{aligned} \nu_1^{p_1}(1) &= \{a\} & \text{and} & & \nu_1^{p_2}(1) &= \{c\} \\ \nu_2^{p_1}(n) &\equiv \{b\} & \text{and} & & \nu_2^{p_2}(1) &= \{c\} \end{aligned}$$

and let  $\nu_1$  and  $\nu_2$  coincide otherwise. Then  $\nu_1$  and  $\nu_2$  are incomparable with respect to  $\sqsubseteq_i$ , but  $\mathcal{G}^{step}(\nu_2)$  consists of one vertex, the initial state, while  $\mathcal{G}^{step}(\nu_1)$  has edges corresponding to the concurrent firing of  $a$  and  $b$ .

The controlling a system by a routing function is – depending on the state of the system – highly sensitive to perturbations of the routing.

It should be noted that, focussing on semantics, we have restricted ourselves to un-timed PNs entirely here. However, the study of timed nets has been a major driving force for the use and study of routed nets. The results in [4] concerning liveness properties are based on a study of timed nets; there, routing decisions may also depend on the moment in time. In [10], the behaviour of routed timed stochastic PNs is studied, using both types of routing; *injective* o.d. routing is used to ensure time reversability of processes.

**Remark 3** *If such a routing is bijective in the sense of [10], i. e. if  $|Fp| = |pF|$  and the transition by which a token enters  $p$  determines 1–1 the transition to which it is routed, a*

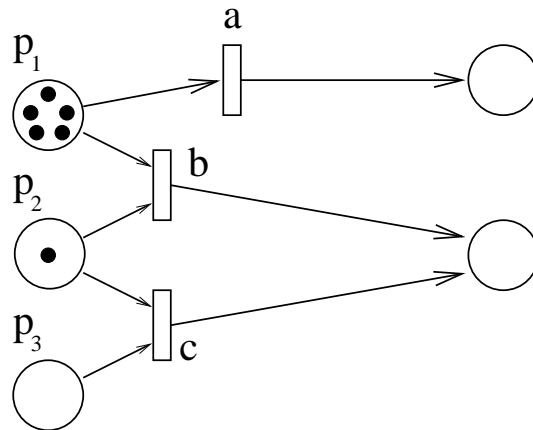


Figure 8: Incomparable Routings with comparable reachability graphs

further refinement of firing semantics is obtained. As the reader may verify, bijective routing for all places entails uniqueness up to isomorphism of the maximal process net according to Definition 8; in general, this uniqueness only holds true for 1-safe, non-branching nets.

Moreover, properties of untimed nets can often be studied as limits of appropriate timed nets ([4]).

### 6.0.1 Representability

O.d. routings that do not depend on  $n$  can be modelled by coloured nets: at any given place  $p$ , take as colours the nonempty subsets of  $pF$ . In the general case, every  $\bar{q} \in Fp$  must be able to count the tokens it has produced so far and equip the newly produced  $n$ th token with the colour corresponding to  $\rho^p(\bar{q}, n)$ , which (in general) exceeds the expressive power of coloured nets. Other possible parameters that may, in future studies, be allowed to influence the routing are:

- *Time*: the routing of a token  $\tau$  on a place  $p$  may depend either on the global time or on the time that has passed since  $\tau$ 's arrival on  $p$ . The latter model allows to express timed arcs or windows of validity for tokens or holding times for places; it is strictly more general than those models combined.
- *Global state*: The routing state of  $\tau$  on  $p$  may be influenced by the tokens on some set  $\mathcal{A} \subset \mathcal{P}$  of places. Thus models with read/inhibitor arcs as well as self-modifying nets are also included (and generalized).

- *Type*:  $\tau$  may belong to one of several pairwise disjoint sets, which may be thought of as colours or types, with the type of  $\tau$  influencing its routing. Coloured nets (where only one firing mode per transition is permitted) use, implicitly, a routing of this kind.

In some of these cases, it is necessary to allow empty routing, i. e.  $\nu = \emptyset$  or  $\rho = \emptyset$  for some set of parameters, to model temporary or permanent non-availability of tokens.

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