Shape Sensitivity Analysis of Variational Problems in Domains with Cracks

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Abstract: We obtain the structure theorem for a differentiable shape functional defined in a domain including curved cracks. The theorem is an extension of the structure theorem given eg. in [15] in the case of smooth domains. Four examples of applications of our result are given for shape functionals defined for elliptic boundary value problems: to nonlinear elliptic boundary value problems, to optimal control problems and to the shape differentiability of the first eigenvalue of the Laplacian.

Key-words: shape optimization, shape functional, shape derivative, cracks, nonlinear boundary problems, optimal control problem

(Résumé : tsvp)

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La dérivation par rapport au domaine dans des ouverts fissurés

Résumé : Nous présentons le théorème de structure pour une fonctionnelle de forme différentiable définie dans un domaine possédant des fissures. Le théorème est une extension du théorème de structure donné dans [15] dans le cas de domaines réguliers. Quatre exemples d'applications du théorème sont donnés: deux applications à des problèmes non linéaires, une application à un problème de contrôle optimal et une apllication liée à la dérivée de la première valeur propre du laplacien.

Mots-clé : optimisation de forme, fonctionnelle de forme, dérivée par rapport à la forme, fissures, problèmes non linéaires, problèmes de contrôle optimal

1 Introduction

Shape sensitivity analysis of boundary value problems defined in domains with cracks is important for applications, we refer the reader to the review paper [3] for some applications in fracture mechanics and a list of references. In the simplest case such results are derived in [8], we refer also to [10] and [2]. Since the structure theorem was not established, the direct approach is used in [8] which requires in the case of the energy functional the existence of the shape derivative of solutions to the elliptic equations defined in the domain with cracks. The same approach is in fact used in [7] in the case of unilateral conditions on the crack faces, however in such a case the shape differentiability result does not seem to be known in the literature, we refer the reader to [15] for the related results on the shape differentiability of solutions to variational inequalities in smooth domains.

The problem with unilateral conditions on the crack faces is considered in [13] for a scalar equation and in [14] for an elasticity system, where the so-called Rice-Cherepanov formula is derived. It is shown in [13], [14] that the result on shape differentiability of solutions to variational inequalities is not required for the proof of the differentiability of the energy functional with respect to the crack's length for problems with unilateral conditions prescribed on the crack faces.

2 The structure theorem

The structure theorem is important for applications in shape optimization, because it allows to obtain the shape differentiability of a specific shape functional by means of simple verifications of hypothesis, usually in the fixed domain setting, by an application of the material derivative method [15].

Let $D \subset \mathbb{R}^2$ be a bounded domain with smooth boundary Γ, and Σ be a part of a smooth curve. We assume that $\overline{\Sigma}$ belongs to the domain D. Therefore, we consider the domain $\Omega = D\backslash\overline{\Sigma}$ with crack Σ. Let us denote by A and B the tips of $\overline{\Sigma}$. Assume that J is a domain functional which is shape differentiable at Ω. We refer the reader to [15] for the definition of the shape differentiability.

The velocity field V is used to construct a family of domains $\Omega_t = T_t(V)(\Omega)$ using the technique described in [15]. Without losing the generality, we can consider the problem with autonomous vector fields. We have the following result on the structure of the Eulerian semiderivative $dJ(\Omega; V)$:

Theorem 1 (Structure theorem) *Let k be a nonnegative integer. Let us assume that the mapping $\mathcal{D}^k(D;\mathbb{R}^2) \ni V \longmapsto dJ(\Omega;V) \in \mathbb{R}$ is linear and continuous. Then there exist two real numbers α_A and α_B, and a linear form ϕ which is continuous on $C^k(\overline{\Sigma})$ $\big(\phi \in (C^k(\overline{\Sigma}))'\big)$ such that:*

$$dJ(\Omega; V) = \alpha_A(V.\tau)(A) + \alpha_B(V.\tau)(B) + \phi(V.n), \quad \forall V \in \mathcal{D}^k(D;\mathbb{R}^2),$$

where $V.\tau$ and $V.n$ denotes the tangential and normal components of field V on $\overline{\Sigma}$, respectively.

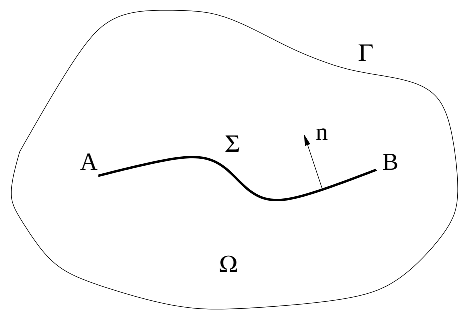

Figure 1: Domain Ω with the curved crack Σ

Proof: We may assume that Σ is the set given by:

$$\Sigma = \{(x_1, x_2) \mid 0 < x_1 < 1,\ x_2 = 0\} \tag{1}$$

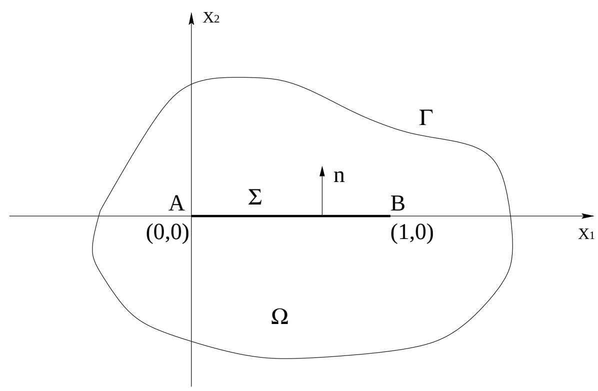

Figure 2: Domain Ω

otherwise we can use an appropriate change of variables.
We need the form of the tangent set

$$T_{\overline{\Sigma}}(x) = \left\{ v \in \mathbb{R}^2 \mid \liminf_{h \to 0^+} \frac{d_{\overline{\Sigma}}(x + hv)}{h} = 0 \right\}.$$

Evaluation of $T_{\overline{\Sigma}}(x)$ for $x \in \overline{\Sigma}$.

- *first case*: $x = (x_1, x_2) \in \Sigma$, *i.e.* $0 < x_1 < 1$, $x_2 = 0$.
  In this case, the normal $n(x)$ to $\overline{\Sigma}$ at $x \in \Sigma$ is well defined and moreover we have

  $$T_{\overline{\Sigma}}(x) = \mathcal{T}_x(\overline{\Sigma}), \text{ tangent space to } \overline{\Sigma} \text{ at } x,$$

  so

  $$V(x) \in T_{\overline{\Sigma}}(x) \quad \text{iff} \quad V(x).n(x) = 0.$$

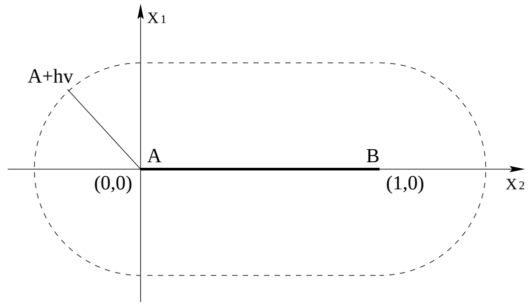

Figure 3: Evaluation of $T_{\overline{\Sigma}}(A)$

- *second case*: $x = A = (0,0)$.

$$T_{\overline{\Sigma}}(x) = T_{\overline{\Sigma}}(A) = \left\{ v = (X_1, X_2) \in \mathbb{R}^2 \mid \liminf_{h \to 0^+} \frac{d_{\overline{\Sigma}}(x+hv)}{h} = 0 \right\}$$

$$d_{\overline{\Sigma}}(x+hv) = \begin{cases} h\|v\| & \text{if } X_1 \leq 0 \\ h|X_2| & \text{if } X_1 \geq 0 \end{cases}$$

thus

$$\liminf_{h \to 0^+} \frac{d_{\overline{\Sigma}}(x+hv)}{h} = \begin{cases} \|v\| & \text{if } X_1 \leq 0 \\ |X_2| & \text{if } X_1 \geq 0 \end{cases}$$

and consequently

$$T_{\overline{\Sigma}}\big((0,0)\big) = T_{\overline{\Sigma}}(A) = \big\{ v = (X_1, X_2) \in \mathbb{R}^2 \mid X_2 = 0 \text{ and } X_1 \geq 0 \big\}. \tag{2}$$

- *third case*: $x = B = (1,0)$.
  In the same way as for $x = A = (0,0)$, we have

$$T_{\overline{\Sigma}}\big((1,0)\big) = T_{\overline{\Sigma}}(B) = \big\{ v = (X_1, X_2) \in \mathbb{R}^2 \mid X_2 = 0 \text{ and } X_1 \leq 0 \big\}. \tag{3}$$

For any $x \in \Sigma$, $T_{\overline{\Sigma}}(x)$ is a vector space, thus $-T_{\overline{\Sigma}}(x) = T_{\overline{\Sigma}}(x)$. On the other hand

$$\big\{T_{\overline{\Sigma}}(A)\big\} \cap \big\{-T_{\overline{\Sigma}}(A)\big\} = \big\{T_{\overline{\Sigma}}(B)\big\} \cap \big\{-T_{\overline{\Sigma}}(B)\big\} = \big\{(0,0)\big\}, \tag{4}$$

so according to *Nagumo's theorem* [1] or to the double viability condition [5],[6] and in view of the relations (2)–(4), if the field $V \in \mathcal{D}^k(D;\mathbb{R}^2)$ satisfies the following conditions

$$\begin{cases} V(x).n(x) = 0, \quad \forall x \in \Sigma \\ V(A) = V(B) = (0,0) \end{cases} \tag{5}$$

then $\overline{\Sigma}$ is globally invariant by the associated transformation $T_t(V)$. The exterior boundary $\Gamma = \partial D$ is also invariant by the transformation $T_t(V)$, since the support of the field V is included in D. So, the boundary of $\Omega = D\backslash\overline{\Sigma}$, *i.e.* $\partial\Omega = \Gamma\cup\overline{\Sigma}$, is globally invariant by the transformation $T_t(V)$. In consequence $\Omega_t \stackrel{def}{=} T_t(V)(\Omega) = \Omega$. Hence

$$dJ(\Omega; V) = 0 \tag{6}$$

for any vector field which satisfies (5). It is not difficult to see that it is possible to extend the notion of tangent and normal vectors to the points A and B of $\overline{\Omega}$. We can formulate again the conditions (5) as follows

$$\begin{cases} V(x).n(x) = 0, \quad \forall x \in \overline{\Sigma} \\ (V.\tau)(A) = (V.\tau)(B) = 0. \end{cases} \tag{7}$$

So, we have established that for given $V \in \mathcal{D}^k(D, \mathbb{R}^2)$ which satisfies conditions (7), it follows that $dJ(\Omega; V) = 0$. Hence, it is natural to consider the following set

$$F(\Omega) = \{V \in \mathcal{D}^k(D, \mathbb{R}^2) \mid V.n = 0 \text{ on } \overline{\Sigma},\ (V.\tau)(A) = (V.\tau)(B) = 0\}. \tag{8}$$

According to the hypothesis that the mapping $V \longmapsto dJ(\Omega; V)$ is linear and continuous from $\mathcal{D}^k(D, \mathbb{R}^2)$ in $\mathbb{R}$, the set $F(\Omega)$ defined by (8) is included in its kernel. Consequently, we have the following lemma.

Lemma 1 *The mapping*

$$\begin{array}{rccl} \psi: & \mathcal{D}^k(D, \mathbb{R}^2)/F(\Omega) & \longrightarrow & C^k(\overline{\Sigma}) \times \mathbb{R} \times \mathbb{R} \\ & \{V\} & \longmapsto & \big(V.n, (V.\tau)(A), (V.\tau)(B)\big) \end{array}$$

is an isomorphism.

Proof: The linear mapping $\psi:\ \{V\} \longmapsto \big(V.n, (V.\tau)(A), (V.\tau)(B)\big)$ is well defined since, if $V_1 - V_2 \in F(\Omega)$ then

$$(V_1 - V_2).n = 0 \quad \text{on } \overline{\Sigma},\ \big((V_1 - V_2).\tau\big)(A) = \big((V_1 - V_2).\tau\big)(B) = 0.$$

Let $\{V\} \in \mathcal{D}^k(D, \mathbb{R}^2)/F(\Omega)$ be such that $\psi(\{V\}) = 0$, *i.e.*

$$V.n = 0 \quad \text{on } \overline{\Sigma},\ (V.\tau)(A) = (V.\tau)(B) = 0,$$

which means that $V \in F(\Omega)$ and then $\{V\} = \{0\}$. Consequently ψ is one-to-one.

Now, let us show that ψ is onto. Let $(v, v_1, v_2) \in C^k(\overline{\Sigma}) \times \mathbb{R} \times \mathbb{R}$. We want to find $V \in \mathcal{D}^k(D, \mathbb{R}^2)$ such that $\psi(\{V\}) = (v, v_1, v_2)$. For any $v \in C^k(\overline{\Sigma}) = C^k([0,1])$, by definition of the space $C^k([0,1])$, there exists $\tilde{v} \in C^k(\mathbb{R})$ such that $\tilde{v}_{|[0,1]} = v$. So we define $\widetilde{V_2}$ by

$$\widetilde{V_2}(x_1, x_2) = \tilde{v}(x_1), \quad \forall x_1,\ x_2 \in \mathbb{R}. \tag{9}$$

Then it is evident that $\widetilde{V_2} \in C^k(\mathbb{R}^2)$. Let $\theta \in \mathcal{D}(D, \mathbb{R}) = C_0^\infty(D, \mathbb{R})$ be such that $\theta \equiv 1$ in a sufficiently small neighbourhood of $\overline{\Sigma}$. Denote

$$V_2(x_1, x_2) = \theta(x_1, x_2)\widetilde{V_2}(x_1, x_2), \quad \forall x_1,\ x_2 \in \mathbb{R}. \tag{10}$$

Let us define the function μ by

$$\mu(x_1) = (1 - x_1)v_1 + x_1 v_2. \tag{11}$$

Then $\mu \in C^k(\mathbb{R})$ (extension by convex combination).
In the same way, we introduce

$$\widetilde{V_1}(x_1, x_2) = \mu(x_1), \quad \forall x_1,\ x_2 \in \mathbb{R}, \tag{12}$$

hence $\widetilde{V_1} \in C^k(\mathbb{R}^2)$. Then, we define V_1 by the formula

$$V_1(x_1, x_2) = \theta(x_1, x_2)\widetilde{V_1}(x_1, x_2). \tag{13}$$

Let V be the vector field whose components are V_1 and V_2 $\big(V_1$ and V_2 being successively given by (9),(10),(11),(12),(13)$\big)$, so

$$V = (V_1, V_2) \in \mathcal{D}^k(D, \mathbb{R}^2),$$

moreover

$$\begin{array}{rclcll} V_1(x_1, x_2) & = & \theta(x_1, x_2)\widetilde{V_1}(x_1, x_2) & = & \widetilde{V_1}(x_1, x_2) & \text{on } \overline{\Sigma} \\ & = & \mu(x_1) & = & (1 - x_1)v_1 + x_1 v_2. & \end{array}$$

In consequence $(V.\tau)(A) = \mu(0) = v_1$ and $(V.\tau)(B) = \mu(1) = v_2$, and it follows that by construction,

$$\begin{array}{rclcll} V_2(x_1, x_2) & = & \theta(x_1, x_2)\widetilde{V_2}(x_1, x_2) & = & \widetilde{V_2}(x_1, x_2) & \text{on } \overline{\Sigma} \\ & = & \tilde{v}(x_1) & = & v(x_1), & \end{array}$$

i.e. $V.n = v$ on $\overline{\Sigma}$. We have defined $V \in \mathcal{D}^k(D, \mathbb{R}^2)$ such that $\psi\big(\{V\}\big) = (v, v_1, v_2)$. This completes the proof of lemma 1. $\square$

Lemma 2 *There exists a linear, continuous mapping* Φ

$$\Phi :\ C^k(\overline{\Sigma}) \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$$

such that for any vector field $V \in \mathcal{D}^k(D, \mathbb{R}^2)$*,*

$$dJ(\Omega; V) = \Phi\big(V.n, (V.\tau)(A), (V.\tau)(B)\big).$$

Proof: We define Φ by the following formula

$$\Phi(\{V\}) = dJ(\Omega; V). \tag{14}$$

Indeed, if $\{V'\} = \{V\}$, *i.e.* $V' \in \{V\}$, we have $V' - V \in F(\Omega)$, since $F(\Omega)$ is included in the kernel of $dJ(\Omega; \cdot)$, it follows that

$$dJ(\Omega; V - V') = 0.$$

The Eulerian semiderivative $dJ(\Omega; \cdot)$ is linear by our assumption, therefore

$$dJ(\Omega; V) = dJ(\Omega; V'). \tag{15}$$

The relation (15) enables us to define Φ. Using lemma 1 and the relation $\mathcal{D}^k(D,\mathbb{R}^2)/F(\Omega) \simeq C^k(\overline{\Sigma}) \times \mathbb{R} \times \mathbb{R}$, it follows that

$$\{V\} = \big(V.n, (V.\tau)(A), (V.\tau)(B)\big) \tag{16}$$

thus

$$dJ(\Omega; V) = \Phi\big(\{V\}\big) = \Phi\big(V.n, (V.\tau)(A), (V.\tau)(B)\big). \tag{17}$$

Furthermore, $dJ(\Omega; \cdot)$ is linear and continuous which implies that Φ is linear and continuous.

Now, we can complete the proof of the structure theorem. Indeed, there exists a linear mapping Φ, which is continuous from $C^k(\overline{\Sigma}) \times \mathbb{R} \times \mathbb{R}$ in $\mathbb{R}$, such that

$$\forall V \in \mathcal{D}^k(D,\mathbb{R}^2), \quad dJ(\Omega; V) = \Phi\big(V.n, (V.\tau)(A), (V.\tau)(B)\big)$$

with

$$\Phi \in \big(C^k(\overline{\Sigma}) \times \mathbb{R} \times \mathbb{R}\big)' = \big(C^k(\overline{\Sigma})\big)' \times \mathbb{R} \times \mathbb{R}.$$

So there exist two real numbers α_A and α_B, and a linear form ϕ which is continuous on $C^k(\overline{\Sigma})$ such that

$$dJ(\Omega; V) = \phi(V.n) + \alpha_A (V.\tau)(A) + \alpha_B (V.\tau)(B), \quad \forall V \in \mathcal{D}^k(D,\mathbb{R}^2) \tag{18}$$

which completes the proof of structure theorem. $\square$

3 Applications to nonlinear problems

We present two applications of the structure theorem for shape functionals defined in the domains with cracks. We refer the reader to [13],[14] for the results on the Griffith formula in the case of unilateral conditions prescribed on the crack faces.

3.1 Nonlinear boundary value problem

We apply the structure theorem in the case of energy functional for a nonlinear boundary value problem. Since the singular part of the solution is unknown in this case (see [10]), we cannot identify the coefficients α_A, α_B in the representation formula.

Let $D \subset \mathbb{R}^2$ be an open bounded set with a smooth boundary Γ. Let Σ_l be the set defined by $\big\{(y_1, y_2) \mid 0 < y_1 < l,\ y_2 = 0\big\}$, A and B denote its tips. We assume that this set belongs to the domain D for $l > 0$ small enough. The domain with the crack is denoted by $\Omega = D \setminus \overline{\Sigma_l}$.
Let us consider the functional

$$I(\varphi) = \int_\Omega F\big(\nabla\varphi(y)\big)dy, \tag{19}$$

where $F : \mathbb{R}^2 \longrightarrow \mathbb{R}$ satisfies assumptions specified below.
Let $\mathcal{W}$ be the subspace of $H^1(\Omega)$ of functions whose trace vanish on Γ:

$$\mathcal{W} = \big\{\varphi \in H^1(\Omega) \mid \varphi_{|_\Gamma} = 0\big\} = H^1_\Gamma(\Omega).$$

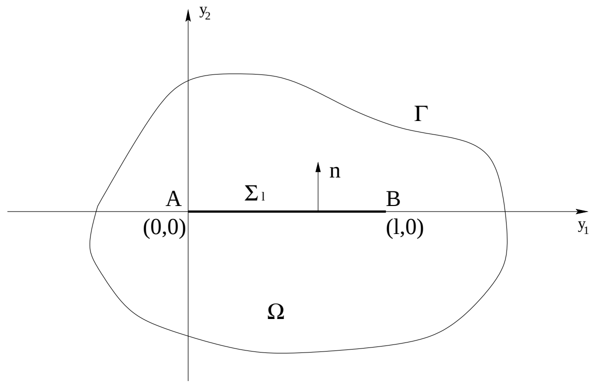

Figure 4: Domain Ω with crack Σ_l

We can show the existence and the uniqueness of the solution to the following minimization problem:

$$\text{Find } u \in \mathcal{W} \text{ such that } I(u) = \inf_{v \in \mathcal{W}} I(v). \tag{20}$$

Let $\theta_1, \theta_2 \in C_0^\infty(D) = \mathcal{D}(D)$. Then we consider the transformation (see [14]) defined by

$$\begin{cases} y_1 = x_1 - \delta\theta_1(x_1, x_2) \\ y_2 = x_2 - \delta\theta_2(x_1, x_2) \quad (\delta > 0). \end{cases} \tag{21}$$

The coordinates of a given point in open sets Ω, Ω_δ are denoted by $(y_1, y_2) \in \Omega$, $(x_1, x_2) \in \Omega_\delta$ respectively.
Let V denote the vector field whose components are θ_1, θ_2, $V \in (\mathcal{D}(D))^2$. The Jacobian of (21) equals to:

$$\begin{aligned} q_\delta &= 1 - \delta(\theta_{1,x_1} + \theta_{2,x_2}) + \delta^2(\theta_{1,x_1}\theta_{2,x_2} - \theta_{1,x_2}\theta_{2,x_1}) \\ &= 1 - \delta \operatorname{div} V + \delta^2 \det(DV). \end{aligned}$$

For δ small enough, $q_\delta > 0$, so the transformation (21) is one-to-one and we denote $y = y(x, \delta)$, $x = x(y, \delta)$. Let Ω_δ be the image of Ω for the transformation (21). Since θ_1, $\theta_2 \in \mathcal{D}(D)$, Γ is invariant for the transformation (21). Denote $\mathcal{W}_\delta = H^1_\Gamma(\Omega_\delta)$. We can show the existence and the uniqueness of $u^\delta \in \mathcal{W}_\delta$ such that

$$I_\delta(u^\delta) = \inf_{w \in \mathcal{W}_\delta} I_\delta(w), \text{ where } I_\delta(\psi) = \int_{\Omega_\delta} F\big(\nabla\psi(x)\big)dx, \quad \forall \psi \in \mathcal{W}_\delta.$$

So we define

$$J(\Omega) = I(u) = \int_\Omega F\big(\nabla u(y)\big)dy, \; J(\Omega_\delta) = I_\delta(u^\delta) = \int_{\Omega_\delta} F\big(\nabla u^\delta(x)\big)dx.$$

In order to apply the structure theorem, the shape differentiability of the shape functional $J(\Omega)$ at Ω is established by proving that there exists the following limit

$$dJ(\Omega; V) = \lim_{\delta \longrightarrow 0^+} \frac{J(\Omega_\delta) - J(\Omega)}{\delta} \tag{22}$$

which is linear and continuous with respect to $V = (\theta_1, \theta_2)$.

The shape differentiability of $J(\Omega)$ is obtained under the following assumptions.

Theorem 2 *Assume that:*

(i) $F \in C^1(\mathbb{R}^2; \mathbb{R})$, *is convex and Lipschitz,*
DF is Lipschitz continuous and strictly monotone.

(ii) I *and* I_δ *are coercive and strictly convex functionals.*

Then:

- *there exists a unique* $u \in \mathcal{W}$ *such that*

$$I(u) = \inf_{v \in \mathcal{W}} I(v) \text{ and, moreover, } u \text{ satisfies } \int_\Omega \langle DF(\nabla u), \nabla \varphi \rangle dy = 0,\ \forall \varphi \in \mathcal{W},$$

- *there exists a unique* $u^\delta \in \mathcal{W}_\delta$ *such that*

$$I_\delta(u^\delta) = \inf_{w \in \mathcal{W}_\delta} I_\delta(w) \text{ and } u^\delta \text{ satifies } \int_{\Omega_\delta} \langle DF(\nabla u^\delta), \nabla \psi \rangle dx = 0,\ \forall \psi \in \mathcal{W}_\delta,$$

- J *is shape differentiable at* Ω *and*

$$dJ(\Omega; V) = \int_\Omega \langle DF(\nabla u), B \cdot \nabla u \rangle dy + \int_\Omega \mathrm{div} V F(\nabla u) dy,$$

where

$$B = -\begin{pmatrix} \theta_{1,y_1} & \theta_{2,y_1} \\ \theta_{1,y_2} & \theta_{2,y_2} \end{pmatrix}.$$

Proof: The proof of the two first points is standard and it is omitted here. We only prove the shape differentiablity of J at Ω.
Let us introduce some notations:

(i) for a matrix function $A : \Omega \longrightarrow \mathcal{M}_{2,2}(\mathbb{R})$, A^T is the transposed mapping defined by $(A^T)(y) = \big(A(y)\big)^T$, $\forall y \in \Omega$, where $\big(A(y)\big)^T$ denotes the transposed matrix of $A(y)$.

(ii) $\|A\|_\infty = \sup_{y \in \Omega} \sup_{i,j=1,2} |A_{ij}(y)|$ is L^∞-norm of the matrix function A.

The proof is divided into some steps.

Step 1:
There exists a constant C independent of δ such that

$$\|u^\delta\|_{\mathcal{W}_\delta} \leq C.$$

Under our assumptions, DF is strictly monotone, *i.e.*

$$\exists \gamma > 0 \text{ such that } \gamma |p-q|^2 \leq \big\langle DF(p) - DF(q), p-q \big\rangle, \quad \forall (p,q) \in \big(\mathbb{R}^2\big)^2.$$

For $p = \nabla u^\delta$ and $q = 0$ we have

$$\gamma |\nabla u^\delta|^2 \leq \big\langle DF(\nabla u^\delta) - DF(0), \nabla u^\delta \big\rangle.$$

Integration over Ω_δ leads to the inequality

$$\gamma \int_{\Omega_\delta} |\nabla u^\delta|^2 dx \leq \int_{\Omega_\delta} \big\langle DF(\nabla u^\delta) - DF(0), \nabla u^\delta \big\rangle dx.$$

Taking into account that $\displaystyle\int_{\Omega_\delta} \big\langle DF(\nabla u^\delta), \nabla u^\delta \big\rangle dx = 0$ for the test function $\psi = u^\delta$, it follows that

$$\gamma \|u^\delta\|^2_{\mathcal{W}_\delta} \leq -\int_{\Omega_\delta} \big\langle DF(0), \nabla u^\delta \big\rangle dx \leq \int_{\Omega_\delta} |DF(0)||\nabla u^\delta| dx = \int_{\Omega_\delta} \frac{|DF(0)|}{\sqrt{\gamma}} \sqrt{\gamma} |\nabla u^\delta| dx,$$

therefore

$$\gamma \|u^\delta\|^2_{\mathcal{W}_\delta} \leq \frac{|DF(0)|^2}{2\gamma} \int_{\Omega_\delta} dx + \frac{\gamma}{2} \|u^\delta\|^2_{\mathcal{W}_\delta}$$

and the required estimate for the norm of u^δ follows

$$\|u^\delta\|_{\mathcal{W}_\delta} \leq \frac{|DF(0)||D|^{1/2}}{\gamma} = C.$$

Denote $u^\delta(x) = u_\delta(y)$, $x = x(y,\delta)$, $u_\delta \in \mathcal{W}$. The following formula is derived

$$\nabla_x u^\delta = A_\delta \cdot \nabla_y u_\delta,$$

where

$$A_\delta = \begin{pmatrix} 1 - \delta\theta_{1,x_1} & -\delta\theta_{2,x_1} \\ -\delta\theta_{1,x_2} & 1 - \delta\theta_{2,x_2} \end{pmatrix}.$$

Now, we are going to show that $\|u_\delta\|_{\mathcal{W}} \leq C'$ for δ small enough.
We have shown that

$$\|u^\delta\|^2_{\mathcal{W}_\delta} = \int_{\Omega_\delta} |\nabla u^\delta(x)|^2 dx \leq C^2,$$

hence by the change of variables,

$$\int_\Omega \frac{1}{q_\delta} |A_\delta \cdot \nabla u_\delta(y)|^2 dy \leq C^2.$$

Taking into account that $A_\delta = I + \delta B$ and

$$q_\delta = 1 - \delta \mathrm{divV} + \delta^2 \mathrm{det(DV)},$$

it is easy to see that

$$|q_\delta - 1| \leq \delta |\mathrm{div} V| + \delta^2 |\det(DV)| \leq \delta \|\mathrm{div} V\|_{L^\infty(\Omega)} + \delta^2 \|\det(DV)\|_{L^\infty(\Omega)} \longrightarrow 0 \quad \text{as } \delta \longrightarrow 0^+,$$

which means that

$$q_\delta \longrightarrow 1 \text{ uniformly on } \overline{\Omega} \text{ as } \delta \longrightarrow 0^+.$$

By the uniform convergence, there exists $\varepsilon_1 > 0$ such that $|q_\delta - 1| \leq \dfrac{1}{2}$ for $\delta \in (0, \varepsilon_1)$, or equivalently

$$\frac{1}{2} \leq q_\delta \leq \frac{3}{2} \quad \text{in } \Omega.$$

Using the estimate for q_δ we can estimate the L^2-norm of $A_\delta \cdot \nabla u_\delta$,

$$\frac{2}{3}\int_\Omega |A_\delta \cdot \nabla u_\delta|^2 \leq \int_\Omega \frac{1}{q_\delta}|A_\delta \cdot \nabla u_\delta(y)|^2 dy \leq C^2,$$

i.e. $\|A_\delta \cdot \nabla u_\delta\|_{(L^2(\Omega))^2} \leq \sqrt{\frac{3}{2}}C$ which can be rewritten in the form

$$\|\nabla u_\delta + \delta B \cdot \nabla u_\delta\|_{(L^2(\Omega))^2} \leq \sqrt{\frac{3}{2}}C.$$

The latter inequality implies that

$$\|\nabla u_\delta\|_{(L^2(\Omega))^2} - \delta \|B \cdot \nabla u_\delta\|_{(L^2(\Omega))^2} \leq \sqrt{\frac{3}{2}}C.$$

In view of

$$\|B \cdot \nabla u_\delta\|_{(L^2(\Omega))^2} \leq \|B\|_\infty \|\nabla u_\delta\|_{(L^2(\Omega))^2},$$

it follows that

$$\|\nabla u_\delta\|_{(L^2(\Omega))^2}(1 - \delta\|B\|_\infty) \leq \sqrt{\frac{3}{2}}C.$$

Assuming that $\|B\|_\infty \neq 0$, otherwise $V \equiv 0$, let us denote $\varepsilon_2 = \dfrac{1}{2\|B\|_\infty}$. Then we can estimate $1 - \delta\|B\|_\infty$ from below,

$$1 - \delta\|B\|_\infty \geq \frac{1}{2} \quad \text{for } \delta \in (0, \varepsilon_2).$$

For any $\delta \in (0, \varepsilon)$, $\varepsilon = \min(\varepsilon_1, \varepsilon_2) > 0$, we have

$$\frac{1}{2}\|\nabla u_\delta\|_{(L^2(\Omega))^2} \leq \sqrt{\frac{3}{2}}C$$

which means that

$$\|\nabla u_\delta\|_{(L^2(\Omega))^2} = \|u_\delta\|_{\mathcal{W}} \leq \sqrt{6}C = C'$$

therefore, there exists $C' \geq 0$ such that

$$\|u_\delta\|_{\mathcal{W}} \leq C' \text{ for } \delta \text{ small enough.} \tag{23}$$

Step 2: $u_\delta \longrightarrow u$ in $\mathcal{W}$ as $\delta \longrightarrow 0^+$.
By the change of variables, it follows that

$$\int_\Omega \langle DF(A_\delta \cdot \nabla u_\delta), A_\delta \cdot \nabla \varphi \rangle \frac{dy}{q_\delta} = 0, \quad \forall \varphi \in \mathcal{W}.$$

By our assumptions, DF is strictly monotone,

$$\gamma|p-q|^2 \le \big\langle DF(p)-DF(q), p-q\big\rangle, \quad \forall p,\ q \in \mathbb{R}^2.$$

Applying the inequality with $p = \nabla u$ and $q = \nabla u_\delta$, it follows that

$$\gamma|\nabla u - \nabla u_\delta|^2 \le \big\langle DF(\nabla u) - DF(\nabla u_\delta), \nabla u - \nabla u_\delta\big\rangle.$$

Integration over Ω leads to

$$\begin{aligned}
\gamma\|u-u_\delta\|^2_{\mathcal{W}} &\le \int_\Omega \big\langle DF(\nabla u) - DF(\nabla u_\delta), \nabla u - \nabla u_\delta\big\rangle \\
&\le \int_\Omega \big\langle DF(\nabla u), \nabla u\big\rangle - \int_\Omega \big\langle DF(\nabla u), \nabla u_\delta\big\rangle + \int_\Omega \big\langle DF(\nabla u_\delta), \nabla u_\delta - \nabla u\big\rangle \\
&\le \int_\Omega \big\langle DF(\nabla u_\delta), \nabla u_\delta - \nabla u\big\rangle.
\end{aligned}$$

Using the variational equation for u_δ we obtain

$$\int_\Omega \frac{1}{q_\delta}\big\langle DF(A_\delta\cdot\nabla u_\delta), A_\delta\cdot\nabla u\big\rangle = 0 = \int_\Omega \frac{1}{q_\delta}\big\langle DF(A_\delta\cdot\nabla u_\delta), A_\delta\cdot\nabla u_\delta\big\rangle,$$

which leads to the equality

$$\int_\Omega \frac{1}{q_\delta}\big\langle DF(A_\delta\cdot\nabla u_\delta), A_\delta\cdot(\nabla u - \nabla u_\delta)\big\rangle = 0.$$

The equality is rewritten in the form

$$\int_\Omega \frac{1}{q_\delta}\big\langle A_\delta^T\cdot DF(A_\delta\cdot\nabla u_\delta), \nabla u - \nabla u_\delta\big\rangle = 0,$$

which allows us to estimate the norm of $u - u_\delta$,

$$\begin{aligned}
\gamma\|u-u_\delta\|^2_{\mathcal{W}} &\le \int_\Omega \big\langle DF(\nabla u_\delta) - \frac{1}{q_\delta}A_\delta^T\cdot DF(A_\delta\cdot\nabla u_\delta), \nabla u_\delta - \nabla u\big\rangle dy \\
&\le \int_\Omega \big\langle \mathcal{F}_\delta(\nabla u_\delta), \nabla u_\delta - \nabla u\big\rangle dy \ \le\ \|\mathcal{F}_\delta(\nabla u_\delta)\|_{(L^2(\Omega))^2}\|u-u_\delta\|_{\mathcal{W}},
\end{aligned}$$

where we denote

$$\mathcal{F}_\delta(v) = DF(v) - \frac{1}{q_\delta}A_\delta^T\cdot DF(A_\delta\cdot v).$$

The L^2-norm of $\mathcal{F}_\delta(v)$ can be estimated in the following way. We have

$$\begin{aligned}
|\mathcal{F}_\delta(v)| &\le \Big|DF(v) - \frac{1}{q_\delta}DF(v)\Big| + \Big|\frac{1}{q_\delta}DF(v) - \frac{1}{q_\delta}A_\delta^T\cdot DF(v)\Big| \\
&\quad + \Big|\frac{1}{q_\delta}A_\delta^T\cdot DF(v) - \frac{1}{q_\delta}A_\delta^T\cdot DF(A_\delta\cdot v)\Big| \\
&\le \alpha_\delta + \beta_\delta + \gamma_\delta
\end{aligned}$$

We proceed with the terms α_δ , β_δ , γ_δ separately,

$$\alpha_\delta = \delta|DF(v)|\left|\frac{-\mathrm{div}V + \delta\det(DV)}{q_\delta}\right| \leq 2\delta|DF(v)|\left(\|\mathrm{div}V\|_{L^\infty(\Omega)} + \varepsilon\|\det(DV)\|_{L^\infty(\Omega)}\right),$$

which shows that $\alpha_\delta \leq C_1\delta|DF(v)|$ for $\delta \in (0,\varepsilon)$. In the same way for β_δ,

$$\beta_\delta = \frac{1}{|q_\delta|}|DF(v) - (I+\delta B)^T \cdot DF(v)| \leq 2|\delta B^T \cdot DF(v)| \leq 2\delta\|B^T\|_\infty|DF(v)|,$$

which leads to the estimate $\beta_\delta \leq C_2\delta|DF(v)|$ for $\delta \in (0,\varepsilon)$. Finally, for γ_δ,

$$\gamma_\delta = \frac{1}{|q_\delta|}|A_\delta^T \cdot DF(v) - A_\delta^T \cdot DF(A_\delta \cdot v)| \leq 2\|A_\delta^T\||DF(v) - DF(A_\delta \cdot v)|.$$

Taking into account that

$$\|A_\delta^T\| = \|I + \delta B^T\| \leq \|I\| + \varepsilon\|B^T\|_\infty \quad \text{for } \delta \in (0,\varepsilon),$$

the inequality for γ_δ follows,

$$\gamma_\delta \leq 2(\|I\| + \varepsilon\|B^T\|_\infty)|DF(v) - DF(A_\delta \cdot v)|.$$

Since DF is Lipschitz continuous,

$$\gamma_\delta \leq 2(\|I\| + \varepsilon\|B^T\|_\infty)L|v - A_\delta \cdot v|$$

and $|v - A_\delta \cdot v| = |\delta B \cdot v| \leq \delta\|B\|_\infty|v|$, the inequality $\gamma_\delta \leq C_3\delta|v|$ follows for $\delta \in (0,\varepsilon)$.

Now we are in the position to complete the estimate for $\mathcal{F}_\delta(v)$. We have

$$|\mathcal{F}_\delta(v)| \leq C_1\delta|DF(v)| + C_2\delta|DF(v)| + C_3\delta|v| = \delta(C_4|DF(v)| + C_3|v|),$$

taking into account that $|DF(v)| \leq |DF(0)| + L|v|$, it follows that $|\mathcal{F}_\delta(v)| \leq \delta(\mu_1 + \mu_2|v|)$, and

$$|\mathcal{F}_\delta(v)|^2 \leq \delta^2(\mu_1 + \mu_2|v|)^2 \leq 2\delta^2(\mu_1^2 + \mu_2^2|v|^2),$$

therefore, for $v = \nabla u_\delta$,

$$|\mathcal{F}_\delta(\nabla u_\delta)|^2 \leq 2\delta^2(\mu_1^2 + \mu_2^2|\nabla u_\delta|^2).$$

Integrating this inequality over Ω,

$$\int_\Omega |\mathcal{F}_\delta(\nabla u_\delta)|^2 \leq 2\delta^2\left(\mu_1^2|D| + \mu_2^2\int_\Omega |\nabla u_\delta|^2\right) \leq 2\delta^2\left(\mu_1^2|D| + \mu_2^2\|u_\delta\|_{\mathcal{W}}^2\right)$$

results in the estimate $\|\mathcal{F}_\delta(\nabla u_\delta)\|_{(L^2(\Omega))^2} \leq \mu\delta$ for $\delta \in (0,\varepsilon)$.
Therefore,

$$\gamma\|u - u_\delta\|_{\mathcal{W}}^2 \leq \mu\delta\|u - u_\delta\|_{\mathcal{W}} \quad \text{for } \delta \in (0,\varepsilon),$$

and we pass to the limit

$$\|u - u_\delta\|_{\mathcal{W}} \leq \frac{\mu}{\gamma}\delta \longrightarrow 0 \quad \text{as } \delta \longrightarrow 0^+,$$

which completes the proof of the continuity with respect to δ,

$$u_\delta \longrightarrow u \text{ in } \mathcal{W} \quad \text{as } \delta \longrightarrow 0^+.$$

Step 3: passage to the limit in (22).
Let us introduce the following notations

$$\pi(\Omega;\varphi)=\int_\Omega F(\nabla\varphi)dy,\quad \pi_\delta(\Omega;\varphi)=\int_\Omega \frac{1}{q_\delta}F(A_\delta\cdot\nabla\varphi)dy,\quad \pi(\Omega_\delta;\varphi)=\int_{\Omega_\delta} F(\nabla\varphi)dx,$$

$$\pi(\Omega;u)=\min_{\varphi\in\mathcal{W}}\pi(\Omega;\varphi)\quad\text{and}\quad \pi(\Omega_\delta;u^\delta)=\min_{\varphi\in\mathcal{W}_\delta}\pi(\Omega_\delta;\varphi).$$

By the change of variables, we have

$$\min_{\varphi\in\mathcal{W}}\pi_\delta(\Omega;\varphi)=\min_{\varphi\in\mathcal{W}_\delta}\pi(\Omega_\delta;\varphi)$$

We denote

$$J(\Omega)=\pi(\Omega;u),\quad J(\Omega_\delta)=\pi(\Omega_\delta;u^\delta).$$

We have

$$\frac{J(\Omega_\delta)-J(\Omega)}{\delta}=\frac{\pi(\Omega_\delta;u^\delta)-\pi(\Omega;u)}{\delta}=\frac{\pi_\delta(\Omega;u_\delta)-\pi(\Omega;u)}{\delta}\le\frac{\pi_\delta(\Omega;u)-\pi(\Omega;u)}{\delta}.$$

Passage to the limit in the both sides leads to

$$\limsup_{\delta\longrightarrow0^+}\frac{J(\Omega_\delta)-J(\Omega)}{\delta}\le\liminf_{\delta\longrightarrow0^+}\frac{\pi_\delta(\Omega;u)-\pi(\Omega;u)}{\delta}\le\limsup_{\delta\longrightarrow0^+}\frac{\pi_\delta(\Omega;u)-\pi(\Omega;u)}{\delta}.$$

By the definition of the energy functionals,

$$\begin{aligned}\frac{\pi_\delta(\Omega;u)-\pi(\Omega;u)}{\delta} &= \int_\Omega\frac{F(A_\delta\cdot\nabla u)\frac{1}{q_\delta}-F(\nabla u)}{\delta}dy\\ &= \int_\Omega\frac{F(A_\delta\cdot\nabla u)-F(\nabla u)}{\delta q_\delta}dy+\int_\Omega\frac{1-q_\delta}{\delta q_\delta}F(\nabla u)dy\\ &= \int_\Omega\frac{F(\nabla u+\delta B\cdot\nabla u)-F(\nabla u)}{\delta q_\delta}dy+\int_\Omega\frac{1-q_\delta}{\delta q_\delta}F(\nabla u)dy.\end{aligned}$$

Under our assumptions there exists a function $\xi,\ 0\le\xi(y)\le\delta$, such that

$$\frac{F(\nabla u+\delta B\cdot\nabla u)-F(\nabla u)}{\delta}=\big\langle DF(\nabla u+\xi B\cdot\nabla u),B\cdot\nabla u\big\rangle.$$

Consequently

$$\frac{\pi_\delta(\Omega;u)-\pi(\Omega;u)}{\delta}=\int_\Omega\frac{1}{q_\delta}\big\langle DF(\nabla u+\xi B\cdot\nabla u),B\cdot\nabla u\big\rangle dy+\int_\Omega\frac{1-q_\delta}{\delta q_\delta}F(\nabla u)dy,$$

and the passage to the limit as $\delta\longrightarrow0^+$ in the right-hand side is performed by an application of the dominated convergence theorem, since F and DF are Lipschitz continuous,

$$\int_\Omega\frac{1}{q_\delta}\big\langle DF(\nabla u+\xi B\cdot\nabla u),B\cdot\nabla u\big\rangle dy\longrightarrow\int_\Omega\big\langle DF(\nabla u),B\cdot\nabla u\big\rangle dy,$$

$$\int_\Omega\frac{1-q_\delta}{\delta q_\delta}F(\nabla u)dy\longrightarrow\int_\Omega \mathrm{div}V F(\nabla u)dy,$$

which allows us to obtain the following limit

$$\begin{aligned}\limsup_{\delta\longrightarrow 0^+}\frac{\pi_\delta(\Omega;u)-\pi(\Omega;u)}{\delta} &= \lim_{\delta\longrightarrow 0^+}\frac{\pi_\delta(\Omega;u)-\pi(\Omega;u)}{\delta}\\ &= \int_\Omega \langle DF(\nabla u), B\cdot\nabla u\rangle dy + \int_\Omega \mathrm{div}V F(\nabla u)dy.\end{aligned}$$

In order to complete the proof of differentiability we derive the estimate from below for $\liminf$ using the following inequality

$$\frac{J(\Omega_\delta)-J(\Omega)}{\delta} = \frac{\pi(\Omega_\delta;u^\delta)-\pi(\Omega;u)}{\delta} = \frac{\pi_\delta(\Omega;u_\delta)-\pi(\Omega;u)}{\delta} \geq \frac{\pi_\delta(\Omega;u_\delta)-\pi(\Omega;u_\delta)}{\delta}.$$

Taking the limit in the both sides leads to

$$\liminf_{\delta\longrightarrow 0^+}\frac{J(\Omega_\delta)-J(\Omega)}{\delta} \geq \liminf_{\delta\longrightarrow 0^+}\frac{\pi_\delta(\Omega;u_\delta)-\pi(\Omega;u_\delta)}{\delta}.$$

Again, by the definition of energy functionals,

$$\begin{aligned}\frac{\pi_\delta(\Omega;u_\delta)-\pi(\Omega;u_\delta)}{\delta} &= \int_\Omega \frac{F(A_\delta\cdot\nabla u_\delta)\frac{1}{q_\delta}-F(\nabla u_\delta)}{\delta}dy\\ &= \int_\Omega \frac{F(A_\delta\cdot\nabla u_\delta)-F(\nabla u_\delta)}{\delta q_\delta}dy + \int_\Omega \frac{1-q_\delta}{\delta q_\delta}F(\nabla u_\delta)dy.\end{aligned}$$

Using the Taylor formula we obtain

$$\frac{F(A_\delta\cdot\nabla u_\delta)-F(\nabla u_\delta)}{\delta} = \frac{F(\nabla u_\delta+\delta B\cdot\nabla u_\delta)-F(\nabla u_\delta)}{\delta} = \langle DF(\nabla u_\delta+\xi B\cdot\nabla u_\delta), B\cdot\nabla u_\delta\rangle$$

for a function $\xi,\ 0\leq\xi(y)\leq h$, which leads to

$$\frac{\pi_\delta(\Omega;u_\delta)-\pi(\Omega;u_\delta)}{\delta} = \int_\Omega \frac{1}{q_\delta}\langle DF(\nabla u_\delta+\xi B\cdot\nabla u_\delta), B\cdot\nabla u_\delta\rangle dy + \int_\Omega \frac{1-q_\delta}{\delta q_\delta}F(\nabla u_\delta)dy.$$

We have the following estimate

$$\begin{aligned}\Big|\int_\Omega \frac{1-q_\delta}{\delta q_\delta}F(\nabla u_\delta)dy - \int_\Omega \frac{1-q_\delta}{\delta q_\delta}F(\nabla u)dy\Big| &= \Big|\int_\Omega \frac{1-q_\delta}{\delta q_\delta}\big(F(\nabla u_\delta)-F(\nabla u)\big)dy\Big|\\ &\leq \int_\Omega\Big|\frac{1-q_\delta}{\delta q_\delta}\Big|M|\nabla u_\delta-\nabla u|dy.\end{aligned}$$

Taking into account the inequality

$$\Big|\frac{1-q_\delta}{\delta q_\delta}\Big| \leq 2\big(\|\mathrm{div}V\|_{L^\infty(\Omega)}+\varepsilon\|\det(DV)\|_{L^\infty(\Omega)}\big) \quad \text{for } \delta\in(0,\varepsilon),$$

the following inequality is obtained,

$$\Big|\int_\Omega\frac{1-q_\delta}{\delta q_\delta}F(\nabla u_\delta)dy-\int_\Omega\frac{1-q_\delta}{\delta q_\delta}F(\nabla u)dy\Big| \leq 2M\big(\|\mathrm{div}V\|_{L^\infty(\Omega)}+\varepsilon\|\det(DV)\|_{L^\infty(\Omega)}\big)\int_\Omega|\nabla u_\delta-\nabla u|dy.$$

From the property

$$\lim_{\delta \longrightarrow 0^+} \|u - u_\delta\|_{\mathcal{W}} = 0$$

if follows that

$$\int_\Omega \frac{1-q_\delta}{\delta q_\delta} F(\nabla u_\delta) dy \longrightarrow \int_\Omega \mathrm{div} V F(\nabla u) dy \quad \text{for } \delta \longrightarrow 0^+.$$

Furthermore

$$\begin{aligned}
\int_\Omega \frac{1}{q_\delta} \langle DF(\nabla u_\delta + \xi B \cdot \nabla u_\delta), B \cdot \nabla u_\delta \rangle dy &= \int_\Omega \frac{1}{q_\delta} \langle DF(\nabla u_\delta + \xi B \cdot \nabla u_\delta), B \cdot \nabla u_\delta \rangle dy \\
- \int_\Omega \frac{1}{q_\delta} \langle DF(\nabla u + \xi B \cdot \nabla u_\delta), B \cdot \nabla u_\delta \rangle dy &+ \int_\Omega \frac{1}{q_\delta} \langle DF(\nabla u + \xi B \cdot \nabla u_\delta), B \cdot \nabla u_\delta \rangle dy \\
- \int_\Omega \frac{1}{q_\delta} \langle DF(\nabla u + \xi B \cdot \nabla u), B \cdot \nabla u_\delta \rangle dy &+ \int_\Omega \frac{1}{q_\delta} \langle DF(\nabla u + \xi B \cdot \nabla u), B \cdot \nabla u_\delta \rangle dy \\
- \int_\Omega \frac{1}{q_\delta} \langle DF(\nabla u + \xi B \cdot \nabla u), B \cdot \nabla u \rangle dy &+ \int_\Omega \frac{1}{q_\delta} \langle DF(\nabla u + \xi B \cdot \nabla u), B \cdot \nabla u \rangle dy \\
&= A_\delta + \int_\Omega \frac{1}{q_\delta} \langle DF(\nabla u + \xi B \cdot \nabla u), B \cdot \nabla u \rangle dy,
\end{aligned}$$

where for the last term

$$\int_\Omega \frac{1}{q_\delta} \langle DF(\nabla u + \xi B \cdot \nabla u), B \cdot \nabla u \rangle dy \longrightarrow \int_\Omega \langle DF(\nabla u), B \cdot \nabla u \rangle dy \quad \text{for } \delta \longrightarrow 0^+.$$

Moreover DF is Lipschitz continuous, therefore, there exists a constant $N \geq 0$ such that

$$\forall \delta \in (0, \varepsilon), \quad |A_\delta| \leq N \|u - u_\delta\|_{\mathcal{W}}$$

which means that $A_\delta \longrightarrow 0$ for $\delta \longrightarrow 0^+$, and we have obtained the convergence

$$\int_\Omega \frac{1}{q_\delta} \langle DF(\nabla u_\delta + \xi B \cdot \nabla u_\delta), B \cdot \nabla u_\delta \rangle dy \longrightarrow \int_\Omega \langle DF(\nabla u), B \cdot \nabla u \rangle dy \quad \text{for } \delta \longrightarrow 0^+,$$

which implies that

$$\begin{aligned}
\liminf_{\delta \longrightarrow 0^+} \frac{\pi_\delta(\Omega; u_\delta) - \pi(\Omega; u_\delta)}{\delta} &= \lim_{\delta \longrightarrow 0^+} \frac{\pi_\delta(\Omega; u_\delta) - \pi(\Omega; u_\delta)}{\delta} \\
&= \int_\Omega \langle DF(\nabla u), B \cdot \nabla u \rangle dy + \int_\Omega \mathrm{div} V F(\nabla u) dy.
\end{aligned}$$

Finally, we have

$$\lim_{\delta \longrightarrow 0^+} \frac{J(\Omega_\delta) - J(\Omega)}{\delta} = \int_\Omega \langle DF(\nabla u), B \cdot \nabla u \rangle dy + \int_\Omega \mathrm{div} V F(\nabla u) dy = dJ(\Omega; V).$$

We have shown that the shape functional $J(\Omega)$ is differentiable, the mapping $V \longmapsto dJ(\Omega; V)$ is linear and continuous, and we can apply the structure theorem in this case, which leads to the following formula

$$dJ(\Omega; V) = \alpha_A(V.\tau)(A) + \alpha_B(V.\tau)(B) + \phi(V.n), \quad \forall V \in \mathcal{D}^k(D; \mathbb{R}^2)$$

where $\phi \in (C^k(\overline{\Sigma_l}))'$, $\alpha_A, \ \alpha_B \in \mathbb{R}$ and k=1. $\square$

3.2 Signorini conditions

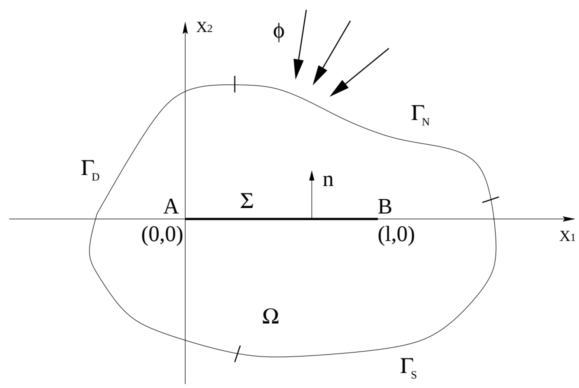

Figure 5: Domain Ω

Let $D \subset \mathbb{R}^2$ be a bounded domain with smooth boundary $\Gamma = \Gamma_D \cup \Gamma_N \cup \Gamma_S$, and Σ be the set $\{(x_1, x_2) \mid 0 < x_1 < 1,\ x_2 = 0\}$. We assume that this set belongs to the domain D. The domain with crack Σ is denoted by $\Omega = D \setminus \overline{\Sigma}$.
In the domain Ω, we consider the following boundary value problem for a function u which satisfies

$$\left\{\begin{array}{ccccc} -\Delta u & = & f & \text{in} & \Omega, \\ u & = & 0 & \text{on} & \Gamma_D, \\ \dfrac{\partial u}{\partial n} & = & \phi & \text{on} & \Gamma_N, \\ \dfrac{\partial u}{\partial n} & = & 0 & \text{on} & \Sigma^{\pm}, \\ u \geq 0, & \dfrac{\partial u}{\partial n} \geq 0, & u\dfrac{\partial u}{\partial n} = 0 & \text{on} & \Gamma_S. \end{array}\right. \tag{24}$$

Here $f \in C^1(\overline{D})$, $\phi \in H^2(\Omega)$ are given functions.

The associated variational formulation of this nonlinear problem, as well as the existence and uniqueness of a solution, are well known. Let us briefly recall two equivalent formulations of problem (24)

$$\left\{\begin{array}{l} u \in K \\ a(u, v-u) \geq L(v-u), \quad \forall v \in K \end{array}\right. \tag{25}$$

or

$$\left\{\begin{array}{l} u \in K \\ I(u) \leq I(v), \quad \forall v \in K, \end{array}\right. \tag{26}$$

where K is the closed convex set of $H^1(\Omega)$ defined by

$$K = \{v \in H^1(\Omega) \mid v = 0 \text{ on } \Gamma_D \text{ and } v \geq 0 \text{ on } \Gamma_S\}.$$

Here the following notations are used, for u and v in $H^1(\Omega)$,

$$a(u,v) = \int_\Omega \langle \nabla u, \nabla v \rangle, \qquad L(v) = \int_\Omega fv + \int_{\Gamma_N} \phi v, \quad I(v) = \frac{1}{2}a(v,v) - L(v).$$

The energy functional for the problem (24) is defined by the formula

$$J(\Omega) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \int_\Omega fu - \int_{\Gamma_N} \phi u, \tag{27}$$

where u is the variational solution to (24).
Let $\theta_1,\ \theta_2 \in C_0^\infty(D)$. We use the same notation as in section 3.1.
For $\delta > 0$, the minimization problem is defined in Ω_δ, with the energy functional

$$J(\Omega_\delta) = \frac{1}{2}\int_{\Omega_\delta} |\nabla u^\delta|^2 - \int_{\Omega_\delta} fu^\delta - \int_{\Gamma_N} \phi u^\delta. \tag{28}$$

We derive the form of the shape derivative

$$\frac{dJ(\Omega_\delta)}{d\delta}|_{\delta=0} = \lim_{\delta \longrightarrow 0+} \frac{J(\Omega_\delta) - J(\Omega)}{\delta} = dJ(\Omega; V) \tag{29}$$

in order to apply the structure theorem. Let $u^\delta(x)$ be the solution of minimization problem in Ω_δ, and $u^\delta(x) = u_\delta(y),\ x = x(y,\delta)$. We have the following formula

$$\nabla_x u^\delta = A_\delta \cdot \nabla_y u_\delta \tag{30}$$

with

$$A_\delta = \begin{pmatrix} 1 - \delta\theta_{1,x_1} & -\delta\theta_{2,x_1} \\ & \\ -\delta\theta_{1,x_2} & 1 - \delta\theta_{2,x_2} \end{pmatrix}.$$

Thus $A_\delta = I + \delta B$. Consequently

$$\int_{\Omega_\delta} |\nabla u^\delta|^2 dx = \int_\Omega \frac{1}{q_\delta} |A_\delta \cdot \nabla u_\delta|^2 dy.$$

By the change of variables, it follows that

$$\int_{\Omega_\delta} fu^\delta dx = \int_\Omega \frac{1}{q_\delta} f\big(x(y,\delta)\big) u_\delta(y) dy.$$

Denote $f^\delta(y) = \dfrac{f\big(x(y,\delta)\big)}{q_\delta}$, then

$$f'(y) = \frac{df^\delta(y)}{d\delta}|_{\delta=0} = \lim_{\delta \longrightarrow 0+} \frac{f^\delta(y) - f^0(y)}{\delta}.$$

Assuming that $y,\ \delta$ are independent variables in (21), we have $x = x(y,\delta)$. Differentiation of (21) with respect to δ yields

$$\begin{cases} 0 = \dfrac{dx_1}{d\delta} - \theta_1 - \delta\theta_{1,x_1}\dfrac{dx_1}{d\delta} - \delta\theta_{1,x_2}\dfrac{dx_2}{d\delta} \\ 0 = \dfrac{dx_2}{d\delta} - \theta_2 - \delta\theta_{2,x_1}\dfrac{dx_1}{d\delta} - \delta\theta_{2,x_2}\dfrac{dx_2}{d\delta}, \end{cases}$$

thus

$$\begin{cases} \dfrac{dx_1}{d\delta} = \dfrac{\theta_1(1-\delta\theta_{2,x_2})+\delta\theta_2\theta_{1,x_2}}{q_\delta} \\ \dfrac{dx_2}{d\delta} = \dfrac{\theta_2(1-\delta\theta_{1,x_1})+\delta\theta_1\theta_{2,x_1}}{q_\delta}. \end{cases} \tag{31}$$

Consequently, by (31),

$$\frac{\partial f\big(x(y,\delta)\big)}{\partial\delta}|_{\delta=0} = f_{x_1}\frac{dx_1}{d\delta}|_{\delta=0} + f_{x_2}\frac{dx_2}{d\delta}|_{\delta=0} = f_{y_1}\theta_1 + f_{y_2}\theta_2. \tag{32}$$

Now we are in a position to find the derivative $f'(y)$. Indeed, by (32),

$$f'(y) = \lim_{\delta\longrightarrow 0^+}\left(\frac{f\big(x(y,\delta)\big)}{q_\delta} - f(y)\right)\frac{1}{q_\delta} = \lim_{\delta\longrightarrow 0^+}\frac{f\big(x(y,\delta)\big)-q_\delta f(y)}{\delta q_\delta}$$

$$= \lim_{\delta\longrightarrow 0^+}\frac{f\big(x(y,\delta)\big)-f(y)}{\delta} + \mathrm{div}V f(y)_{|\delta=0} = f_{y_1}\theta_1 + f_{y_2}\theta_2 + (\theta_{1y_1}+\theta_{2y_2})f,$$

i.e.

$$f'(y) = \frac{\partial}{\partial y_1}(\theta_1 f) + \frac{\partial}{\partial y_2}(\theta_2 f) = \mathrm{div}(Vf). \tag{33}$$

Since $f\in C^1(\overline{\Omega})$ we can see that as $\delta\longrightarrow 0^+$

$$\frac{f^\delta(y)-f^0(y)}{\delta} \longrightarrow f'(y) \text{ in } L^\infty(\Omega). \tag{34}$$

The sets of admissible functions for the minimization problems under considerations are defined by

$$K_\delta = \{w\in H^1(\Omega_\delta) \mid w=0 \text{ on } \Gamma_D \text{ and } w\geq 0 \text{ on } \Gamma_S\},$$

$$K_0 = K = \{w\in H^1(\Omega) \mid w=0 \text{ on } \Gamma_D \text{ and } w\geq 0 \text{ on } \Gamma_S\},$$

respectively.

In view of (21), let $x=x(y,\delta)$. Then $w^\delta(x)=w_\delta(y)$. The inclusion $w^\delta\in K_\delta$ implies $w_\delta\in K_0$, and, conversely, $w_\delta\in K_0$ implies $w^\delta\in K_\delta$. This means that the transformation (21) maps K_δ into K_0, and it is one-to-one. Now we shall prove the continuity of u_δ with respect to δ

$$\|u_\delta - u\|_{H^1(\Omega)} \longrightarrow 0 \quad \text{as } \delta\longrightarrow 0^+.$$

The function $u^\delta\in K_\delta$ is the solution of the variational inequality

$$\int_{\Omega_\delta}\langle\nabla u^\delta, \nabla v - \nabla u^\delta\rangle \geq \int_{\Omega_\delta} f(v-u^\delta) + \int_{\Gamma_N}\phi(v-u^\delta), \quad \forall v\in K_\delta. \tag{35}$$

But by substituting $v=0$ in (35), it follows that

$$\|u^\delta\|_{H^1(\Omega_\delta)} \leq C \text{ uniformly in } \delta.$$

Consequently

$$\|u_\delta\|_{H^1(\Omega)} \leq C \text{ uniformly in } \delta.$$

By the change of variables in (35), it follows that

$$\int_\Omega \langle A_\delta \cdot \nabla u_\delta, A_\delta \cdot \nabla \tilde{v} - A_\delta \cdot \nabla u_\delta\rangle \frac{dy}{q_\delta} \geq \int_\Omega f^\delta(\tilde{v} - u_\delta)dy + \int_{\Gamma_N} \phi(\tilde{v} - u_\delta)d\sigma(y), \quad \forall \tilde{v} \in K_0 \quad (36)$$

or $A_\delta = I + \delta B$. Hence, according to (36), we have

$$\int_\Omega \langle \nabla u_\delta + \delta B \cdot \nabla u_\delta, \nabla \tilde{v} + \delta B \cdot \nabla \tilde{v} - \nabla u_\delta - \delta B \cdot \nabla u_\delta\rangle \frac{dy}{q_\delta} \geq \int_\Omega f^\delta(\tilde{v} - u_\delta)dy$$

$$+ \int_{\Gamma_N} \phi(\tilde{v} - u_\delta)d\sigma(y), \quad \forall \tilde{v} \in K_0. \quad (37)$$

We can substitute $\tilde{v} = u$ in (37) and we obtain

$$\int_\Omega \langle \nabla u_\delta, \nabla u - \nabla u_\delta\rangle \frac{dy}{q_\delta} + P(\delta, u, u_\delta) \geq \int_\Omega f^\delta(u - u_\delta)dy + \int_{\Gamma_N} \phi(u - u_\delta)d\sigma(y) \quad (38)$$

$$\text{where } P(\delta, u, u_\delta) \longrightarrow 0 \quad \text{as } \delta \longrightarrow 0^+.$$

The solution of the problem (24) is the solution of the variational inequality

$$u \in K_0 : \int_\Omega \langle \nabla u, \nabla v - \nabla u\rangle \geq \int_\Omega f(v - u) + \int_{\Gamma_N} \phi(v - u)d\sigma(y), \quad \forall v \in K_0.$$

We can sustitute $v = u_\delta$ which provides the inequality

$$\int_\Omega \langle \nabla u, \nabla u_\delta - \nabla u\rangle \geq \int_\Omega f(u_\delta - u) + \int_{\Gamma_N} \phi(u_\delta - u). \quad (39)$$

Summing up the relations (38),(39) implies that

$$\|u_\delta - u\|_{H^1(\Omega)} \longrightarrow 0 \quad \text{as } \delta \longrightarrow 0^+. \quad (40)$$

Denote

$$J(\Omega) = \pi(\Omega; u), \quad J(\Omega_\delta) = \pi(\Omega_\delta; u^\delta).$$

We have

$$\frac{J(\Omega_\delta) - J(\Omega)}{\delta} = \frac{\pi(\Omega_\delta; u^\delta) - \pi(\Omega; u)}{\delta} = \frac{\pi_\delta(\Omega; u_\delta) - \pi(\Omega; u)}{\delta} \leq \frac{\pi_\delta(\Omega; u) - \pi(\Omega; u)}{\delta}$$

and consequently

$$\begin{aligned} \limsup_{\delta \longrightarrow 0^+} \frac{J(\Omega_\delta) - J(\Omega)}{\delta} &\leq \limsup_{\delta \longrightarrow 0^+} \frac{\pi_\delta(\Omega; u) - \pi(\Omega; u)}{\delta} \\ &\leq \int_\Omega \langle B \cdot \nabla u, \nabla u\rangle dy + \frac{1}{2}\int_\Omega \mathrm{div}V |\nabla u|^2 dy - \int_\Omega f' u dy. \quad (41) \end{aligned}$$

On the other hand
$$\frac{J(\Omega_\delta)-J(\Omega)}{\delta}=\frac{\pi(\Omega_\delta;u^\delta)-\pi(\Omega;u)}{\delta}=\frac{\pi_\delta(\Omega;u_\delta)-\pi(\Omega;u)}{\delta}\geq\frac{\pi_\delta(\Omega;u_\delta)-\pi(\Omega;u_\delta)}{\delta}$$
therefore, by using (40)
$$\begin{aligned}\liminf_{\delta\longrightarrow 0^+}\frac{J(\Omega_\delta)-J(\Omega)}{\delta} &\geq \liminf_{\delta\longrightarrow 0^+}\frac{\pi_\delta(\Omega;u_\delta)-\pi(\Omega;u_\delta)}{\delta}\\ &\geq \int_\Omega\langle B\cdot\nabla u,\nabla u\rangle dy+\frac{1}{2}\int_\Omega \mathrm{div}V|\nabla u|^2dy-\int_\Omega f'udy. \end{aligned}\tag{42}$$
Using (41) and in view of (42) it follows that
$$\lim_{\delta\longrightarrow 0^+}\frac{J(\Omega_\delta)-J(\Omega)}{\delta}=\int_\Omega\langle B\nabla u,\nabla u\rangle dy+\frac{1}{2}\int_\Omega\mathrm{div}V|\nabla u|^2dy-\int_\Omega f'udy. \tag{43}$$
The substitution of B and f' in (43) leads to
$$dJ(\Omega;V)=-\frac{1}{2}\int_\Omega\Big((\theta_{1y_1}-\theta_{2y_2})\big((u_{y_1})^2-(u_{y_2})^2\big)+2(\theta_{1y_2}+\theta_{2y_1})u_{y_1}u_{y_2}\Big)-\int_\Omega\big((\theta_1 f)_{y_1}+(\theta_2 f)_{y_2}\big)u. \tag{44}$$

The mapping $V\longmapsto dJ(\Omega;V)$ is linear and continuous for $k=1$. The functional $J(\Omega)$ is shape differentiable and therefore, the structure theorem applies in this case.

Now, we are going to establish the relation between coefficients $\alpha_A,\ \alpha_B$ in the formula (18) and the singularity coefficients of the solution to the problem (24).

We are interested in perturbations of Ω such that the crack tip moves while the other part of the boundary does not change. In addition, we assume that crack tip moves without changing the direction, therefore, the perturbation of the boundary is defined by the following vector field
$$V(x)=(\theta_1(x),0), \tag{45}$$
where θ_1 has the support in D and $\theta_1(x)=-1$ in the vicinity of the origin A.
According to the relation (44), we have
$$dJ(\Omega;V)=-\frac{1}{2}\int_\Omega\Big(\theta_{1y_1}\big((u_{y_1})^2-(u_{y_2})^2\big)+2\theta_{1y_2}u_{y_1}u_{y_2}\Big)-\int_\Omega(\theta_1 f)_{y_1}u.$$
On the other hand,
$$dJ(\Omega;V)=\lim_{\varepsilon\to 0}\left(-\frac{1}{2}\int_{\Omega_\varepsilon}\Big(\theta_{1y_1}\big((u_{y_1})^2-(u_{y_2})^2\big)+2\theta_{1y_2}u_{y_1}u_{y_2}\Big)\right)-\int_\Omega(\theta_1 f)_{y_1}u.$$
where Ω_ε is the subset of Ω defined by $r>\varepsilon$. Let γ_ε be the curve given by $r=\varepsilon$ and $0<\theta<2\pi$.

Let us introduce the notation
$$A_\varepsilon=-\frac{1}{2}\int_{\Omega_\varepsilon}\Big(\theta_{1y_1}\big((u_{y_1})^2-(u_{y_2})^2\big)+2\theta_{1y_2}u_{y_1}u_{y_2}\Big). \tag{46}$$

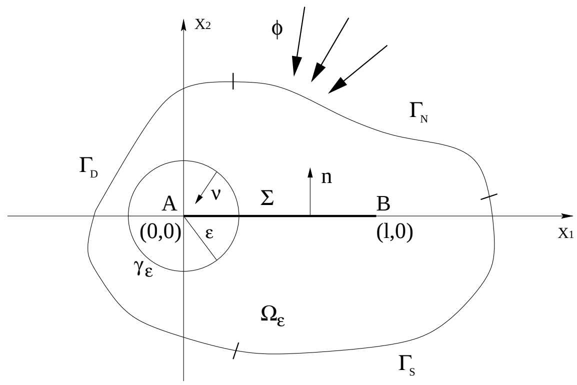

Figure 6: Domain Ω_ε

By integration by parts, we have

$$A_\varepsilon = \int_{\Omega_\varepsilon} \theta_1 \left(\frac{1}{2} \big((u_{y_1})^2 - (u_{y_2})^2 \big)_{y_1} + (u_{y_1} u_{y_2})_{y_2} \right) dy + \int_{\gamma_\varepsilon} \theta_1 \left(\frac{1}{2} \cos\theta \big((u_{y_1})^2 - (u_{y_2})^2 \big) + \sin\theta u_{y_1} u_{y_2} \right) d\sigma.$$

Using the identity for the solution u,

$$\frac{1}{2} \big((u_{y_1})^2 - (u_{y_2})^2 \big)_{y_1} + (u_{y_1} u_{y_2})_{y_2} = u_{y_1} (u_{y_1 y_1} + u_{y_2 y_2}) = u_{y_1} \Delta u = -f u_{y_1},$$

we obtain the formula

$$A_\varepsilon = -\int_{\Omega_\varepsilon} \theta_1 f u_{y_1} dy + \int_{\gamma_\varepsilon} \theta_1 \left(\frac{1}{2} \cos\theta \big((u_{y_1})^2 - (u_{y_2})^2 \big) + \sin\theta u_{y_1} u_{y_2} \right) d\sigma. \tag{47}$$

Moreover, we have

$$\int_{\Omega_\varepsilon} \theta_1 f u_{y_1} dx \longrightarrow \int_{\Omega} \theta_1 f u_{y_1} dx \quad \text{as } \varepsilon \to 0^+.$$

Let

$$B_\varepsilon = \int_{\gamma_\varepsilon} \theta_1 \left(\frac{1}{2} \cos\theta \big((u_{y_1})^2 - (u_{y_2})^2 \big) + \sin\theta u_{y_1} u_{y_2} \right) d\sigma.$$

Moreover, we know that

$$u = u^R + cS, \tag{48}$$

where $S = \sqrt{r}\cos(\frac{\theta}{2})$ and $u^R \in H^2(U)$ (U is a small neighbourhood of A in Ω such that $\overline{U} \cap \Gamma = \emptyset$) and c denotes the coefficient of singularity for the solution to the variational inequality (25). For ε small enough, since $\theta_1 \equiv -1$ on γ_ϵ and taking into account the decomposition (48),

$$\begin{aligned} B_\varepsilon &= c^2 \int_0^{2\pi} \varepsilon \left(\frac{1}{2} \cos\theta \big((S_{y_1})^2 - (S_{y_2})^2 \big) + \sin\theta (S_{y_1} S_{y_2}) \right) d\theta \\ &\quad + c \int_0^{2\pi} \varepsilon \big(\cos\theta (u^R_{y_1} S_{y_1} - u^R_{y_2} S_{y_2}) + \sin\theta (u^R_{y_1} S_{y_2} + u^R_{y_2} S_{y_1}) \big) d\theta \\ &\quad + \int_0^{2\pi} \varepsilon \left(\frac{1}{2} \cos\theta \big((u^R_{y_1})^2 - (u^R_{y_2})^2 \big) + \sin\theta (u^R_{y_1} u^R_{y_2}) \right) d\theta \\ &= B^{(1)}_\varepsilon + B^{(2)}_\varepsilon + B^{(3)}_\varepsilon. \end{aligned}$$

The form of singular functions is known in this case,

$$S_{y_1} = \frac{1}{2\sqrt{r}} \cos\left(\frac{\theta}{2}\right) \text{ and } S_{y_2} = \frac{1}{2\sqrt{r}} \sin\left(\frac{\theta}{2}\right).$$

The first integral in B_ε takes the form

$$\begin{aligned} B_\varepsilon^{(1)} &= c^2 \int_0^{2\pi} \left(\frac{\cos\theta}{2} \left(\frac{1}{4}\cos^2\left(\frac{\theta}{2}\right) - \frac{1}{4}\sin^2\left(\frac{\theta}{2}\right) \right) + \frac{\sin\theta}{4} \cos\left(\frac{\theta}{2}\right) \sin\left(\frac{\theta}{2}\right) \right) d\theta \\ &= \frac{c^2}{8} \int_0^{2\pi} (\cos^2\theta + \sin^2\theta) d\theta = \frac{2\pi c^2}{8} = \frac{\pi c^2}{4}. \end{aligned}$$

We have

$$B_\varepsilon^{(1)} = \frac{\pi c^2}{4}. \tag{49}$$

It is not difficult to see that $B_\varepsilon^{(2)} \longrightarrow 0$ and $B_\varepsilon^{(3)} \longrightarrow 0$ as $\varepsilon \to 0^+$ $\big($in fact, we have the estimates $B_\varepsilon^{(2)} = O(\sqrt{\varepsilon}),\ B_\varepsilon^{(3)} = O(\varepsilon)\big)$. Thus

$$B_\varepsilon \longrightarrow \frac{\pi c^2}{4},\ A_\varepsilon \longrightarrow -\int_\Omega \theta_1 f u_{y_1} dy + \frac{\pi c^2}{4} \quad \text{as } \varepsilon \to 0^+,$$

hence

$$dJ(\Omega; V) = -\int_\Omega \big((\theta_1 f)_{y_1} u\big) dy - \int_\Omega \theta_1 f u_{y_1} dy + \frac{\pi c^2}{4} = -\lim_{\varepsilon \to 0^+} \left(\int_{\Omega_\epsilon} \big((\theta_1 f)_{y_1} u + \theta_1 f u_{y_1}\big) dy \right) + \frac{\pi c^2}{4}$$

and finally

$$dJ(\Omega; V) = \frac{\pi c^2}{4} = \alpha_A. \tag{50}$$

We have identified the coefficent $\alpha_A = \dfrac{\pi c^2}{4}$ in the expression of the Eulerian semiderivative $dJ(\Omega; V)$ given by the structure theorem. □

4 The Griffith formula for optimal control problems

We apply the structure theorem in the case of cost functionals for control problems. The optimal control problem considered in this section is defined for the elliptic equation modelling the deflection of an elastic membrane with crack.

Let us consider the domain D in $\mathbb{R}^2$ with smooth boundary Γ. Let Σ_l be the set defined by $\big\{(y_1, y_2) \mid 0 < y_1 < l,\ y_2 = 0\big\}$, A and B denote its tips. We assume this set belongs to the domain D for $l > 0$ small enough. The domain with the crack is denoted by $\Omega = D \setminus \overline{\Sigma_l}$. Let K be an open subset of D (also with smooth boundary) and moreover we assume that $\overline{K} \cap \overline{\Sigma_l} = \emptyset$. The state equation for the control problem is of the form

$$\left\{ \begin{aligned} -\Delta q &= u\chi_K && \text{in} && \Omega, \\ q &= 0 && \text{on} && \Gamma = \partial D, \\ \frac{\partial q}{\partial n} &= 0 && \text{on} && {\Sigma_l}^{\pm}, \end{aligned} \right. \tag{51}$$

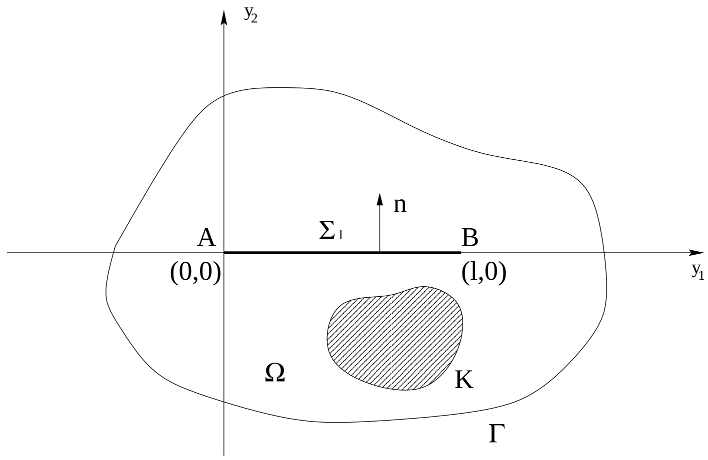

Figure 7: Domain Ω with the crack Σ_l

where χ_K denotes the characteristic function of K.
For given $u \in L^2(K)$, $q = q(u)$ represents the deflection of an elastic membrane loaded by the vertical force u concentrated on K. For the system (51), we define the cost functional

$$I(u) = \frac{1}{2} \int_K [(q - q_d)^2 + \alpha u^2] d\Omega, \tag{52}$$

which is minimized over the space of controls $u \in L^2(K)$, $\alpha > 0$, and $q_d \in L^2(K)$ is a given function. The minimization of the functional (52) with respect to u means approximation of a given function q_d in the region K by the deflection of an elastic membrane, using the smallest possible load u applied in K. The minimal value of the cost functional for this control problem defines the shape functional, depending on the geometrical domain Ω,

$$J(\Omega) = \min_{u \in L^2(K)} I(u).$$

Variation of the state $q'(v)$, corresponding to the variation v of the control

$$q(u + sv) = q(u) + sq'(v), \tag{53}$$

satisfies the equation

$$\begin{cases} -\Delta q' & = & v\chi_K & \text{in } \Omega, \\ q' & = & 0 & \text{on } \Gamma, \\ \dfrac{\partial q'}{\partial n} & = & 0 & \text{on } {\Sigma_l}^{\pm}, \end{cases} \tag{54}$$

and, according to (54), the variation $dI(u; v)$ of the cost functional is given by

$$dI(u; v) = \int_K [(q(u) - q_d) q'(v) + \alpha uv] d\Omega. \tag{55}$$

Let us introduce the adjoint state p, which is defined by the following equation

$$\left\{\begin{array}{rcll} -\Delta p &=& (q-q_d)\chi_K & \text{in } \Omega, \\ p &=& 0 & \text{on } \Gamma, \\ \dfrac{\partial p}{\partial n} &=& 0 & \text{on } \Sigma_l{}^{\pm}. \end{array}\right. \tag{56}$$

The adjoint state p given by (56) allows us to have another expression for (55)

$$\begin{aligned} dI(u;v) &= \int_K [(q(u)-q_d)q'(v)+\alpha uv]d\Omega \\ &= \int_K [\langle \nabla p, \nabla q'\rangle + \alpha uv]d\Omega, \end{aligned}$$

and using the equation (54) satisfied by q', we obtain

$$dI(u;v) = \int_K [p+\alpha u]v d\Omega.$$

Thus the stationarity condition

$$dI(u;v) = 0, \quad \forall v \in L^2(K)$$

leads to the following equality

$$u(y) = -\frac{1}{\alpha}p(y), \text{ a.e. in } K. \tag{57}$$

And consequently, by using (57), the minimal value of the cost functional for the control problem takes the form

$$J(\Omega) = \frac{1}{2}\int_K [(q-q_d)^2 + \frac{1}{\alpha}p^2]d\Omega, \tag{58}$$

where p, q are given as a solution of the coupled system of equations:

$$\left\{\begin{array}{rcll} -\Delta q &=& -\dfrac{1}{\alpha}p\chi_K & \text{in } \Omega, \\ -\Delta p &=& (q-q_d)\chi_K & \text{in } \Omega, \\ q &=& 0 & \text{on } \Gamma, \\ p &=& 0 & \text{on } \Gamma, \\ \dfrac{\partial q}{\partial n} &=& 0 & \text{on } \Sigma_l{}^{\pm}, \\ \dfrac{\partial p}{\partial n} &=& 0 & \text{on } \Sigma_l{}^{\pm}. \end{array}\right. \tag{59}$$

Let us consider the perturbations of Ω by the vector field $V = (\theta_1, \theta_2)$ with $\theta_1,\ \theta_2 \in C_0^\infty(D) = \mathcal{D}(D)$. We use the same notations as in section 3.1. Moreover we assume that $\overline{K} \cap \mathrm{supp}\{V\} = \emptyset$. For $\delta > 0$, the minimization problem is defined in Ω_δ, with the cost functional

$$J(\Omega_\delta) = \frac{1}{2}\int_K [(q_\delta - q_d)^2 + \frac{1}{\alpha}p_\delta^2]d\Omega, \tag{60}$$

where p_δ, q_δ are the solutions of the following coupled equations (after the change of variables in order to transport the problem in Ω)

$$\left\{\begin{array}{lcll} \displaystyle\int_\Omega \langle C_\delta \cdot \nabla q_\delta, \nabla\varphi\rangle d\Omega & = & \displaystyle -\frac{1}{\alpha}\int_K p_\delta \varphi d\Omega, & \forall\varphi \in H^1_\Gamma(\Omega), \\ \displaystyle\int_\Omega \langle C_\delta \cdot \nabla p_\delta, \nabla\psi\rangle d\Omega & = & \displaystyle\int_K (q_\delta - q_d)\psi d\Omega, & \forall\psi \in H^1_\Gamma(\Omega), \end{array}\right. \tag{61}$$

where $C_\delta = \dfrac{1}{q_\delta} A_\delta^T \cdot A_\delta$ and $A_\delta,\ q_\delta$ take the following form

$$A_\delta = \begin{pmatrix} 1 - \delta\theta_{1,y_1} & -\delta\theta_{2,y_1} \\ -\delta\theta_{1,y_2} & 1 - \delta\theta_{2,y_2} \end{pmatrix}.$$

and

$$q_\delta = 1 - \delta \mathrm{div} V + \delta^2 \det(DV).$$

Then we have the following result:

Theorem 3 (Griffith formula) *We have the Griffith formula*

$$\frac{dJ(\Omega_\delta)}{d\delta}|_{\delta=0} = -\frac{\pi}{2}(c_p c_\eta + c_q c_\xi)$$

where c_p, c_η, c_q, c_ξ are the coefficients of singularity of solutions to the systems (59),(65).

Proof: Applying the implicite functions theorem gives us the existence of the material derivatives $\dot p$, $\dot q$. Moreover, we obtain the variational equalities satisfied by $\dot p$ and $\dot q$:

$$\int_\Omega \langle C' \cdot \nabla q, \nabla\varphi\rangle d\Omega + \int_\Omega \langle \nabla \dot q, \nabla\varphi\rangle d\Omega = -\frac{1}{\alpha}\int_K \dot p \varphi d\Omega, \quad \forall\varphi \in H^1_\Gamma(\Omega), \tag{62}$$

$$\int_\Omega \langle C' \cdot \nabla p, \nabla\psi\rangle d\Omega + \int_\Omega \langle \nabla \dot p, \nabla\psi\rangle d\Omega = \int_K \dot q \psi d\Omega, \quad \forall\psi \in H^1_\Gamma(\Omega). \tag{63}$$

On the other hand, the previous results show that the cost functional is shape differentiable with the formula

$$dJ(\Omega; V) = \int_K [(q - q_d)\dot q + \frac{1}{\alpha} p \dot p] d\Omega. \tag{64}$$

In consequence, we can apply the structure theorem, which leads to the following formula for the derivative (64)

$$dJ(\Omega; V) = \alpha_A (V.\tau)(A) + \alpha_B (V.\tau)(B) + \phi(V.n),$$

where $\phi \in (C^1(\overline{\Sigma_l}))'$, $\alpha_A,\ \alpha_B \in \mathbb{R}$.
Now, our main is specifying the form of the coefficient α_A. In order to find the coefficient α_A, we can consider the variations of Ω such that the crack's length changes while the rest of the

boundary does not move. In addition, we assume the crack grows without changing direction that's why the vector field takes the form

$$V(y) = (\theta_1(y), 0),$$

where θ_1 is supported in D, $\overline{K} \cap \text{supp}\{\theta_1\} = \emptyset$ and $\theta_1(y) = -1$ in the vicinity of the origin.
In order to express the Eulerian semiderivative $dJ(\Omega; V)$ without the material derivatives $\dot{p}$ and $\dot{q}$, we have to introduce the second level adjoint variables ξ, η (see [16]), defined by the following equations:

$$\left\{\begin{array}{rcll} -\Delta\xi - \eta\chi_K &=& (q-q_d)\chi_K & \text{in } \Omega, \\ -\Delta\eta + \dfrac{1}{\alpha}\xi\chi_K &=& \dfrac{1}{\alpha}p\chi_K & \text{in } \Omega, \\ \xi &=& 0 & \text{on } \Gamma, \\ \eta &=& 0 & \text{on } \Gamma, \\ \dfrac{\partial\xi}{\partial n} &=& 0 & \text{on } \Sigma^\pm \\ \dfrac{\partial\eta}{\partial n} &=& 0 & \text{on } \Sigma^\pm, \end{array}\right. \tag{65}$$

or, in the weak form:

$$\int_\Omega \langle\nabla\xi, \nabla\varphi\rangle d\Omega - \int_K \eta\varphi d\Omega = \int_K (q-q_d)\varphi d\Omega, \quad \forall\varphi \in H^1_\Gamma(\Omega), \tag{66}$$

$$\int_\Omega \langle\nabla\eta, \nabla\psi\rangle d\Omega + \frac{1}{\alpha}\int_K \xi\psi d\Omega = \frac{1}{\alpha}\int_K p\psi d\Omega, \quad \forall\psi \in H^1_\Gamma(\Omega). \tag{67}$$

Applying (66) with $\varphi = \dot{q}$ leads to

$$\int_\Omega \langle\nabla\xi, \nabla\dot{q}\rangle d\Omega - \int_K \eta\dot{q} d\Omega = \int_K (q-q_d)\dot{q} d\Omega \tag{68}$$

and taking $\psi = \dot{p}$ in (67) gives us

$$\int_\Omega \langle\nabla\eta, \nabla\dot{p}\rangle d\Omega + \frac{1}{\alpha}\int_K \xi\dot{p} d\Omega = \frac{1}{\alpha}\int_K p\dot{p} d\Omega. \tag{69}$$

Moreover, we can substitute $\varphi = \xi$ and $\psi = \eta$ in the variational equalities (62),(63) respectively and we obtain

$$\int_\Omega \langle C' \cdot \nabla q, \nabla\xi\rangle d\Omega + \int_\Omega \langle\nabla\dot{q}, \nabla\xi\rangle d\Omega = -\frac{1}{\alpha}\int_K \dot{p}\xi d\Omega, \tag{70}$$

and

$$\int_\Omega \langle C' \cdot \nabla p, \nabla\eta\rangle d\Omega + \int_\Omega \langle\nabla\dot{p}, \nabla\eta\rangle d\Omega = \int_K \dot{q}\eta d\Omega. \tag{71}$$

Thus by combining the equalities (68),(69),(70),(71), it follows that

$$
\begin{aligned}
dJ(\Omega;V) &= \int_\Omega \langle \nabla\xi, \nabla\dot{q}\rangle d\Omega - \int_K \eta\dot{q} d\Omega + \int_\Omega \langle \nabla\eta, \nabla\dot{p}\rangle d\Omega + \frac{1}{\alpha}\int_K \xi\dot{p} d\Omega \\
&= -\int_\Omega \langle C'\cdot\nabla q, \nabla\xi\rangle d\Omega - \int_\Omega \langle C'\cdot\nabla p, \nabla\eta\rangle d\Omega \\
&= -\lim_{\varepsilon\to 0^+}\left(\int_{\Omega_\varepsilon} \langle C'\cdot\nabla q, \nabla\xi\rangle d\Omega + \int_{\Omega_\varepsilon} \langle C'\cdot\nabla p, \nabla\eta\rangle d\Omega\right)
\end{aligned}
$$

where Ω_ε is the subset of Ω defined by $r > \varepsilon$. Let γ_ε be the curve given by $r = \varepsilon$ and $0 < \theta < 2\pi$. Let us introduce the notation

$$
B_\varepsilon = \int_{\Omega_\varepsilon} \langle C'\cdot\nabla q, \nabla\xi\rangle d\Omega + \int_{\Omega_\varepsilon} \langle C'\cdot\nabla p, \nabla\eta\rangle d\Omega.
$$

Moreover we have

$$
A_\delta = \begin{pmatrix} 1-\delta\theta_{1,y_1} & 0 \\ -\delta\theta_{1,y_2} & 1 \end{pmatrix}, \; q_\delta = 1-\delta\theta_{1,y_1}
$$

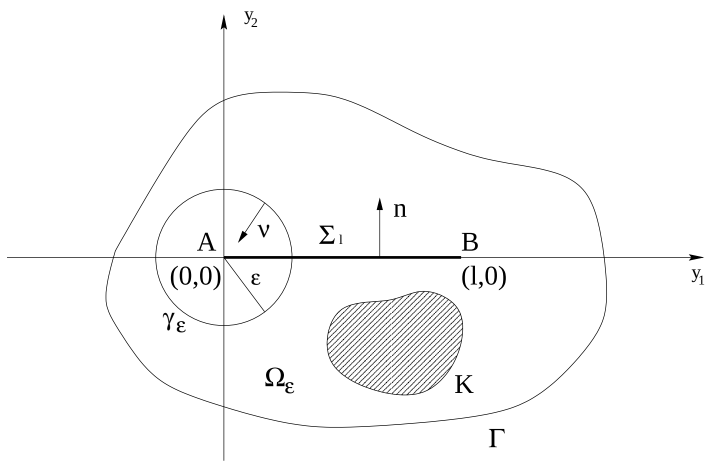

Figure 8: Domain Ω_ε with the crack Σ_l

which leads to

$$
C_\delta = \frac{1}{1-\delta\theta_{1,y_1}}\begin{pmatrix} (1-\delta\theta_{1,y_1})^2 + \delta^2\theta_{1,y_2}^2 & -\delta\theta_{1,y_2} \\ -\delta\theta_{1,y_2} & 1 \end{pmatrix},
$$

and finally we obtain

$$
C' = \frac{dC_\delta}{d\delta}|_{\delta=0} = \begin{pmatrix} -\theta_{1,y_1} & -\theta_{1,y_2} \\ -\theta_{1,y_2} & \theta_{1,y_1} \end{pmatrix}.
$$

By using this relation, it follows that

$$\begin{aligned} B_\varepsilon &= \int_{\Omega_\varepsilon} \theta_{1,y_1}(-q_{y_1}\xi_{y_1} + q_{y_2}\xi_{y_2} - p_{y_1}\eta_{y_1} + p_{y_2}\eta_{y_2})d\Omega \\ &+ \int_{\Omega_\varepsilon} \theta_{1,y_2}(-q_{y_2}\xi_{y_1} - q_{y_1}\xi_{y_2} - p_{y_2}\eta_{y_1} - p_{y_1}\eta_{y_2})d\Omega. \end{aligned} \tag{72}$$

By integrating by parts in (72), we have

$$\begin{aligned} B_\varepsilon &= \int_{\gamma_\varepsilon} \theta_1(-q_{y_1}\xi_{y_1} + q_{y_2}\xi_{y_2} - p_{y_1}\eta_{y_1} + p_{y_2}\eta_{y_2})\nu_1 d\sigma \\ &+ \int_{\gamma_\varepsilon} \theta_1(-q_{y_2}\xi_{y_1} - q_{y_1}\xi_{y_2} - p_{y_2}\eta_{y_1} - p_{y_1}\eta_{y_2})\nu_2 d\sigma \\ &+ \int_{\Omega_\varepsilon} \theta_1(\xi_{y_1}\Delta q + q_{y_1}\Delta\xi + \eta_{y_1}\Delta p + p_{y_1}\Delta\eta)d\Omega. \end{aligned}$$

But, for ε small enough, $K \subset \Omega_\varepsilon$ and moreover $\overline{K} \cap \mathrm{supp}\{\theta_1\} = \emptyset$ that's why

$$\Delta q = \Delta\xi = \Delta p = \Delta\eta = 0 \quad \text{on } \Omega_\varepsilon$$

and in consequence

$$\begin{aligned} B_\varepsilon &= \int_{\gamma_\varepsilon} \theta_1(-q_{y_1}\xi_{y_1} + q_{y_2}\xi_{y_2} - p_{y_1}\eta_{y_1} + p_{y_2}\eta_{y_2})\nu_1 d\sigma \\ &+ \int_{\gamma_\varepsilon} \theta_1(-q_{y_2}\xi_{y_1} - q_{y_1}\xi_{y_2} - p_{y_2}\eta_{y_1} - p_{y_1}\eta_{y_2})\nu_2 d\sigma. \end{aligned} \tag{73}$$

For ε small enough, $\theta_1 \equiv -1$ on γ_ε and in view of (73)

$$\begin{aligned} B_\varepsilon &= \int_{\gamma_\varepsilon} (q_{y_1}\xi_{y_1} - q_{y_2}\xi_{y_2} + p_{y_1}\eta_{y_1} - p_{y_2}\eta_{y_2})\nu_1 d\sigma \\ &+ \int_{\gamma_\varepsilon} (q_{y_2}\xi_{y_1} + q_{y_1}\xi_{y_2} + p_{y_2}\eta_{y_1} + p_{y_1}\eta_{y_2})\nu_2 d\sigma. \end{aligned} \tag{74}$$

Moreover according to [10] we know that

$$\begin{cases} p = p^R + c_p S, \\ q = q^R + c_q S, \\ \eta = \eta^R + c_\eta S, \\ \xi = \xi^R + c_\xi S, \end{cases} \tag{75}$$

where $S = \sqrt{r}\cos(\frac{\theta}{2})$ is the singular function, $p^R,\ q^R,\ \eta^R,\ \xi^R \in H^2(U)$ (U being any neighbourhood of $(0,0)$ in Ω such that $\overline{U} \cap \Gamma = \emptyset$) and $c_p,\ c_q,\ c_\eta,\ c_\xi$ denote the coefficients of singularity of solutions to (59), (65), respectively.
Taking into account the decomposition (75) and developping in (74) we have

$$B_\varepsilon = B_\varepsilon^{(1)} + B_\varepsilon^{(2)} + B_\varepsilon^{(3)}$$

where $B_\varepsilon^{(1)}, B_\varepsilon^{(2)}, B_\varepsilon^{(3)}$ are defined by

$$B_\varepsilon^{(1)} = (c_p c_\eta + c_q c_\xi) \int_{\gamma_\varepsilon} [((S_{y_1})^2 - (S_{y_2})^2)\nu_1 + 2\nu_2 S_{y_1} S_{y_2}] d\sigma,$$

$$\begin{aligned} B_\varepsilon^{(2)} &= \int_{\gamma_\varepsilon} (c_\xi q^R_{y_1} + c_q \xi^R_{y_1} + c_\eta p^R_{y_1} + c_p \eta^R_{y_1})(\nu_1 S_{y_1} + \nu_2 S_{y_2}) d\sigma \\ &\quad + \int_{\gamma_\varepsilon} (c_\xi q^R_{y_2} + c_q \xi^R_{y_2} + c_\eta p^R_{y_2} + c_p \eta^R_{y_2})(\nu_2 S_{y_1} - \nu_1 S_{y_2}) d\sigma, \end{aligned}$$

$$\begin{aligned} B_\varepsilon^{(3)} &= \int_{\gamma_\varepsilon} \nu_1 (q^R_{y_1}\xi^R_{y_1} - q^R_{y_2}\xi^R_{y_2} + p^R_{y_1}\eta^R_{y_1} - p^R_{y_2}\eta^R_{y_2}) d\sigma \\ &\quad + \int_{\gamma_\varepsilon} \nu_2 (q^R_{y_2}\xi^R_{y_1} + q^R_{y_1}\xi^R_{y_2} + p^R_{y_2}\eta^R_{y_1} + p^R_{y_1}\eta^R_{y_2}) d\sigma. \end{aligned}$$

Moreover the form of singular function is known in this case,

$$S_{y_1} = \frac{1}{2\sqrt{r}} \cos\left(\frac{\theta}{2}\right) \text{ and } S_{y_2} = \frac{1}{2\sqrt{r}} \sin\left(\frac{\theta}{2}\right)$$

and after using polar coordinates, we have

$$\begin{aligned} B_\varepsilon^{(1)} &= (c_p c_\eta + c_q c_\xi) \int_0^{2\pi} \left(\frac{1}{4}\cos^2\left(\frac{\theta}{2}\right) - \frac{1}{4}\sin^2\left(\frac{\theta}{2}\right)\right) \cos\theta d\theta \\ &\quad + 2(c_p c_\eta + c_q c_\xi) \int_0^{2\pi} \frac{1}{4} \cos\left(\frac{\theta}{2}\right) \sin\left(\frac{\theta}{2}\right) \sin\theta d\theta \\ &= \frac{1}{4}(c_p c_\eta + c_q c_\xi) \int_0^{2\pi} (\cos^2\theta + \sin^2\theta) d\theta \\ &= \frac{\pi}{2}(c_p c_\eta + c_q c_\xi) \end{aligned}$$

and it is not difficult to see that $B_\varepsilon^{(2)} \to 0$ and $B_\varepsilon^{(3)} \to 0$ as $\varepsilon \to 0^+$ (in fact we have the estimations $B_\varepsilon^{(2)} = O(\sqrt{\varepsilon}),\ B_\varepsilon^{(3)} = O(\varepsilon)$). And finally, we have

$$dJ(\Omega; V) = -\lim_{\varepsilon \to 0^+} B_\varepsilon = -\frac{\pi}{2}(c_p c_\eta + c_q c_\xi).$$

We have identified the coefficient $\alpha_A = -\frac{\pi}{2}(c_p c_\eta + c_q c_\xi)$ in the expression of the Eulerian semiderivative $dJ(\Omega; V)$ given by the structure theorem. □

5 Shape derivative of the first eigenvalue of the Laplacian in domains with cracks

For the convenience of the reader, the proof of Auchmuty's principle is given in section 5.2. We use this principle in order to derive the shape derivative of the first eigenvalue of the Laplacian in domains with cracks.

Let us introduce the problem and the notations.

5.1 Introduction and notations

Given:

(i) two Hilbert spaces V and H (over $\mathbb{R}$) with infinite dimensions and satisfying the following properties

$$V \subset H \text{ with continuous injection,} \tag{76}$$
$$V \text{ is dense in } H \tag{77}$$

(ii) a bilinear form $(u, v) \mapsto a(u, v)$ continuous on $V \times V$.

We will denote by $(\cdot, \cdot)$ the scalar product in H and by $|\cdot|$ the norm which corresponds. We will write $\|\cdot\|$ for the norm in V.
We consider the spectral problem: find $\lambda \in \mathbb{R}$ such that there exists a solution $u \in V$, $u \neq 0$, to the equation

$$\forall v \in V, \quad a(u, v) = \lambda(u, v). \tag{78}$$

In the remainder of this section, we assume $a(\cdot, \cdot)$ to be a symmetric bilinear form. If the bilinear form $a(\cdot, \cdot)$ is V-elliptic, we can define the operator $T \in \mathcal{L}(H; V)$ by

$$\forall v \in V, \quad a(Tf, v) = (f, v) \tag{79}$$

and (78) reads

$$u = \lambda T u. \tag{80}$$

In order to solve (80), we need some properties of T, given by the following theorem (see [17]).

Theorem 4 *We assume that the canonic injection $V \hookrightarrow H$ is compact and that the bilinear form $a(\cdot, \cdot)$ is symmetric and V-elliptic. Then, under the above assumptions, the eigenvalues of the problem (78) represent an increasing sequence which converges to $+\infty$,*

$$0 < \lambda_1 \le \lambda_2 \le \ldots \le \lambda_m \le \ldots, \tag{81}$$

and moreover, there exists an orthonormal hilbertian basis of V of eigenvectors v_m such that

$$T v_m = \mu_m v_m \quad \textit{with} \quad \mu_m = \frac{1}{\lambda_m}. \tag{82}$$

After recalling these results, we can give a very useful characterization of the eigenvalue λ_1.
It is convenient to introduce the following functional defined by

$$\forall v \in V, \quad G(v) = \frac{1}{2} a(v, v) - |v|.$$

Let z_m the element of V given by

$$z_m = \frac{v_m}{\sqrt{\lambda_m}}, \quad \forall m = 1, 2, \ldots$$

Consequently

$$a(z_m, z_n) = a\left(\frac{v_m}{\sqrt{\lambda_m}}, \frac{v_n}{\sqrt{\lambda_n}}\right) = \frac{1}{\sqrt{\lambda_m \lambda_n}} a(v_m, v_n) = \frac{\delta_{mn}}{\sqrt{\lambda_m \lambda_n}},$$

where δ_{mn} denotes the Kronecker's symbol, that's why

$$\text{if } m \neq n, \quad a(z_m, z_n) = 0, \quad \text{and} \quad a(z_m, z_m) = \frac{1}{\lambda_m}. \tag{83}$$

Moreover

$$(z_m, z_n) = \left(\frac{v_m}{\sqrt{\lambda_m}}, \frac{v_n}{\sqrt{\lambda_n}}\right) = \frac{1}{\sqrt{\lambda_m \lambda_n}} (v_m, v_n),$$

but we know that

$$a(v_m, v) = \lambda_m (v_m, v), \quad \forall v \in V$$

and substituting $v = v_n$ in the latter equality leads to

$$\lambda_m (v_m, v_n) = a(v_m, v_n) = \delta_{mn},$$

which implies that

$$(z_m, z_n) = \frac{\delta_{mn}}{\lambda_m \sqrt{\lambda_m \lambda_n}},$$

and in consequence

$$\text{if } m \neq n, \quad (z_m, z_n) = 0, \quad \text{and} \quad |z_m| = \frac{1}{\lambda_m}. \ \square \tag{84}$$

5.2 Auchmuty's principle

The relations (83),(84) are used in order to prove the following theorem which gives a characterization of the first eigenvalue different compared to the classical minimization of Rayleigh's ratio.

Theorem 5 (Auchmuty's principle) *The first eigenvalue λ_1 is given by*

$$-\frac{1}{2\lambda_1} = \min_{v \in V} G(v).$$

Proof: For the convenience of reader the proof is given. First, let us calculate $G(z_1)$,

$$G(z_1) = \frac{1}{2} a(z_1, z_1) - |z_1| = \frac{1}{2\lambda_1} - \frac{1}{\lambda_1} = -\frac{1}{2\lambda_1}.$$

The second step consists in showing the following inequality

$$\forall v \in V, \quad G(v) \geq -\frac{1}{2\lambda_1}. \tag{85}$$

Let $v = \sum_{i\geq 1} \alpha_i z_i$ be an element of V $(\alpha_i \in \mathbb{R},\ \forall i \in \mathbb{N}^*)$. In order to prove the inequality (85), we have to evaluate $G(v)$:

$$a(v,v) = a\Big(\sum_{i\geq 1}\alpha_i z_i, \sum_{j\geq 1}\alpha_j z_j\Big) = \sum_{i,j\geq 1}\alpha_i\alpha_j a(z_i,z_j) = \sum_{i\geq 1}\frac{\alpha_i^2}{\lambda_i},$$

$$|v|^2 = (v,v) = \Big(\sum_{i\geq 1}\alpha_i z_i, \sum_{j\geq 1}\alpha_j z_j\Big) = \sum_{i,j\geq 1}\alpha_i\alpha_j (z_i,z_j) = \sum_{i\geq 1}\frac{\alpha_i^2}{\lambda_i^2},$$

and in consequence

$$\begin{aligned} G(v) &= \frac{1}{2}a(v,v) - |v| \\ &= \frac{1}{2}\sum_{i\geq 1}\frac{\alpha_i^2}{\lambda_i} - \sqrt{\sum_{i\geq 1}\frac{\alpha_i^2}{\lambda_i^2}}. \end{aligned} \tag{86}$$

Let us introduce $y_i \in \mathbb{R}$ defined by

$$\forall i \geq 1, \quad \alpha_i = \lambda_i y_i. \tag{87}$$

Combining (86),(87) leads to

$$G(v) = \frac{1}{2}\sum_{i\geq 1}\lambda_i y_i^2 - \sqrt{\sum_{i\geq 1} y_i^2}.$$

Moreover, according to theorem 4, we have

$$\forall i \in \mathbb{N}^*, \quad \lambda_i \geq \lambda_1,$$

and consequently

$$\begin{aligned} G(v) &\geq \frac{\lambda_1}{2}\sum_{i\geq 1} y_i^2 - \sqrt{\sum_{i\geq 1} y_i^2} \\ &\geq \frac{\lambda_1}{2}S^2 - S \end{aligned}$$

by taking $S = \sqrt{\sum_{i\geq 1} y_i^2}$.

This inequality uses the properties of the following function

$$\begin{aligned} g:\ [0,+\infty[&\rightarrow \mathbb{R} \\ x &\mapsto \frac{\lambda_1}{2}x^2 - x. \end{aligned}$$

Studying g shows this function has a minimum which is $-\dfrac{1}{2\lambda_1}$ (for $x = \frac{1}{\lambda_1}$).
And finally, we obtain the inequality (85). $\square$

5.3 Shape derivative of the first eigenvalue of the Laplace operator

Now, we can obtain the form of the shape derivative of the first eigenvalue of the Laplace operator. Let $D \subset \mathbb{R}^2$ be a bounded domain with a smooth boundary Γ. Let Σ_l be the set defined by $\{(y_1, y_2) \mid 0 < y_1 < l,\ y_2 = 0\}$, A and B denote its tips. We assume Σ_l belongs to the domain D for $l > 0$ small enough. The domain with the crack is denoted by $\Omega = D \setminus \overline{\Sigma_l}$.

According to Auchmuty's principle, the smallest eigenvalue $\lambda(\Omega)$ of the Laplacian in Ω is given by

$$-\frac{1}{2\lambda(\Omega)} = \min_{\varphi \in H^1_\Gamma(\Omega)} G(\varphi),$$

with

$$G(\varphi) = \frac{1}{2}\int_\Omega |\nabla\varphi|^2 dy - \sqrt{\int_\Omega \varphi^2 dy}, \quad \forall \varphi \in H^1_\Gamma(\Omega).$$

In this case, we have

$$V = H^1_\Gamma(\Omega),\ H = L^2(\Omega)$$

and

$$a(\varphi, \psi) = \int_\Omega \langle \nabla\varphi, \nabla\psi \rangle dy, \quad \forall \varphi,\ \psi \in H^1_\Gamma(\Omega).$$

Let us introduce $\mu(\Omega)$ defined by

$$\mu(\Omega) = -\frac{1}{2\lambda(\Omega)}, \tag{88}$$

and consequently

$$\mu(\Omega) = \min_{\varphi \in H^1_\Gamma(\Omega)} G(\varphi).$$

Let us consider the perturbations of Ω by the vector field $V = (\theta_1, \theta_2)$ with $\theta_1,\ \theta_2 \in C_0^\infty(D) = \mathcal{D}(D)$. We refer to section 3.1 for details.
For $\delta > 0$, the minimization problem is defined in Ω_δ by

$$-\frac{1}{2\lambda(\Omega_\delta)} = \mu(\Omega_\delta) = \min_{\psi \in H^1_\Gamma(\Omega_\delta)} G_\delta(\psi), \tag{89}$$

where $\lambda(\Omega_\delta)$ denotes the smallest eigenvalue of the Laplacian in Ω_δ and

$$G_\delta(\psi) = \frac{1}{2}\int_{\Omega_\delta} |\nabla\psi|^2 dx - \sqrt{\int_{\Omega_\delta} \psi^2 dx}, \quad \forall \psi \in H^1_\Gamma(\Omega_\delta).$$

Moreover, we use the following notations

$$\lambda_\delta = \lambda(\Omega_\delta),\ \mu_\delta = \mu(\Omega_\delta),$$

and in particular, if $\delta = 0$,

$$\lambda_0 = \lambda(\Omega_0) = \lambda(\Omega),\ \mu_0 = \mu(\Omega_0) = \mu(\Omega).$$

Our aim is to find the directional derivative

$$d\mu(\Omega; V) = \lim_{\delta \downarrow 0} \frac{\mu(\Omega_\delta) - \mu(\Omega)}{\delta} = \lim_{\delta \downarrow 0} \frac{\mu_\delta - \mu_0}{\delta},$$

indeed, the directional differentiability of μ_δ at $\delta = 0$ is equivalent to the directional differentiability of λ_δ at $\delta = 0$ because $\lambda_\delta \neq 0$.
Denote
$$M(\Omega) = \left\{\phi \in H^1_\Gamma(\Omega) \mid \mu(\Omega) = G(\phi)\right\},$$
$M(\Omega)$ is the set of minimizers of G over $H^1_\Gamma(\Omega)$.
By changing the variables in (89), in order to transport the problem to Ω, we obtain

$$\mu_\delta = \min_{\psi \in H^1_\Gamma(\Omega_\delta)} G_\delta(\psi) = \min_{\varphi \in H^1_\Gamma(\Omega)} G(\delta, \varphi),$$

with
$$G(\delta, \varphi) = \frac{1}{2}\int_\Omega |A_\delta \cdot \nabla\varphi|^2 \frac{dy}{q_\delta} - \sqrt{\int_\Omega \varphi^2 \frac{dy}{q_\delta}}, \quad \forall \varphi \in H^1_\Gamma(\Omega),$$

and where $A_\delta = I + \delta B$, B is the matrix defined in theorem 2, $q_\delta = 1 - \delta \mathrm{div}V + \delta^2 \det(DV)$.
Denote
$$M_\delta = M(\Omega_\delta) = \left\{\phi \in H^1_\Gamma(\Omega) \mid \mu_\delta = G(\delta, \phi)\right\},$$
in particular we have $M_0 = M(\Omega)$.
Before showing the differentiability of μ_δ at $\delta = 0^+$, we prove a preliminary lemma.

Lemma 3 *Let $\{\delta_k\}_{k=1}^{+\infty}$ be a sequence such that $\delta_k \downarrow 0$ as $k \to +\infty$. Then for any sequence $\{z_k\}$ with $z_k \in M_{\delta_k}$, there exists a subsequence also denoted by $\{z_k\}$ and an element $z^* \in M_0$ such that*
$$z_k \to z^* \text{ in } H^1_\Gamma(\Omega) \quad \text{as } k \to +\infty.$$

Proof: The proof is divided into small steps.

Step 1: $\{z_k\}$ is bounded in $H^1_\Gamma(\Omega)$.
$z_k \in M_{\delta_k}$ *i.e.* z_k minimizes $G(\delta_k, \cdot)$ over $H^1_\Gamma(\Omega)$ and in consequence

$$G(\delta_k, z_k) \leq G(\delta_k, 0) = 0$$

which leads to

$$\frac{1}{2}\int_\Omega |A_{\delta_k} \cdot \nabla z_k|^2 \frac{dy}{q_{\delta_k}} - \sqrt{\int_\Omega z_k^2 \frac{dy}{q_{\delta_k}}} \leq 0 \tag{90}$$

and taking into account that for k large enough, we have the following estimations

$$\frac{1}{2} \leq q_{\delta_k} \leq \frac{3}{2} \quad \text{and} \quad |A_{\delta_k} \cdot \nabla z_k|^2 \geq \frac{1}{4}|\nabla z_k|^2,$$

the inequality (90) leads to

$$\frac{1}{12}\int_\Omega |\nabla z_k|^2 dy - \sqrt{2}\sqrt{\int_\Omega z_k^2 dy} \leq 0,$$

and by Poincaré's inequality there exists a constant C such that

$$\int_\Omega |\nabla z_k|^2 dy \le C.$$

$\{z_k\}$ is a bounded sequence of $H^1_\Gamma(\Omega)$ and moreover the injection $H^1_\Gamma(\Omega) \hookrightarrow L^2(\Omega)$ is compact, therefore there exists a subsequence, still denoted by $\{z_k\}$, such that

$$\begin{aligned} z_k &\rightharpoonup z^* \text{ weakly in } H^1_\Gamma(\Omega), \\ z_k &\to z^* \text{ strongly in } L^2(\Omega) \quad \text{as } k \to +\infty. \end{aligned}$$

Step 2: $z^* \in M_0$.
z_k minimizes $G(\delta_k, \cdot)$ over $H^1_\Gamma(\Omega)$, therefore,

$$G(\delta_k, z_k) \le G(\delta_k, \varphi), \quad \forall \varphi \in H^1_\Gamma(\Omega). \tag{91}$$

First, it is not difficult to see that for any fixed function φ,

$$G(\delta_k, \varphi) \to G(0, \varphi) \quad \text{as } k \to +\infty.$$

Moreover, by the sequential lower semi-continuity of the functional $\varphi \longmapsto \int_\Omega |\nabla \varphi|^2 dy$ for the weak topology of $H^1_\Gamma(\Omega)$ and the strong convergence of the sequence $z_k \to z^*$ in $L^2(\Omega)$, it follows that

$$\liminf_{k\to+\infty} G(\delta_k, z_k) \ge G(0, z^*).$$

Using the inequality (91), we have

$$G(0, z^*) \le \liminf_{k\to+\infty} G(\delta_k, z_k) \le \liminf_{k\to+\infty} G(\delta_k, \varphi) = \lim_{k\to+\infty} G(\delta_k, \varphi) = G(0, \varphi)$$

which implies that

$$G(0, z^*) \le G(0, \varphi), \quad \forall \varphi \in H^1_\Gamma(\Omega). \tag{92}$$

The inequality (92) means that $z^* \in M_0$.

Step 3: strong convergence $z_k \to z^*$ in $H^1_\Gamma(\Omega)$.
Showing this property is based on the stationarity conditions satisfied by z_k and z^*, respectively. Indeed, let $z_k \in M_{\delta_k}$ and $z* \in M_0$ which means that the following equations are satisfied

$$\int_\Omega \langle A_{\delta_k} \cdot \nabla z_k, A_{\delta_k} \cdot \nabla \varphi \rangle \frac{dy}{q_{\delta_k}} = \frac{\displaystyle\int_\Omega z_k \varphi \frac{dy}{q_{\delta_k}}}{\sqrt{\displaystyle\int_\Omega z_k^2 \frac{dy}{q_{\delta_k}}}}, \quad \forall \varphi \in H^1_\Gamma(\Omega), \tag{93}$$

and

$$\int_\Omega \langle \nabla z^*, \nabla \psi \rangle \, dy = \frac{\displaystyle\int_\Omega z^* \psi dy}{\sqrt{\displaystyle\int_\Omega {z^*}^2 dy}}, \quad \forall \psi \in H^1_\Gamma(\Omega). \tag{94}$$

By substituting $\varphi = z_k$ in (93) and $\psi = z^*$ in (94), we obtain

$$\int_\Omega |A_{\delta_k} \cdot \nabla z_k|^2 \frac{dy}{q_{\delta_k}} = \sqrt{\int_\Omega z_k^2 \frac{dy}{q_{\delta_k}}}, \tag{95}$$

$$\int_\Omega |\nabla z^*|^2 dy = \sqrt{\int_\Omega z^{*2} dy}. \tag{96}$$

Consequently, by using (95),(96),

$$\begin{aligned}
\int_\Omega |\nabla z_k|^2 dy &= \int_\Omega |\nabla z_k|^2 dy - \int_\Omega |A_{\delta_k} \cdot \nabla z_k|^2 \frac{dy}{q_{\delta_k}} + \int_\Omega |A_{\delta_k} \cdot \nabla z_k|^2 \frac{dy}{q_{\delta_k}} \\
&= \alpha_k + \int_\Omega |A_{\delta_k} \cdot \nabla z_k|^2 \frac{dy}{q_{\delta_k}} \\
&= \alpha_k + \sqrt{\int_\Omega z_k^2 \frac{dy}{q_{\delta_k}}} \\
&= \alpha_k + \sqrt{\int_\Omega z_k^2 \frac{dy}{q_{\delta_k}}} - \sqrt{\int_\Omega z^{*2} dy} + \sqrt{\int_\Omega z^{*2} dy} \\
&= \alpha_k + \beta_k + \sqrt{\int_\Omega z^{*2} dy} \\
&= \alpha_k + \beta_k + \int_\Omega |\nabla z^*|^2 dy,
\end{aligned}$$

where

$$\begin{aligned}
\alpha_k &= \int_\Omega |\nabla z_k|^2 dy - \int_\Omega |A_{\delta_k} \cdot \nabla z_k|^2 \frac{dy}{q_{\delta_k}}, \\
\beta_k &= \sqrt{\int_\Omega z_k^2 \frac{dy}{q_{\delta_k}}} - \sqrt{\int_\Omega z^{*2} dy}
\end{aligned}$$

and

$$\alpha_k,\ \beta_k \to 0 \quad \text{as } k \to +\infty.$$

We have shown that

$$\int_\Omega |\nabla z_k|^2 dy \to \int_\Omega |\nabla z^*|^2 dy \quad \text{as } k \to +\infty. \tag{97}$$

But, the convergence

$$z_k \rightharpoonup z^* \text{ in } H^1_\Gamma(\Omega) \quad \text{as } k \to +\infty$$

implies that

$$\nabla z_k \rightharpoonup \nabla z^* \text{ in } (L^2(\Omega))^2 \quad \text{as } k \to +\infty. \tag{98}$$

And finally, by using (97) and (98), it follows that

$$z_k \to z^* \text{ in } H^1_\Gamma(\Omega) \quad \text{as } k \to +\infty. \ \square$$

Remark 1 *$G(\cdot,\phi)$ is right-differentiable at $\delta = 0$ for any fixed $\phi \in H^1_\Gamma(\Omega)$. And moreover, we have the following formula*

$$\begin{aligned}
\frac{\partial G}{\partial \delta}(0,\phi) &= \lim_{s\downarrow 0} \frac{G(s,\phi)-G(0,\phi)}{s} \\
&= \frac{1}{2}\int_\Omega |\nabla\phi|^2 \mathrm{div} V dy + \int_\Omega \langle \nabla\phi, B\cdot\nabla\phi\rangle dy - \frac{\int_\Omega \phi^2 \mathrm{div} V dy}{2\|\phi\|_{L^2(\Omega)}}.
\end{aligned}$$

Now, we can prove the differentiability of μ_δ at $\delta = 0^+$.

$$\begin{aligned}
\mu_\delta - \mu_0 &= G(\delta,\varphi_\delta) - G(0,\varphi_0), \quad \forall \varphi_\delta \in M_\delta,\ \forall\varphi_0 \in M_0 \\
&\leq G(\delta,\varphi_0) - G(0,\varphi_0), \quad \forall\varphi_0 \in M_0,
\end{aligned}$$

and if $\delta > 0$,

$$\frac{\mu_\delta - \mu_0}{\delta} \leq \frac{G(\delta,\varphi_0)-G(0,\varphi_0)}{\delta}$$

which implies that

$$\begin{aligned}
\limsup_{\delta\downarrow 0} \frac{\mu_\delta-\mu_0}{\delta} &\leq \limsup_{\delta\downarrow 0} \frac{G(\delta,\varphi_0)-G(0,\varphi_0)}{\delta} \\
&\leq \lim_{\delta\downarrow 0} \frac{G(\delta,\varphi_0)-G(0,\varphi_0)}{\delta} \\
&\leq \frac{\partial G}{\partial\delta}(0,\varphi_0)
\end{aligned}$$

according to remark 1.
We have obtained the inequality

$$\limsup_{\delta\downarrow 0} \frac{\mu_\delta-\mu_0}{\delta} \leq \frac{\partial G}{\partial\delta}(0,\varphi_0), \quad \forall\varphi_0 \in M_0. \tag{99}$$

On the other hand

$$\begin{aligned}
\mu_\delta - \mu_0 &= G(\delta,\varphi_\delta) - G(0,\varphi_0), \quad \forall \varphi_\delta \in M_\delta,\ \forall\varphi_0 \in M_0 \\
&\geq G(\delta,\varphi_\delta) - G(0,\varphi_\delta), \quad \forall\varphi_\delta \in M_\delta.
\end{aligned}$$

By using Taylor's expansion, there exists s, $0 \leq s \leq \delta$, such that

$$\frac{G(\delta,\varphi_\delta)-G(0,\varphi_\delta)}{\delta} = \frac{\partial G}{\partial\delta}(s,\varphi_\delta),$$

and it follows that

$$\frac{\mu_\delta-\mu_0}{\delta} \geq \frac{\partial G}{\partial\delta}(s,\varphi_\delta). \tag{100}$$

For $\delta\downarrow 0$, $s\downarrow 0$ and in view of lemma 3, there exists $\varphi^* \in M_0$ such that

$$\varphi_\delta \to \varphi^* \text{ in } H^1_\Gamma(\Omega) \quad \text{with } \delta \to 0.$$

In consequence

$$\frac{\partial G}{\partial \delta}(s,\varphi_\delta) \to \frac{\partial G}{\partial \delta}(0,\varphi^*) \quad \text{as } \delta \to 0,$$

and passage to the limit in (100) gives us

$$\frac{\partial G}{\partial \delta}(0,\varphi^*) \le \liminf_{\delta\downarrow 0} \frac{\mu_\delta - \mu_0}{\delta}. \tag{101}$$

Finally, combining (99),(101) leads to

$$\frac{\partial G}{\partial \delta}(0,\varphi^*) \le \liminf_{\delta\downarrow 0} \frac{\mu_\delta - \mu_0}{\delta} \le \limsup_{\delta\downarrow 0} \frac{\mu_\delta - \mu_0}{\delta} \le \frac{\partial G}{\partial \delta}(0,\varphi_0), \quad \forall \varphi_0 \in M_0$$

with $\varphi^* \in M_0$.
We can conclude that μ_δ is differentiable at $\delta = 0^+$ with the following formula for the derivative

$$\begin{aligned} d\mu(\Omega;V) &= \min\left\{\frac{\partial G}{\partial \delta}(0,\phi) \mid \phi \in M_0\right\} \\ &= \frac{\partial G}{\partial \delta}(0,\varphi^*), \end{aligned}$$

where

$$\frac{\partial G}{\partial \delta}(0,\phi) = \frac{1}{2}\int_\Omega |\nabla\phi|^2 \mathrm{div} V dy + \int_\Omega \langle \nabla\phi, B\cdot\nabla\phi\rangle dy - \frac{\displaystyle\int_\Omega \phi^2 \mathrm{div} V dy}{2\|\phi\|_{L^2(\Omega)}}.$$

The differentiability of μ_δ at $\delta = 0$ leads to the differentiability of λ_δ at $\delta = 0$. Using (88),(89), we obtain the formula

$$\begin{aligned} d\lambda(\Omega;V) &= 2\lambda^2(\Omega) d\mu(\Omega;V) \\ &= 2\lambda^2(\Omega) \min\left\{\frac{\partial G}{\partial \delta}(0,\phi) \mid \phi \in M_0\right\} \\ &= 2\lambda^2(\Omega)\frac{\partial G}{\partial \delta}(0,\varphi^*). \end{aligned}$$

By applying the structure theorem, it follows that

$$d\lambda(\Omega;V) = \alpha_A(V.\tau)(A) + \alpha_B(V.\tau)(B) + \phi(V.n), \quad \forall V \in \mathcal{D}^k(D;\mathbb{R}^2)$$

where $\phi \in (C^k(\overline{\Sigma_l}))'$, $\alpha_A,\ \alpha_B \in \mathbb{R}$ and $k = 1$. We derive the explicite form of α_A.
If $V(y) = (\theta_1(y), 0)$ where θ_1 has the support in D and $\theta_1(y) = -1$ in the vicinity of the origin A, by using the same method as in previous sections, *i.e.* integrating on Ω_ε (where Ω_ε is the subset of Ω defined by $r > \varepsilon$) and passing to the limit as $\varepsilon \to 0^+$, we obtain

$$\begin{aligned} \alpha_A &= 2\lambda^2(\Omega)\frac{\pi c_{\varphi^*}^2}{4} \\ &= 2\lambda^2(\Omega)\min\left\{\frac{\pi c_\phi^2}{4} \mid \phi \in M_0\right\}, \end{aligned}$$

where c_{φ^*} and c_ϕ respectively denote the coefficients of singularity of functions φ^* and ϕ.
And in consequence

$$c_{\varphi^*}^2 = \min_{\phi\in M_0} c_\phi^2. \ \square$$

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