The Uncertain Volatility Model and American Options

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Abstract: We study standard option prices in the one-dimensional Avellaneda-Lyons model with unknown volatility and no interest rate. We show that the price is that of an american option in the Black-Scholes model at the maximum volatility. Relying on the viscosity formulation of the problem and the qualitative property of the boundary between the regions of concavity and convexity of the solution we manage to design explicit solutions in some cases. The key idea is an embedding of the initial problem in a standard european one related to another payoff.

Key-words: Volatility, option pricing, free boundary, explicit calculations, nonlinear parabolic equations.

(Résumé : tsvp)

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Modèle UVM et options américaines

Résumé : On étudie les prix d'options dans le modèle à volatilité inconnue d'Avellaneda et Lyons, en dimension 1. On montre que le prix de l'option est celui d'une option américaine dans le modèle de Black-Scholes à volatilité maximum. A partir de la formulation viscosité du problème et des propriétés qualitatives de la frontière entre les régions de convexité et de concavité de la solution on parvient à obtenir des formules explicites pour certains payoffs. L'idée principale est de plonger le problème initial dans un problème européen standard associé à un autre payoff.

Mots-clé : Volatilité, évaluation d'options, frontière libre, solutions explicites, équations non-linéaires paraboliques.

1 Introduction

The starting point of this work is the quest of closed-form solutions for the non-linear PDE:

$$\frac{1}{2}\overline{\sigma}^{2}x^{2}\left(\frac{\partial^{2}u}{\partial x^{2}}\right)^{+}-\frac{1}{2}\underline{\sigma}^{2}x^{2}\left(\frac{\partial^{2}u}{\partial x^{2}}\right)^{-}=\frac{\partial u}{\partial t}\text{ in }\left]0,\infty\right[\ \times\mathbb{R}_{*}^{+}\tag{1}$$

where $y^{+}=\max(y,0)$, $y^{-}=-\min(y,0)$ with initial data $u(0,.)=\varphi(.)$ which is convex on one side (say, above) and concave on the other side of some value x_c. This PDE has been first introduced by Avellaneda ([Av]) and Lyons ([Ly]) in the context of the management of volatility risk for non-convex or non-concave standard european options. It may be rewritten as a classical Hamilton-Jacobi-Bellman PDE:

$$\sup_{\alpha\in[\underline{\sigma},\overline{\sigma}]}\frac{1}{2}\alpha^{2}x^{2}\frac{\partial^{2}u}{\partial x^{2}}=\frac{\partial u}{\partial t}\tag{2}$$

In (1) it is usually assumed $0<\underline{\sigma}^{2}<\overline{\sigma}^{2}$ (It reduces to the standard Black&Scholes PDE in case $\underline{\sigma}^{2}=\overline{\sigma}^{2}$). Then we know (cf [FlSo] ch IV theorem 4.2) that for $u(0,.)$ smooth enough (more precisely C_b^3) (1) has a $C^{1,2}$ solution (this result is due to Krylov), mainly because of the non degeneracy assumption $\underline{\sigma}^{2}>0$. What can be said about the solution? After the hypothesis on $u(0,.)$ the intuition is that there should be some boundary $t\mapsto\widetilde{x}(t)$ such that $u(t,.)$ is convex above and concave below $\widetilde{x}(t)$, with u classical solution of the Black&Scholes PDE at volatility $\overline{\sigma}$ (resp. $\underline{\sigma}$) above (resp. below) $\widetilde{x}$, u being thus fully determined by the boundary $\widetilde{x}$.

It seems a bit naive to hope to find out $\widetilde{x}$, even for some particular initial data. But let us now turn to the extreme case $\underline{\sigma}=0$. We know from standard stochastic control results that u is well-defined as the unique bounded viscosity solution of (1). Since $\frac{\partial u}{\partial t}=0$ in the concavity region of $u(t,.)$ it is easily seen then that in case $\widetilde{x}$ is decreasing the value of u below $\widetilde{x}$ should be $u(0,.)$. More precisely we show that if $\widetilde{x}$ exists, it is decreasing if and only if $u(t,.)$ equals $u(0,.)$ below $\widetilde{x}$ (section 3). It is also easy to see that u cannot be a $C^{1,2}$ function, so that the notion of viscosity solution is required.

Consider now a function $u(t,.)$ with an associated boundary $\widetilde{x}$ which is (strictly) decreasing and goes to zero at infinity (remember that the state space is $\mathbb{R}_*^+$). Then the value of $u(0,.)$ below x_c may be recovered from the value of u along $\widetilde{x}$.

This leads to a natural idea to design explicit solutions, at least for some functions $u(0,.)$ in the following way: start from an auxiliary function $v(0,.)$ which is convex above and concave below some value x_c. Let $v(t,x)$ denote the solution of the Black&Scholes PDE at volatility $\overline{\sigma}$ with initial data $v(0,.)$. Assume $v(0,.)$ is chosen in such a way that the "gamma-zero" curve $t \mapsto \widehat{x}(t)$ (ie the set where $\frac{\partial^2 v}{\partial x^2} = 0$; gamma stands for the second-order derivative $\frac{\partial^2 v}{\partial x^2}$ in the derivatives dialect) is well-defined, strictly decreasing and goes to zero at infinity. Define a function $\widehat{\varphi}$ for $x < x_c$ by $\widehat{\varphi}(\widehat{x}(t)) = v(t, \widehat{x}(t))$. Then it is easily seen that $\widehat{\varphi}$ is a concave function, and that if you replace v below $\widehat{x}$ by this (constant in time) function, that the new function is C^1 in x and t for $t > 0$ (proposition 12). Therefore it is a good candidate to solve (1) with the adequate intial data ($\widehat{\varphi}$ below x_c and $v(0,.)$ above). This may be checked by characterizing the viscosity property at a boundary point, this is the matter of the verification proposition 7.

What is the connection with american options? It is clear by a time-change argument that in full generality the solution of (1) with $\underline{\sigma} = 0$ is in fact the price of the american option in the Black&Scholes model at volatility $\overline{\sigma}$.

The paper is organized as follows: in section 2 we prove the equality with the american price and we give basic properties of the solution of (1) with $\underline{\sigma} = 0$, without particular assumptions on the initial data. In the next section we consider the case of PutSpread-like initial data (which means concave below and convex above some value), the first ingredient is the verification proposition 7. Section 4 is devoted to the design of explicit solutions. The last section deals with numerous applications yielding close and "almost-close" formulas.

2 Volatility uncertainty and american options

Let us consider the one-dimensional fully nonlinear second order parabolic PDE:

$$\frac{1}{2}\overline{\sigma}^2 x^2 \left(\frac{\partial^2 u}{\partial x^2}\right)^+ = \frac{\partial u}{\partial t} \text{ in }]0,\infty[\ \times \mathbb{R}_*^+ \tag{3}$$

with initial condition:

$$u(0,x) = \varphi(x)$$

where until further notice φ is assumed to be in $C^0(\mathbb{R}_*^+) \cap L^\infty(\mathbb{R}_*^+)$, that is continuous and bounded.

Note that (3) may be rewritten

$$\sup_{\alpha \in [0,\overline{\sigma}]} \frac{1}{2}\alpha^2 x^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} \tag{4}$$

which is a standard second order HJB equation.

Despite the degeneracy of the operator we know from standard stochastic control results (cf [FlSo] ch. V theorem 9.1) that there exists a unique bounded viscosity solution for (4) given at any point (t,x) by

$$u(t,x) = \sup_{\sigma} E^P \left[\varphi \left(x \exp(\int_0^t \sigma_u dB_u - \frac{1}{2} \int_0^t \sigma_u^2 du) \right) \right] \tag{5}$$

where σ runs across the set of progressively measurable processes with respect to the filtration F of the standard Brownian motion B such that

$$0 \leq \sigma_u \leq \overline{\sigma} \;\; du\, dP \text{ almost surely}$$

We also introduce the american price at volatility $\overline{\sigma}$:

$$u^*(t,x) = \sup_{\tau} E^P \left[\varphi \left(x \exp(\overline{\sigma} B_\tau - \frac{1}{2}\overline{\sigma}^2 \tau) \right) \right] \tag{6}$$

where τ runs across the set of stopping times of F such that

$$\tau \leq t \;\; dP \text{ almost surely}$$

The following is easy to prove:

Proposition 1 *Let $uce(\varphi)$ denote the smallest concave function above φ. Then for every (t,x),*

$$\varphi(x) \leq u(t,x) = u^*(t,x) \leq uce(\varphi)(x)$$

In addition for every x, the function $t \mapsto u(t,x)$ is increasing with

$$\lim_{t \to \infty} u(t,x) = uce(\varphi)(x) \tag{7}$$

Proof. The inequality $\varphi(x) \leq u(t,x)$ follows by choosing $\sigma_u \equiv 0$ in (5) whereas $u^*(t,x) \leq uce(\varphi)(x)$ comes from the variational form:

$$uce(\varphi)(x) = \inf\left\{\alpha \in \mathbb{R} / \exists\beta, \forall y \in \mathbb{R}^+,\ \alpha + \beta(y-x) \geq \varphi(x)\right\}$$

the optional sampling theorem and (6).

Observe now that

$$\begin{aligned} u(t,x) &= \sup_\sigma E^P\left[\varphi\left(x\exp(\int_0^t \sigma_u dB_u - \frac{1}{2}\int_0^t \sigma_u^2 du)\right)\right] \\ &= \sup_\sigma E^P\left[\varphi\left(x\exp(\beta_{\int_0^t \sigma_u^2 du} - \frac{1}{2}\int_0^t \sigma_u^2 du)\right)\right] \end{aligned}$$

for some Brownian motion β by time-change. The random variable $\int_0^t \sigma_u^2 du$ is a stopping time of the filtration $\mathcal{B}$ of β which is less than $\overline{\sigma}^2 t$, therefore

$$u(t,x) \leq \sup_\rho E^P\left[\varphi\left(x\exp(\beta_\rho - \frac{1}{2}\rho)\right)\right]$$

where ρ runs across the set of $\mathcal{B}$ -stopping time less than $\overline{\sigma}^2 t$. By scaling

$$\sup_\rho E^P\left[\varphi\left(x\exp(\beta_\rho - \frac{1}{2}\rho)\right)\right] = \sup_\tau E^P\left[\varphi\left(x\exp(\overline{\sigma}B_\tau - \frac{1}{2}\overline{\sigma}^2\tau)\right)\right] = u^*(t,x)$$

Conversely

$$\begin{aligned} u(t,x) &= \sup_\sigma E^P\left[\varphi\left(x\exp(\int_0^t \sigma_u dB_u - \frac{1}{2}\int_0^t \sigma_u^2 du)\right)\right] \\ &\geq \sup_\tau E^P\left[\varphi\left(x\exp(\int_0^t \overline{\sigma}\mathbf{1}(u \leq \tau)\, dB_u - \frac{1}{2}\int_0^t [\overline{\sigma}\mathbf{1}(u \leq \tau)]^2 du)\right)\right] \\ &= \sup_\tau E^P\left[\varphi\left(x\exp(\overline{\sigma}B_\tau - \frac{1}{2}\overline{\sigma}^2\tau)\right)\right] \\ &= u^*(t,x) \end{aligned}$$

The fact that $t \mapsto u^*(t,x)$ is increasing follows readily from (6). Lastly (7) maybe found for instance in ([Ma]). ■

Notice that as a direct consequence of the proposition and the semigroup property of the solution of the homogeneous equation (3):

Proposition 2 *(i) For every $t > 0$, $uce(\varphi) = uce(u(t,.))$.*
(ii) The contact set $X(t) = \{x / u(t,x) = uce(u(t,.))(x)\}$ is increasing.

Let us turn to the study of a particular class of payoffs φ.
In the sequel we shall set

$$S_t = \exp(\overline{\sigma} B_t - \frac{1}{2}\overline{\sigma}^2 t)$$

3 PutSpread-like payoffs

Let us first set:

Definition 3 *A PutSpread-like payoff is a function $\varphi : \mathbb{R}_*^+ \to \mathbb{R}$ which is bounded, continuous decreasing, concave below and convex above some value $x_c > 0$ (maybe not unique).*

Consider from now on the case where φ is a PutSpread-like payoff. In such a situation we have the following:

Proposition 4 *(i) $uce(\varphi) \equiv \sup \varphi$*
(ii) $X(t) \equiv X(0) =]0, x_d]$ for some $x_d \geq 0$ (with the convention $\emptyset =]0,0]$).

Proof. The first assertion is obvious. For the second one, if $\varphi(x) < \sup \varphi$ then $u^*(t,x) < \sup \varphi$: indeed let $\eta > 0$ be such that $\varphi(y) < \frac{\varphi(x) + \sup \varphi}{2}$ for $y \in]x(1-\eta), x(1+\eta)[$.
Then

$$\begin{aligned} E^P[\varphi(xS_\tau)\ ] &= E^P[\varphi(xS_\tau)\ 1(|S_\tau - 1| \leq \eta)] + E^P[\varphi(xS_\tau)\ 1(|S_\tau - 1| > \eta)] \\ &\leq \left(\frac{\varphi(x) + \sup \varphi}{2}\right) E^P[\ 1(|S_\tau - 1| \leq \eta)] + \sup \varphi\, E^P[\ 1(|S_\tau - 1| > \eta)] \\ &\leq \sup \varphi + \left(\frac{\varphi(x) - \sup \varphi}{2}\right) \inf_\tau E^P[\ 1(|S_\tau - 1| \leq \eta)] \\ &\leq \sup \varphi + \left(\frac{\varphi(x) - \sup \varphi}{2}\right) P\left(\sup_{0<u<t} |S_u - 1| \leq \eta\right) \end{aligned}$$

which is a constant with respect to τ srtictly less than $\sup \varphi$ because of $P(\sup_{0<u<t} |S_u - 1| \leq \eta) > 0$. ■

3.1 The case $x_c = x_d$ (evidences)

In such a case the solution is easy to compute:

Proposition 5 *Assume $X(0) =]0, x_c]$ for some x_c. Then*

$$u(t,x) = u^*(t,x) = E^P\left[\varphi\left(x\exp(\overline{\sigma}\, B_{\widehat{\tau}^{x,t}} - \frac{1}{2}\overline{\sigma}^2\widehat{\tau}^{x,t})\right)\ \right]$$

where $\widehat{\tau}^{x,t} = \inf\{u > 0, xS_u \leq\ x_c\} \wedge t.$

Proof. It suffices to prove the last equality. For any stopping time τ less than t :

$$\begin{aligned}
E^P[\varphi(xS_\tau)\ ] &= E^P\left[\varphi(xS_\tau)\, 1\left(\tau \leq \widehat{\tau}^{x,t}\right) + \varphi(xS_\tau)\, 1\left(\tau > \widehat{\tau}^{x,t}\right)\right] \\
&\leq E^P\left[\varphi(xS_\tau)\, 1\left(\tau \leq \widehat{\tau}^{x,t}\right) + \sup\varphi 1\left(\tau > \widehat{\tau}^{x,t}\right)\right] \\
&= E^P\left[\varphi(xS_\tau)\, 1\left(\tau \leq \widehat{\tau}^{x,t}\right) + \varphi(xS_{\widehat{\tau}^{x,t}})\, 1\left(\tau > \widehat{\tau}^{x,t}\right)\right] \\
&= E^P\left[\varphi(xS_{\tau\wedge\widehat{\tau}^{x,t}})\ \right] \\
&= E^P\left[\varphi\left(x\, E^P\left[S_{\widehat{\tau}^{x,t}} \mid F_{\tau\wedge\widehat{\tau}^{x,t}}\right]\right)\ \right] \\
&\leq E^P\left[E^P\left[\varphi(xS_{\widehat{\tau}^{x,t}}) \mid F_{\tau\wedge\widehat{\tau}^{x,t}}\right]\ \right] = E^P\left[\varphi(xS_{\widehat{\tau}^{x,t}})\right]
\end{aligned}$$

by Jensen inequality and the convexity of φ above x_c. ■

3.2 The general case

Note that we can always suppose $x_d \leq x_c$ for some x_c since φ is constant below x_d. In the general case where $x_d < x_c$ we expect that $u(t,.)$ be convex above and concave below some critical value $\widehat{x}(t)$. In order to get (7) we expect that $\lim_{t\to\infty}\widehat{x}(t) = x_d$.

Should $\widehat{x}$ be a continuous decreasing function, then as stated in the introduction $u(t,x)$ would be fully determined by $\widehat{x}$ through the formula:

$$\begin{aligned}
u(t,x) &= \varphi(x)\ \text{ if } x \leq \widehat{x}(t) \qquad (8)\\
u(t,x) &= E^P\left[\varphi\left(x\exp(\overline{\sigma}\, B_{\tau^{\widehat{x},t}} - \frac{1}{2}\overline{\sigma}^2\tau^{\widehat{x},t})\right)\ \right] \text{ otherwise}
\end{aligned}$$

with $\tau^{\widehat{x},t} = \inf \{u > 0, xS_u \le \widehat{x}(t-u)\}$.

Note that the converse is true, at least under a strict concavity-convexity assumption:

Proposition 6 *Assume that $\varphi(.)$ is strictly convex above x_c and strictly concave below x_c and that there exists a continuous function $t \mapsto \widehat{x}(t)$ such that:*

(i) $\lim_{t\to\infty} \widehat{x}(t) = x_d$

(ii) For each t, $u(t,.)$ is concave below equal to $\varphi(.)$ and convex above $\widehat{x}(t)$.

Then $t \mapsto \widehat{x}(t)$ is decreasing.

Proof. Let $t > 0$, $t > h > 0$ such that $\widehat{x}(t) > \widehat{x}(t-h)$. Define $\widehat{t}(h)$ by

$$\widehat{t}(h) = \sup\{w \ge 0 \,/\, \widehat{x}(w) = \widehat{x}(t-h)\}$$

Then obviously $\widehat{t}(h) > t$. Observe now that

$$u(t-h, \widehat{x}(t-h)) = u\left(\widehat{t}(h), \widehat{x}(t-h)\right) = \varphi(\widehat{x}(t-h))$$

which entails since $t \mapsto u(t,x)$ is increasing that $u(v, \widehat{x}(t-h))$ is equal to $\varphi(\widehat{x}(t-h))$ for any v between $t-h$ and $\widehat{t}(h)$. In particular $u(t, \widehat{x}(t-h)) = \varphi(\widehat{x}(t-h))$. By the same argument for any h' such that $t-h < t-h' < t$ and $\widehat{x}(t) > \widehat{x}(t-h') > \widehat{x}(t-h)$ we get that $u(t,x') = \varphi(x')$ for any x' between $\widehat{x}(t-h)$ and $\widehat{x}(t)$. Since φ is convex above $\widehat{x}(t-h)$, concave below $\widehat{x}(t)$, $\widehat{x}(t-h) = x_c = \widehat{x}(t)$ which contradicts $\widehat{x}(t) > \widehat{x}(t-h)$. ■

Of course it seems difficult to find out $\widehat{x}$. A related question is to study the behavior of u near the boudary $\widehat{x}$. Note that u cannot be a $C^{1,2}$ function: indeed $\frac{\partial^2 u}{\partial x^2} \ge 0$ above $\widehat{x}(t)$ and $\varphi''(x) \le 0$ below $\widehat{x}(t)$ so that $\varphi''(\widehat{x}(t)) = 0$ necessarily which entails $\widehat{x}(t) \equiv x_c$. From (8) we get $u(t, \widehat{x}(t)) \equiv \varphi(x_c)$, in particular $\lim_{t\to\infty} u(t, x_c) = \varphi(x_c)$ which contradicts (7) since $\varphi(x_c) < \sup \varphi$ by assumption.

3.3 Viscosity property at the boundary

The following proposition will be the key tool for the design of explicit solutions:

Proposition 7 *Let $\widehat{\varphi}$ a bounded continuous function concave below x_c, convex above x_c, C^2 in $]0, x_c[$ and a bounded continuous function $W : [0, \infty[\times \mathbb{R}_*^+ \to \mathbb{R}$ satisfy:*

(i) W is continuous in $[0, \infty[\times \mathbb{R}_*^+$ with $W(0, x) \equiv \widehat{\varphi}(x)$.

(ii) There is a function $t \mapsto \widehat{x}(t)$ continuous and strictly decreasing such that for each t, $W(t, .)$ is convex above $\widehat{x}(t)$, concave equal to $\widehat{\varphi}$ below $\widehat{x}(t)$.

(iii) W is $C^{1,2}$ and satisfies (3) in the classical sense in $D = \{(t, x) \ / \ x > \widehat{x}(t)\}$.

(iv) For each $t > 0$ the one-sided derivatives $\frac{\partial W}{\partial x}\left(t, \widehat{x}(t)_+\right)$, $\frac{\partial W}{\partial t}(t_+, \widehat{x}(t))$ and $\frac{\partial^2 W}{\partial x^2}\left(t, \widehat{x}(t)_+\right)$ exist (then it is necesary that $\frac{\partial W}{\partial t}(t_+, \widehat{x}(t)) \geq 0$ and $\frac{\partial^2 W}{\partial x^2}\left(t, \widehat{x}(t)_+\right) \geq 0$).

Then W is a viscosity solution of (3) if and only if for each $t > 0$:

$$\begin{aligned} \frac{\partial W}{\partial x}\left(t, \widehat{x}(t)_+\right) &= \widehat{\varphi}'(\widehat{x}(t)) \\ \frac{\partial W}{\partial t}(t_+, \widehat{x}(t)) &= 0 \end{aligned}$$

or alternatively if it is a $C^{1,1}$ function in $]0, \infty[\times \mathbb{R}_*^+$.

Obviously W satisfies (3) in the classical sense at every point of $]0, \infty[\times \mathbb{R}_*^+ \setminus \widehat{x}$ so that the whole issue is the viscosity property at the boundary. We make use of the characterization of the viscosity property in terms of the parabolic sub and superdifferential sets (cf [FlSo] ch V proposition 4.1) and of the following lemmas:

Lemma 8 *The set of parabolic superdifferential at $(t, \widehat{x}(t))$ is given by*

$$\begin{aligned} D_+^{1,2}(t, \widehat{x}(t)) = & \left\{ \begin{array}{c} (q, p, A) \ / \ q \geq \frac{\partial W}{\partial t}(t_+, \widehat{x}(t)), \ p > \frac{\partial W}{\partial x}\left(t, \widehat{x}(t)_+\right) \\ \text{or } q \geq \frac{\partial W}{\partial t}(t_+, \widehat{x}(t)), \ p = \frac{\partial W}{\partial x}\left(t, \widehat{x}(t)_+\right), \ A \geq \frac{\partial^2 W}{\partial x^2}\left(t, \widehat{x}(t)_+\right) \end{array} \right\} \\ & \cap \left\{ \begin{array}{c} (q, p, A) \ / \ q \leq 0, \ p < \widehat{\varphi}'(\widehat{x}(t)) \\ \text{or } q \leq 0, \ p = \widehat{\varphi}'(\widehat{x}(t)), \ A \geq \widehat{\varphi}''(\widehat{x}(t)) \end{array} \right\} \end{aligned} \tag{9}$$

Proof. By definition

$$D_+^{1,2}(t, \widehat{x}(t)) = \left\{ (q, p, A) \ / \ \lim \sup_{h \to 0, y \to 0} \frac{W(t + h, \widehat{x}(t) + y) - qh - py - \frac{1}{2}Ay^2}{|h| + y^2} \geq 0 \right\}$$

The first set in (9) follows then from

$$W(t+h,x+y) = W(t,\widehat{x}(t)) + \frac{\partial W}{\partial t}(t_+,\widehat{x}(t))\,h + \frac{\partial W}{\partial x}\left(t,\widehat{x}(t)_+\right)y + \tfrac{1}{2}Ay^2 + o\left(|h|+y^2\right)$$

for $h>0$ and $y>0$, the second one from

$$W(t+h,x+y) = W(t,\widehat{x}(t)) + \widehat{\varphi}'(\widehat{x}(t))\,y + \tfrac{1}{2}\widehat{\varphi}''(\widehat{x}(t))\,y^2 + o\left(|h|+y^2\right)$$

for $h<0$ and $y<0$. ■

Corollary 9 *W is a viscosity supersolution of* (3) *if and only if for each $t>0$ $\frac{\partial W}{\partial x}\left(t,\widehat{x}(t)_+\right) \le \widehat{\varphi}'(\widehat{x}(t))$ and $\frac{\partial W}{\partial t}(t_+,\widehat{x}(t)) = 0$.*

Proof. W is a viscosity supersolution if and only if for each $t>0$ the set $D_+^{1,2}(t,\widehat{x}(t))$ is not empty and for every (q,p,A)in $D_+^{1,2}(t,\widehat{x}(t))$,

$$\frac{1}{2}\overline{\sigma}^2\widehat{x}(t)^2A^+ - q \ge 0 \tag{10}$$

Assume W is a viscosity supersolution. Since $\frac{\partial W}{\partial t}(t_+,\widehat{x}(t)) \ge 0$, it is necessary that $q=0$ for (q,p,A)in $D_+^{1,2}(t,\widehat{x}(t))$, therefore (10) is always in force. As $D_+^{1,2}(t,\widehat{x}(t))$ is not empty the conditions on p yield $\frac{\partial W}{\partial x}\left(t,\widehat{x}(t)_+\right) \le \widehat{\varphi}'(\widehat{x}(t))$. Conversely, if $\frac{\partial W}{\partial x}\left(t,\widehat{x}(t)_+\right) \le \widehat{\varphi}'(\widehat{x}(t))$ and $\frac{\partial W}{\partial t}(t_+,\widehat{x}(t)) = 0$ then $D_+^{1,2}(t,\widehat{x}(t))$ is not empty and (10) is easily checked. ■

On the other side:

Lemma 10 *The set of parbolic subdifferential at $(t,\widehat{x}(t))$ is given by*

$$\begin{aligned} D_-^{1,2}(t,\widehat{x}(t)) \;=\; & \left\{ \begin{array}{c} (q,p,A)\ /\ q \le \frac{\partial W}{\partial t}(t_+,\widehat{x}(t)),\ p < \frac{\partial W}{\partial x}\left(t,\widehat{x}(t)_+\right) \\ \textit{or}\ q \le \frac{\partial W}{\partial t}(t_+,\widehat{x}(t)),\ p = \frac{\partial W}{\partial x}\left(t,\widehat{x}(t)_+\right),\ A \le \frac{\partial^2 W}{\partial x^2}\left(t,\widehat{x}(t)_+\right) \end{array} \right\} \\ & \cap \left\{ \begin{array}{c} (q,p,A)\ /\ q \ge 0,\ p > \widehat{\varphi}'(\widehat{x}(t)), \\ \textit{or}\ q \ge 0,\ p = \widehat{\varphi}'(\widehat{x}(t)),\ A \le \widehat{\varphi}''(\widehat{x}(t)) \end{array} \right\} \end{aligned}$$

Corollary 11 *W is a viscosity subsolution of* (3) *if and only if for each $t>0$ $\frac{\partial W}{\partial x}\left(t,\widehat{x}(t)_+\right) = \widehat{\varphi}'(\widehat{x}(t))$.*

Proof. Assume W is a viscosity subsolution. Then for each $t > 0$ the set $D_{-}^{1,2}(t, \widehat{x}(t))$ is not empty and whatever (q,p,A)in $D_{-}^{1,2}(t, \widehat{x}(t))$,

$$\frac{1}{2}\overline{\sigma}^{2}\widehat{x}(t)^{2} A^{+} - q \leq 0 \tag{11}$$

The conditions on p and q yield $\widehat{\varphi}'(\widehat{x}(t)) \leq p \leq \frac{\partial W}{\partial x}\left(t, \widehat{x}(t)_{+}\right)$and $0 \leq q \leq \frac{\partial W}{\partial t}(t_{+}, \widehat{x}(t))$. In case $\widehat{\varphi}'(\widehat{x}(t)) < p < \frac{\partial W}{\partial x}\left(t, \widehat{x}(t)_{+}\right)$, (11) should be in force for any A, which is impossible. Therefore $\frac{\partial W}{\partial x}\left(t, \widehat{x}(t)_{+}\right) = \widehat{\varphi}'(\widehat{x}(t))$. The condition on A is then $A \leq \widehat{\varphi}''(\widehat{x}(t)) \wedge \frac{\partial^{2} W}{\partial x^{2}}\left(t, \widehat{x}(t)_{+}\right) = \widehat{\varphi}''(\widehat{x}(t)) < 0$ whence (11) is always in force. Conversely if $\frac{\partial W}{\partial x}\left(t, \widehat{x}(t)_{+}\right) = \widehat{\varphi}'(\widehat{x}(t))$ then $D_{-}^{1,2}(t, \widehat{x}(t))$ is not empty and (11) is in force. ■

4 The design of explicit solutions

In this section we shall design explicit solutions for (3) in the case of some PutSpread like payoffs with $x_d = 0$. The idea is very simple: first start from a PutSpread-like payoff ψ. Consider then its Black-Scholes price at volatility $\overline{\sigma}$ $v(t,x)$. In many cases there maybe a unique "gamma-zero" curve defined for each $t > 0$ by

$$\frac{\partial^{2} v}{\partial x^{2}}(t, \widetilde{x}(t)) = 0$$

Should $\widetilde{x}$ be a continuous strictly decreasing function, then it goes to zero at infinity: this comes from $\lim_{t\to\infty} v(t,x) = \sup\psi$ (by dominated convergence) and from the fact that $t \to v(t,x)$ is increasing above $\widetilde{x}$ and decreasing below.

Now look at the price $v(t, \widetilde{x}(t))$ along $t \mapsto \widetilde{x}(t)$ and more precisely define a function $\widehat{\varphi}$ for $x \leq \widetilde{x}(0)$ by

$$\widehat{\varphi}(\widetilde{x}(t)) = v(t, \widetilde{x}(t)) \tag{12}$$

Then we may hope that for the function φ defined by:

$$\begin{aligned} \varphi(x) &= \psi(x) \;\; \text{for}\, x \geq \widetilde{x}(0) \\ &= \widehat{\varphi}(x) \;\; \text{for}\, x < \widetilde{x}(0) \end{aligned}$$

the solution to (3) is:

$$\begin{aligned} u(t,x) &= v(t,x) \text{ for } x \geq \widetilde{x}(t) \\ &= \widehat{\varphi}(x) \text{ for } x < \widetilde{x}(t) \end{aligned} \tag{13}$$

Why is it a good candidate? Obviously the question is the pasting along $\widetilde{x}$ and the concavity of $\widehat{\varphi}$. Everything works fine because of the gamma-zero property:

Proposition 12 *Let ψ a PutSpread-like payoff. Assume there is a continuous function $t \to \widetilde{x}(t)$ such that $v(t,.)$ is concave below and convex above $\widetilde{x}$. Then if $\widetilde{x}$ is C^1 strictly decreasing for $t > 0$, the function $\widehat{\varphi}$ defined by (12) is concave C^2 for $x < \widetilde{x}(0)$ and the function u defined by (13) is C^0 in (t,x) and C^1 for $t > 0$.*

Proof.

$$\begin{aligned} \widehat{\varphi}'(\widetilde{x}(t))\,\widetilde{x}'(t) &= \frac{\partial v}{\partial t}(t,\widetilde{x}(t)) + \frac{\partial v}{\partial x}(t,\widetilde{x}(t))\,\widetilde{x}'(t) \\ &= \frac{1}{2}\overline{\sigma}^2\widetilde{x}(t)^2\frac{\partial^2 v}{\partial x^2}(t,\widetilde{x}(t)) + \frac{\partial v}{\partial x}(t,\widetilde{x}(t))\,\widetilde{x}'(t) \\ &= \frac{\partial v}{\partial x}(t,\widetilde{x}(t))\,\widetilde{x}'(t) \end{aligned}$$

whence $\widehat{\varphi}'(\widetilde{x}(t)) = \frac{\partial v}{\partial x}(t,\widetilde{x}(t))$ for almost every t since $\widetilde{x}'(t) < 0$ almost everywhere, and for every $t > 0$ by continuity. Then

$$\begin{aligned} \widehat{\varphi}''(\widetilde{x}(t))\,\widetilde{x}'(t) &= \frac{\partial^2 v}{\partial x \partial t}(t,\widetilde{x}(t)) + \frac{\partial^2 v}{\partial x^2}(t,\widetilde{x}(t))\,\widetilde{x}'(t) \\ &= \frac{\partial^2 v}{\partial x \partial t}(t,\widetilde{x}(t)) \end{aligned}$$

But by taking the derivative in the Black&Scholes PDE:

$$\begin{aligned} \frac{\partial^2 v}{\partial x \partial t}(t,\widetilde{x}(t)) &= \frac{\partial}{\partial x}\left[\frac{1}{2}\overline{\sigma}^2 x^2 \frac{\partial^2 v}{\partial x^2}(t,x)\right]_{x=\widetilde{x}(t)} \\ &= \frac{1}{2}\overline{\sigma}^2\widetilde{x}(t)^2\frac{\partial^3 v}{\partial x^3}(t,\widetilde{x}(t)) \geq 0 \end{aligned}$$

since $v(t,.)$ is concave below and convex above $\widetilde{x}(t)$. Therefore $\widehat{\varphi}$ is C^2 with $\widehat{\varphi}''(\widetilde{x}(t)) \leq 0$, as above. ■

Remark 1 *The choice of PutSpreads and not CallSpreads relies on the remark that for a CallSpread* $\lim_{t\to\infty} v(t,x) = 0$ *so that there is little hope to get (7) on one side or the other of the gamma-zero curve (the fact that the CallSpread payoff is not bounded doesn't seem to be an insuperable difficulty).*

4.1 Main result

We have proved the following:

Theorem 13 *Let* $\psi : \mathbb{R}_*^+ \to \mathbb{R}^+$ *a continuous bounded decreasing function concave below and convex above some value* x_c. *Let*

$$v(t,x) = E^P\left[\psi\left(x\exp(\overline{\sigma}B_t - \frac{1}{2}\overline{\sigma}^2 t)\right)\right]$$

the Black-Scholes price at volatility $\overline{\sigma}$ *with no interest rate of the european option with maturity* t *and payoff* ψ. *Assume there is a continuous function* $t \to \widetilde{x}(t)$, C^1*strictly decreasing for* $t > 0$, *such that* $v(t,.)$ *is concave below and convex above* $\widetilde{x}(t)$. *Then* $\lim_{t\to\infty}\widetilde{x}(t) = 0$ *and the Black-Scholes price (at volatility* $\overline{\sigma}$ *with no interest rate) of the american option with payoff:*

$$x \to \widehat{\varphi}(x) = v\left(\widetilde{x}^{-1}(x), x\right)\mathbf{1}(x < \widetilde{x}(0)) + \psi(x)\mathbf{1}(x \geq \widetilde{x}(0))$$

and maturity t *is given by*

$$u(t,x) = v\left(\widetilde{x}^{-1}(x), x\right)\mathbf{1}(x < \widetilde{x}(t)) + v(t,x)\mathbf{1}(x \geq \widetilde{x}(t))$$

5 Applications

5.1 Starting from a classical PutSpread

Let us start from a classical PutSpread:

$$\psi(x) = (K_2 - x)^+ - (K_1 - x)^+$$

with $0 < K_1 < K_2$. Let

$$p_t(x,y) = \frac{\exp\left(\frac{-\left(\ln\left(\frac{y}{x}\right)+\frac{1}{2}\overline{\sigma}^2 t\right)^2}{\overline{\sigma}\sqrt{t}}\right)}{y\sqrt{2\pi\overline{\sigma}^2 t}} \tag{14}$$

denote the density of the probability transition of the Black&Scholes model.

Then from the well-known equality

$$E^P\left[\left(K - x\exp(\overline{\sigma}B_t - \frac{1}{2}\overline{\sigma}^2 t)\right)^+\right] = E^P\left[\left(K\exp(\overline{\sigma}B_t - \frac{1}{2}\overline{\sigma}^2 t) - x\right)^+\right]$$

or by a direct computation it is easy to get:

$$\frac{\partial^2 v}{\partial x^2}(t,x) = p_t(K_2,x) - p_t(K_1,x)$$

which yields

$$\begin{aligned} &\frac{\partial^2 v}{\partial x^2}(t,x) = 0 \\ \Leftrightarrow \quad & x = \sqrt{K_1K_2}\exp\left(-\frac{\overline{\sigma}^2 t}{2}\right) \stackrel{def}{=} \widetilde{x}(t) \end{aligned}$$

therefore $\widetilde{x}$ is C^1 strictly decreasing and also

$$\begin{aligned} \widetilde{x}^{-1}(x) &= -\frac{2\ln\left(x/\sqrt{K_1K_2}\right)}{\overline{\sigma}^2} \\ \widetilde{x}(0) &= \sqrt{K_1K_2} \end{aligned}$$

Let now $Put_K(t,x)$ denote the put price in the Black&Scholes model. Then in this case:

$$\begin{aligned} \widehat{\varphi}(x) = & \left[Put_{K_2}\left(-\frac{2\ln\left(x/\sqrt{K_1K_2}\right)}{\overline{\sigma}^2}, x\right) - Put_{K_1}\left(-\frac{2\ln\left(x/\sqrt{K_1K_2}\right)}{\overline{\sigma}^2}, x\right)\right]\mathbf{1}\left(x < \sqrt{K_1K_2}\right) \\ & + \left[(K_2 - x)^+ - (K_1 - x)^+\right]\mathbf{1}\left(x \geq \sqrt{K_1K_2}\right) \end{aligned}$$

with

$$\begin{aligned} u(t,x) &= \left[Put_{K_2}\left(-\frac{2\ln\left(x/\sqrt{K_1K_2}\right)}{\overline{\sigma}^2},x\right)-Put_{K_1}\left(-\frac{2\ln\left(x/\sqrt{K_1K_2}\right)}{\overline{\sigma}^2},x\right)\right] \\ &\quad\times\mathbf{1}\left(x<\sqrt{K_1K_2}\exp\left(-\frac{\overline{\sigma}^2t}{2}\right)\right) \\ &\quad+\left[Put_{K_2}(t,x)-Put_{K_1}(t,x)\right]\mathbf{1}\left(x\geq\sqrt{K_1K_2}\exp\left(-\frac{\overline{\sigma}^2t}{2}\right)\right) \end{aligned}$$

5.2 A larger family

Let us look for functions φ of the form

$$\varphi(x)=\int\alpha(k)(k-x)^+\,dk-\int\beta(k)(k-x)^+\,dk$$

with $\alpha(.)$ and $\beta(.)$ positive measurable functions. Then

$$x^2\frac{\partial^2 v}{\partial x^2}(t,x)=E\left[x^2S_t^2\alpha(xS_t)\right]-E\left[x^2S_t^2\beta(xS_t)\right]$$

Define now $A,B:\mathbb{R}\to\mathbb{R}^+$ by

$$y^2\alpha(y)\equiv A\circ\ln(y)\,,\,y^2\beta(y)\equiv B\circ\ln(y)$$

Then

$$x^2\frac{\partial^2 v}{\partial x^2}(t,x)=E\left[A\left(\ln x+\overline{\sigma}B_t-\frac{1}{2}\overline{\sigma}^2t\right)\right]-E\left[B\left(\ln x+\overline{\sigma}B_t-\frac{1}{2}\overline{\sigma}^2t\right)\right]$$

so that we look for A and B such that:

(i) For each $t>0,$ there exists $z^*(t)$ with

$$\begin{aligned} E\left[A\left(z+\overline{\sigma}B_t-\frac{1}{2}\overline{\sigma}^2t\right)\right] &\leq E\left[B\left(z+\overline{\sigma}B_t-\frac{1}{2}\overline{\sigma}^2t\right)\right] \\ \Leftrightarrow z &\leq z^*(t) \end{aligned}$$

(ii) $t\mapsto z^*(t)$ is C^1 strictly decreasing.

In order to get closed-form solutions, it is natural to look for special parametric families for A and B. We manage to get the required properties (i) and (ii) for the following family:

Lemma 14 *Let a a positive number, ν a positive measure with compact support which is not the null measure and l a real number strictly greater than the support of ν. Set*

$A(z) = \exp\left(-\frac{1}{2}\left(\frac{z-l}{a}\right)^2\right)$, $B(z) = \int \nu(dm)\exp\left(-\frac{1}{2}\left(\frac{z-m}{a}\right)^2\right)$

Then (i) and (ii) are in force with $z^*(t)$ characterized by

$$\int \nu(dm)\exp\left(-\frac{(m-l)\left(z^*(t)-\frac{l+m}{2}\right)}{2(a^2+t)}\right) = \exp\left(\frac{\overline{\sigma}^2 t}{4(a^2+t)}\right)$$

Proof. By calculus

$$\begin{aligned} E\left[A\left(z+\overline{\sigma}B_t-\frac{1}{2}\overline{\sigma}^2 t\right)\right] &\leq E\left[B\left(z+\overline{\sigma}B_t-\frac{1}{2}\overline{\sigma}^2 t\right)\right] \\ \Leftrightarrow \exp\left(-\frac{\left(z-\frac{1}{2}\overline{\sigma}^2 t-l\right)^2}{2(a^2+t)}\right) &\leq \int \nu(dm)\exp\left(-\frac{\left(z-\frac{1}{2}\overline{\sigma}^2 t-m\right)^2}{2(a^2+t)}\right) \\ \Leftrightarrow \int \nu(dm)\exp\left(-\frac{(m-l)\left(z-\frac{l+m}{2}\right)}{2(a^2+t)}\right) &\leq \exp\left(\frac{\overline{\sigma}^2 t}{4(a^2+t)}\right) \end{aligned}$$

Since $l > m$ $\nu(dm)$-almost surely the function on the left-hand-side is strictly decreasing and scans the whole real line since ν is compact supported. Therefore (i) is all right. Now

$$\int \nu(dm)\exp\left(-\frac{(m-l)\left(z^*(t)-\frac{l+m}{2}\right)}{2(a^2+t)}\right) = \exp\left(\frac{\overline{\sigma}^2 t}{4(a^2+t)}\right)$$

and by convexity for $u < t$:

$$\begin{aligned} &\int \nu(dm)\exp\left(-\frac{(m-l)\left(z^*(t)-\frac{l+m}{2}\right)}{2(a^2+u)}\right) \\ = &\int \nu(dm)\left[\exp\left(-\frac{(m-l)\left(z^*(t)-\frac{l+m}{2}\right)}{2(a^2+t)}\right)\right]^{\frac{(a^2+t)}{(a^2+u)}} \end{aligned}$$

$$
\begin{aligned}
&\geq \left[\int \nu\left(dm\right)\exp\left(-\frac{\left(m-l\right)\left(z^*\left(t\right)-\frac{l+m}{2}\right)}{2\left(a^2+t\right)}\right)\right]^{\frac{\left(a^2+t\right)}{\left(a^2+u\right)}} \\
&= \exp\left(\frac{\overline{\sigma}^2 t}{4\left(a^2+u\right)}\right) > \exp\left(\frac{\overline{\sigma}^2 u}{4\left(a^2+u\right)}\right) \\
&= \int \nu\left(dm\right)\exp\left(-\frac{\left(m-l\right)\left(z^*\left(u\right)-\frac{l+m}{2}\right)}{2\left(a^2+u\right)}\right)
\end{aligned}
$$

Observe now that the function $z \mapsto \int \nu\left(dm\right)\exp\left(-\frac{(m-l)\left(z-\frac{l+m}{2}\right)}{2(a^2+u)}\right)$ is strictly decreasing so that the map $t \mapsto z^*\left(t\right)$ is also strictly decreasing. ■

The corresponding objects of interest are easily computed:

$$
\begin{aligned}
\varphi\left(x\right) &= \int \frac{A\left(\ln k\right)}{k^2}\left(k-x\right)^+ dk - \int \frac{B\left(\ln k\right)}{k^2}\left(k-x\right)^+ dk \qquad (15)\\
&= \int \exp\left(-\frac{1}{2}\left(\frac{\ln k - l}{a}\right)^2\right)\left(k-x\right)^+\frac{dk}{k^2} \\
&\quad - \int\int \nu\left(dm\right)\exp\left(-\frac{1}{2}\left(\frac{\ln k - m}{a}\right)^2\right)\left(k-x\right)^+\frac{dk}{k^2}
\end{aligned}
$$

The map $x \mapsto \widetilde{x}^{-1}\left(x\right)$ is characterized by

$$
\int \nu\left(dm\right)\exp\left(-\frac{\left(m-l\right)\left(\ln x-\frac{l+m}{2}\right)}{2\left(a^2+\widetilde{x}^{-1}\left(x\right)\right)}\right) = \exp\left(\frac{\overline{\sigma}^2\widetilde{x}^{-1}\left(x\right)}{4\left(a^2+\widetilde{x}^{-1}\left(x\right)\right)}\right)
$$

for each $x < \widetilde{x}\left(0\right)$ with

$$
\int \nu\left(dm\right)\exp\left(-\frac{\left(m-l\right)\left(\ln\widetilde{x}\left(0\right)-\frac{l+m}{2}\right)}{2a^2}\right) = 1
$$

Making use of (14) it is possible to write (15) under a more pleasant form and we may state the following:

Proposition 15 *Let a a positive number, ν a positive measure with compact support which is not the null measure and l a real number strictly greater than the support of ν. Let*

$$\begin{aligned}\varphi(x) &= aPut_1\left(\frac{a^2}{\overline{\sigma}^2}, x\exp\left(\frac{a^2}{2}-l\right)\right) \\ &\quad -a\int \nu(dm)\, Put_1\left(\frac{a^2}{\overline{\sigma}^2}, x\exp\left(\frac{a^2}{2}-m\right)\right)\end{aligned} \tag{16}$$

Then the conditions of theorem 13 are in force with for each $t \geq 0$ $\widetilde{x}(t)$ characterized by

$$\int \nu(dm)\exp\left(-\frac{(m-l)\left(\ln\widetilde{x}(t)-\frac{l+m}{2}\right)}{2(a^2+t)}\right) = \exp\left(\frac{\overline{\sigma}^2 t}{4(a^2+t)}\right)$$

Moreover

$$\begin{aligned}v(t,x) &= aPut_1\left(\frac{a^2}{\overline{\sigma}^2}+t, x\exp\left(\frac{a^2}{2}-l\right)\right) \\ &\quad -a\int \nu(dm)\, Put_1\left(\frac{a^2}{\overline{\sigma}^2}+t, x\exp\left(\frac{a^2}{2}-m\right)\right)\end{aligned}$$

so that for any $x \leq \widetilde{x}(0)$

$$\widetilde{x}^{-1}(x) = \inf\left\{y>0, \int \nu(dm)\exp\left(-\frac{(m-l)\left(\ln x-\frac{l+m}{2}\right)}{2(a^2+y)}\right) = \exp\left(\frac{\overline{\sigma}^2 y}{4(a^2+y)}\right)\right\}$$

$$\begin{aligned}\widehat{\varphi}(x) &= aPut_1\left(\frac{a^2}{\overline{\sigma}^2}+\widetilde{x}^{-1}(x), x\exp\left(\frac{a^2}{2}-l\right)\right) \\ &\quad -a\int \nu(dm)\, Put_1\left(\frac{a^2}{\overline{\sigma}^2}+\widetilde{x}^{-1}(x), x\exp\left(\frac{a^2}{2}-m\right)\right)\end{aligned}$$

Note that in order to get a positive payoff φ it follows from (16) that the additional assumption $\int \nu(dm) = 1$ is enough.

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