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***On the Marginal Laws of One-Dimensional
Stochastic Integrals with Uniformly Elliptic
Integrand***

Claude MARTINI

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On the Marginal Laws of One-Dimensional Stochastic Integrals with Uniformly Elliptic Integrand

Claude MARTINI

Thème 4 — Simulation et optimisation
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Abstract: We show that the marginal law of the stochastic integral with respect to a standard Brownian motion of a uniformly elliptic and bounded progressive process doesn't weight points.

Key-words: Absolute continuity, marginal law, stochastic integral

(Résumé : tsvp)

To appear in *Annales de l'I.H.P. Probabilités et Statistiques*

Sur les lois marginales d'une intégrale stochastique a intégrande uniformément elliptique en dimension 1

Résumé : On montre que la loi marginale de l'intégrale stochastique par rapport au mouvement Brownien standard d'un processus progressif uniformément elliptique et borné est diffuse.

Mots-clé : Absolue continuité, loi marginale, intégrale stochastique

1 Introduction

Let us consider on some P -complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F})_{u \geq 0}, P)$ the stochastic integral

$$M_t = \int_0^t \sigma_u dB_u$$

where B is a one-dimensional standard P - $(\mathcal{F})_{u \geq 0}$ Brownian motion and σ is a $(\mathcal{F})_{u \geq 0}$ -progressive process such that

$$0 < \underline{\sigma} < \sigma_u < \bar{\sigma} < \infty \quad du \, dP - a.s. \quad (1)$$

for two real numbers $(\underline{\sigma}, \bar{\sigma})$, and $t > 0$.

Let A be a Lebesgue null set. Then

$$\begin{aligned} 0 \leq \underline{\sigma}^2 \int_0^T 1_A(M_u) \, du &\leq \int_0^T 1_A(M_u) \, d\langle M \rangle_u \\ &= \int 1_A(a) L_T^a(M) \, da = 0 \quad P \, a.s. \end{aligned}$$

where $L_T^a(M)$ denotes the local time of M . This entails

$$\int_0^T P(M_u \in A) \, du = 0$$

which says that the set of time indexes u at which the law of M_u gives a weight to a given Lebesgue null set A is not a big one, at least of zero measure with respect to the Lebesgue measure on $[0, T]$.

Thus the following question is very natural: for every $t > 0$, is the law of M_t absolutely continuous? This is a much stronger statement: in the reasoning above it could happen that for a fixed u the law of M_u weights A . In fact such a phenomenon does happen for some processes σ : Fabes and Kenig¹ in [2] have designed a uniformly continuous function $\sigma : [0, T] \times \mathbb{R}_+ \rightarrow [1, 2]$ such that the law of the solution of the s.d.e.

$$X_0 = 0, \quad dX_u = \sigma(u, X_u) dB_u$$

is singular at time T , such yielding a negative answer to the issue of absolute continuity.

The purpose of this paper is to show the following:

Theorem 1.1 *For every $t > 0$, the law of M_t doesn't weight points.*

¹I thank Yuyin Hu for this reference.

First of all let us remark that because of assumption (1), σ is a progressive process with respect to the natural filtration of M and B a standard Brownian motion with respect to the same filtration. Therefore we may “normalize” the situation by considering the image law of M on the canonical space $(C(\mathbb{R}_+, \mathbb{R}), \mathcal{B}(C(\mathbb{R}_+, \mathbb{R})), (G_u)_{u \geq 0})$ where $(G_u)_{u \geq 0}$ is the coordinate filtration. Let still denote by P the image law of M . Then under \bar{P} the canonical process $(\omega_u)_{u \geq 0}$ is a $(G_u)_{u \geq 0}$ -martingale such that for some $(G_u)_{u \geq 0}$ -progressive process σ and some $(G_u)_{u \geq 0}$ -standard Brownian motion B

$$\omega = \int_0^\cdot \sigma_u dB_u$$

Let us fix $(\underline{\sigma}, \bar{\sigma})$ and denote by $\mathcal{P}(\underline{\sigma}^2, \bar{\sigma}^2)$ the set of probability laws on

$$(C(\mathbb{R}_+, \mathbb{R}), \mathcal{B}(C(\mathbb{R}_+, \mathbb{R})), (G_u)_{u \geq 0})$$

for which the canonical process may be written this way, or equivalently (cf [3] chapter 5) for which the canonical process is a martingale with bracket almost surely equivalent to du satisfying:

$$0 < \underline{\sigma}^2 < \frac{d\langle \omega \rangle_u}{du} < \bar{\sigma}^2 < \infty \quad du \quad dP - a.s.$$

We take a stochastic control-looking route. We shall work with the symmetrized laws of $\mathcal{P}(\underline{\sigma}^2, \bar{\sigma}^2)$: consider for any P in $\mathcal{P}(\underline{\sigma}^2, \bar{\sigma}^2)$ the law \hat{P} which is the image law of the process

$$u \mapsto \frac{1}{\sqrt{2}}(\omega_u - \omega'_u)$$

defined on the product of the canonical spaces. Notice that thanks to the $\frac{1}{\sqrt{2}}$ factor, $\hat{P} \in \mathcal{P}(\underline{\sigma}^2, \bar{\sigma}^2)$. Set for a function f bounded and Borel

$$\hat{C}(f)(t, x) = \sup_{P \in \mathcal{P}(\underline{\sigma}^2, \bar{\sigma}^2)} E^{\hat{P}}[f(x + \omega_t)]$$

If a law P gives some weight to a point in \mathbb{R} , then \hat{P} weights $\{0\}$ so that we must prove for any $t > 0$ $\hat{C}(1_{\{0\}})(t, 0) = 0$.

Observe that by Brownian scaling, for every t, x , $\hat{C}(f)(t, x) = \hat{C}(f)\left(1, \frac{x}{\sqrt{t}}\right)$, in particular for $x = 0$

$$\hat{C}(1_{\{0\}})(t, 0) = \hat{C}(1_{\{0\}})(1, 0) \tag{2}$$

We proceed as follows: in the next section we show that if f satisfies

$$\begin{aligned} \text{For every } x, f(x) &\leq f(0) \\ \lim_{|x| \rightarrow \infty} f(x) &= 0 \end{aligned} \tag{3}$$

then $\sup_{P \in \mathcal{P}(\underline{\sigma}^2, \bar{\sigma}^2)} E^P [f(\omega_t)]$ (with no hat) for big enough t is smaller than $\lambda^* f(0)$ with $\lambda^* < 1$. This relies on a rough upper bound for $\sup_{P \in \mathcal{P}(\underline{\sigma}^2, \bar{\sigma}^2)} E^P [f(\omega_t)]$ obtained from the Dambins-Dubins-Schwarz representation theorem.

The second ingredient is to show that the function $x \mapsto \widehat{C}(1_{\{0\}})(1, x)$ satisfies (3). The third ingredient (section 3) is the following super-harmonic type property of $\widehat{C}(1_{\{0\}})$:

$$\widehat{C}(1_{\{0\}})(t, 0) \leq \sup_{P \in \mathcal{P}(\underline{\sigma}^2, \bar{\sigma}^2)} E^P \left[\widehat{C}(1_{\{0\}})(1, \omega_t) \right]$$

The result then readily follows from the scaling property (2).

Throughout the paper f stands for a bounded Borel function.

2 A rough upper bound

By the Dambins-Dubins-Schwarz theorem

$$M_t = \int_0^t \sigma_u dB_u = \beta_{\int_0^t \sigma_u^2 du}$$

for some Brownian motion β . Note that $\int_0^t \sigma_u^2 du$ is a stopping time T_t of the filtration G with respect to which β is a Brownian motion with

$$\underline{\sigma}^2 t \leq T_t = \int_0^t \sigma_u^2 du \leq \bar{\sigma}^2 t \text{ a.s.}$$

One can wonder if this property of range of T_t is enough to grant that the law of β_{T_t} doesn't weight points. Of course it is not: take for instance the crossing time of a fixed level by β between times $\underline{\sigma}^2 t$ and $\bar{\sigma}^2 t$. Clearly the stopping times T_t are very particular stopping times: they satisfy for instance the property that $\bar{\sigma}^2 t - T_t = \int_0^t (\bar{\sigma}^2 - \sigma_u^2) du$ is also a stopping time of the same filtration.

Nevertheless, we derive in this section a control of $E^P [f(\beta_T)]$ where T ranges over all stopping times between times $\underline{\sigma}^2 t$ and $\bar{\sigma}^2 t$, with a few assumptions on f , which gives in particular a contraction property for big times (corollary 2.3).

Proposition 2.1 *The following holds:*

$$\sup_{P \in \mathcal{P}(\underline{\sigma}^2, \bar{\sigma}^2)} E^P [f(x + \omega_t)] \leq \sup_G \sup_{\tau \in T(G, \underline{\sigma}^2 t, \bar{\sigma}^2 t)} E [f(x + \beta_\tau)]$$

where $T(G, \underline{\sigma}^2 t, \bar{\sigma}^2 t)$ is the set of the G -stopping times with values almost surely in $[\underline{\sigma}^2 t, \bar{\sigma}^2 t]$

It is easy to give a more explicit bound for this upper bound:

Proposition 2.2 *Let $-\infty < a \leq b < \infty$, $M \geq 0, \varepsilon \geq 0$. Assume $|f| \leq M\mathbf{1}_{[a,b]} + \varepsilon\mathbf{1}_{[a,b]^c}$. Then*

$$\begin{aligned} \sup_G \sup_{\tau \in T(G, \underline{\sigma}^2 t, \bar{\sigma}^2 t)} |E[f(\beta_\tau)]| &\leq \varepsilon + MP(\beta_{\underline{\sigma}^2 t} \in [a, b]) \\ &\quad + 2MP(\beta_{\underline{\sigma}^2 t} < a < \beta_{\bar{\sigma}^2 t}) + 2MP(\beta_{\underline{\sigma}^2 t} < b < \beta_{\bar{\sigma}^2 t}) \end{aligned}$$

We have

$$\sup_{\tau \in T(G, \underline{\sigma}^2 t, \bar{\sigma}^2 t)} |E[f(\beta_\tau)]| \leq M \sup_{\tau \in T(G, \underline{\sigma}^2 t, \bar{\sigma}^2 t)} E[\mathbf{1}_{[a,b]}(\beta_\tau)] + \varepsilon$$

Observe now that

$$\begin{aligned} &\sup_{\tau \in T(G, \underline{\sigma}^2 t, \bar{\sigma}^2 t)} E[\mathbf{1}_{[a,b]}(\beta_\tau)] \\ &= E[\mathbf{1}(T_{[a,b]} \leq \bar{\sigma}^2 t)] = P(T_{[a,b]} \leq \bar{\sigma}^2 t) \end{aligned}$$

where $T_{[a,b]} = \inf\{u \geq \underline{\sigma}^2 t, \beta_u \in [a, b]\}$. Now by the reflection principle

$$\begin{aligned} P(T_{[a,b]} \leq \bar{\sigma}^2 t) &\leq P(\beta_{\underline{\sigma}^2 t} \in [a, b]) \\ &\quad + 2P(\beta_{\underline{\sigma}^2 t} < a < \beta_{\bar{\sigma}^2 t}) + 2P(\beta_{\bar{\sigma}^2 t} < b < \beta_{\underline{\sigma}^2 t}) \end{aligned}$$

whence the result.

As an application:

Corollary 2.3 *Assume $0 \leq f \leq f(0)$ and $f(x) \xrightarrow{|x| \rightarrow \infty} 0$. Then*

$$\limsup_{t \rightarrow \infty} \left(\sup_G \sup_{\tau \in T(G, \underline{\sigma}^2 t, \bar{\sigma}^2 t)} |E[f(\beta_\tau)]| \right) \leq \lambda^* f(0)$$

where

$$\lambda^* = 1 - \frac{2}{\pi} \arctan\left(\frac{\sigma}{\sqrt{\bar{\sigma}^2 - \underline{\sigma}^2}}\right)$$

In particular $\lambda^* < 1$.

By setting $a = b = 0$ in the proposition the result is clear with $2P(\beta_{\underline{\sigma}^2 t} < 0 < \beta_{\bar{\sigma}^2 t}) + 2P(\beta_{\bar{\sigma}^2 t} < 0 < \beta_{\underline{\sigma}^2 t})$ for the value of λ^* . Now

$$\begin{aligned} P(\beta_{\underline{\sigma}^2 t} < 0 < \beta_{\bar{\sigma}^2 t}) &= P(\beta_{\underline{\sigma}^2 t} < 0, -\beta_{\underline{\sigma}^2 t} < \beta_{\bar{\sigma}^2 t} - \beta_{\underline{\sigma}^2 t}) \\ &= P\left(X < 0, -\underline{\sigma}\sqrt{t}X < \sqrt{\bar{\sigma}^2 t - \underline{\sigma}^2 t}Y\right) \end{aligned}$$

whence

$$\begin{aligned} \frac{\lambda^*}{2} &= P\left(X < 0, -\underline{\sigma}\sqrt{t}X < \sqrt{\overline{\sigma}^2t - \underline{\sigma}^2t}Y\right) \\ &\quad + P\left(X > 0, -\underline{\sigma}\sqrt{t}X > \sqrt{\overline{\sigma}^2t - \underline{\sigma}^2t}Y\right) \\ &= P\left(\frac{Y}{X} < -\frac{\underline{\sigma}}{\sqrt{\overline{\sigma}^2 - \underline{\sigma}^2}}\right) \end{aligned}$$

where X and Y are two independent standard gaussian random variables. Then $\frac{Y}{X}$ is Cauchy, whence

$$\begin{aligned} \frac{\lambda^*}{2} &= \int_{-\infty}^{-\frac{\underline{\sigma}}{\sqrt{\overline{\sigma}^2 - \underline{\sigma}^2}}} \frac{dx}{\pi(1+x^2)} \\ &= \frac{1}{\pi} \left[\frac{\pi}{2} - \arctan\left(\frac{\underline{\sigma}}{\sqrt{\overline{\sigma}^2 - \underline{\sigma}^2}}\right) \right] \end{aligned}$$

We also get from the proposition, by replacing $f(\cdot)$ by $f(x + \cdot)$:

Lemma 2.4 *If $f(x) \xrightarrow{|x| \rightarrow \infty} 0$ then for every t*

$$\sup_G \sup_{\tau \in T(G, \underline{\sigma}^2t, \overline{\sigma}^2t)} |E[f(x + \beta_\tau)]| \rightarrow 0 \text{ as } |x| \rightarrow \infty.$$

3 Superharmonic-type property of $\widehat{C}(1_{\{0\}})$

In fact we prove in this section the announced property for any f .

Notice first:

Lemma 3.1 *For every $f, t > 0$, the function $x \mapsto \widehat{C}(f)(t, x)$ is Borel.*

Take first a (bounded) Lipschitz f with Lipschitz constant k . Then for any P in $\mathcal{P}(\underline{\sigma}^2, \overline{\sigma}^2)$

$$\left| E^{\widehat{P}}[f(x + \omega_t)] - E^{\widehat{P}}[f(x + \varepsilon + \omega_t)] \right| \leq k\varepsilon$$

which entails

$$\left| \widehat{C}(f)(t, x) - \widehat{C}(f)(t, x + \varepsilon) \right| \leq k\varepsilon$$

so that the function $\widehat{C}(f)$ is Lipschitz, therefore Borel. The result follows by a monotone class argument, more precisely by the version given in theorem 21, chapter 2 of [1] of the monotone class theorem.

Lemma 3.2 For any $f, x, s > 0, t > 0$

$$\widehat{C}(f)(t+s, x) \leq \sup_{P \in \mathcal{P}(\underline{\sigma}^2, \overline{\sigma}^2)} E^P \left[\widehat{C}(f)(t, x + \omega_s) \right]$$

For any P in $\mathcal{P}(\underline{\sigma}^2, \overline{\sigma}^2)$, with transparent notations

$$\begin{aligned} & E^{\widehat{P}} [f(x + \omega_{t+s})] \\ &= E^{P \otimes P} [f(x + \omega'_{t+s} - \omega''_{t+s})] \\ &= E^{P \otimes P} [E^{P \otimes P} [f(x + (\omega'_s - \omega''_s) + (\omega'_{t+s} - \omega''_{t+s}) - (\omega'_s - \omega''_s)) \mid F'_s \otimes F''_s]] \end{aligned}$$

Observe now that the conditional law with respect to $F'_s \otimes F''_s$ is the product of the conditional laws, therefore it is a symmetrized law of $\mathcal{P}(\underline{\sigma}^2, \overline{\sigma}^2)$ so that

$$\begin{aligned} & E^{P \otimes P} [f(x + (\omega'_s - \omega''_s) + (\omega'_{t+s} - \omega''_{t+s}) - (\omega'_s - \omega''_s)) \mid F'_s \otimes F''_s] \\ & \leq \widehat{C}(f)(t, x + (\omega'_s - \omega''_s)) \quad P \otimes P \text{ a.s.} \end{aligned}$$

Whence $E^{\widehat{P}} [f(x + \omega_{t+s})] \leq E^{\widehat{P}} [\widehat{C}(f)(t, x + \omega_s)]$ so that

$$E^{\widehat{P}} [f(x + \omega_{t+s})] \leq \sup_{P \in \mathcal{P}(\underline{\sigma}^2, \overline{\sigma}^2)} E^P \left[\widehat{C}(f)(t, x + \omega_s) \right]$$

The result follows.

4 Conclusion

In order to apply corollary 2.3 to the function $\widehat{C}(1_{\{0\}})$ we need the following property:

Lemma 4.1 For every $t > 0$, $\widehat{C}(1_{\{0\}})(t, x) \leq \widehat{C}(1_{\{0\}})(t, 0)$

Indeed the result is true if $\widehat{C}(1_{\{0\}})(t, x) = 0$. If not, take P such that P weights some points, let I be the (denumerable) set of such points. Then $\widehat{P}(x) = \sum_{y \in I / y-x \in I} P(y-x) P(y)$ so that

$$\begin{aligned} 2\widehat{P}(x) &= 2 \sum_{y \in I / y-x \in I} P(y-x) P(y) \\ &\leq \sum_{y \in I / y-x \in I} P^2(y) + \sum_{y-x \in I / y \in I} P^2(y-x) \\ &\leq \sum_{y \in I} P^2(y) + \sum_{z \in I} P^2(z) \\ &= 2\widehat{P}(0) \end{aligned}$$

The result follows.

In fact, the above property of the set of symmetrized laws explains why we work with these instead of $\mathcal{P}(\underline{\sigma}^2, \overline{\sigma}^2)$: we were not able to prove the corresponding a priori inequality for $\mathcal{P}(\underline{\sigma}^2, \overline{\sigma}^2)$.

It is easy to conclude now: by lemma 2.4 and lemma 4.1 we can apply corollary 2.3 to the function $x \mapsto C(1_{\{0\}})(1, x)$. Next lemma 3.2 altogether with the scaling property (2) yield

$$C(1_{\{0\}})(1, 0) = \limsup_{t \rightarrow \infty} C(1_{\{0\}})(t, 0) \leq \lambda^* C(1_{\{0\}})(1, 0)$$

whence $C(1_{\{0\}})(1, 0) = 0$ since $\lambda^* < 1$.

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