# Using Viscosity Solution for Approximations in Piecewise Deterministic Control Systems

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# Using Viscosity Solution for Approximations in Piecewise Deterministic Control Systems

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**Abstract:** In [15] a numerical approximation scheme is proposed for the solution of piecewise deterministic control systems (PDCS). This approximation scheme is based on a time discretization which reformulates the original PDCS into a stochastic program. In this paper we prove the convergence of the approximating vector value function to the vector value function of the original PDCS as the discretization's step tends to zero.

**Key-words:** piecewise deterministic control systems, stochastic jump, vectorial value function, approximation, viscosity solution

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# Utilisation des solutions de viscosité pour l'approximation dans les systèmes contrôlés continus par morceaux

**Résumé :** L'article [15] présente un schéma d'approximation numérique pour la solution d'un problème de contrôle déterministe par morceaux. Cette approximation, basée sur une discrétisation du temps, permet de reformuler le problème de contrôle en un problème de programmation stochastique. Dans ce papier on prouve la convergence de la fonction vectorielle approximante vers la fonction valeur vectorielle du problème de contrôle initial quand le pas de discretisation converge vers zéro

**Mots-clés :** Contrôle déterministe par morceaux, sauts stochastiques, fonction valeur vectorielle, approximation, solution de viscosité

# 1 Introduction

Piecewise deterministic control systems (PDCS) are stochastic controlled processes that evolve according to a deterministic dynamic except at some stochastic time where a stochastic jump of the state variable occurs. For three decades, PDCS have been the object of considerable investigation in Control Theory (see for example [9], [10],[19], [20] and [21]). They offer a wide variety of applications for the modeling of industrial and economic processes; we can quote, amongst other things, manufacturing (see *e.g.* [5], [2] and [14]), optimal exploration and consumption of renewable and non-renewable resources [12], permanent health insurance [11] and capacity expansion [17].

In [15], Haurie and Moresino proposed a numerical approximation scheme for the solution of a special class of PDCS. The state variable of the PDCS studied is composed of a continuous state variable that evolves according to a controlled differential equation and a discrete state variable that evolves according to an uncontrolled jump process taking values in a finite set. This approximation scheme is based on a time discretization which reformulates the original PDCS into a stochastic program. The main scope of this paper is to prove, using viscosity techniques, the convergence of this discretization scheme. As the value function of the PDCS is a vector valued function, it is necessary to extend the classical viscosity solution first introduced by Crandall, Ishii and Lions (see *e.g.* [8]). Such an extension was already done in [7] and [16] in a slightly different context. The scheme of proof used in this paper is an extension of some techniques first used for deterministic control system by Capuzzo-Dolcetta in [6].

The paper is organized as follows. In Section 2 we formulate the PDCS and the admissible strategies and give the hypotheses needed throughout the paper. We recall the dynamic programming principle stated in [19] and prove regularity properties satisfied by the vector value function of the PDCS. In Section 3 we define the vector viscosity solution and show that it is consistent with the classical solution. We prove, using fixed point arguments, that the vector value function of the PDCS is the unique Lipschitz continuous vector viscosity solution of a system of coupled Hamilton-Jacobi-Bellman's (HJB) type equations. In Section 4 we recall briefly the approximation scheme studied in [15] and its interpretation in terms of stochastic program. We prove that the sequence of vector value functions associated to the approximating stochastic programs converges to the vector viscosity solution of the HJB system, and consequently to the vector value function of the initial PDCS, as the discretization's step tends to zero.

# 2 The piecewise deterministic control system

## 2.1 Dynamics of the PDCS

We consider a piecewise deterministic control system, with hybrid state $(y(t), \xi(t))$, where $y(t) \in I\!R^m$ denotes the continuous part of the state while $\xi(t)$ denotes the discrete part. The discrete state, $\xi(t)$, belongs to a finite set $\mathcal{I} = \{1, 2, \ldots, I\}$, and evolves according to a continuous time Markov jump process with transition rates defined by

$$\begin{aligned} P[\xi(t+dt) = j | \xi(t) = i] &= q_{ij} dt + o(dt) \quad i, j \in \mathcal{I}, \quad i \neq j \\ P[\xi(t+dt) = i | \xi(t) = i] &= 1 + q_{ii} dt + o(dt)\,, \end{aligned} \tag{1}$$

with

$$q_{ii} = -\sum_{i \neq j} q_{ij} < 0 \tag{2}$$

and

$$\lim_{dt\to 0} \frac{o(dt)}{dt} = 0 \tag{3}$$

The continuous state evolves according to a differential equation that depends on the value taken by the discrete state. More precisely, if the discrete state at time $t$ is $\xi(t) = i$, then the continuous state evolves according to the following state equation from time $t$ on until the next jump of the discrete state occurs

$$\dot{y}(t) = f^i(y(t), u(t)), \quad u(t) \in U^i, \tag{4}$$

where the control $u(t)$ takes value in a set $U^i$.

**Assumption 1.** *$U^i$ is a closed compact set for all $i$ in $\mathcal{I}$.*

**Assumption 2.** *The functions $f^i$, $i \in \mathcal{I}$ are Lipschitz continuous in $x$, continuous in $(x,u)$ and bounded, i.e.*

- $||f^i(x,u) - f^i(y,u)|| \le C_f ||x-y||, \quad \forall i \in \mathcal{I}, x, y \in I\!R^m, u \in U^i,$
- *$f^i(x,u)$ is continuous in $(x,u)$,*
- $||f^i(x,u)|| \le M_f, \quad \forall (x,u) \in I\!R^m \times U^i.$

Assumptions 1 and 2 insure the existence and uniqueness of the solution of (4) for each possible initial point $(x,i)$ at time $t$.

Denote $X_0 \subset I\!R^m$ the compact set of possible initial continuous state. Let $\mathcal{X} \subset I\!R^m \times [0,T]$ be the set of points $(y,t)$ such that there exists a trajectory starting at time $t=0$ from a point $x \in X_0$ and reaching the continuous state $y$ at time $t$. It is clear from Assumptions 1 and 2 that the reachable set $\mathcal{X}$ is a closed bounded subset of $I\!R^m \times [0,T]$.

Define $\lambda_0$ as follows:

$$\lambda_0 = \sup_{\substack{u\in\mathcal{U}\ i\in\mathcal{I} \\ x\neq y}} \frac{\langle f^i(x,u) - f^i(y,u), x-y\rangle}{|x-y|^2}. \tag{5}$$

**Assumption 3.** $\lambda_0 \le C_\lambda, \quad C_\lambda > 0.$

**Lemma 1.** *Let $(x(s), \xi(t))$ and $(y(s), \xi(t))$ be two trajectories starting respectively at time $t=0$ at $(x_0, \xi_0)$ and $(y_0, \xi_0)$ associated with the same control $u(\cdot)$ and the same realization of the jump process. Then we have for all $s > 0$*

$$|x(s) - y(s)| \le |x_0 - y_0| e^{\lambda_0 s}.$$

**Proof :** Since the evolution of the discrete state does not depend on the value of the continuous state, the lemma follows directly from Gronwald theorem (see *e.g.* [1]).

$\diamond\diamond$

## 2.2 Admissible controls and piecewise open loop strategies

For a given initial hybrid state of the system $(x, i)$ at time $\tau$, an admissible open loop control, will be a measurable mapping

$$u(\cdot) : [\tau, T[\to U^i$$

such that the solution of (4), with initial condition $y(\tau) = x$ exists and is unique. Let us denote $\mathcal{U}^i$ the set of such mappings.

In this context a strategy of the controller is described in the following way. At each jump time of the system, *i.e.* at each instant $\tau$ when a jump of the discrete state occurs, the controller observes the new hybrid state $(y(\tau), \xi(\tau)) = (x, i)$ and chooses an admissible open loop control in the set $\mathcal{U}^i$. This open loop control will be applied, until time $\tau'$, when either a new jump of the system occurs, or the final time is reached, *i.e.* $\tau' = T$. During the interval of time $[\tau, \tau'[$, the associated trajectory $y(\cdot)$ of the continuous state is solution of

$$\dot{y}(t) = f^i(y(t), u(t)), \text{ with } y(\tau) = x.$$

To sum up a strategy $\gamma$ is thus a mapping from $[\tau, T[\times \mathbb{R}^m \times \mathcal{I}$ to $\cup_{i \in \mathcal{I}} \mathcal{U}^i$.

Notice that since we have a deterministic control problem between two successive jumps, open loop and closed loop strategies are equivalent between these two jumps. So in this case piecewise open loop and piecewise closed loop strategies are equivalent.

A complete and precise description of admissible strategies involves the use of a concept of solution of an ordinary differential equation with discontinuous right-hand side, since the control $u(\cdot)$ and consequently the functions $f^i(y(t), u(t))$ can be discontinuous. This can be found in [19] in a more general setup.

## 2.3 Value function and optimality equations

Suppose that, at time $t$, the state of the system is $(x, i) \in X_t \times \mathcal{I}$, where $X_t = \{x | (x, t) \in \mathcal{X}\}$, and that the controller uses a given strategy $\gamma$. The trajectory $y(\cdot)$ of the continuous state together with the trajectory of the control $u(\cdot)$ is a stochastic processes with measure $P$ induced by $\gamma$. We define the evaluation function associated to the strategy $\gamma$ and initial state $(x, i)$ as

$$J^i_\gamma(x, t) = E_P[\int_t^T L^{\xi(s)}(y(s), u(s))\, ds \mid y(t) = x, \xi(t) = i], \tag{6}$$

**Assumption 4.** *The instantaneous cost is Lipschitz continuous in $x$, continuous in $(x, u)$ and bounded, i.e*

- $|L^i(x, u) - L^i(y, u)| \leq C_L ||x - y||, \quad \forall x, y \in \mathcal{X}, u \in U^i,$
- *$L^i(x, u)$ is continuous in $(x, u)$,*
- $|L^i(x, u)| \leq M_L \, \forall x \in \mathbb{R}^m, u \in U^i.$

Throughout the paper we assume that Assumptions 1 to 4 are satisfied.

We are interested to find the optimal vector value function $(\mathcal{V}^i(x, t))_{i=1,\ldots,I}$ where

$$\mathcal{V}^i(x, t) = \inf_\gamma J^i_\gamma(x, t), \quad i = 1, \ldots, I. \tag{7}$$

Now we state the dynamic programming principle verified by $\mathcal{V}^i$. This results can be for example found in [13], in a slightly different formulation, or in a more general set up in [19].

**Proposition 1.** *For any $(x,t)$ in $\mathcal{X}$ and positive $\epsilon$ we have*

$$\mathcal{V}^i(x,t) = \inf_{u(\cdot)\in\mathcal{U}^i} \Bigg\{ \mathcal{V}^i(y(t+\epsilon), t+\epsilon) + \int_t^{t+\epsilon} L^i(y(s),u(s)) + \sum_{j\in\mathcal{I}} q_{ij}\mathcal{V}^j(y(s),s)ds \Bigg\} \tag{8}$$

*where $y(\cdot)$ is given by (4) and $i = 1,\ldots,I$.*

The proof of this proposition can be found in [19] or in [13] for a slightly different version.

**Theorem 1.** *If Assumptions 2 to 4 hold then the functions $\mathcal{V}^i(x,t)$ are bounded and Lipschitz continuous in $(x,t)$.*

**Proof :** The boundedness is straightforward from the definition (6) of the cost function together with Assumption 4. Let us compute the difference $\mathcal{V}^i(\bar{x},\bar{t}) - \mathcal{V}^i(x,t)$. We have

$$\begin{aligned} &|\mathcal{V}^i(\bar{x},\bar{t}) - \mathcal{V}^i(x,t)| \\ &\leq |\mathcal{V}^i(\bar{x},\bar{t}) - \mathcal{V}^i(\bar{x},t)| + |\mathcal{V}^i(\bar{x},t) - \mathcal{V}^i(x,t)| \end{aligned} \tag{9}$$

On the one hand, by definition (6) of the cost function, we have

$$\begin{aligned} &|\mathcal{V}^i(\bar{x},t) - \mathcal{V}^i(x,t)| \\ &= \left| \inf_\gamma E_P \left[ \int_t^T L^{\xi(s)}(\bar{y}(s),u(s))ds | \xi(t)=i, \bar{y}(t)=\bar{x} \right] \right. \\ &\qquad \left. - \inf_\gamma E_P \left[ \int_t^T L^{\xi(s)}(y(s),u(s))ds | \xi(t)=i, y(t)=x \right] \right| \\ &= \sup_\gamma E_P \Bigg[ \int_t^T |L^{\xi(s)}(\bar{y}(s),u(s))ds - L^{\xi(s)}(y(s),u(s))| ds \\ &\qquad\qquad |\xi(t)=i, \bar{y}(t)=\bar{x}, y(t)=x \Bigg]. \end{aligned}$$

The Lipschitz property of $L$ and lemma 1 imply

$$|\mathcal{V}^i(\bar{x},t) - \mathcal{V}^i(x,t)| \leq ||\bar{x}-x|| C_L C_1 \leq ||(\bar{x},\bar{t}) - (x,t)|| C_L C_1 \tag{10}$$

where $C_1 = \frac{e^{C_\lambda T}}{C_\lambda}$ is a constant.

On the other hand,

$$\begin{aligned} |\mathcal{V}^i(\bar{x},\bar{t}) - \mathcal{V}^i(\bar{x},t)| \quad = \quad & \left| \inf_\gamma E_P \left[ \int_{\bar{t}}^T L^{\overline{\xi}(s)}(\bar{y}(s),u(s))ds \mid \overline{\xi}(\bar{t})=i, \bar{y}(\bar{t})=\bar{x} \right] \right. \\ & \left. - \inf_\gamma E_P \left[ \int_t^T L^{\xi(s)}(y(s),u(s))ds | \xi(t)=i, y(t)=\bar{x} \right] \right| \end{aligned}$$

Without loss of generality, we may suppose that $\bar{t} > t$. Using the fact that the instantaneous cost $L$ is bounded it comes that

$$\begin{aligned}
&|\mathcal{V}^i(\bar{x},\bar{t}) - \mathcal{V}^i(\bar{x},t)| \\
&\quad \leq \Bigg| \inf_{\gamma} E_P \left[ \int_{\bar{t}}^{T-(\bar{t}-t)} L^{\bar{\xi}(s)}(\bar{y}(s),u(s))ds \mid \bar{\xi}(\bar{t}) = i, \bar{y}(\bar{t}) = \bar{x} \right] + \int_{T-(\bar{t}-t)}^{T} M_L ds \\
&\qquad - \inf_{\gamma} E_P \left[ \int_t^T L^{\xi(s)}(y(s),u(s))ds|\xi(t) = i, y(t) = \bar{x} \right] \Bigg|.
\end{aligned}$$

The first and the last terms in the previous expression involve stochastic replicas and are therefore equal. We thus obtain

$$|\mathcal{V}^i(\bar{x},\bar{t}) - \mathcal{V}^i(\bar{x},t)| \leq M_L|\bar{t} - t| \leq M_L||(\bar{x},\bar{t}) - (x,t)||. \tag{11}$$

Taking together inequalities (9), (10) and (11) leads to the following result

$$|\mathcal{V}^i(\bar{x},\bar{t}) - \mathcal{V}^i(x,t)| \leq (M_L + C_L C_1)||(\bar{x},\bar{t}) - (x,t)||, \tag{12}$$

which concludes the proof.

$\diamond\diamond$

# 3 Vectorial viscosity solution

In this section we extend the notion of classical viscosity solution for a system of first order partial differential equations.

## 3.1 Definition and properties

We consider the following system of coupled first order differential equations with boundary conditions

$$\begin{cases}
H^1(x,t,V^1(x,t),\ldots,V^I(x,t),\nabla V^1(x,t)) = 0 \\
H^2(x,t,V^1(x,t),\ldots,V^I(x,t),\nabla V^2(x,t)) = 0 \\
\vdots \\
H^I(x,t,V^1(x,t),\ldots,V^I(x,t),\nabla V^I(x,t)) = 0, \quad \forall (x,t) \in \mathcal{X} \\
\\
V^1(x,T) = V^2(x,T) \cdots = V^I(x,T) = 0 \quad \forall x,
\end{cases} \tag{13}$$

where $H^i$ are continuous functions from $I\!R^m \times I\!R \times I\!R^I \times I\!R^{m+1}$ to $I\!R$.

**Definition 1.** *The continuous vector function of $\mathcal{X}$ $V(x,t) = (V^1(x,t),\ldots,V^I(x,t))$ is said to be a continuous vector viscosity solution of the system (13) if:*

- *for all $\phi \in C^1(\mathcal{X})$, if $(y,s)$ is a local maximum of $V^i - \phi$ for some $i \in \mathcal{I}$, we have*

$$H^i(y,s,V^1(y,s),\ldots,V^I(y,s),\nabla\phi(y,s)) \leq 0, \tag{14}$$

*and*

- *for all $\phi \in C^1(\mathcal{X})$, if $(y,s)$ is a local minimum of $V^i - \phi$ for some $i \in \mathcal{I}$, we have*

$$H^i(y,s,V^1(y,s),\ldots,V^I(y,s),\nabla\phi(y,s)) \geq 0. \tag{15}$$

*If $V(x,t)$ verifies only (14) (respectively (15)), we call it a vector viscosity sub-solution (resp. super-solution).*

A similar definition was proposed in [7] and in [16]. This definition is a straightforward extension of the continuous viscosity solution for first order partial differential equation first developed by Crandall, Ishii and Lions (see *e.g.* [8]).

The following theorem links classical solution and viscosity solution of (13), in the case where (13) admits a classical solution (*i.e.* a continuously differentiable solution).

**Theorem 2.** *Suppose $V^i(x,t)$ is $C^1$ in $\mathcal{X}$, for all $i$ in $\mathcal{I}$, then $V(x,t)$ is a classical solution of (13) if and only if it is a vector viscosity solution of (13).*

**Proof :** Let $V$ be a viscosity solution of (13). We take $\phi = V^i$ in the definition 1. Since each point of $\mathcal{X}$ is a maximum and a minimum of $V^i - \phi$ we obtain the two inequalities

$$H^i(x,t,V^1(x,t),\dots,V^I(x,t),\nabla V^i(x,t)) \le 0$$

and

$$H^i(x,t,V^1(x,t),\dots,V^I(x,t),\nabla V^i(x,t)) \ge 0$$

over $\mathcal{X}$, and conclude that $V$ is a classical solution.
Conversely, if $V$ is a classical solution of (13) and if $y$ is a local maximum (*resp.* minimum) of $V^i - \phi$, we have $\nabla V^i(y) = \nabla\phi(y)$ and so $H(y,V(y),\nabla\phi(y)) = H(y,V(y),\nabla V^i(y)) = 0$, from which we conclude that $V$ is a viscosity sub-solution (*resp.* super-solution).

◊◊

## 3.2 The viscosity solution of the PDCS

The dynamic programming principle applied to PDCS defines a system of coupled Hamilton-Jacobi-Bellman's (HJB) equations.

**Theorem 3.** *The vector value function of the PDCS, $\mathcal{V} = (\mathcal{V}^1,\dots,\mathcal{V}^I)$, defined by (7) with (6) is a viscosity solution of the equation (13) where the Hamiltonians are defined by*

$$\begin{aligned}&H^i(x,t,V^1(x,t),\dots,V^I(x,t),\nabla V^i(x,t)) = \\ &\min_{u\in U^i}\{-L^i(x,u) - \sum_{j\in\mathcal{I}} q_{ij}V^j(x,t) - \frac{\partial}{\partial t}V^i(x,t) - \nabla V^i(x,t)f^i(x,u)\} \quad i=1,\dots,I.\end{aligned} \tag{16}$$

**Proof :** We first prove that the value function $\mathcal{V}$ is sub-solution. Consider $\varphi$ a $C^1(\mathcal{X})$ function. Suppose that a local minimum of $\mathcal{V}^i - \varphi$ is attained on $(\tilde{x},\tilde{t})$. Without any loss of generality we can assume that $\mathcal{V}^i(\tilde{x},\tilde{t}) = \varphi(\tilde{x},\tilde{t})$. Consequently, there exists a real positive number, $r$, such that for all $(y,s)$ such that $||(\tilde{x},\tilde{t}) - (y,z)|| \le r$ we have

$$\mathcal{V}^i(y,s) \le \varphi(y,s),$$

and then for $\epsilon$ sufficiently small, using proposition 1 we can write,

$$\begin{aligned}&\varphi(\tilde{x},\tilde{t}) - \varphi(\tilde{y}(\tilde{t}+\epsilon),\tilde{t}+\epsilon) \le \mathcal{V}^i(\tilde{x},\tilde{t}) - \mathcal{V}^i(\tilde{y}(\tilde{t}+\epsilon),\tilde{t}+\epsilon) \\ &\le \int_{\tilde{t}}^{\tilde{t}+\epsilon} L^i(\tilde{y}(s),u)\,ds + \sum_j \int_{\tilde{t}}^{\tilde{t}+\epsilon} q_{ij}\mathcal{V}^j(\tilde{y}(s),s)\,ds,\end{aligned} \tag{17}$$

where $\tilde{y}(\cdot)$ is the continuous state trajectory corresponding to initial conditions $(\tilde{x}, i)$ at time $\tilde{t}$, when the constant control $u(\cdot) = u \in U^i$ is applied. Dividing equation (17) by $\epsilon$ and letting $\epsilon$ tend to zero we obtain:

$$-\frac{\partial}{\partial t}\varphi(\tilde{x},\tilde{t}) - \nabla\varphi(\tilde{x},\tilde{t})f^i(\tilde{x},u) - L^i(\tilde{x},u) - \sum_{j\neq i} q_{ij}[\mathcal{V}^j(\tilde{x},\tilde{t}) - \mathcal{V}^i(\tilde{x},\tilde{t})] \leq 0.$$

This inequality is true for all $u$, and in particular for the control $u$ that minimizes the left hand side of the last inequality. We thus can conclude that $\mathcal{V} = (\mathcal{V}^1, \ldots, \mathcal{V}^I)$ is a sub-solution of (13) with (16), in the sense of the definition 1.

We now prove that $\mathcal{V}$ is a super-solution. Let $\varphi \in C^1(\mathcal{X})$ and $(\tilde{x}, \tilde{t})$ be a local maximum of $\mathcal{V}^i - \varphi$. Again we assume that $\mathcal{V}^i(\tilde{x},\tilde{t}) = \varphi(\tilde{x},\tilde{t})$. So, there exists $r > 0$ such that, for any $(y, s)$ satisfying $||(y,s) - (\tilde{x},\tilde{t})|| \leq r$, we have :

$$\mathcal{V}^i(y,s) \geq \varphi(y,s).$$

>From (8) we obtain the following equality:

$$\begin{aligned}\mathcal{V}^i(\tilde{x},\tilde{t}) \quad = \quad & \min_{u(\cdot)\in\mathcal{U}^i}\{\mathcal{V}^i(\tilde{y}(\tilde{t}+\epsilon),\tilde{t}+\epsilon) + \\ & \int_{\tilde{t}}^{\tilde{t}+\epsilon} L^i(\tilde{y}(s),u(s)) + \sum_{j\in\mathcal{I}} q_{ij}\mathcal{V}^j(\tilde{y}(s),s)ds\}\end{aligned}$$

and consequently

$$\begin{aligned}\varphi(\tilde{x},\tilde{t}) \quad \geq \quad & \min_{u(\cdot)\in\mathcal{U}^i}\{\varphi(\tilde{y}(\tilde{t}+\epsilon),\tilde{t}+\epsilon) + \\ & \int_{\tilde{t}}^{\tilde{t}+\epsilon} L^i(\tilde{y}(s),u(s)) + \sum_{j\in\mathcal{I}} q_{ij}\mathcal{V}^j(\tilde{y}(s),s)\, ds\}.\end{aligned}$$

Using the equality

$$\begin{aligned}&\varphi(\tilde{y}(\tilde{t}+\epsilon),\tilde{t}+\epsilon) = \varphi(\tilde{x},\tilde{t}) \\ &+ \int_{\tilde{t}}^{\tilde{t}+\epsilon} \frac{\partial}{\partial t}\varphi(\tilde{y}(s),s) + \nabla\varphi(\tilde{y}(s),s)f^i(\tilde{y}(s),u(s))ds,\end{aligned}$$

we rewrite the last inequality

$$\begin{aligned}0 \quad \geq \quad & \min_{u(\cdot)\in\mathcal{U}^i}\{\int_{\tilde{t}}^{\tilde{t}+\epsilon} L^i(\tilde{y}(s),u(s)) + \sum_{j\in\mathcal{I}} q_{ij}\mathcal{V}^j(\tilde{y}(s),s) \\ & +\frac{\partial}{\partial t}\varphi(\tilde{y}(s),s) + \nabla\varphi(\tilde{y}(s),s)f^i(\tilde{y}(s),u(s))ds\}\end{aligned}$$

which implies

$$\begin{aligned}0 \quad \geq \quad & \min_{u(\cdot)\in\mathcal{U}^i}\{\int_{\tilde{t}}^{\tilde{t}+\epsilon} \min_{w\in\mathcal{U}^i}[L^i(\tilde{y}(s),w) + \sum_{j} q_{ij}\mathcal{V}^j(\tilde{y}(s),s) \\ & +\frac{\partial}{\partial t}\varphi(\tilde{y}(s),s) + \nabla\varphi(\tilde{y}(s),s)f^i(y(s),w)]ds\}\end{aligned}$$

or again

$$0 \geq \min_{u(\cdot)\in\mathcal{U}^i}\{\int_{\tilde{t}}^{\tilde{t}+\epsilon} -H^i(\tilde{y}(s),s,\mathcal{V}(\tilde{y}(s),s),\nabla\varphi(\tilde{y}(s),s))\, ds\}.$$

Dividing by $-\epsilon$ and letting $\epsilon$ tend to zero we obtain

$$0 \leq H^i(\tilde{x}, \tilde{t}, \mathcal{V}(\tilde{x}, \tilde{t}), \nabla\varphi(\tilde{x}, \tilde{t})), \tag{18}$$

which concludes the proof of the fact that $\mathcal{V}$ is a super-solution and finishes the proof of the theorem.

$\diamond\diamond$

Let us denote $C^{0,1}(\mathcal{X})$ the set of the Lipschitz continuous functions defined on $\mathcal{X}$.

**Theorem 4.** *In $\Pi_{i\in\mathcal{I}}C^{0,1}(\mathcal{X})$ there exists a unique viscosity solution of (13,16).*

To prove theorem 4 we need a result that links "classical" viscosity solution with the solution of an optimal control problem, and a result that gives the uniqueness of "classical" viscosity solution. These results can be found for example in [4] (theorem 3.6) that we recall below for the sake of completeness :

**Theorem 5.** *Consider the finite horizon optimal control problem given by its dynamics*

$$\dot{y} = f(y(t), u(t)), \quad y(0) = x$$

*and the cost function*

$$J(x, u(\cdot)) = \int_{t=0}^{T} L(y(s), u(s))ds.$$

*Suppose that the dynamics and instantaneous cost functions $f$ and $L$ are Lipschitz continuous with respect to their first argument, bounded and continuous in $(y, u)$ and that $V_0$ is bounded uniformly continuous. Then the value function of the optimal control problem is the unique bounded uniformly continuous viscosity solution of the Bellman equation*

$$\frac{\partial V}{\partial t} + H(x, V, \nabla V) = 0$$

*associated to the boundaries conditions*

$$V(x, 0) = V_0(x)$$

*with*

$$H(x, V, p) = \sup_{u\in U} \{-pf(x, u) - L(x, u)\}.$$

**Proof of Theorem 4 :** Existence follows directly from theorem 3, since we have exhibited such a solution.

In order to prove uniqueness, we introduce an operator $\mathcal{T}$ from the set $\Pi_{i\in\mathcal{I}}C^{0,1}(\mathcal{X})$ to itself, and will show that it is contractive and consequently admits a unique fixed point.

Define the operator $\mathcal{T}$ in the following way :

$$\begin{aligned} \mathcal{T}: \quad & \Pi_{i\in\mathcal{I}}C^{0,1}(\mathcal{X}) \to \Pi_{i\in\mathcal{I}}C^{0,1}(\mathcal{X}) \\ & (\mathcal{W}^1, \dots, \mathcal{W}^I) \longrightarrow (\mathcal{V}^1, \dots, \mathcal{V}^I), \end{aligned}$$

where $\mathcal{V}^i$ is the standard viscosity solution of the following equation

$$\begin{aligned} 0 \;=\; & \min_{u\in U^i}\{-\tilde{L}^i(x, u) \\ & -\frac{\partial}{\partial t}\mathcal{V}^i(x, t) - \nabla\mathcal{V}^i(x, t)f^i(x, u) - q_{ii}\mathcal{V}^i(x, t)\}, \end{aligned} \tag{19}$$

where

$$\tilde{L}^i(x,u) = L^i(x,u) + \sum_{j \neq i} q_{ij} \mathcal{W}^j(x,t) \tag{20}$$

with terminal conditions $\mathcal{V}^i(x,T) = 0$ for all $x$.

Notice that if for $i \neq j$, the functions $\mathcal{W}^j$ are Lipschitz continuous, then the instantaneous cost functions $\tilde{L}^i$ satisfy Assumption 4.
Notice also that, according to theorem 5, the viscosity solution of this equation can be interpreted as the solution of a problem of control with finite horizon $T$, where the dynamics is given by the function $f^i$, the instantaneous cost is given by the function $\tilde{L}$ and the discount rate is given by $q_{ii}$.

Let us now prove that $\mathcal{T}$ is well defined, *i.e.* that $\mathcal{T}(\mathcal{W}^1,\ldots,\mathcal{W}^I)$ exists, is unique, and belongs to $\Pi_{i\in\mathcal{I}} C^{0,1}(\mathcal{X})$ for $(\mathcal{W}^1,\ldots,\mathcal{W}^I) \in \Pi_{i\in\mathcal{I}} C^{0,1}(\mathcal{X})$. The existence follows directly from the previous remark. $\mathcal{T}(\mathcal{W}^1,\ldots,\mathcal{W}^I)$ can be interpreted as a vector of classical viscosity solutions for decoupled optimal control problems. According to Theorem 5 we have existence and uniqueness. The fact that each component of $\mathcal{T}(\mathcal{W}^1,\ldots,\mathcal{W}^I)$ belongs to $C^{0,1}$, follows directly from the fact that each component can be interpreted as the solution of a finite horizon optimal control problem without final cost and Lipschitz continuous instantaneous cost function.
Now, from the definition of a vector viscosity solution and "classical" viscosity solution, it is straightforward to see that a fixed point of the operator $\mathcal{T}$ is a vector viscosity solution of equation (13-16), and conversely, any viscosity solution of (13-16) is a fixed point of the operator $\mathcal{T}$.

Again we use theorem 5 to interpret each component of $\mathcal{T}(\mathcal{W}^1,\ldots,\mathcal{W}^I)$ as the value function of an optimal control problem. Now to prove that $\mathcal{T}$ is contractive let us compute a upper bound for $||\mathcal{T}\mathcal{W} - \mathcal{T}\tilde{\mathcal{W}}||$, where the norm is defined as

$$||\mathcal{W}|| = ||\mathcal{W}^1,\ldots,\mathcal{W}^I|| = \max_{i\in\mathcal{I}} \max_{(x,t)\in\mathcal{X}} |\mathcal{W}^i(x,t)|$$

$$\begin{aligned}
&\mathcal{T}\tilde{\mathcal{W}}^i(x,t) - \mathcal{T}\mathcal{W}^i(x,t)\\
&\quad = \inf_{u(\cdot)} \int_t^T e^{q_{ii}(s-t)} L^i(x(s),u(s))\,ds + \sum_{j\neq i} q_{ij} \int_t^T e^{q_{ii}(s-t)} \tilde{\mathcal{W}}^j(x(s),s)\,ds\\
&\quad -\inf_{u(\cdot)} \int_t^T e^{q_{ii}(s-t)} L^i(x(s),u(s))\,ds + \sum_{j\neq i} q_{ij} \int_t^T e^{q_{ii}(s-t)} \mathcal{W}^j(x(s),s)\,ds\\
&\quad \leq \sup_{u(\cdot)} \sum_{j\neq i} q_{ij} \int_t^T e^{q_{ii}(s-t)} [\tilde{\mathcal{W}}^j(x(s),s) - \mathcal{W}^j(x(s),s)]\,ds\\
&\quad \leq \int_t^T \sum_{j\neq i} q_{ij} e^{q_{ii}(s-t)}\,ds \quad ||\tilde{\mathcal{W}} - \mathcal{W}||\\
&\quad = \int_t^T -q_{ii} e^{q_{ii}(s-t)}\,ds \quad ||\tilde{\mathcal{W}} - \mathcal{W}||\\
&\quad = (1 - e^{q_{ii}(T-t)}) \quad ||\tilde{\mathcal{W}} - \mathcal{W}||\\
&\quad = (1 - e^{q_{ii}T}) ||\tilde{\mathcal{W}} - \mathcal{W}|| = \theta \ \ ||\tilde{\mathcal{W}} - \mathcal{W}||
\end{aligned}$$

With $0 < \theta < 1$ as $q_{ii} < 0$ and $0 < T < \infty$. In the same way we could have proved that

$$\mathcal{T}\tilde{\mathcal{W}}^i(x,t) - \mathcal{T}\mathcal{W}^i(x,t) \leq \theta \ \ ||\tilde{\mathcal{W}} - \mathcal{W}||.$$

The property of contraction follows since the previous inequalities are true for any $(x,t)$ in $\mathcal{X}$.

$\diamondsuit\diamondsuit$

# 4 Approximation of the value function

## 4.1 The discrete time problem

We now turn to the approximation of the viscosity solution of the coupled system of HJB equations (13,16). We use time discretization. Let us denote $\delta_K = T/K$ the time step of the time interval $[0,T]$. In order to obtain an approximation $\mathcal{V}^i_K$ of the function $\mathcal{V}^i$ associated to the time interval discretization, K, we approximate the dynamics $\dot{x}(t) = f^i(x,u)$ by

$$x(t+\delta_K) = x(t) + \delta_K f^i(x,u),$$

and the time derivative of the value functions

$$\frac{d}{dt}\mathcal{V}^i(x(t),t) = \frac{\partial}{\partial t}\mathcal{V}^i(x(t),t) + \nabla\mathcal{V}^i(x(t),t)f^i(x,u),$$

by

$$\frac{\mathcal{V}^i_K(x + f^i(x,u)\delta_K, t+\delta_K) - \mathcal{V}^i_K(x,t)}{\delta_K}.$$

Plugging these approximations in the system of equations (13-16), we obtain for any time $t \in \{0, \delta_K, 2\delta_K, (K-1)\delta_K\}$ :

$$\mathcal{V}^i_K(x,t) \quad = \quad \min_{u\in U^i}\left\{ L^i(x + f^i(x,u)\delta_K, u)\delta_K \right. \tag{21}$$

$$\left. + \sum_{j\neq i} q_{ij}\delta_K \ \mathcal{V}^j_K(x + f^i(x,u)\delta_K, t+\delta_K) + (1+q_{ii}\delta_K)\mathcal{V}^i_K(x + f^i(x,u)\delta_K, t+\delta_K) \right\}$$

and terminal conditions

$$\mathcal{V}^i_K(x,T) = 0. \tag{22}$$

The approximating HJB system of equations (21-22) can be interpreted as the dynamic programming equations for a stochastic discrete time problem starting at time $t_0 = 0$, and given by

- the dynamics of the continuous state $x$

$$\begin{array}{l} x((k+1)\delta_K) = x(k\delta_K) + f^{\xi^K(k\delta_K)}(x(k\delta_K), u(k\delta_K))\delta_K, \\ x(0) = x_0 \in X_0 \end{array} \tag{23}$$

- the discrete time Markov jump process $\xi^K$,

$$\begin{array}{l} P[\xi^K((k+1)\delta_K) = j | \xi^K(k\delta_K) = i] = q_{ij}\delta_K \quad i \neq j, \\ P[\xi^K((k+1)\delta_K) = i | \xi^K(k\delta_K) = i] = 1 + q_{ii}\delta_K, \end{array} \tag{24}$$

- the evaluation function

$$J^i_K(x,0) = E[\sum_{k=0}^{K-1} L^{\xi^K(k\delta_K)}(x(k\delta_K), u(k\delta_K))\delta_K]. \tag{25}$$

Analogously to a strategy in the continuous problem, a strategy of this discrete problem is defined as a mapping from $\{0, \delta_K, 2\delta_K, \dots, (K-1)\delta_K) \times I\!R^m \times \mathcal{I}$ to the set $\cup_{i\in\mathcal{I}} U^i$.

A way to solve numerically this problem now is the following. Since the stochastic process does not depend on the continuous state or on the control it is possible to construct an event tree associated to this process. To each arc of this tree (24) gives a probability. Equations (25-23) can thus be interpreted as a stochastic program, where (25) is the function to be minimized and (23) is the set of constraints. In [15], Haurie and Moresino use this technique with a linear problem.

## 4.2 Interpolation of the discrete time value function

In order to get convergence results we need to define the value function $\mathcal{V}^i_K$ for any $t$ in the interval $[0, T]$. We define $\mathcal{V}^i_K(x,t)$ for any $t$ in $[0,T]$ by using equation (21) and (22) together with the following terminal conditions for any $t$ in the last segment $](K-1)\delta_K, T[$

$$\mathcal{V}^i_K(x,t) = \min_{u\in U^i} \left\{ L^i(x + f^i(x,u)(T-t), u)(T-t) \right\}. \tag{26}$$

A most natural way to get an interpolation of $\mathcal{V}^i_K$ on any $t$ would have been to use linear interpolation techniques. Nevertheless the proofs of the results given on the last section are much simpler using the interpolation introduced above.

**Theorem 6.** *The functions $(\mathcal{V}^i_K(x,t))_K$ are bounded by a constant $M$, and Lipschitz continuous in $(x,t)$ with the same Lipschitz constant for any $K$.*

**Proof :** The proof is similar to the proof of theorem 1. The fact that the functions are bounded comes from boundedness of the instantaneous cost, which is independent of $K$.

$\diamond\diamond$

## 4.3 Convergence result

We now turn to the approximation theorem :

**Theorem 7.** *For any $i \in \mathcal{I}$, $\mathcal{V}^i_K(x,t)$ converges to $\mathcal{V}^i(x,t)$, locally uniformly in $\mathcal{X}$ as $K$ tends to infinity, where $\mathcal{V}$ is the viscosity solution of (13,16 ).*

**Proof :** By Theorem 6 the family $(\mathcal{V}^i_K(x,t))_K$ is equicontinuous. Therefore, by Ascoli Theorem, there exists a subsequence $\{\mathcal{V}^i_p\}_p$ of the sequence $\{\mathcal{V}^i_K\}_K$ that converges locally uniformly to some function $V$. Let us prove that the vector function $V = (V^1, V^2, \dots, V^I)$ is a vector viscosity solution of system (13-16). This will prove the theorem, since $\mathcal{V}$ has been proved to be the unique Lipschitz continuous vector viscosity solution of (13-16).

Let $\varphi$ be a $C^1(\mathcal{X})$ function, and $(\tilde{x}, \tilde{t})$ a local maximum of $\mathcal{V}^i - \varphi$ for some $i$ in $\mathcal{I}$. Without loss of generality we can suppose that this maximum is strict, Hence there exists a positive integer $r > 0$ such that for any $(x,t)$ in the ball $B$ centered in $(\tilde{x}, \tilde{t})$ and of radius $r$ we have

$$(\mathcal{V}^i - \varphi)(\tilde{x}, \tilde{t}) > (\mathcal{V}^i - \varphi)(x,t).$$

Define now the point $\{\tilde{x}_p, \tilde{t}_p\}_p$, such that $(\tilde{x}_p, \tilde{t}_p)$ is a maximum of $\mathcal{V}_p^i - \varphi$ on the closed set $B$. Since $\mathcal{V}_p^i$ converges uniformly to $V^i$ on $B$, it follows that $(\tilde{x}_p, \tilde{t}_p)$ converges to $(\tilde{x}, \tilde{t})$. Since $f^i(x,u)$ is bounded, for $p$ large enough, $(\tilde{x}_p + f^i(\tilde{x}_p, u)\delta_p, \tilde{t}_p + \delta_p)$, belongs to $B$, and consequently from the definition of $(\tilde{x}_p, \tilde{t}_p)$ we have

$$\begin{aligned} &\mathcal{V}_p^i(\tilde{x}_p, \tilde{t}_p) - \varphi(\tilde{x}_p, \tilde{t}_p) \geq \\ &\quad \mathcal{V}_p^i(\tilde{x}_p + f^i(\tilde{x}_p, u)\delta_p, \tilde{t}_p + \delta_p) - \varphi(\tilde{x}_p + f^i(\tilde{x}_p, u)\delta_p, \tilde{t}_p + \delta_p). \end{aligned} \tag{27}$$

Using the expression (21), and keeping in mind the equation (2), we obtain

$$\begin{aligned} 0 &= \min_{u \in \mathcal{U}^i} \{ -\mathcal{V}_p^i(\tilde{x}_p, \tilde{t}_p) + L^i(\tilde{x}_p + f^i(\tilde{x}_p, u)\delta_p, u)\delta_p \\ &\qquad + \sum_{j \neq i} q_{ij}\delta_p \mathcal{V}_p^j(\tilde{x}_p + f^i(\tilde{x}_p, u)\delta_p, \tilde{t}_p + \delta_p) \\ &\qquad + (1 + q_{ii}\delta_p)\mathcal{V}_p^i(\tilde{x}_p + f^i(\tilde{x}_p, u)\delta_p, \tilde{t}_p + \delta_p)\} \\ &\leq \min_{u \in \mathcal{U}^i} \{ \varphi(\tilde{x}_p + f^i(\tilde{x}_p, u)\delta_p, \tilde{t}_p + \delta_p) - \varphi(\tilde{x}_p, \tilde{t}_p) + L^i(\tilde{x}_p + f^i(\tilde{x}_p, u)\delta_p, u)\delta_p \\ &\qquad + \sum_{j \neq i} q_{ij}\delta_p [\mathcal{V}_p^j(\tilde{x}_p + f^i(\tilde{x}_p, u)\delta_p, \tilde{t}_p + \delta_p) \\ &\qquad - \mathcal{V}_p^i(\tilde{x}_p + f^i(\tilde{x}_p, u)\delta_p, \tilde{t}_p + \delta_p)].\} \end{aligned} \tag{28}$$

Since $\varphi$ is a $C^1(\mathcal{X})$ function we have for some $\theta$ in $[0,1]$,

$$\begin{aligned} &\varphi(\tilde{x}_p + f^i(\tilde{x}_p, u)\delta_p, \tilde{t}_p + \delta_p) - \varphi(\tilde{x}_p, \tilde{t}_p) = \\ &\delta_p \nabla\varphi(\tilde{x}_p + \theta f^i(\tilde{x}_p, u)\delta_p, \tilde{t}_p + \theta\delta_p) f^i(\tilde{x}_p, u) + \delta_p \frac{\partial}{\partial t}\varphi(\tilde{x}_p + \theta f^i(\tilde{x}_p, u)\delta_p, \tilde{t}_p + \theta\delta_p). \end{aligned}$$

Substituting this last expression in (28) and dividing by $\delta_p$ we obtain

$$\begin{aligned} 0 &\leq \min_{u \in U^i} \{ \nabla\varphi(\tilde{x}_p + \theta f^i(\tilde{x}_p, u)\delta_p, \tilde{t}_p + \theta\delta_p) f^i(\tilde{x}_p, u) \\ &\quad + \frac{\partial}{\partial t}\varphi(\tilde{x}_p + \theta f^i(\tilde{x}_p, u)\delta_p, \tilde{t}_p + \theta\delta_p) + L^i(\tilde{x}_p + f^i(\tilde{x}_p, u)\delta_p, u) \\ &\quad + \sum_{j \neq i} q_{ij} [\mathcal{V}_p^j(\tilde{x}_p + f^i(\tilde{x}_p, u)\delta_p, \tilde{t}_p + \delta_p) - \mathcal{V}_p^i(\tilde{x}_p + f^i(\tilde{x}_p, u)\delta_p, \tilde{t}_p + \delta_p)]. \end{aligned}$$

Now we let $p$ tend to infinity to get

$$0 \leq \min_{u \in U^i} \{ \nabla\varphi(\tilde{x}, \tilde{t}) \cdot f^i(\tilde{x}_p, u) + \frac{\partial}{\partial t}\varphi(\tilde{x}, \tilde{t}) + L^i(\tilde{x}, u) + \sum_{j \neq i} q_{ij} [\mathcal{V}^j(\tilde{x}, \tilde{t}) - \mathcal{V}^i(\tilde{x}, \tilde{t})],$$

which establishes that $V$ is a viscosity subsolution of (13,16). The same arguments can be used to prove that $V$ is also a viscosity super-solution. This ends the proof of the theorem.

$\diamond\diamond$

# 5 Conclusions

In this paper we proved, using viscosity techniques, the convergence of the numerical approximation proposed in [15]. For this purpose, it was necessary to extend the classical (scalar) viscosity solution to a vector viscosity solution as it was already done in [16] and [7] in a slightly different context.

It would be interesting to see if this work and methodology could be applied in the context of game theory. Classical viscosity techniques has already been used to prove convergence of discrete scheme for pursuit evasion games by Bardi, Falcone and Soravia (see *e.g.* [3]) and by Pourtallier and Tidball (see [18]). For these games, since the value function is a scalar function, it was possible to use classical viscosity solution. Possible extensions of this work could be done for games with vector value function; in this case the use of vector viscosity solution would be necessary. We think more precisely to the case of two players zero-sum piecewise deterministic games and the case of non-zero sum games.

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