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N° 3624

Février 1999

THÈME 1

Rapport de recherche

ISSN 0249-6399

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Thème 1 — Réseaux et systèmes
Projet SLOOP

Rapport de recherche n° 3624 — Février 1999 — 23 pages

Abstract: As it is introduced by Bermond, Pérennes, and Kodate and by Fragopoulou and Akl, some Cayley graphs, including most popular models for interconnection networks, admit a special automorphism, called *complete rotation*. Such an automorphism is often used to derive algorithms or properties of the underlying graph. For example, some optimal gossiping algorithms can be easily designed by using a complete rotation, and the constructions of the best known edge disjoint spanning trees in the toroidal meshes and the hypercubes are based on such an automorphism. Our purpose is to investigate such Cayley graphs. We relate some symmetries of a graph with potential algebraic symmetries appearing in its definition as a Cayley graph on a group. In the case of Cayley graphs defined on a group generated by transpositions, we characterize the ones admitting a complete rotation.

Key-words: networks, Cayley graphs, rotations, transposition graphs, group

This work was partially supported by the french "Opération RUMEUR" and the project SLOOP (INRIA-CNRS-I3S).

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Rotations Complètes dans les Graphes de Cayley

Résumé : Certains graphes de Cayley, parmi les plus populaires des modèles pour les réseaux d'interconnexion, admettent des automorphismes particuliers, appelés *rotations complètes* Bermond, Pérennes, et Kodate et également étudiés Fragopoulou et Akl. Ces automorphismes sont souvent utilisés pour construire des algorithmes ou montrer des propriétés du graphe. Ils permettent, par exemple, de construire facilement des algorithmes optimaux d'échange total et la meilleure construction connue d'arbres couvrants arête disjoints dans le tore ou l'hypercube repose sur l'utilisation d'un tel automorphisme. Nous étudions ici ces graphes de Cayley qui possèdent des symétries particulières. Pour cela, nous relions certaines symétries d'un graphe à des propriétés du groupe sur lequel est définie sa structure de graphe de Cayley. Nous donnons en particulier une caractérisation des graphes de Cayley pour lesquels les générateurs sont des transpositions et qui admettent une rotation complète.

Mots-clés : réseaux, graphes de Cayley, rotations, graphes de transposition, groupe

1 Introduction

Cayley graphs are good models for interconnection networks and have been intensively studied for this reason during the last few years. Articles [1], [20] and [17] give a survey.
Bermond, Kodate and Perennes define in [4] the concept of complete rotation in Cayley graphs in order to construct a gossip algorithm from a broadcast protocol applied to each vertex simultaneously. Given particular conditions on the orbits of the vertices under the complete rotation, they provide an optimal gossip algorithm. They build such an algorithm in the hypercube, the squared toroidal mesh and the star-graph (see the definitions in Appendix A).
Fragopoulou and Akl consider in [12] and [13] a similar concept of rotation in Cayley graphs to construct a spanning subgraph used as a basic tool for the design of communication algorithms (gossiping, scattering). The class of graphs they consider contains most popular Cayley graphs for interconnection networks, such as cycles, hypercubes, generalized hypercubes, star graphs and the square n-dimensional torus.
Hence Cayley graphs admitting a complete rotation have specific symmetry properties which enable efficient and simple algorithmic schemes. In this paper, we study this class of Cayley graphs and derive some of their properties. More precisely, we relate some symmetries of a graph with potential algebraic symmetries appearing in its definition as a Cayley graph on a group. In the case of Cayley graphs defined on a group generated by transpositions, we characterize the ones admitting a complete rotation.
This paper is organized as follows. In Section 2, after recalling some basic definitions and properties of Cayley graphs, we give the definitions and some properties of rotations and complete rotations. In Section 3, we study several conditions for the existence of a rotation. First, a characterization of graphs having a complete rotation is given in terms of representation and relators for the group and the set of generators (Section 3.1). Then, we introduce the rotation-translation group of a Cayley graph and consider some necessary conditions of the rotational property (Section 3.3). In Section 3.4, we consider complete rotations on Cartesian products of graphs. The last part, Section 4, is devoted to the Cayley graphs defined by transpositions. Generalized star graphs are introduced (Section 4.3) and the characterization of rotational Cayley graphs defined on a group generated by transpositions is given (Section 4.4). Finally, Appendix A contains the definitions and drawings of some Cayley graphs and Appendix B summarizes the notation.

2 Preliminaries

2.1 Cayley graphs

Appendix B summarizes the notation given below.
All groups considered are finite. By abuse of notation, we use the same letter to denote a group and the set of its elements and specify the operation of the group only when confusion can arise. We use multiplicative notation except in the case of Abelian groups. We denote by $\mathbb{Z}$ the additive group of integers, and by $\mathbb{Z}_n$ the group of integers modulo n. For G a group and $S \subset G$, the group generated by S is denoted by $\langle S \rangle$. The automorphism group of G (set of one-to-one mappings from G to G which preserve the composition law) is denoted by $Aut(G)$.
A permutation σ on the set $X = \{1, \cdots, n\}$ is a one-to-one mapping from X to X. As usual, it is denoted by the images $(\sigma(1), \cdots, \sigma(n))$.
For a permutation σ on X, $Supp\ \sigma$ is the set of elements i of X such that $\sigma(i) \neq i$.
A product of permutations $\sigma\tau$ means that we apply first mapping τ on the set $\{1, \cdots, n\}$ and then mapping σ, i.e., $\sigma\tau = (\sigma(\tau(1)), \cdots, \sigma(\tau(n)))$.
We denote by $\mathfrak{S}_X$ the group of all permutations on X and, for short, by $\mathfrak{S}_n$ if $X = \{1 \cdots n\}$.
A cycle σ such that $\sigma(i_1) = i_2, \dots \sigma(i_{k-1}) = i_k, \sigma(i_k) = i_1$ is denoted by $\langle i_1, i_2, \dots, i_k \rangle$. In particular, $\langle i, j \rangle$ denotes the transposition of elements i and j.

We will consider mainly simple undirected graphs. A graph Γ is defined by its vertex set $V\Gamma$ and its edge set $E\Gamma$. The edge between two vertices u and v is denoted by $[u, v]$ or simply by uv if no confusion is possible. If necessary, we consider the symmetric digraph Γ^* associated to a graph Γ and obtained by replacing any edge uv by two opposite arcs (u, v) and (v, u). We denote by $A\Gamma$ the set of arcs of Γ^*.
We denote by $Aut(\Gamma)$ the automorphism group of a graph Γ.
A graph Γ is said to be *arc-transitive* (*symmetric* in [5]) if for any given pair of directed edges $(u, v), (u', v')$

there exists an automorphism $f \in Aut(\Gamma)$ such that $f(u) = u'$ and $f(v) = v'$. In other words Γ is said to be arc-transitive if $Aut(\Gamma)$ acts transitively on $A\Gamma$.

Definition 2.1 (see for example [5]) Let G be a group with unit I and S a subset of G such that $I \notin S$ and the inverse of elements of S belong to S. The *Cayley graph* $Cay(G, S)$ is the graph with vertex set G and with edge set $\{[g, gs] : g \in G, s \in S\}$.

We will say that the edge $[g, gs]$, $s \in S$, is labeled by s. Notice that the edge $[g, gs]$ can also be labeled by s^{-1} since it is equal to the edge $[gs, gss^{-1}]$.
Examples of well-known Cayley graphs are given in Section 4.1 and Appendix A. We recall some well known results on Cayley graphs we will use later.
If G is generated by S, i.e. $G = \langle S \rangle$, then $Cay(G, S)$ is connected.
By analogy with geometry, for $a \in G$, the mapping $t_a : G \rightarrow G$, defined by $t_a(x) = ax$, is called a *translation* of $Cay(G, S)$. The mappings $t_a, a \in G$, form a subgroup T of $Aut(Cay(G, S))$ which is isomorphic to the group G and acts regularly on G. The following characterization of Cayley graphs is well-known.

Theorem 2.2 [23]*Let Γ be a connected graph. The automorphism group $Aut(\Gamma)$ has a subgroup G which acts regularly on $V\Gamma$ if and only if Γ is a Cayley graph $Cay(G, S)$, for some set S generating G.*

2.2 S-stabilizers and rotations

Let G be a group. Note that any internal mapping of G can be considered as an action on the vertices of the graph $Cay(G, S)$. So some symmetries of the group G give naturally rise to symmetries in the graph $Cay(G, S)$. For commodity, we introduce:

Definition 2.3 Let G be a finite group and S a set of generators of G. A homomorphism ω of the group G is called a *S-stabilizer* if $\omega(S) = S$.

Notice that since G is finite, a S-stabilizer is bijective and therefore a group automorphism. We denote by $Stab(G, S)$ the set of S-stabilizers of G which is a subgroup of $Aut(G)$. A S-stabilizer different from the identity is said to be *non-trivial*.

In the following, we will study graph automorphisms of $Cay(G, S)$ which are induced by S-stabilizers of G using the following proposition, a proof of which can be found in [5], Proposition 16.2.

Proposition 2.4 [27] *If ω is an automorphism of the group G generated by S such that $\omega(S) = S$, then ω is a graph automorphism of $Cay(G, S)$ which fixes the vertex I.*

By proposition 2.4, a S-stabilizer induces a graph automorphism of $Cay(G, S)$ we simply call a *rotation.*

When applying Proposition 2.4, we will use the same letter to denote the group automorphism and the graph automorphism it induces.
If H is a subgroup of $Stab(G, S)$, we will denote by $\underline{H}$ its corresponding isomorphic subgroup of $Aut(Cay(G, S))$, or simply by H when no confusion will arise.

2.3 Definitions of complete rotations

The notion of rotation in graph theory was first used in the context of embeddings (see for example [6], [28]). In this context, a *rotation* of a graph Γ at a vertex i is a cyclic ordering of the neighbors of i, and a *rotation scheme* is a collection $\{r_i, i \in V\Gamma\}$, where r_i is a rotation at the vertex i. It is used to embed the graph Γ into a surface. For a Cayley graph, any cyclic permutation r of the generators allows us to define a rotation scheme by $r_i(j) = ir(i^{-1}j)$ for any edge ij (see [6], page 117).

The notion of complete rotation in Cayley graphs we will use is related, but different. The original definition of complete rotation is given in [4] as follows:

Definition 2.5 [4] Let $Cay(G,S)$ be a Cayley graph with $G = \langle S \rangle$. A mapping $\omega : G \rightarrow G$ is a complete rotation of $Cay(G,S)$ if it is bijective and satisfies the following two properties for some ordering of $S = \{s_i, 0 \leq i \leq d-1\}$:

$$\omega(I) = I \quad (1)$$
$$\omega(xs_i) = \omega(x)s_{i+1} \quad (2)$$

for any $x \in G$ and any $i \in \mathbb{Z}_d$.

It is a particular case of the concept of rotation. As we will see below, a complete rotation of $Cay(G,S)$ is a rotation of $Cay(G,S)$ such that the permutation induced on S is a cycle of length $|S|$. More precisely, let us first consider the S-stabilizers of G which cyclically permutes the generators in S.

Definition 2.6 A S-stabilizer of G, $\omega : G \rightarrow G$, is said to be *cyclic* if, for some ordering of $S = \{s_i, 0 \leq i \leq d-1\}$, $\omega(s_i) = s_{i+1}$, for any $i \in \mathbb{Z}_d$.

Then, we get:

Property 2.7 A mapping $\omega : G \rightarrow G$ is a complete rotation of $Cay(G,S)$ if and only if it is the graph automorphism induced by a cyclic S-stabilizer of G.

Proof. Clearly, any cyclic S-stabilizer of G induces a complete rotation of $Cay(G,S)$ as defined in Definition 2.5. The converse is a corollary of the following proposition 2.8 listing some properties of complete rotations (some of them are used in [13] and [4]). □

Proposition 2.8 *Let ω be a complete rotation of the Cayley graph $Cay(G,S)$, with $G = \langle S \rangle$. Then, for some order of $S = \{s_i, 0 \leq i \leq d-1\}$, the following properties are satisfied.*

(i) For any $i \in \mathbb{Z}_d$, $\omega(s_i) = s_{i+1}$;

(ii) For any $i, j \in \mathbb{Z}_d$ and any $x \in G$, $\omega^{j-i}(xs_i) = \omega^{j-i}(x)s_j$;

(iii) ω is a group automorphism of order d;

(iv) ω is a graph automorphism; and

(v) ω^p is a group automorphism for any $p \in \mathbb{Z}$ and a complete rotation for p prime with d. In particular, ω^{-1} is a complete rotation.

Proof. (i) By taking $x = I$ in Equation (2) of Definition 2.5.
(ii) By induction on $j - i$ using Equation (2).
(iii) By induction on the number of factors of an element written as a product of generators, we get from definition 2.5, for any $x, y \in G$,

$$\omega(xy) = \omega(x)\omega(y)$$

Thus the bijective mapping ω is a group automorphism. Furthermore, for any generator s_i, by (ii), $\omega^d(s_i) = s_i$ and $\omega^j(s_i) \neq s_i$ for $0 \leq j < d$, so that $\omega^d = I$ and $\omega^k \neq I$ for $1 \leq k < d$.
(iv) By Proposition 2.4 and (iii), ω is a graph automorphism.
(v) By induction on p, for any $x, y \in G$, $\omega^p(xy) = \omega^p(x)\omega^p(y)$. If p and d are co-prime then $p\mathbb{Z}_d = \mathbb{Z}_d$ and the sequence $s_0, s_p, s_{2p}, \cdots, s_{(d-1)p}$ defines a new ordering of the generators so that ω^p is a complete rotation. □

The simplest automorphisms of a group G are inner automorphisms : $x \rightarrow \sigma x \sigma^{-1}$, where $\sigma \in G$. Therefore it is natural to consider the following property which defines the notion of rotation considered in [13]:

Property 2.9 Let $Cay(G,S)$ be a Cayley graph where $G = \langle S \rangle$. If there exist an element $\sigma \in G$ and an ordering of $S = \{s_i, 0 \leq i \leq d-1\}$ such that for any $i \in \mathbb{Z}_d$,

$$s_{i+1} = \sigma s_i \sigma^{-1}, \quad (3)$$

then the mapping $\omega : G \rightarrow G$, such that $\omega(x) = \sigma x \sigma^{-1}$, is a complete rotation of $Cay(G,S)$.

Proof. An inner automorphism of G defined by $\omega(x) = \sigma x \sigma^{-1}$ and satisfying Equation 3 is a cyclic S-stabilizer. By Property 2.7, it induces a complete rotation of $Cay(G, S)$. □

In [13], the authors give the generators s_i, $0 \leq i \leq d-1$, and a permutation $\sigma \in \mathfrak{S}_n$ for cycles, hypercubes, square torus, star graphs, modified bubble-sort graphs, bisectional networks, and two generalizations of hypercubes, showing by Property 2.9 that all these graphs have a complete rotation (see Appendix A). Thus most of the popular Cayley graphs for interconnection networks have a complete rotation.

Property 2.9 suggests the following problem.

Problem 2.10 For which Cayley graphs $Cay(G, S)$ is the existence of a complete rotation equivalent to the existence of an inner automorphism of G which cyclically permutes the generators in S ?

We give a partial answer to this problem in Proposition 4.13.
Notice that it is a classical result of group theory that if $G = \mathfrak{S}_n$ with $n \neq 2, 6$, then the only group automorphisms of G are the inner automorphisms. But this result is not sufficient since, for example, the hypercube $H(d)$ is a Cayley graph on a proper subgroup of $\mathfrak{S}_d$ (see Appendix A).

2.4 Rotational graphs

We say for short that a graph Γ is *rotational* if there exist a group G and a set of generators S such that $\Gamma = Cay(G, S)$ and G has a cyclic S-stabilizer.

Remark 2.11 The existence of a complete rotation in a given Cayley graph depends on the choice of the group and the set of generators as the following proposition and theorem show.

Proposition 2.12 *The additive group $\mathbb{Z}_n$ has a cyclic $\mathbb{Z}_n \setminus \{0\}$-stabilizer if and only if n is prime.*

Proof. The additive group $\mathbb{Z}_n$ is generated by ${\mathbb{Z}_n}^* = \mathbb{Z}_n \setminus \{0\}$. For $x \in \mathbb{Z}$, any group homomorphism ω satisfies $\omega(x) = \omega(1 + 1 + \cdots + 1) = x\omega(1)$. Thus, if $\omega(1) = a$, then $\omega(x) = ax$. If ω is a complete rotation, then the generators are $1, a, a^2, \cdots a^{n-2}$ and thus $\mathbb{Z}_n^* = \{1, a, a^2, \cdots a^{n-2}\}$ is cyclic. Thus, n is prime. Conversely, if n is prime, there is an integer a such that $\mathbb{Z}_n^* = \{1, a, a^2, \cdots a^{n-2}\}$ and then $\omega(x) = ax$ is a complete rotation. □
Thus $Cay(\mathbb{Z}_n, {\mathbb{Z}_n}^*)$ has a complete rotation if and only if n is prime. On the other hand, we have the following result.

Theorem 2.13 *The complete graph K_n is rotational if and only if n is a power of a prime number.*

Proof. [21] First note that $K_n = Cay(G, S)$ if and only if the order of G is n and $S = G \setminus I$. It means that every element of G except the identity is a generator.
If n is not a prime power, then there exist two different prime numbers p and q which divide n. Then the group G has at least an element of order p and an element of order q with $p \neq q$. By Corollary 3.6, K_n is not rotational.
If n is a prime power, then there exists a field F with n elements (see for example [2], page 445) and $F \setminus \{0\}$ is a cyclic multiplicative group. For any generator r of $F \setminus \{0\}$, the mapping ω, defined by $\omega(x) = rx$, is a complete rotation of $K_n = Cay(F, F \setminus \{0\})$ (F is considered as an additive group). □
Notice that a similar result has already been proved in the context of maps, in a different way ([6], page 128).

Theorem 2.14 [6] *There is a rotation on K_n which gives rise to a symmetrical map if and only if n is a prime power.*

The next proposition shows that one can construct new rotational Cayley graphs by taking a quotient according to a normal subgroup which is invariant by the rotation.

Proposition 2.15 *If $Cay(G,S)$ has a complete rotation which is a K-stabilizer for a normal subgroup K of G, then the quotient Cayley graph $Cay(G/K,S')$ is also rotational, where S' is the image of S by the canonical epimorphism from G onto G/K.*

Proof. Let ω be a complete rotation of $Cay(G,S)$ such that $\omega(K) = K$. Since K is stabilized by ω, we can define the automorphism of G/K induced by ω denoted by ω'. Let S' be the set of the images of S in G/K by the canonical epimorphism. Then ω' is a group-automorphism of G/K which is also a graph-automorphism of $Cay(G/K,S')$. Furthermore, ω' induces a cyclic permutation of the generators. Thus ω' is a complete rotation of $Cay(G/K,S')$. □

Example 2.16 Let K be a cyclic binary code (that is a subgroup of ${\mathbb{Z}_2}^n$ invariant by cyclic shift of the coordinates). Then the graph (also called quotient) obtained from the hypercube $H(n)$ by identifying all the vertices $\{x+k : k \in K\}$ to one vertex, for every $x \in {\mathbb{Z}_2}^n$, is a rotational Cayley graph.

Proof. The hypercube $H(n)$, considered as a Cayley graph on the additive group ${\mathbb{Z}_2}^n$, admits the cyclic shift of the coordinates as a complete rotation (see Appendix A.3 and Example 3.7). By definition a binary cyclic code K is a subgroup of ${\mathbb{Z}_2}^n$ invariant by the cyclic shift and K is a normal subgroup since ${\mathbb{Z}_2}^n$ is Abelian. By Proposition 2.15, $Cay({\mathbb{Z}_2}^n/K,S')$ is a rotational Cayley graph. □

In the following sections, we will give other examples of rotational graphs belonging to particular classes of Cayley graphs, the ones defined on Abelian groups and on permutation groups generated by transpositions. To finish this section, we present an example which does not belong to these classes.

Example 2.17 *Knödel graph.*
The Knödel graphs are defined in [14] and are based on the Knödel construction of an optimal gossiping algorithm [19]. They can also be defined as Cayley graphs on the semi-direct product $G = \mathbb{Z}_p \rtimes \mathbb{Z}_2$ for the multiplicative law:

$$(x,y)(x',y') = (x + (-1)^y x', y+y'),\ \ x,x' \in \mathbb{Z}_p,\ \ y,y' \in \mathbb{Z}_2.$$

and

$$S = \{(2^i,1), 0 \le i \le d-1\}.$$

We consider here the particular case $p = 2^n - 1$ and $S = \{s_i, 0 \le i \le n-1\}$, with $s_i = (2^i, 1)$. Let us consider the mapping ω defined by $\omega[(x,y)] = (2x,y)$. Since $(x,y)s_i = (x + (-1)^y 2^i, y+1)$, for $0 \le i \le n-1$, we get:

$$\omega(0,0) = (0,0)$$
$$\omega[(x,y)s_i] = \omega[(x,y)]s_{i+1}.$$

By Definition 2.5, ω is a complete rotation of $Cay(\mathbb{Z}_p \rtimes \mathbb{Z}_2, S)$.

3 Study of conditions for the existence of a rotation

3.1 A characterization of rotations

An attractive way to define a group generated by a set S is to consider the elements of the group as words on the alphabet S modulo some well chosen set of equalities satisfied by the set S. For example, the additive group $\mathbb{Z}_n \times \mathbb{Z}_n$ is generated by $(1,0)$ and $(0,1)$. Notice that
$(1,0)+(0,1) = (0,1)+(1,0)$, and $n(0,1) = n(1,0) = (0,0)$. This group can also be defined as a multiplicative group generated by $S = \{s_1, s_2\}$ satisfying the equalities (called relations in group theory): $s_1^n = I,\ s_2^n = I$ and $s_1 s_2 = s_2 s_1$ or $s_1 s_2 s_1^{-1} s_2^{-1} = I$. Equivalently, in order to define the group, one can use a set of relators $R = \{s_1 s_2 s_1^{-1} s_2^{-1}, s_1^n, s_2^n\}$. In the above example the mapping $(x,y) \to (y,x)$ belongs to $Stab(G,S)$ and this fact clearly appears in the set of relations which is symmetric in s_1 and s_2.
More precisely any group G generated by a set S can be seen as the quotient of the free group generated by S by a set of relations between the generators (see for example [7], [18] or [22]) for the definitions on

presentations of groups). As in [18], we denote by $F(S)$ the free group generated by S and by R the subset of $F(S)$ of the elements which are called *relators* (thus consisting of words on the elements of S). Let $N(R)$ be the normal closure of R in $F(S)$, that is the smallest normal subgroup of $F(S)$ containing R. It is also the subgroup of $F(S)$ generated by the elements grg^{-1}, $g \in F(S), r \in R$ (see for example [22], page 16). Then G is the quotient group $F(S)/N(R)$. We denote by α the canonical epimorphism from $F(S)$ onto G and by e the empty word of $F(S)$. Thus $\alpha(e) = I$ and, for any $x \in F(S)$, $\alpha(x) = I$ if and only if $x \in N(R)$. As usual, we do not distinguish s from $\alpha(s)$ for $s \in S$.

Recall also that any free group automorphism of $F(S)$ can be defined by the images of the elements of S.

Definition 3.1 For any S-stabilizer f of a group G with presentation $G = (S|R)$, we denote by $\tilde{f}$ the automorphism of $F(S)$ defined by $\tilde{f}(s) = f(s)$, for any $s \in S$.

The following proposition shows the relation between a non-trivial group $Stab(G,S)$ and a presentation of G with a set of relators admitting symmetries.

Proposition 3.2 *Let G be a group generated by a subset S. Then the following properties are equivalent:*

(i) the group G admits a non trivial S-stabilizer, i.e. the subgroup $Stab(G,S)$ is non trivial;

(ii) for any subset R of $F(S)$ such that $G = (S|R)$ is a presentation of G, the free group $F(S)$ has a non trivial $N(R)$-stabilizer, where $N(R)$ is the normal closure of R, which is also a S-stabilizer; and

(iii) there exists a presentation of G, $G = (S|R)$, such that $F(S)$ has a non trivial R-stabilizer which is also a S-stabilizer.

Remark 3.3 In other words the existence of a S-stabilizer is equivalent to the existence of a permutation on the set of generators S letting the set of relators R invariant.

Proof. (i)$\Rightarrow$ (ii) Assume f is an S-stabilizer of the group G generated by S. Then for any presentation $G = (S|R)$, let us define a group automorphism $\tilde{f}$ of $F(S)$, as explained above, by $\tilde{f}(s) = f(s)$, for any $s \in S$. This implies $\alpha\tilde{f} = f\alpha$. Furthermore, if $x \in N(R)$, then $\alpha(x) = I$ and $f(\alpha(x)) = I = \alpha(\tilde{f}(x))$, and thus $\tilde{f}(x) \in N(R)$. This proves that $\tilde{f}$ is a $N(R)$-stabilizer. It is also a non-trivial S-stabilizer.
(ii)$\Rightarrow$ (iii) Evident by taking the canonical presentation $G = (S|N(R))$.
(iii)$\Rightarrow$ (i) Let $G = (S|R)$ be a presentation of G and $\tilde{f}$ a R-stabilizer. Since every element x of $N(R)$ is a product of elements of the form grg^{-1} with $r \in R$, $g \in F(S)$ and $\tilde{f}$ is a R-stabilizer, using $\tilde{f}(grg^{-1}) = \tilde{f}(g)\tilde{f}(r)\tilde{f}(g)^{-1} = g'r'g'^{-1}$ with $r' \in R$, $g' \in F(S)$, we get that $\tilde{f}$ is also a $N(R)$-stabilizer. Therefore it is possible to define a group automorphism f of the quotient $F(S)/N(R)$ such that $\alpha\tilde{f} = f\alpha$. Furthermore S is invariant by f. □

Corollary 3.4 *Let G be a group generated by a subset S. Then the following properties are equivalent:*

(i) the Cayley graph $Cay(G,S)$ has a complete rotation;

(ii) for any presentation $G = (S|R)$, the free group $F(S)$ has a $N(R)$-stabilizer, where $N(R)$ is the normal closure of R, which induces a cyclic permutation of S; and

(iii) there exists a presentation of G, $G = (S|R)$, such that $F(S)$ has a R-stabilizer which induces a cyclic permutation of S.

Proof. The proof is similar to the proof of Proposition 3.2 using the definition of a complete rotation and the fact that the action of f on S is the same as the action of $\tilde{f}$. □

Remark 3.5 Once again the existence of a complete rotation of $Cay(G,S)$ is equivalent to the existence of a presentation of $G = (S|R)$ such that the set of relators R is invariant by a cyclic permutation of the generators.

Corollary 3.6 *If $Cay(G, S)$ has a complete rotation, then all the generators in S have the same order.*

Proof. This result is a consequence of Corollary 3.4 since, if the generator s_i is of order p, the relation $(s_i)^p = I$ has to be fixed by a cyclic permutation on the generators. □

The following table gives presentations $(S|R)$ for some well known Cayley graphs $Cay(G, S)$ with $G = (S|R)$. These presentations are already known (see for example [7] and [9]). By applying Corollary 3.4, this proves that the considered graphs are rotational (see [13] and Appendix A for another proof using Property 2.9).

Example 3.7

Graph	S	R
Hypercube $H(n)$	$\{s_1, s_2, \dots, s_n\}$	$\{s_i^2, s_i s_j s_i^{-1} s_j^{-1}\}$
Squared toroidal mesh TM_p^d	$\{s_1, s_2, \dots, s_d$ $s_1^{-1}, s_2^{-1}, \dots, s_d^{-1}\}$	$\{s_i^p, s_i s_j s_i^{-1} s_j^{-1}\}$
Modified bubble-sort graph $MBS(n)$	$\{s_1, s_2, \dots, s_n\}$	$\{s_i^2, (s_i.s_{i+1})^3, s_i s_j s_i^{-1} s_j^{-1}\ (j \neq i+1,\ j \neq i-1)$ $s_n s_1 s_2 s_3 \dots s_{n-2} s_{n-1} s_{n-2} \dots s_2 s_1\}$
Star graph ST(n)	$\{s_1, s_2, \dots, s_{n-1}\}$	$\{s_i^2, (s_i s_j)^3, (s_i s_j s_k s_j)^2\}$

Let us notice that despite we only work here on graphs the same notion of complete rotation can be considered for digraphs. In that case, the generating set S do not need to be symmetric ($S = S^{-1}$). With this definition similar result can be derived. In particular Corollary 3.4 can be applied to digraphs. For example, the digraphs defined as *arrowheads* in [10] have a complete rotation since they can be defined as the Cayley digraphs on the groups $G_n = (S|R_n)$ with $S = \{s_1, s_2, s_3\}$ and $R_n = \{s_1 s_2 s_3, s_1 s_2 s_1^{-1} s_2^{-1}, s_2 s_3 s_2^{-1} s_3^{-1}, s_3 s_1 s_3^{-1} s_1^{-1}, s_1^{2^n}, s_2^{2^n}, s_3^{2^n}\}$ for any $n \geq 0$.

In Proposition 3.2 and Corollary 3.4 a symmetric presentation of G is provided when the associated Cayley graph admits a rotation. One can think about asking the following question : if $Cay(G, S)$ is rotational, is it possible to find a symmetric presentation which is also minimal with respect to the inclusion? For example, in the case of arrowheads the presentation of G_n given in [10] is minimal but not symmetric: $(S|R_n')$, with $R_n' = \{s_1 s_2 s_3, s_1 s_2 s_1^{-1} s_2^{-1}, s_1^{2^n}, s_2^{2^n}, s_3^{2^n}\}$.

3.2 Abelian groups

One can give more details in the case of Cayley graphs on Abelian groups.
Let us recall that a *circulant graph* (also called multi-loop graph) is a Cayley graph $Cay(\mathbb{Z}_n, S)$ on the additive group $\mathbb{Z}_n$ with symmetric generating set $S = \{\pm s_1, \pm s_2, \dots, \pm s_k\}$, for some integers $n, s_1, s_2, \dots, s_k$. These graphs have been intensively studied as models of interconnection networks (see the survey given in [3]).

Lemma 3.8 *A circulant graph $Cay(\mathbb{Z}_n, S)$ has a complete rotation if and only if there exists integers a and p prime with n such that $S = \{ap^\alpha : \alpha \in \mathbb{N}\}$.*

Proof. The if part is evident by taking $\omega(x) = px$.
The only if part follows from the fact that every automorphism of the additive group $\mathbb{Z}_n$ is of the kind $x \to px$ for some integer p (see the proof of Proposition 2.12). □

Lemma 3.9 *Let $\omega : \mathbb{Z}^n \to \mathbb{Z}^n$ be defined by $\omega(x_1, \cdots, x_n) = (x_2, \cdots, x_n, x_1)$. A Cayley graph on a (finite) Abelian group G has a complete rotation if there exists an integer n and a subgroup Q of $\mathbb{Z}^n$ such that $\omega(Q) = Q$ and G is isomorphic to the quotient $\mathbb{Z}^n/Q$.*

Proof. By Corollary 3.4, we get the result. □
Example 2.16 is also an illustration of this lemma.

3.3 Rotation-translation group

We will consider some properties of Cayley graphs and compare them to the rotational property.

Proposition 3.10 *Given $\Gamma = Cay(G,S)$, let H be a subgroup of $Stab(G,S)$ and $\underline{H}$ be the induced subgroup of $Aut(\Gamma)$. Let T be the subgroup of translations of Γ. Then the subgroup of $Aut(\Gamma)$ generated by T and $\underline{H}$, $<\underline{H},T>$, is a semi-direct product $T \rtimes H$ and therefore has cardinal $|G||H|$. Moreover, the set $A_h = \{th \mid t \in T\}$ for $h \in \underline{H}$, acts regularly on the vertices of Γ and maps any arc labeled s on any arc labeled $h(s)$.*

Proof. Let us recall conditions which are sufficient to have a (inner) semi-direct product $H \ltimes T = T \rtimes H$ ([22], page 27) : (i) T is a normal subgroup of $<\underline{H},T>$, (ii) $<\underline{H},T>=T\underline{H}$, (iii) $T \cap \underline{H} = I$.
We prove that all these conditions are fulfilled.
(i) Let h be a S-stabilizer and t_a a translation. For any $x \in G$, we get $ht_a(x) = h(ax) = h(a)h(x) = t_{h(a)}h(x) = t_{h(a)}h(x)$. Thus $ht_a = t_{h(a)}h$ and T is a normal subgroup of $<\underline{H},T>$.
(ii) Every element of $<\underline{H},T>$ is a product of elements of $\underline{H}$ and T and using equality of (i) can be written as a product of $T\underline{H}$ or $\underline{H}T$.
(iii) If $t_a \in T$ belongs to $\underline{H}$, then $t_a(I) = aI = I$, thus $a = I$ and $t_a = I$.
We now prove that, for any given $h \in \underline{H}$, $A_h = \{th \mid t \in T\}$ acts regularly on the vertices. Let x and x' be two given vertices of Γ. $x' = t_a h(x)$ implies $a = x'h(x)^{-1}$ and $x' = t_{x'h(x)^{-1}}(x)$. Thus there exists a unique automorphism $t_a h \in A_h$ such that $x' = t_a h(x)$. Furthermore, if $y = xs$, then $t_a h(y) = t_a(h(x)h(s)) = ah(x)h(s) = t_a(h(x))h(s) = t_a h(x)h(s)$.

Thus, if (x,y) is an arc labeled s, then $(t_a h(x), t_a h(y))$ is an arc labeled $h(s)$. This achieves the proof. □
By taking $H = Stab(G,S)$ in Proposition 3.10, we can introduce the following definition :

Definition 3.11 Let $\Gamma = Cay(G,S)$. The subgroup of automorphisms of Γ defined by the (inner) semi-direct product $T \rtimes \underline{Stab(G,S)}$ is called the *rotation-translation group* of Γ.

In the case of complete rotation we obtain the following result.

Corollary 3.12 *For any rotational Cayley graph Γ, there exists a subgroup of $Aut(\Gamma)$ which acts regularly on $A\Gamma$ and is isomorphic to the semi-direct product $T \rtimes \mathbb{Z}_d$, where d is the degree and T is the translation group of Γ.*

Proof. Let ω be a complete rotation of $Cay(G,S)$. We apply Proposition 3.10 when $\underline{H}$ is the cyclic group $<\omega>$ which is isomorphic to $\mathbb{Z}_d$.
Let x, y, x', y' be vertices of Γ such that $y = xs$ and $y' = x's'$, with $s, s' \in S$. Since ω is a complete rotation there exists an integer $i \in \mathbb{Z}_n$ such that $\omega^i(s) = s'$.

By applying Proposition 3.10 with $h = \omega^i$, we obtain an automorphism $f = t_a\omega^i \in A_h$ such that $f(x) = x'$ and $f(y) = f(xs) = x'\omega^i(s) = x's' = y'$. Furthermore f is unique, for if $y' = t_a\omega^i(y)$ and $x' = t_a\omega^i(x)$, then $t_a\omega^i(x)s' = y' = t_a\omega^i(x)\omega^i(s)$, thus $s' = \omega^i(s)$. Since ω is a complete rotation, i is unique in $\mathbb{Z}_d$. By Proposition 3.10, a is also unique. □

For the hypercube $H(d)$, the subgroup of Corollary 3.12 is $(\mathbb{Z}_2)^d \rtimes \mathbb{Z}_d$. Let us notice that the butterfly graph and the cube-connected cycles graph (see for example their definitions in [17]) are two Cayley graphs defined on this group. Notice also we will see later (see Proposition 4.14) that the rotation-translation group of $H(d)$ is equal to $\mathbb{Z}_2^d \rtimes \mathfrak{S}_d$.

Corollary 3.13 *Any rotational Cayley graph is arc-transitive.*

Notice that, in particular, the pancake graph, the cube-connected cycles graph and the butterfly graph are not rotational since they are not arc-transitive (see [20]). Let us recall that the edge-connectivity of a vertex-transitive graph (in particular a Cayley graph) is maximal and that the vertex-connectivity of an edge-transitive Cayley graph is equal to its degree and therefore maximal [26]. By Corollary 3.13, we get the next result.

Corollary 3.14 *The vertex-connectivity of a rotational Cayley graph is maximal.*

Remark 3.15 Since K_n is arc-transitive, Proposition 2.12 shows that not every arc-transitive Cayley graph is rotational. We will also deduce from Section 4 that the complete transposition graph which is arc-transitive ([20]) is not rotational.

By Corollary 3.6, if $Cay(G, S)$ has a complete rotation, then all generators of S have the same order in the finite group G. This condition is not sufficient to insure the existence of a complete rotation.

Remark 3.16 [21] There exist non-rotational Cayley graphs $Cay(G, S)$ such that all generators of S have the same order in the group G. The *Möbius graph* (depicted on Figure 1) is an example of such a graph.

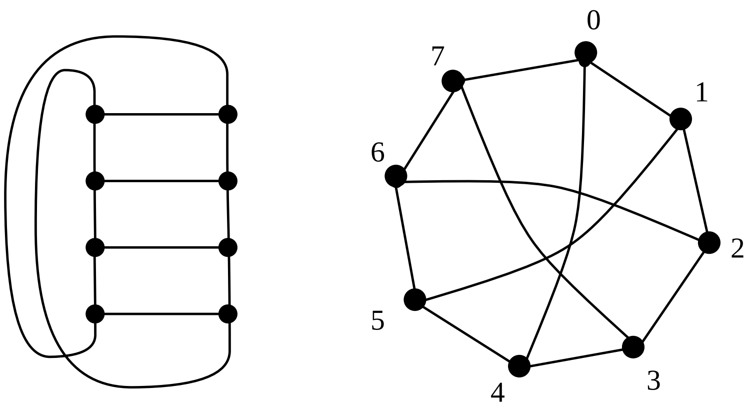

Figure 1: Möbius graph

The Möbius graph can be defined as the circulant Cayley graph $Cay(G, S)$ with $G = \mathbb{Z}_8$ and $S = \{-1,\ +1,\ +4\}$ $(-4 \equiv +4 \bmod 8)$. The generators are of orders 8, 8 and 2, respectively. By Corollary 3.6, we cannot find a complete rotation for this structure.
Furthermore, this graph is not arc-transitive. In fact, consider its vertices as labeled by $\mathbb{Z}_8$. It is easy to verify that the edge 01 belongs to only one 4-cycle $(0, 1, 5, 4)$, but the edge 04 belongs to two 4-cycles $(0, 1, 5, 4)$ and $(0, 4, 3, 7)$. Thus by Corollary 3.13, this graph is not rotational.
Since the Möbius graph is isomorphic to $Cay(G', S')$ with $G' = (S'|R')$, $S' = \{x, y, z\}$ and $R' = \{xyxyz^{-1}, x^2, y^2, z^2\}$, this graph is an example of non rotational Cayley graph with all the generators of S' having the same order in G'.

3.4 Complete rotations on Cartesian products

The Cartesian product of two graphs Γ and Γ', denoted by $\Gamma \Box \Gamma'$, is the graph with vertex set $V\Gamma \times V\Gamma'$ and edge set $\{[(i, j), (k, j)],\ [i, k] \in E\Gamma\} \cup \{[(i, j), (i, l)],\ [j, l] \in E\Gamma'\}$.
We recall the following well known result.

Proposition 3.17 *If* $\Gamma = Cay(G, S)$ *and* $\Gamma' = Cay(G', S')$*, then* $\Gamma \Box \Gamma'$ *is the Cayley graph on the group* $G \times G'$ *with set of generators* $(S \times I) \cup (I \times S')$*.*

In [13] the following question is settled. If Γ and Γ' are two graphs having a (complete) rotation, how about the Cartesian product $\Gamma \Box \Gamma'$?

Proposition 3.18 (also found independently by D. Barth) *Let* $\Gamma = Cay(G, S)$ *be a Cayley graph with a complete rotation. Then the Cartesian product* $\Gamma^n = \Gamma \Box \Gamma \Box \cdots \Box \Gamma$ *also has a complete rotation with the induced Cayley structure.*

Proof. Assume ω is a complete rotation of Γ. We denote the vertices of Γ^n by $(x_0, x_1, \cdots, x_{n-1})$. The nd generators of Γ^n can be ordered as

$$t_{jn+i} = (I, I, \ldots, s_j, \ldots I, \ldots, I)$$

(where i symbols I precede s_j), for $0 \leq i \leq n-1$ and $0 \leq j \leq d-1$.
A complete rotation ρ on Γ^n is given by

$$\rho(x_0, x_1, \cdots, x_{n-1}) = (\omega(x_{n-1}), x_0, \cdots, x_{n-2}).$$

Now ρ is a group homomorphism since

$$\rho[(x_0, x_1, \cdots x_{n-1})(y_0, y_1, \cdots, y_{n-1})] = \rho(x_0y_0, x_1y_1, \cdots, x_{n-1}y_{n-1}) = (\omega(x_{n-1}y_{n-1}), x_0y_0, \cdots, x_{n-2}y_{n-2})$$
$$= (\omega(x_{n-1})\omega(y_{n-1}), x_0y_0, \cdots, x_{n-2}y_{n-2}) = \rho(x_0, x_1, \cdots, x_{n-1})\rho(y_0, y_1, \cdots, y_{n-1}).$$

Furthermore, $\rho(t_i) = t_{i+1}$ for $0 \leq i \leq dn-1$ ($t_{nd} = t_0$). □

Notice that one can derive the same result by using Corollary 3.4 and considering a presentation $G = (S|R)$ such that R is invariant by a cyclic permutation of S. Then one obtains a presentation $(S'|R')$ of the Cartesian product by taking n disjoint copies of this presentation $(S_1|R_1), (S_2|R_2), \ldots, (S_n|R_n)$, with $S_j = \{s_i^j, 1 \leq i \leq d\}$ and $1 \leq j \leq n$. The mapping ω defined by $\omega(s_i^j) = s_i^{j+1}$ for $1 \leq j < n$ and $\omega(s_i^n) = s_{i+1}^1$ is a cyclic permutation of $S' = \cup S_j$ which is a R'-stabilizer.

Definition 3.19 A graph Γ is said to be *prime* if there exist no non-trivial graphs γ and γ' such that Γ is isomorphic to $\gamma \square \gamma'$. Two graphs Γ and Γ' are said to be *relatively prime* if there exist no non-trivial graph H, and graphs γ and γ', such that Γ is isomorphic to $H \square \gamma$ and Γ' is isomorphic to $H \square \gamma'$.

Lemma 3.20 *If γ and γ' are two relatively prime graphs, then $\gamma \square \gamma'$ is not arc-transitive, and thus not rotational.*

Proof. Applying the result of Sabidussi ([24]) to relatively prime γ and γ' , we get

$$Aut(\gamma \square \gamma') = Aut(\gamma) \times Aut(\gamma'). \tag{4}$$

Consider an arc $[(x, y), (x', y)]$ of $\gamma \square \gamma'$ (where $x \neq x'$ and $[x, x']$ is an arc of γ). Its image by any graph automorphism of $\gamma \square \gamma'$ is $[(h(x), g(y)), (h(x'), g(y))]$ where $h \in Aut(\gamma)$ and $g \in Aut(\gamma')$. This image will never be an arc $[(z, t), (z, t')]$ (with $t \neq t'$ and $[t, t']$ an arc in γ').

This proves that $\gamma \square \gamma'$ is not arc-transitive and by Corollary 3.13 not rotational. □
Thus we get,

Corollary 3.21 *If Γ is a rotational Cayley graph, then there exists a prime graph γ and an integer $n \geq 1$ such that $\Gamma = \gamma^n$.*

Corollary 3.21 shows that if a Cayley graph is rotational and is a Cartesian product, then all its prime factors are isomorphic. But we do not know at the present time if these factors are rotational and even Cayley graphs. Thus we can formulate the following problem.

Problem 3.22 *If the graph $\Gamma = \gamma^n$ is rotational, is γ also*

(i) a Cayley graph ?

(ii) a rotational graph ?

Notice that, as far as we know, it is even not evident that if γ^n is a Cayley graph, then γ is also a Cayley graph.

4 Cayley graphs defined by transpositions

In this part we consider only Cayley graphs $Cay(G, S)$ where $G = \langle S \rangle$ and S is a set of transpositions on $\{1,\ 2, \ldots,\ n\}$. For short we say that such a Cayley graph is *defined by transpositions.*
It involves many well studied interconnection Cayley graphs such as hypercubes, star graphs, complete transposition graphs, and modified bubble-sort graphs.
Notice that if s is a transposition, then $s = s^{-1}$.

4.1 Transposition graph

The transposition graph associated to a set of transpositions S is defined in [1] and [20] as follows.

Definition 4.1 Let S be a set of transpositions of $\{1, 2,\ \cdots,\ n\}$. The *transposition graph* of S, denoted by TS, is the graph with vertex set $\{1, 2,\ \cdots,\ n\}$ and edges ij for all $\langle i, j\rangle \in S$.

The following table give some examples considered in [20] except the generalized star graph which is defined in Section 4.3.

Graph	S	TS
Hypercube $H(n)$	$< 2p-1, 2p >,\ p = 1 \ldots n$	n vertex disjoint K_2
Star graph $ST(n)$	$< 1, i >,\ i = 2 \ldots n$	a star $K_{1,n-1}$
Generalized star graph $GST(n, k)$	$< i, j >,\ i = 1 \ldots k,\ j = k+1 \ldots n$	$K_{k,n-k}$
Bubble-sort graph $BS(n)$	$< i, i+1 >,\ i = 1 \ldots n-1$	a Hamiltonian path
Modified bubble-sort graph $MBS(n)$	$< n, 1 >,\ \ < i, i+1 >,\ i = 1 \ldots n-1$	a Hamiltonian cycle
Complete transposition graph CT_n	$< i, j >,\ i, j = 1 \ldots n,\ i \neq j$	complete graph K_n

We recall without proof some results we will use in Section 4.4. The following proposition shows that a Cayley graph $Cay(\langle S \rangle, S)$ generated by transpositions is characterized by the transposition graph TS.

Proposition 4.2 [20] *Let S and S' be two sets of transpositions of $\{1, 2,\ \cdots,\ n\}$. If the two graphs TS and TS' are isomorphic, then the Cayley graphs $Cay(\langle S \rangle, S)$ and $Cay(\langle S' \rangle, S')$ are also isomorphic.*

The converse of Proposition 4.2 has been proved recently by C. Delorme and J. Fournier.

Proposition 4.3 [9, 11] *Let S and S' be two sets of transpositions of $\{1, 2,\ \cdots,\ n\}$. If the Cayley graphs $Cay(\langle S \rangle, S)$ and $Cay(\langle S' \rangle, S')$ are isomorphic, then the two graphs TS and TS' are also isomorphic.*

Proposition 4.4 [20] *If the transposition graph TS is edge-transitive, then the Cayley graph $Cay(\langle S \rangle, S)$ is arc-transitive.*

The converse of Proposition 4.4 has been proved recently by C. Delorme.

Proposition 4.5 [9] *If the Cayley graph $Cay(\langle S \rangle, S)$ is arc-transitive, then the transposition graph TS is edge-transitive.*

Proposition 4.6 *If TS has r, $r > 1$, connected components corresponding to the subsets $S_l, 1 \leq l \leq r$, of S, then $Cay(\langle S \rangle, S)$ is the Cartesian product of the r Cayley graphs $Cay(\langle S_l \rangle, S_l)$, for $1 \leq l \leq r$.*

Proof. The group $\langle S \rangle$ is isomorphic to the direct product of the subgroups $\langle S_l \rangle$, $1 \leq l \leq r$, since two permutations with disjoint supports commute. Use Proposition 3.17 to finish the proof. □

Corollary 4.7 *If TS has r, $r > 1$, connected components and the graph $Cay(\langle S \rangle, S)$ has a complete rotation, then the r connected components of TS are isomorphic and $Cay(\langle S \rangle, S)$ is the Cartesian product of r isomorphic Cayley graphs.*

Proof. Let us assume that TS has r, $r > 1$, connected components corresponding to the subsets $S_l, 1 \leq l \leq r$. By Proposition 4.6, $Cay(\langle S\rangle, S)$ is the Cartesian product of the r Cayley graphs $Cay(\langle S_l\rangle, S_l)$, $1 \leq l \leq r$, generated by transpositions. Since $Cay(\langle S\rangle, S)$ has a complete rotation, it is arc-transitive and, by Proposition 4.5, the transposition graph TS is edge-transitive. This implies that its connected components TS_l, $1 \leq l \leq r$, are isomorphic. Thus the r Cayley graphs $Cay(\langle S_l\rangle, S_l)$, $1 \leq l \leq r$, are also isomorphic by Proposition 4.2. □

4.2 Rotations of Cayley graphs generated by transpositions

In this section we study the S-stabilizers of a permutation group generated by a set S of transpositions on $\{1, 2, \ldots, n\}$. We will assume that TS has *no isolated vertices* (otherwise we can consider permutations defined on a smaller set). We will show that complete rotations in Cayley graphs defined by transpositions are exactly those of Property 2.9.
Let Γ be a graph. We first recall some definitions from [16].
A *star* of Γ is any set of edges incident to a vertex of a graph. An automorphism f of Γ is said to be *star-preserving* if the image by f of any star of Γ is a star.

Lemma 4.8 *If h is a S-stabilizer of $\langle S\rangle$, then h induces a permutation ϕ of the edges of TS which is a star-preserving graph automorphism of the line-graph $L(TS)$.*

Proof. By definition, h is a group automorphism of $\langle S\rangle$. If we associate the edge $[i, j]$ of TS with the transposition $\langle i, j\rangle \in S$, then h induces a natural permutation ϕ of the edges of TS. First we prove that ϕ is a graph automorphism of $L(TS)$. Note that two transpositions commute in $\langle S\rangle$ if and only if their supports are disjoint. If $Supp\, \sigma \cap Supp\, \sigma' = \emptyset$, then $\sigma\sigma' = \sigma'\sigma$ and $\phi(\sigma)\phi(\sigma') = \phi(\sigma')\phi(\sigma)$, thus $Supp\, \phi(\sigma) \cap Supp\, \phi(\sigma') = \emptyset$. Conversely, by applying ϕ^{-1}.
This implies that ϕ maps adjacent vertices of $L(TS)$ onto adjacent vertices of $L(TS)$.

Now we prove that ϕ maps stars onto stars (and therefore triangles onto triangles). Assume that TS contains the three edges $[i, j], [i, k], [i, l]$ and that $\phi([i, j]) = [i', j']$, $\phi([i, k]) = [i', k']$. If $\phi([i, l]) \neq [i', l']$, then ϕ being an automorphism of $L(TS)$, $\phi([i, l]) = [j', k']$. Since h is a group homomorphism of $\langle S\rangle$, $h(\langle i, j\rangle\langle i, k\rangle\langle i, l\rangle\langle i, k\rangle) = \langle i', j'\rangle\langle i', k'\rangle\langle j', k'\rangle\langle i', k'\rangle = I$. But $\langle i, k\rangle\langle i, l\rangle\langle i, k\rangle = \langle k, l\rangle$, so that $h(\langle i, j\rangle\langle i, k\rangle\langle i, l\rangle\langle i, k\rangle) = \langle i', j'\rangle h(\langle k, l\rangle)$. This implies $I = \langle i', j'\rangle h(\langle k, l\rangle)$, thus $h(\langle k, l\rangle) = \langle i', j'\rangle = h(\langle i, j\rangle)$. This is impossible, since h is bijective on $\langle S\rangle$. □

Notice that if h is a S-stabilizer of $\langle S\rangle$, then h^{-1} is also a S-stabilizer and the induced permutation is ϕ^{-1} which is also star-preserving automorphism of $L(TS)$.
Let us recall a well-known result of Whitney on line-graphs.

Proposition 4.9 [16]
If Γ and Γ' are connected graphs and $f : E\Gamma \rightarrow E\Gamma'$ is a bijection, then f is induced by an isomorphism of Γ onto Γ' if and only if f and f^{-1} are star-preserving.

In fact the proof of Proposition 4.9 given in [16], uses only the hypothesis that Γ and Γ' have no isolated vertices and that each vertex of degree 1 is adjacent to a vertex of degree at least 2 (in other words the line graphs $L(\Gamma)$ and $L(\Gamma')$ have also no isolated vertices). Thus, we get :

Lemma 4.10 *Let S be a set of transpositions on $\{1,\ 2, \ldots,\ n\}$ such that the graphs TS and $L(TS)$ have no isolated vertices. Let ϕ and its reverse ϕ^{-1} be star-preserving graph automorphisms of $L(TS)$. Then there exists a graph automorphism σ_ϕ of TS such that $\phi([i, j]) = [\sigma_\phi(i), \sigma_\phi(j)]$*

Remark 4.11 The automorphism σ_ϕ of TS induces a permutation of the vertices $\{1, \ldots, n\}$ of TS also denoted by σ_ϕ.

We are now able to prove the next result.

Lemma 4.12 *Let S be a set of transpositions on $\{1,\ 2, \dots,\ n\}$ such that the graphs TS and $L(TS)$ have no isolated vertices. Then the S-stabilizers of $\langle S\rangle$ are exactly the mappings $x \to \sigma x \sigma^{-1}$, where the permutation $\sigma \in \mathfrak{S}_n$ is a graph automorphism of TS.*

Proof. Let h be a given S-stabilizer of $< S >$. It induces a permutation ϕ of the edges of TS. By Lemma 4.8, ϕ is a star-preserving automorphism of $L(TS)$. Lemma 4.10 then shows that there exists an automorphism σ_ϕ of the vertices of TS which induces ϕ on the edges of TS (vertices of $L(TS)$). So h maps the transposition $\langle i,j\rangle$ onto $\langle \sigma_\phi(i), \sigma_\phi(j)\rangle$. As $\langle \sigma_\phi(i), \sigma_\phi(j)\rangle = \sigma_\phi \langle i,j\rangle \sigma_\phi^{-1}$, the group automorphism of $< S >$: $x \to \sigma_\phi x \sigma_\phi^{-1}$ gives the same images to the generators as the S-stabilizer h does. It follows that this automorphism is indeed exactly h. □

By lemma 4.8 we get that, if the Cayley graph Γ has a complete rotation ω, then $L(TS)$ is vertex-transitive. Indeed, the subgroup of automorphisms $< \omega >$ induces a subgroup of automorphisms of the line-graph which acts transitively on the vertices. Thus if there is one isolated vertex in $L(TS)$ then TS is a union of isolated edges and Γ is a hypercube. We know in that case that there exists $\sigma \in \mathfrak{S}_n$ such that $\omega(x) = \sigma x \sigma^{-1}$ (where ω is the complete rotation). Thus, we get

Proposition 4.13 *Let S be a set of transpositions on $\{1,\ 2, \dots,\ n\}$. The complete rotations of $Cay(\langle S\rangle, S)$ are exactly those of Property 2.9.*

Notice that this proposition is a partial answer to Problem 2.10.

During the writing of this article, we were advised that J. Fournier proved the following generalization of Lemma 4.12.

Proposition 4.14 [11] *Let S be a set of transpositions on $\{1,\ 2, \dots,\ n\}$. If the graph TS is connected and is neither the cycle C_4 nor a complete graph, then any automorphism of $Cay(\langle S\rangle, S)$ which stabilizes the vertex I is induced by a group automorphism of $\mathfrak{S}_n$ $x \to \sigma x \sigma^{-1}$, where the permutation $\sigma \in \mathfrak{S}_n$ is a graph automorphism of TS.*

Since any automorphism of $Cay(\langle S\rangle, S)$ can be seen as the product of a translation by an automorphism which stabilizes I, Proposition 4.14 implies that the rotation-translation group of Cayley graphs defined by transpositions turns out to be the whole automorphism group except for $MBS(4) = GST(4,2)$ and the complete transposition graph CT_n.

4.3 Generalized star graphs

We now introduce a family of graphs which generalize star graphs. We consider Cayley graphs defined by transpositions of $X = \{1, \cdots, n\}$ involving elements of two complementary subsets of X.

Definition 4.15 Let k and n be two integers such that $1 \le k < n$. The *generalized star graph* $GST(n,k)$ is defined as the Cayley graph $Cay(\langle S\rangle, S)$ where S is the set of all the transpositions $\langle i,j\rangle$ of X, with $i \in \{1, \cdots, k\}$ and $j \in \{k+1, \cdots, n\}$.

Property 4.16 The transposition graph TS is the complete bipartite graph $K_{k,n-k}$.

In the case $k = 1$, $GST(n,1)$ is isomorphic to the star graph $ST(n)$ and $GST(2,1)$ to the graph K_2. Notice that $GST(4,2)$ (see Figure 6) is isomorphic to the modified bubble-sort graph $MBS(4)$ since these two graphs have the same associated transposition graph C_4 (see Proposition 4.2).
As defined in [8] the *arrangement graph* $A(n,k)$ is defined as the graph with vertex set the arrangements of k elements chosen out of n elements ; its edges connect vertices which correspond to arrangements differing in exactly one position. It is a quotient of $GST(n,k)$ obtained by contracting in one vertex all vertices of $GST(n,k)$ which are permutations of X giving the same image for all the elements i with $1 \le i \le k$ (and deleting loops and multiple edges).

Proposition 4.17 *For $n-k$ and k relatively prime the generalized star graph $GST(n,k)$ has a rotation.*

Proof. By hypothesis, S is the set of the transpositions $\langle i,j\rangle$, for $1 \le i \le k$ and $k+1 \le j \le n$. Consider the permutation σ defined on X by

$$\sigma = (2, \cdots, k, 1, k+2, k+3, \cdots, n, k+1).$$

Then $\sigma^{-1} = (k, 1, 2, \cdots, k-1, n, k+1, k+2, \cdots, n-1)$. For $1 \le i \le k$ and $k+1 \le j \le n$, $\sigma\langle i,j\rangle\sigma^{-1} = \langle 1+(i \bmod k), k+1+(j-k \mod (n-k))\rangle$. Since $n-k$ and k are relatively prime, the permutation $\langle 1, k+1\rangle$'s orbit under conjugation by σ is S. Using Property 2.9 we know that the graph has a complete rotation. □

Notice that the diameter of $GST(n,k)$ is studied in [25].

4.4 Characterization of rotational Cayley graphs defined by transpositions

We will now characterize the rotational Cayley graphs defined by transpositions by proving the following theorem.

Theorem 4.18 *The only Cayley graphs $Cay(\langle S\rangle, S)$ with S a set of transpositions, which have a complete rotation are*

(i) the modified bubble-sort graphs and the Cartesian products of isomorphic modified bubble-sort graphs,

(ii) the generalized star graphs $GST(t+q,q)$ with t and q relatively prime and the Cartesian products of isomorphic generalized star graphs $GST(t+q,q)$ with t and q relatively prime.

Notice that hypercubes are particular cases of Cartesian products of isomorphic generalized star graphs $GST(2,1)$.
In order to prove this theorem we need several lemmas which we present now.

Let ω be a cyclic S-stabilizer of $\langle S\rangle$ and $\overline{\omega}$ the automorphism of TS given by Lemma 4.10. If $\langle i,j\rangle \in S$, then $\omega(\langle i,j\rangle) = \langle\overline{\omega}(i), \overline{\omega}(j)\rangle$. Furthermore, the cyclic subgroup, $\langle\overline{\omega}\rangle$, of automorphisms of TS generated by $\overline{\omega}$ acts transitively on the edges of TS, and thus TS is edge-transitive. We now consider the orbits defined by the action of the group $\langle\overline{\omega}\rangle$ on the vertices of TS. The following lemma of Elayne Dauber is well-known (see [15], page 172).

Lemma 4.19 *If a graph Γ is edge-transitive without isolated vertices, then either Γ is vertex-transitive or Γ is bipartite and Γ has two vertex orbits which form the bipartition of Γ.*

By Lemma 4.19, there exist two cases : either $\langle\overline{\omega}\rangle$ is transitive on VTS and TS is vertex- and edge-transitive, or the action of $\langle\overline{\omega}\rangle$ on VTS defines two orbits and TS is edge-transitive but not necessarily vertex-transitive. We now consider these two cases separately.

Lemma 4.20 *Let TS be connected of maximum degree at least 2. If $\langle\overline{\omega}\rangle$ acts transitively on the vertices of TS, then TS is a vertex-disjoint union of isomorphic cycles.*

Proof. Since TS is vertex-transitive every vertex of TS has the same degree and this degree is at least 2. Let ij be any edge of TS. Since VTS is the orbit of i under $\langle\overline{\omega}\rangle$, there exists an integer $\alpha > 0$ such that $j = \overline{\omega}^{\alpha}(i)$. For any other edge kl of TS, there exists $\beta \in \mathbb{Z}$ such that $kl = \omega^{\beta}(ij)$. But $\omega^{\beta}(ij) = \overline{\omega}^{\beta}(i)\,\overline{\omega}^{\beta}(j)$ and $\overline{\omega}^{\beta}(j) = \overline{\omega}^{\beta}(\overline{\omega}^{\alpha}(i)) = \overline{\omega}^{\beta+\alpha}(i) = \overline{\omega}^{\alpha}(\overline{\omega}^{\beta}(i))$. Thus any edge of TS is of the form $u\,\overline{\omega}^{\alpha}(u)$, $u \in VTS$. This implies that any vertex of TS is of degree at most two. Therefore TS is regular of degree 2. Thus TS is an union of cycles and these cycles are isomorphic since TS is vertex-transitive. □

Lemma 4.21 *Let TS be connected of maximum degree at least 2. If the action of $\langle\overline{\omega}\rangle$ on the vertices of TS defines two orbits, then TS is a vertex disjoint union of $m \ge 1$ complete bipartite graphs isomorphic to $K_{t,q}$ ($mtq = d$) with t and q relatively prime.*

Proof. By Lemma 4.19, TS is bipartite with two independent sets Y, Z, $Y \cup Z = X$. Let $|Y| = p$, $|Z| = n-p$. For any vertex i in Y (resp. Z), the set of its image by the automorphisms of $\langle\varpi\rangle$ is Y (resp. Z). Thus any vertex of Y (resp. Z) has the same degree t (resp. q) and $d = tp = q(n-p)$.
Let $i \in Y$. The stabilizer of i for $\langle\varpi\rangle$ is by definition the subgroup of automorphisms h of $\langle\varpi\rangle$ such that $h(i) = i$. As a subgroup of a cyclic group, it is also cyclic and generated by an element ϖ^α. Since $\{\varpi^k(i), k \in \mathbb{Z}\} = Y$, $\alpha = p$. Furthermore all vertices of Y have isomorphic stabilizers.
Similarly, for any vertex j of Z, the stabilizer is generated by the element ϖ^{n-p}.
Let ij be any edge of TS, $i \in Y$, $j \in Z$. We will show that the connected component of i is isomorphic to $K_{t,q}$. By $\omega^{kp}(ij) = \varpi^{kp}(i)\ \varpi^{kp}(j) = i\ \varpi^{kp}(j)$ any vertex $\varpi^{kp}(j)$ is adjacent to i. On the other hand, for any edge $il, l \in Z$, the automorphism which sends the edge ij onto the edge il must belong to the stabilizer of i, so that $l = \varpi^{kp}(j)$ for some integer k. This shows that the neighborhood of i is $\{\varpi^{kp}(j), k \in \mathbb{Z}\}$. By symmetry, the neighborhood of j is $\{\varpi^{k(n-p)}(i), k \in \mathbb{Z}\}$. It remains to prove that any neighbor of i and any neighbor of j are adjacent. But, for any integers l, k, $\varpi^{l(n-p)}(i) = \varpi^{l(n-p)}(\varpi^{kp}(i)) = \varpi^{kp}(\varpi^{l(n-p)}(i))$, so that $\varpi^{l(n-p)}(i)\ \varpi^{kp}(j) = \omega^{kp}(\varpi^{l(n-p)}(i)j)$, and $\varpi^{l(n-p)}(i)\ \varpi^{kp}(j)$ is an edge of TS.
Thus TS is a disjoint vertex union of say $m \geq 1$ complete bipartite graph isomorphic to $K_{t,q}$, with $d = mtq$. It remains to prove that t and q are co-prime.
But the least common multiple of p and $n-p$ must be d, otherwise all the edges of TS could not be obtained from a given edge ij by the powers of ω as $\omega^\alpha(ij) = \varpi^\alpha(i)\varpi^\alpha(j)$. But since $d = tp = q(n-p)$, this implies that t and q are relatively prime. □

We can summarize the results obtained so far as follows.

Corollary 4.22 *Let S be a set of transpositions such that TS has no isolated vertices. Then $Cay(G, S)$ has a complete rotation if and only if its transposition graph TS is*

(i) the union of vertex disjoint isomorphic cycles, or

(ii) the union of vertex disjoint complete bipartite graphs isomorphic to $K_{t,q}$ for some t and q relatively prime.

Proof. The if part is proved by Proposition 4.6, Example 3.7 and Proposition 3.18.
Assume $Cay(G, S)$ has a complete rotation. We first consider the case where TS is not connected and of maximum degree 1. By Corollary 4.7, this implies that all the connected components of TS are isomorphic to K_2. Thus TS satisfies the condition (ii) in the particular case of vertex disjoint complete bipartite graphs $K_{1,1}$.
Now assume that the maximum degree of TS is at least 2. Either the hypothesis of Lemma 4.20 or the hypothesis of Lemma 4.21 is satisfied. In the first case, TS satisfies the condition (i) and in the second one, the condition (ii). □

We are now able to prove the theorem.

Proof of Theorem 4.18. By Proposition 3.18, Example 3.7 and Proposition 4.17, Cartesian products of isomorphic modified bubble-sort graphs and Cartesian products of isomorphic generalized star graphs $GST(n, k)$, with $n-k$ and k relatively prime, have a complete rotation. Conversely, assume that $Cay(\langle S\rangle, S)$ has a complete rotation and S is a set of transpositions. We can assume that TS is a graph on n vertices without isolated vertex (otherwise, we replace n by $n-1$). By Corollary 4.22, Proposition 4.6 and Example 3.7, $Cay(\langle S\rangle, S)$ is the Cartesian product of isomorphic modified bubble-sort graphs or the Cartesian product of isomorphic generalized star graphs. □

5 Conclusion

In this article we have studied some Cayley graphs $Cay(G, S)$ which are interesting as models of interconnection networks, since they behave well for communication algorithms. They have particular automorphisms called rotations which are induced by automorphisms of the group G defining the structure of Cayley graph. Such a group automorphism leaves invariant the set of generators S and in the particular case of a complete

rotation cyclically permutes the generators. Not all Cayley graphs have such complete rotations and we have studied some characterizations. We have characterized the complete graphs which have a complete rotation. Our more general characterization is given in terms of representation and relators for the group and the set of generators, but this result is not easy to handle for a general graph. Nevertheless we have completely characterized Cayley graphs generated by transpositions which have a complete rotation.
We have also studied conditions for the existence of a rotation and proved that some necessary conditions are not sufficient. Conversely, we do not know if some sufficient conditions we give, like for Cartesian products, are also necessary. Thus, we have pointed some problems, the most exciting being probably the equivalence of the existence of a complete rotation ω on $Cay(G,S)$ and the existence of an inner group automorphism of G, $x \to \sigma x \sigma^{-1}$, which cyclically permutes the generators.

Acknowledgments
The authors thank Dominique Barth, Charles Delorme and Gert Sabidussi for helpful discussions and references.
This work has its origin in part in discussions the authors have had at the NATO ASI on Graph Symmetry in Montreal, 1996. They also wish to express their thanks to the organizers of that meeting.

A Definitions of some Cayley graphs

In this section we recall the definition of some classical Cayley graphs defined on permutation groups which are rotational (see also [13]).

A.1 Cycle

The cycle C_n is the Cayley graph on $\mathfrak{S}_n$ and the subset of the two cycles $\langle 1,2,\ldots,n\rangle$ and $\langle n,n-1,\ldots,1\rangle$. In this case a complete rotation ω is defined by $\omega(x) = \sigma x \sigma^{-1}$, where the permutation σ is given by $(n,n-1,\ldots,2,1)$.

A.2 Multidimensional torus

The *multidimensional torus* TM_p^d is the Cartesian product of d cycles of length p and therefore TM_p^d is rotational by Proposition 3.18.

A.3 Hypercube

The hypercube $H(d)$ is the graph with vertex set $\{x_1x_2\ldots x_d : x_i \in \{0,1\}\}$, two vertices $x_1x_2\ldots x_d$ and $y_1y_2\ldots y_d$ being adjacent if and only $x_i = y_i$ for all but one i.
$H(d)$ is the Cartesian product of d complete graphs K_2 and the Cayley graph of the additive product group $\mathbb{Z}_2^d$ generated by the d generators $\underbrace{0\ldots0}_{i}1\underbrace{0\ldots0}_{d-i-1}$, $0 \le i \le d-1$.
$H(d)$ is also the Cayley graph of the permutation group G generated by the d transpositions $\langle 2i-1, 2i\rangle$, $1 \le i \le d$, defined on the set of $2d$ elements $X = \{1\ldots 2d\}$ ($H(4)$ is shown in Figure 2 and the associated transposition graph in Figure 3). Indeed, each vertex $x_1x_2\ldots x_d$, $x_i \in \{0,1\}$, can be renamed as the permutation $(a_1,a_2,\ldots,a_{2d})$ where $(a_{2i-1},a_{2i}) = (2i-1,2i)$ if $x_i = 0$ and $(a_{2i-1},a_{2i}) = (2i,2i-1)$ if $x_i = 1$.
$H(d)$ is rotational. A complete rotation ω is defined on $H(d)$ by $\omega(x) = \sigma x\sigma^{-1}$, where σ is the permutation given by $\sigma = (3,4,\ldots,2d-1,\ 2d,1,2) = \langle 1,3,\ldots,2d-1\rangle\langle 2,4,\ldots,2d\rangle$.
Thus, $\sigma^{-1} = (2d-1,2d,1,2,\ldots,2d-3,2d-2)$ and $\sigma^i\langle 1,2\rangle\sigma^{-i} = \langle 2i+1,2i+2\rangle$.

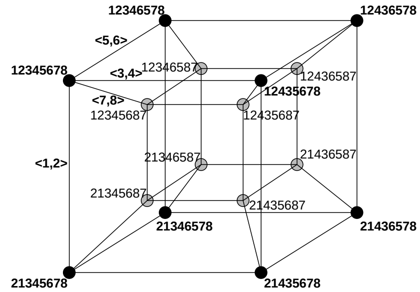

Figure 2: $H(4)$

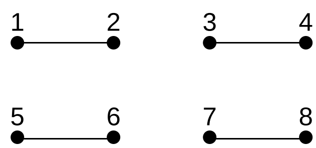

Figure 3: Transposition graph for $H(4)$

A.4 Star graph

The *star graph* $ST(n)$ is defined as the Cayley graph of the group $\mathfrak{S}_n$ generated by the $n-1$ transpositions $S = \{\langle 1, i\rangle, 1 < i \leq n\}$. The associated transposition graph is the star $K_{1,n-1}$ (see $ST(4)$ depicted on Figure 4 and the associated transposition graph depicted on Figure 5). A complete rotation ω is defined on $ST(n)$ by $\omega(x) = \sigma x \sigma^{-1}$, where the permutation σ is given by $\sigma = (1, 3, 4, \ldots, n, 2) = \langle 2, 3, \ldots, n\rangle$.

A.5 Generalized star graph

The *generalized star graph* $GST(n, k)$ is defined as the Cayley graph of the group $\mathfrak{S}_n$ generated by the set of all the transpositions $\langle i, j\rangle$ of X, with $i \in \{1, \cdots, k\}$ and $j \in \{k+1, \cdots, n\}$. It is proved in Section 4.3 that this graph is rotational if and only if k and $n-k$ are co-prime.

A.6 Modified bubble sort graph

The *modified bubble sort graph* of dimension n, $MBS(n)$, is defined as the Cayley graph of the group $\mathfrak{S}_n$ generated by the n transpositions $\{\langle i, i+1\rangle, 1 \leq i < n\} \cup \{\langle n, 1\rangle\}$. The associated transposition graph is the cycle on n vertices C_n. $MBS(n)$ has a complete rotation ω defined by $\omega(x) = \sigma x \sigma^{-1}$ where σ is the cyclic permutation given by $\langle 1, 2, \ldots, n\rangle$.

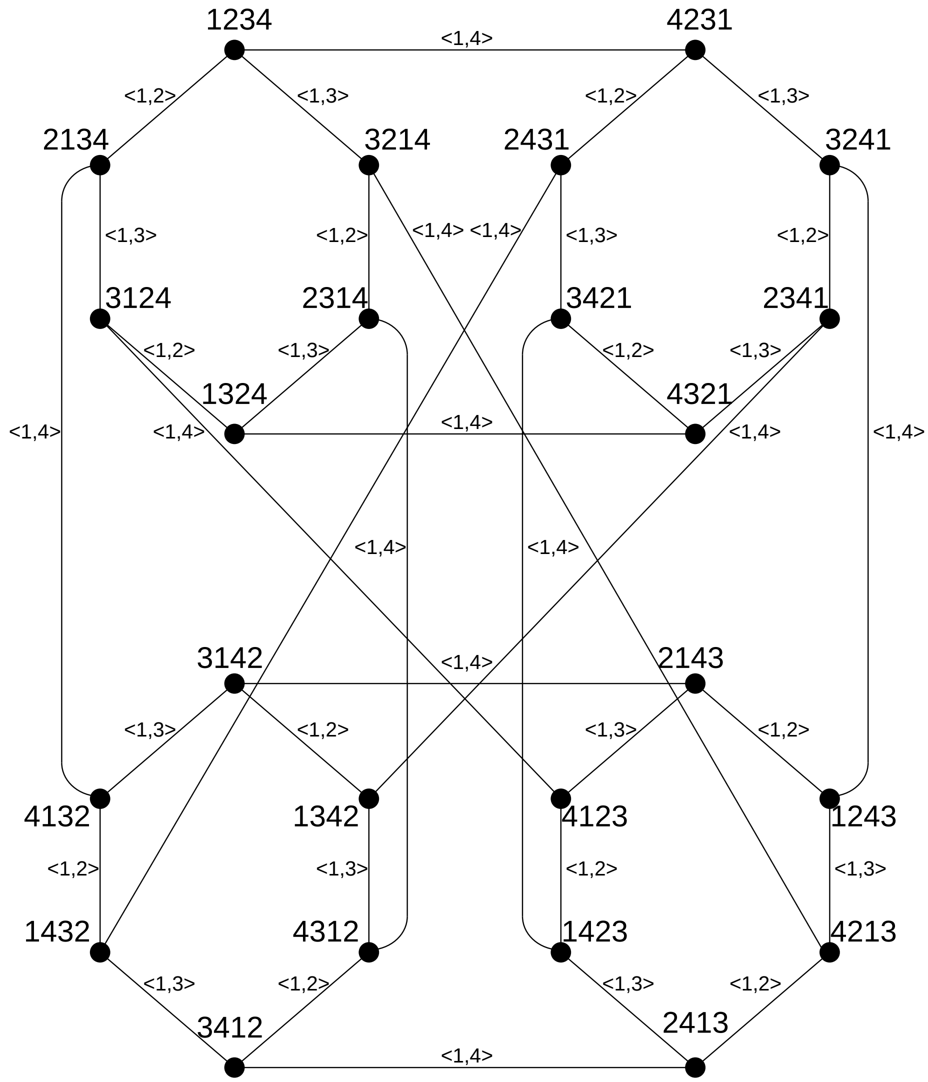

Figure 4: Star graph $ST(4)$.

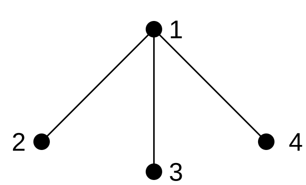

Figure 5: Transposition graph for $ST(4)$.

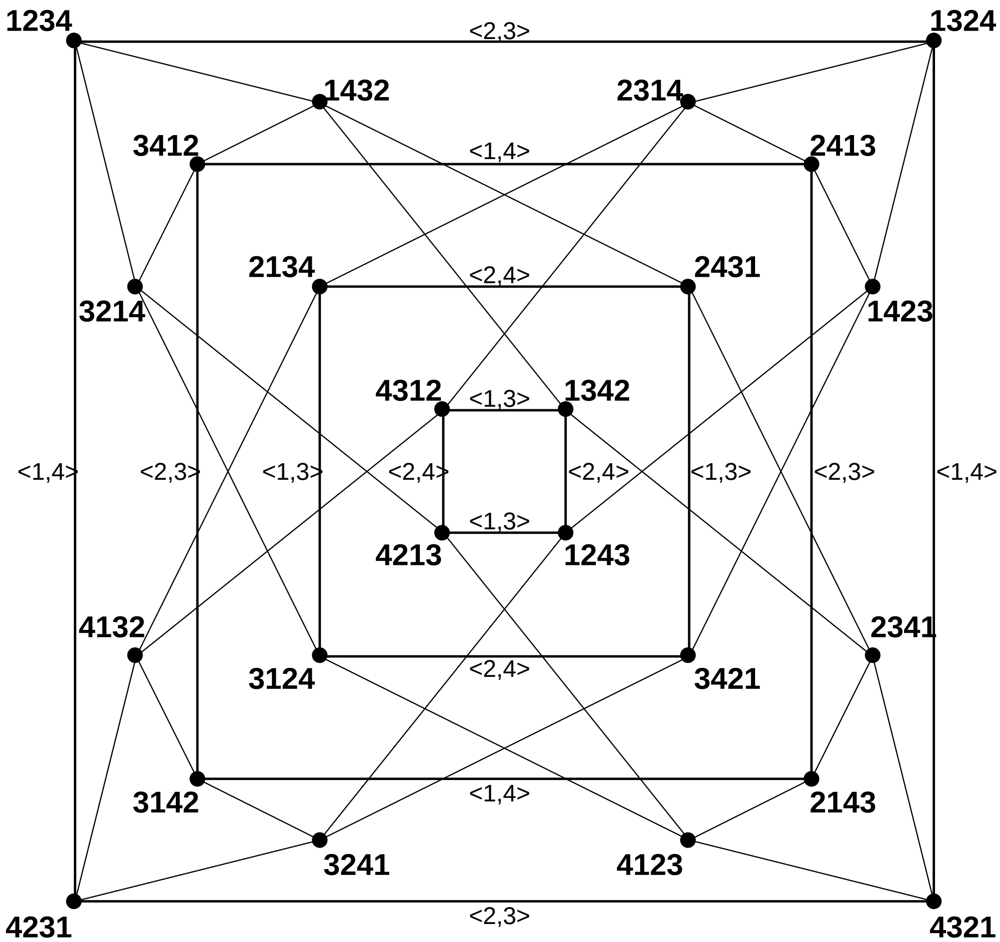

Figure 6: $GST(4,2) = MBS(4)$

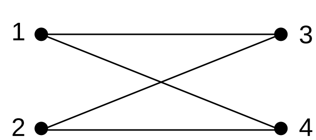

Figure 7: Transposition graph for $GST(4,2)$ and $MBS(4)$.

B Notation

Γ	a graph
$V\Gamma$	its vertex set
$E\Gamma$	its edge set
$[x, y]$	an edge
$A\Gamma$	the arc set $= \{(x, y)$ s.t. $[x, y]$ is an edge$\} \subset V\Gamma \times V\Gamma$
$L(\Gamma)$	the line-graph of Γ
$Aut(\Gamma)$	the graph-automorphism group of Γ
$\mathbb{Z}_n$	the group of integers modulo n
G	a group
I	unit
$Aut(G)$	the automorphism group of the group G
$S \subset G$	a subset
$\langle S \rangle$	the group generated by S
$Stab(G, S)$	subgroup of $Aut(G) = \{h \in \mathrm{Aut(G)}, h(S) = S\}$
$Cay(G, S)$	the Cayley graph of the group G and the subset S
H	a subgroup of $Stab(G, S)$
$\underline{H}$	the induced subgroup of $Aut(Cay(G, S))$
$(\sigma(1), \cdots, \sigma(n))$	a permutation σ on $X = \{1, \cdots, n\}$
$\sigma\tau$	$(\sigma(\tau(1)), \cdots, \sigma(\tau(n)))$
$\mathfrak{S}_X$	the group of permutations on X
$\mathfrak{S}_n$	the group of permutations on $\{1 \cdots n\}$
$\sigma = \langle i_1, i_2, \ldots, i_k \rangle$	the cycle (or cyclic permutation) defined by $\sigma(i_1) = i_2, \ldots \sigma(i_{k-1}) = i_k, \sigma(i_k) = i_1$
$\langle i, j \rangle$	transposition
$Supp\ \sigma$	$\{i \in X,\ \sigma(i) \neq i\}$

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ISSN 0249-6399