A Finite Element Method for Domain Decomposition with Non-Matching Grids

Roland Becker, Peter Hansbo

INSTITUT NATIONAL DE RECHERCHE EN INFORMATIQUE ET EN AUTOMATIQUE

A finite element method for domain decomposition with non-matching grids

Roland Becker — Peter Hansbo[1]

N° 3613

Janvier 1999

THÈME 4

Rapport de recherche

ISSN 0249-6399

A finite element method for domain decomposition with non-matching grids

Roland Becker , Peter Hansbo*

Thème 4 — Simulation et optimisation
de systèmes complexes

Projet Caiman

Rapport de recherche n° 3613 — Janvier 1999 — 17 pages

Abstract: In this note we propose and analyse a new method for handling interfaces between non-matching grids based on an approach suggested by Nitsche [12] for the approximation of Dirichlet boundary conditions. The exposition is limited to self-adjoint elliptic problems, using Poisson's equation as a model. A priori and a posteriori error estimates are given, and we discuss the relation to the mortar finite element method. Some numerical results are included.

Key-words: adaptivity, domain decomposition, finite elements

* Department of Solid Mechanics, Chalmers University of Technology, Göteborg, Sweden

Unité de recherche INRIA Sophia Antipolis
2004, route des Lucioles, B.P. 93, 06902 Sophia Antipolis Cedex (France)
Téléphone : 04 92 38 77 77 - International : +33 4 92 38 77 77 — Fax : 04 92 38 77 65 - International : +33 4 92 38 77 65

Une méthode éléments finis pour la décomposition de domaines avec maillages incompatibles

Résumé : Dans cet article nous présentons une nouvelle méthode pour le traitement d'interfaces entre deux maillages incompatibles. Cette méthode est basée sur une approche proposée par Nitsche [12] dans le cadre de l'approximation des conditions aux limites de type Dirichlet. Ici, nous nous restreignons aux cas d'équations elliptiques symétriques, en prenant l'équation de Poisson comme équation model. Nous établissons des estimations a priori et a posteriori. En suite, nous discutons le lien avec la "méthode des mortiers". Finalement, nous présontons quelques résultats numériques.

Mots-clés : adaptativité, décomposition de domaines, éléments finis

1. Introduction

In any domain decomposition method, one has to define how the continuity between the subdomains is to be enforced. Different approaches have been proposed:

- Iterative procedures, enforcing that the approximate solution or its normal derivative or combinations thereof should be continuous across interfaces. This forms the basis for the standard Schwarz alternating method as defined, e.g., by Lions [11].
- Direct procedures, using Lagrange multiplier techniques to achieve continuity. Different variants have been proposed, e.g., by Le Tallec and Sassi [10], and Bernadi, Maday, and Patera [7].

The multiplier method has the advantage of directly yielding a soluble global system. However, in the latter method, new unknowns (the multipliers) must be introduced and solved for, and must either satisfy the inf-sup condition, which necessitates special choices of multiplier spaces (such as mortar elements, cf. [7]), or use special stabilization techniques (cf. Baiocchi, Brezzi, and Marini [2]).

In this note, we propose a third possibility inspired by Nitsche's method [12], which was originally introduced for the purpose of solving Dirichlet problems without enforcing the boundary conditions in the definition of the finite element spaces. A modern setting for this method, and its relation to multiplier methods, is given by Stenberg [13] and by Brezzi, Franca, Marini, and Russo [8]. This method allows for independent approximations on the different subdomains. The continuity of the solution across interfaces is enforced weakly, but in such a way that the resulting discrete scheme is consistent with the original partial differential equation.

Though we discuss the new method in the context of domain decomposition, it is also suited for other applications, e.g.,

- to handle diffusion terms in the discontinuous Galerkin method [5];
- to simplify mesh generation (different parts can be meshed independently from each other);
- finite element methods with different polynomial degree on adjacent elements;
- new finite element methods such as linear approximations on quadrilaterals.

2. A Nitsche type method

2.1. **Model problem.** For simplicity, we consider only the Poisson problem of solving the partial differential equation

$$\left\{ \begin{array}{rcll} -\Delta u & = & f & \text{in } \Omega, \\ u & = & 0 & \text{on } \partial\Omega, \end{array} \right. \tag{1}$$

which can be written in weak form as: find $u \in H^1_0(\Omega)$ such that

$$a(u,v) = (f,v), \quad \forall v \in H^1_0(\Omega), \tag{2}$$

where

$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v \, d\Omega, \quad (f,v) = \int_\Omega f \, v \, d\Omega,$$

and $H^1_0(\Omega)$ is the space of square-integrable functions, with square-integrable first derivatives, that are zero on the boundary $\partial\Omega$ of Ω.

2.2. **Definition of the method.** For ease of presentation, we consider only the case where Ω is divided into two non-overlapping subdomains Ω_1 and Ω_2. We consider the interface Γ between Ω_1 and Ω_2 to be a straight line.

For the consistency of our numerical method we need some additional regularity of the solution to (2). More precisely, for the solution $u \in H^1_0(\Omega)$ and $v \in \{H^1(\Omega_1) \times H^1(\Omega_2) : \; v = 0 \quad \text{on } \partial\Omega\}$ there must hold:

$$\int_\Omega \nabla u \cdot \nabla v \, d\Omega = \int_\Omega f v \, d\Omega + \int_\Gamma \frac{\partial u}{\partial n} \left(v|_{\Omega_1} - v|_{\Omega_2} \right) ds, \tag{3}$$

which says that the normal derivative of the solution is continuous in a weak sense. This is true for example if $\Delta u \in L^2(\Omega)$, see [1]. For simplicity, we also assume that u is continuous at the interface Γ.

Our discrete method for the approximate solution of (1) is a nonconforming finite element method which is continuous within each Ω_i and discontinuous across Γ.

To formulate our method, we introduce the following spaces

$$V_i = \left\{ v_i : v_i|_{\Omega_i} \in H^1(\Omega_i), \; v_i|_{\partial\Omega \cap \partial\Omega_i} = 0 \right\}, \quad i = 1, 2,$$

and approximating spaces

$$V_i^h = \{ v_i \in V_i : \; v_i \text{ is a piecewise polynomial of degree } p \text{ on } \Omega_i \}.$$

For simplicity, we assume that the polynomial degree p is the same for all subdomains. We seek our approximative solution $\boldsymbol{U} = (U_1, U_2)$ in the product space $V^h = V_1^h \times V_2^h$

and for comparison we introduce the vector $\boldsymbol{u}$ corresponding to the solution of (2) via

$$\boldsymbol{u} = \big(u \cdot \chi(\bar{\Omega}_1), u \cdot \chi(\bar{\Omega}_2)\big) \in V = V_1 \times V_2.$$

where $\chi(\bar{\Omega}_i)$ is the characteristic function on $\bar{\Omega}_i$. Furthermore, we introduce the following notations:

$$\langle U, v\rangle_\Gamma := \int_\Gamma U\, v\, ds, \quad [u] := u_1 - u_2, \quad \{u\} := (u_1 + u_2)/2,$$

and denote by h the size of the smallest element on Ω_1 and Ω_2 neighboring to Γ. The outward pointing normal to Ω_i is denoted $\boldsymbol{n}_i$, and we let $\boldsymbol{n} := \boldsymbol{n}_1 = -\boldsymbol{n}_2$.

Our method can now be defined as follows: find $\boldsymbol{U} \in V^h$ such that

$$a_h(\boldsymbol{U}, \boldsymbol{v}) = ((f, \boldsymbol{v})) \quad \forall \boldsymbol{v} \in V^h \tag{4}$$

where

$$\begin{aligned} a_h(\boldsymbol{U}, \boldsymbol{v}) \quad := \quad & (\nabla U_1, \nabla v_1)_{\Omega_1} + (\nabla U_2, \nabla v_2)_{\Omega_2} - \left\langle [U], \left\{\frac{\partial v}{\partial n}\right\}\right\rangle_\Gamma \\ & - \left\langle \left\{\frac{\partial U}{\partial n}\right\}, [v]\right\rangle_\Gamma + \frac{\gamma}{h}\langle [U], [v]\rangle_\Gamma, \end{aligned}$$

and

$$((f, \boldsymbol{v})) := (f, v_1 + v_2).$$

Here, $\gamma > 0$ is a fixed number that must be chosen sufficiently large to ensure stability of the method (see Section 3) Note that the form $a_h(\cdot, \cdot)$ is symmetric, which may be important in order to find fast solution methods for the resulting matrix problem.

3. A priori analysis of the method

For the purpose of analysis, we will use the following norm on V:

$$|||\boldsymbol{u}|||^2 = \sum_i \|\nabla u_i\|^2_{L_2(\Omega_i)} + \left\|h^{-1/2}[u]\right\|^2_{L_2(\Gamma)} \tag{5}$$

We will need the following Lemmas:

Lemma 3.1. *The method* (4) *is consistent with* (1) *under the assumption* (3).

Proof. Insert the solution to (1) into $a_h(\cdot,\cdot)$ and use Green's formula to obtain

$$\begin{aligned}((f,\boldsymbol{v})) - a_h(\boldsymbol{u},\boldsymbol{v}) &= \sum_i (f,v_i) - \sum_i (\nabla u, \nabla v_i) + \left\langle \frac{\partial u}{\partial n}, [v] \right\rangle_\Gamma \\ &= \sum_i (f + \Delta u, v_i) = 0,\end{aligned}$$

which shows consistency with (1). □

We note in particular that Lemma 3.1 implies the Galerkin orthogonality relation

$$a_h(\boldsymbol{U} - \boldsymbol{u}, \boldsymbol{v}) = 0 \quad \forall \boldsymbol{v} \in V^h. \tag{6}$$

Lemma 3.2. *The method* (4) *is stable if* γ *is chosen large enough.*

Proof. We will show that $a_h(\boldsymbol{U},\boldsymbol{U}) \geq C|||\boldsymbol{U}|||^2$ for some constant C, which implies stability. For this purpose, we shall use the following well known inverse property in V_i^h:

$$\left\| h^{1/2} \frac{\partial v_i}{\partial n_i} \right\|^2_{L_2(\Gamma)} \leq C_0 \left\| \nabla v_i \right\|^2_{L_2(\Omega_i)} \quad \forall v_i \in V_i^h, \tag{7}$$

(see, e.g., Thomée [14]). We then have that

$$a_h(\boldsymbol{U},\boldsymbol{U}) = \sum_i \left\| \nabla U_i \right\|^2_{L_2(\Omega_i)} + \left\langle \frac{\partial U_1}{\partial n} + \frac{\partial U_2}{\partial n}, U_2 - U_1 \right\rangle_\Gamma + \gamma \left\| h^{-1/2}(U_2 - U_1) \right\|^2_{L_2(\Gamma)}.$$

Now, since

$$\begin{aligned}\left\langle \frac{\partial U_i}{\partial n_i}, U_2 - U_1 \right\rangle_\Gamma &\geq -\left\| h^{1/2} \frac{\partial U_i}{\partial n_i} \right\|_{L_2(\Gamma)} \left\| h^{-1/2}(U_2 - U_1) \right\|_{L_2(\Gamma)} \\ &\geq -\frac{1}{\varepsilon} \left\| h^{1/2} \frac{\partial U_i}{\partial n_i} \right\|^2_{L_2(\Gamma)} - \frac{\varepsilon}{4} \left\| h^{-1/2}(U_2 - U_1) \right\|^2_{L_2(\Gamma)},\end{aligned}$$

where we used Young's inequality, we have, using (7) and the triangle inequality, that

$$\begin{aligned}a_h(\boldsymbol{U},\boldsymbol{U}) &\geq \left(1 - \frac{2C_0}{\varepsilon}\right) \sum_i \left\| \nabla U_i \right\|^2_{L_2(\Omega_i)} + \left(\gamma - \frac{\varepsilon}{2}\right) \left\| h^{-1/2} \; [U] \right\|^2_{L_2(\Gamma)} \\ &\geq C|||\boldsymbol{U}|||^2\end{aligned}$$

as long as $\gamma > \varepsilon/2 > C_0$. □

Lemma 3.3. *The following interpolation estimate holds for the subspace V^h with $2 \leq k \leq p+1$:*

$$\inf_{\boldsymbol{v}\in V^h} |||\boldsymbol{u}-\boldsymbol{v}||| \leq C h^{k-1} \sum_i \|\boldsymbol{u}\|_{H^k(\Omega_i)}, \quad u_i \in H^k(\Omega),\ u_i|_{\partial\Omega} = 0. \tag{8}$$

Proof. Since the norm $|||\boldsymbol{u}-\boldsymbol{v}|||$ consists of contributions from the two domains Ω_1 and Ω_2, the Lemma reduces to the question whether

$$\inf_{v_i \in V_i^h} \left(\|\nabla u - \nabla v_i\|^2_{L_2(\Omega_i)} + \|h^{-1/2}\,(u-v_i)\,\|^2_{L_2(\Gamma)} \right)^{1/2} \leq C h^{k-1}\, \|u\|_{H^k(\Omega_i)}$$

for $u \in H^k(\Omega_i)$, $u|_{\partial\Omega} = 0$. This assertion follows from Lemma 2.3 in [14]. □

By use of the preceding Lemmas, we are now able to show the following a priori error estimate.

Theorem 3.1. *With $\boldsymbol{U}$ the solution of (4), u the solution of (1), and $\boldsymbol{u}$ the extension of u into V, we have that*

$$|||\boldsymbol{U}-\boldsymbol{u}||| \leq C h^{k-1}\, \|u\|_{H^k(\Omega)}, \quad 2 \leq k \leq p+1.$$

Proof. For any $\boldsymbol{v} \in V^h$, we have that

$$|||\boldsymbol{U}-\boldsymbol{u}||| \leq |||\boldsymbol{U}-\boldsymbol{v}||| + |||\boldsymbol{v}-\boldsymbol{u}|||$$

and, by Lemma 3.2,

$$\begin{aligned} |||\boldsymbol{U}-\boldsymbol{v}|||^2 &\leq C\ a_h(\boldsymbol{U}-\boldsymbol{v}, \boldsymbol{U}-\boldsymbol{v}) = C\ a_h(\boldsymbol{u}-\boldsymbol{v}, \boldsymbol{U}-\boldsymbol{v}) \\ &\leq C|||\boldsymbol{U}-\boldsymbol{v}||| \left(|||\boldsymbol{u}-\boldsymbol{v}||| + \left\| h^{1/2} \left\{ \frac{\partial u}{\partial n} - \frac{\partial v}{\partial n} \right\} \right\|_{L_2(\Gamma)} \right). \end{aligned}$$

Thus, Lemma 3.3 yields

$$\begin{aligned} |||\boldsymbol{U}-\boldsymbol{u}||| &\leq C \left(|||\boldsymbol{u}-\boldsymbol{v}||| + \left\| h^{1/2} \left\{ \frac{\partial u}{\partial n} - \frac{\partial v}{\partial n} \right\} \right\|_{L_2(\Gamma)} \right) + |||\boldsymbol{v}-\boldsymbol{u}||| \\ &\leq C h^{k-1}\, \|u\|_{H^k(\Omega)} \end{aligned}$$

which is the statement of the theorem. □

Remark 3.1. *Note that it follows that $\|\,[U]\,\|_{L_2(\Gamma)} = O(h^{k-1/2})$.*

Remark 3.2. *Instead of using the norm $|||\cdot|||$ for the a priori analysis, one can follow the analysis by Thomée* [14] *for Nitsche's method, and instead use the norm*

$$|||\boldsymbol{u}|||_{\mathrm{T}} := \sum_i \|\nabla u\|_{L_2(\Omega_i)} + \left\|h^{-1/2}\,[u]\right\|_{L_2(\Gamma)} + \left\|h^{1/2}\left\{\frac{\partial u}{\partial n}\right\}\right\|_{L_2(\Gamma)}$$

to obtain $|||\boldsymbol{U}-\boldsymbol{u}|||_{\mathrm{T}} \le Ch^{k-1}\,\|u\|_{H^k(\Omega)}$.

Remark 3.3. *The presented method resembles a mesh-dependent penalty method, but with added consistency terms involving normal derivatives across the interface. Note that the formulation allows us to deduce optimal order error estimates* with preserved condition number *of $O(h^{-2})$ for the resulting discrete scheme. Pure penalty methods, in contrast, are not consistent, and optimal error estimates require degrading the condition number for higher polynomial approximation (cf.* [4]*).*

4. A posteriori error estimates

4.1. **An a posteriori error estimate in the natural norm.** We first consider control of the error $\boldsymbol{e} = \boldsymbol{u} - \boldsymbol{U}$ in the natural norm $|||\cdot|||$. To be able to control the normal derivatives across the interface, we introduce the following ("saturation") assumption: For the error, there holds

$$\left\|h^{1/2}\left\{\frac{\partial e}{\partial n}\right\}\right\|_{L_2(\Gamma)} \le C|||\boldsymbol{e}||| \tag{9}$$

for some constant C. To support this assumption, we refer to Remark 3.2. (A similar assumption was used by Wohlmuth [15] in the context of a posteriori error estimates for the mortar element method.)

We have that

$$\begin{aligned} |||\boldsymbol{e}|||^2 &= a_h(\boldsymbol{e},\boldsymbol{e}) + 2\left\langle [e], \left\{\frac{\partial e}{\partial n}\right\}\right\rangle_\Gamma + \frac{1-\gamma}{h}\langle [e],[e]\rangle_\Gamma \\ &= ((f,\boldsymbol{e})) - a_h(\boldsymbol{U},\boldsymbol{e}) + 2\left\langle [e], \left\{\frac{\partial e}{\partial n}\right\}\right\rangle_\Gamma + \frac{1-\gamma}{h}\langle [e],[e]\rangle_\Gamma . \end{aligned}$$

We thus have to estimate two different terms,

$$R_1 = ((f,\boldsymbol{e})) - a_h(\boldsymbol{U},\boldsymbol{e}),$$

and

$$R_2 = 2\left\langle [e], \left\{\frac{\partial e}{\partial n}\right\}\right\rangle_\Gamma - \frac{\gamma-1}{h}\langle [e],[e]\rangle_\Gamma .$$

We first note that, for $\gamma \geq 1$ (which is needed for stability, see Section 6),

$$\begin{aligned} R_2 &\leq 2\left\langle [e], \left\{\frac{\partial e}{\partial n}\right\}\right\rangle_\Gamma \\ &\leq 2\left\langle |[U]|, \left|\left\{\frac{\partial e}{\partial n}\right\}\right|\right\rangle_\Gamma \\ &\leq C\left\|h^{-1/2}[U]\right\|_{L_2(\Gamma)} |||\boldsymbol{e}||| \end{aligned}$$

under the assumption (9). Next, using the orthogonality relation (6),

$$a_h(\boldsymbol{U}-\boldsymbol{u}, \boldsymbol{v}) = 0 \quad \forall \boldsymbol{v} \in V^h,$$

with $\boldsymbol{v} = \pi_h \boldsymbol{e} \in V^h$ a suitable interpolant of $\boldsymbol{e}$, we find that

$$R_1 = ((f, \boldsymbol{e} - \pi_h \boldsymbol{e})) - a_h(\boldsymbol{U}, \boldsymbol{e} - \pi_h \boldsymbol{e}),$$

Setting $\boldsymbol{\phi} = \boldsymbol{e} - \pi_h \boldsymbol{e}$, the integrals on the right-hand side may be written as

$$\begin{aligned} R_1 \;=\; & \sum_i \sum_K \left(\int_K f\,\phi_i dK - \int_K \nabla U_i \cdot \nabla \phi_i dK\right) + \\ & \sum_K \int_{\partial K \cap \Gamma} \left\{\frac{\partial U}{\partial n}\right\} [\phi] ds + \sum_K \int_{\partial K \cap \Gamma} \left\{\frac{\partial \phi}{\partial n}\right\} [U] ds - \\ & \sum_K \int_{\partial K \cap \Gamma} \frac{\gamma}{h} [\phi][U] ds. \end{aligned}$$

Using Green's formula we obtain

$$\int_K f\,\phi_i dK - \int_K \nabla U_i \cdot \nabla \phi_i dK = \int_K (f + \Delta U_i)\ \phi_i dK - \int_{\partial K} \frac{\partial U_i}{\partial n_K} \phi_i\ ds$$

where $\boldsymbol{n}_K$ denotes the outward unit normal to the boundary ∂K of the element K. If we now sum up all element contributions, and use the fact that each *interior* element side S is shared by two elements, we obtain

$$\begin{aligned} R_1 \;=\; & \sum_i \sum_K \int_K (f + \Delta U_i)\ \phi_i dK + \sum_i \sum_{S \not\subset \Gamma} \int_S \left[\frac{\partial U_i}{\partial n_S}\right]\ \phi_i\ \mathrm{d}s + \\ & \left\langle \left[\frac{\partial U}{\partial n}\right], \{\phi\}\right\rangle_\Gamma + \left\langle [U], \left\{\frac{\partial \phi}{\partial n}\right\}\right\rangle_\Gamma + \frac{\gamma}{h}\langle [\phi], [U]\rangle_\Gamma . \end{aligned}$$

We thus arrive at

Theorem 4.1. *Under the assumption* (9), *the error in the norm* $|||\cdot|||$ *can be estimated by*

$$|||\boldsymbol{e}||| \leq \sum_{K\subset\Omega} \rho_K + \sum_{S\subset\Gamma} \rho_S \tag{10}$$

where

$$\rho_K = C_1^i h_K \|f+\Delta U\|_{L_2(K)} + C_2^i h_K^{1/2} \left\| \left[\frac{\partial U}{\partial n_K}\right] \right\|_{L_2(\partial K)}$$

and

$$\rho_S = C h_S^{-1/2} \, \| \, [U] \, \|_{L_2(S)} \, ,$$

where C_1^i *and* C_2^i *are interpolation constants resulting from interpolation between* $\|\cdot\|_{L_2(K)}$ *and* $|||\cdot|||$, *and between* $\|\cdot\|_{L_2(\partial K)}$ *and* $|||\cdot|||$, *respectively.*

4.2. **A general a posteriori estimate.** Following [9, 6] we introduce the dual problem of finding $z \in H_0^1(\Omega)$ such that

$$(\nabla v, \nabla z) = j(v) \quad \forall v \in H_0^1(\Omega). \tag{11}$$

where j is a given functional (tailored for the desired error control), and where we assume that z fulfills the continuity requirement (3). Note that for control of quantities not involving derivatives of the solution, we do not need assumption (9).

We have that

$$\begin{aligned} j(e) &= (\nabla e, \nabla z) - \left\langle \left\{ \frac{\partial z}{\partial n} \right\}, [e] \right\rangle_\Gamma \\ &= a_h(\boldsymbol{e}, \boldsymbol{z}) + \left\langle \left\{ \frac{\partial z}{\partial n} \right\}, [e] \right\rangle_\Gamma - \left\langle \left\{ \frac{\partial z}{\partial n} \right\}, [e] \right\rangle_\Gamma \\ &= a_h(\boldsymbol{e}, \boldsymbol{z} - \pi_h \boldsymbol{z}) \\ &= a_h(\boldsymbol{u}, \boldsymbol{z} - \pi_h \boldsymbol{z}) - a_h(\boldsymbol{U}, \boldsymbol{z} - \pi_h \boldsymbol{z}) \\ &= ((f, \boldsymbol{z} - \pi_h \boldsymbol{z})) - a_h(\boldsymbol{U}, \boldsymbol{z} - \pi_h \boldsymbol{z}), \end{aligned}$$

so that

$$j(e) \leq \sum_{K\subset\Omega} \rho_K \omega_K + \sum_{S\subset\Gamma} \rho_S \omega_S \tag{12}$$

where

$$\rho_K = h_K \|f+\Delta U\|_{L_2(K)} + \frac{1}{2} h_K^{1/2} \left\| \left[\frac{\partial U}{\partial n}\right] \right\|_{L_2(\partial K)},$$

$$\omega_K = \max\left(h_K^{-1}\left\|z-\pi_h z\right\|_{L_2(K)}, h_K^{-1/2}\left\|z-\pi_h z\right\|_{L_2(\partial K)}\right),$$

$$\rho_S = (\gamma+1)h_S^{-1/2}\left\|[U]\right\|_{L_2(S)},$$

and

$$\omega_S = h_S^{-1/2}\left\|z-\pi_h z\right\|_{L_2(S)} + h_S^{1/2}\left\|\frac{\partial(z-\pi_h z)}{\partial n}\right\|_{L_2(S)}.$$

Remark 4.1. *From the general a posteriori error estimate, it is easy to prove that we have an optimal convergence rate in the L_2-norm, since, with the choice $j(\boldsymbol{e}) := \boldsymbol{e}$, we have by elliptic regularity that $\|z\|_{H^2(\Omega)} \leq C\|\boldsymbol{e}\|_{L_2(\Omega)}$. In consequence, since $|||\boldsymbol{z}-\pi_h\boldsymbol{z}||| \leq Ch\|z\|_{H^2(\Omega)}$, using the same arguments as in the proof of Theorem 1 shows that*

$$\|\boldsymbol{e}\|_{L_2(\Omega)} \leq Ch^k\|u\|_{H^k(\Omega)}.$$

5. Comparison with the mortar element method

Let us give a different interpretation of our method. We start with a classical formulation for imposing weak continuity on the interface by the use of a Lagrange multiplier: Find $(u,\lambda) \in V \times \Lambda$, such that:

$$(13)\qquad \left\{\begin{array}{rcl} \displaystyle\int_{\Omega_1}\nabla u\cdot\nabla v\,d\Omega_1 + \int_{\Omega_2}\nabla u\cdot\nabla v\,d\Omega_2 + \int_\Gamma [v]\,\lambda\,ds &=& F(v),\\ \displaystyle\int_\Gamma [u]\,\mu\,ds &=& 0,\end{array}\right.$$

for all $(v,\lambda) \in V \times \Lambda$ with appropriate spaces V and Λ.

A finite element discretization of (13) by choosing subsapce $V_h \subset V$ and $\Lambda_h \subset \Lambda$ must be designed carefully in order to be stable. In domain decomposition, it is merely the multiplier space which is of interest, since we would like to use given methods on the subdomains.

There are two possible strategies to obtain a stable discretization: either choose spaces which satisfy the inf-sup condition or change the discrete bilinear form to increase stability.

The first possibility is followed by the mortar element method, see [7]. Here one basically takes the traces of the finite element functions on one specified side of the interface. At the interface boundaries, special conditions have to be satisfied.

The second possibility is to add least squares terms in order to achieve more freedom on the choice of Λ_h. This was first proposed for the inhomogenuous Dirichlet problem by Barbosa and Hughes [3] (see also [13]), and was extended to domain decomposition problems by Baiocchi, Brezzi, and Marini [2]. A straightforward generalization leads to adding the terms:

$$-\delta\left\langle \lambda_h + \left\{\frac{\partial u_h}{\partial n}\right\}, \mu + \left\{\frac{\partial v}{\partial n}\right\}\right\rangle_\Gamma .$$

For a properly choosen parameter $\delta \approx h$, stability of this method follows from an inverse estimate. For instance, we can choose the space of piecewise constants on an arbitrary mesh on the interface. Using the augmented equations, the Lagrange multiplier can be eliminated,

$$\lambda_h = P_h\left(-\left\{\frac{\partial u_h}{\partial n}\right\} + \frac{1}{\delta}\left[u_h\right]\right),$$

denoting by P_h the L_2-projection on the discrete multiplier space.

We now formally let Λ_h tend to Λ and insert the resulting equation in (13). This gives us our method as presented above, since

$$\begin{aligned}\langle [v]\,, \lambda\rangle_\Gamma - \delta\left\langle \lambda + \left\{\frac{\partial u}{\partial n}\right\}, \left\{\frac{\partial v}{\partial n}\right\}\right\rangle_\Gamma &= \\ = -\left\langle [v]\,, \left\{\frac{\partial u}{\partial n}\right\}\right\rangle_\Gamma + \frac{1}{\delta}\langle [u]\,,[v]\rangle_\Gamma - \left\langle [u]\,, \left\{\frac{\partial v}{\partial n}\right\}\right\rangle_\Gamma .\end{aligned}$$

The new method can therefore be interpreted as a stabilized multiplier method with a continuous Lagrange multiplier space (cf. [13, 2]).

Remark 5.1. *We can immediately derive two variants of the method. First, we can choose one side of the interface Γ^+, and set $\lambda = -\frac{\partial u^+}{\partial n}$. This leads to:*

$$\begin{aligned}\widetilde{a}_h(\boldsymbol{U},\boldsymbol{v}) \quad := \quad & (\nabla U_1, \nabla v_1)_{\Omega_1} + (\nabla U_2, \nabla v_2)_{\Omega_2} - \left\langle [U]\,, \frac{\partial v^+}{\partial n}\right\rangle_\Gamma \\ & - \left\langle \frac{\partial U^+}{\partial n}, [v]\right\rangle_\Gamma + \frac{\gamma}{h}\langle [U]\,,[v]\rangle_\Gamma .\end{aligned}$$

Second, the simpler stabilization $\delta\int_\Gamma(\lambda + \{\frac{\partial u}{\partial n}\})\mu\,ds$ could be used to obtain the (unsymmetric) bilinear form:

$$\begin{aligned}\widehat{a}_h(\boldsymbol{U},\boldsymbol{v}) \quad := \quad & (\nabla U_1, \nabla v_1)_{\Omega_1} + (\nabla U_2, \nabla v_2)_{\Omega_2} \\ & - \left\langle \left\{\frac{\partial U}{\partial n}\right\}, [v]\right\rangle_\Gamma + \frac{\gamma}{h}\langle [U]\,,[v]\rangle_\Gamma .\end{aligned}$$

The analysis of these variants closely follows the one given before.

Remark 5.2. *Let us compare the new method with the mortar method from a practical point of view. The advantage is that we get rid of the global coupling on the interface which comes from the fact that the mortar element method uses a multiplier space of continuous functions. The price to pay for this are the additional terms in the bilinear form; in particular, the evaluation of the integrals involving the normal derivative leads to a coupling over one layer of elements.*

6. Computation of C_0 and γ

For any element type, it is important to have a sharp estimate of C_0 in order to be able to safely choose γ. We propose to compute C_0 in the following way (we consider the case of triangular elements in two dimensions, but the concept is completely general).

An arbitrary triangular element K is put in a local coordinate system so that one side lies along the y–axis, $0 \leq y \leq h$, and the third node at an arbitrary position (x_1, y_1) in the first quadrant. Then

$$\frac{\partial u}{\partial n} = -\frac{\partial u}{\partial x}$$

and we look for solutions to the eigenvalue problem of seeking u and C_0 so that

$$\int_0^h h \frac{\partial u}{\partial x} \frac{\partial v}{\partial x}\, dy - C_0 \int_K \nabla u \cdot \nabla v \, dx dy = 0$$

for all v on the element. This problem can be solved analytically to yield $C_0 \geq 2h/x_1$, so we have that in the general case

$$\gamma > 2\frac{h}{h_\perp},$$

where $h_\perp$ is the perpendicular distance from the boundary segment to the third (interior) node. (Assuming equal–sized triangles, we simply put $\gamma > \sqrt{3}$.) Now, we choose the largest such C_0 computed along the interior edge Γ so that

$$C_0 \sum_K \|\nabla v_i\|^2_{L_2(K)} \geq \left\| h^{-1/2} \frac{\partial v_i}{\partial n_i} \right\|^2_\Gamma .$$

Since $\|\nabla v_i\|^2_{L_2(\Omega_i)} = \|\nabla v_i\|^2_{L_2(\Omega_i')} + \|\nabla v_i\|^2_{L_2(\Omega_i \setminus \Omega_i')}$ where Ω_i' denotes the subset of Ω_i covered by triangles neighboring to the boundary so that

$$\sum_K \|\nabla v_i\|^2_{L_2(K)} = \|\nabla v_i\|^2_{L_2(\Omega_i')}$$

we see that

$$C_0 \|\nabla v_i\|^2_{L_2(\Omega_i)} \geq C_0 \|\nabla v_i\|^2_{L_2(\Omega_i')} \geq \left\| h^{-1/2} \frac{\partial v_i}{\partial n_i} \right\|^2_\Gamma ,$$

as required for stability.

7. Numerical examples

7.1. **Effect of γ upon stability.** This example shows the effect of the choice of the parameter γ on the stability of the solution. The domain $(0,3)\times(0.3)$ has an interior part $(1,2)\times(1/2,3/2)$ which is meshed independently of the outer part. We seek the solution to $-\Delta u = 1$ on this domain, with zero Dirichlet boundary conditions. In Figure 1, we show an elevation of the numerical solution using $\gamma = 0$ and for $\gamma = \sqrt{3}$. The instability for $\gamma = 0$ is not very pronounced. While one might expect the system matrix to be singular in this case (and it is likely that this will happen in some situations), our experience is that geometrical restrictions may alleviate this problem in many cases. At any rate, the use of $\gamma = 0$ is not to be recommended.

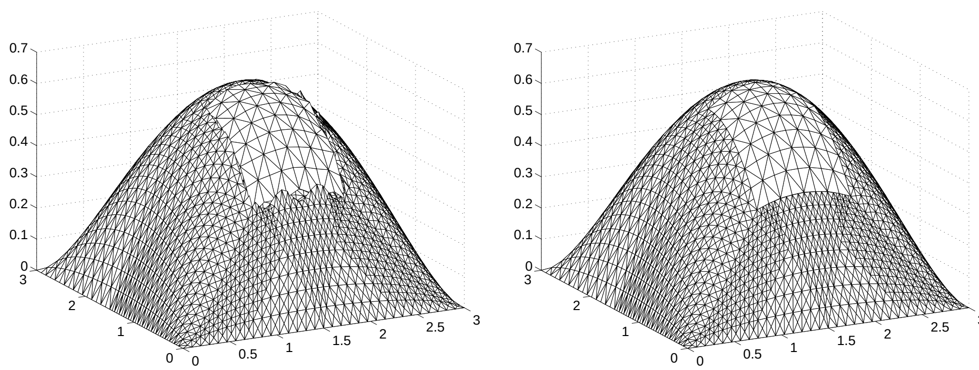

FIGURE 1. Results for $\gamma = 0$ (left) and $\gamma = \sqrt{3}$ (right).

7.2. **Numerical verification of the a priori estimates.** To verify the a priori estimates, we choose the model problem of a unit square with exact solution

$$u = x\, y\, (1-x)\, (1-y)$$

corresponding to a right-hand side of $f = x - x^2 + y - y^2$. The domain is divided by a vertical slit at $x = 0.7$. Two different triangulations were used: one matching and one non-matching, see Figure 2.

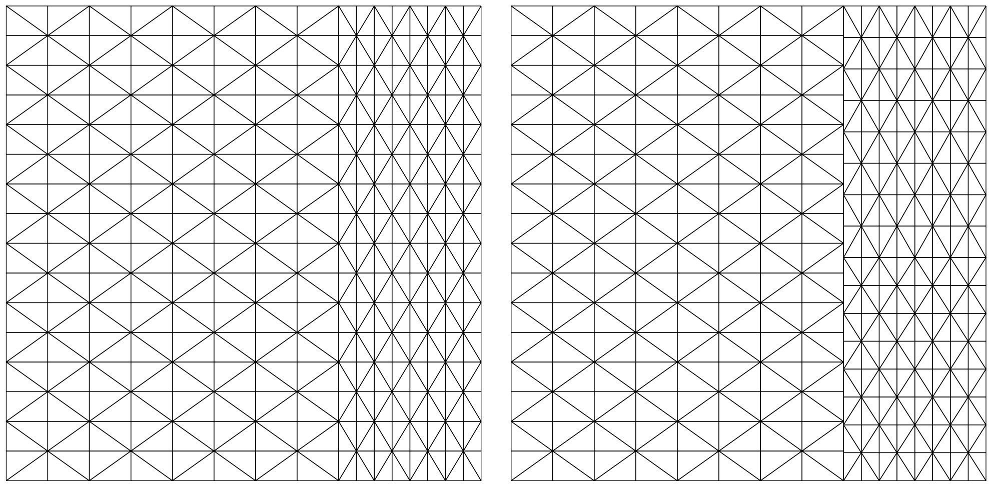

FIGURE 2. Matching and non-matching grids.

In Figure 3 (left-hand side) we give the convergence in the broken energy norm. The dashed line is the non-matching grid computation. Both meshes show the same convergence with slope 0.95. which is close to the theoretical value of 1. On the right-hand side we show the convergence of the L_2-norm of the jump term (dashed line for the non-matching grid). Here we obtain a better convergence (slope 2.15) for the matching grids than for the non-matching grids (slope 1.57, close to the theoretical value of 3/2).

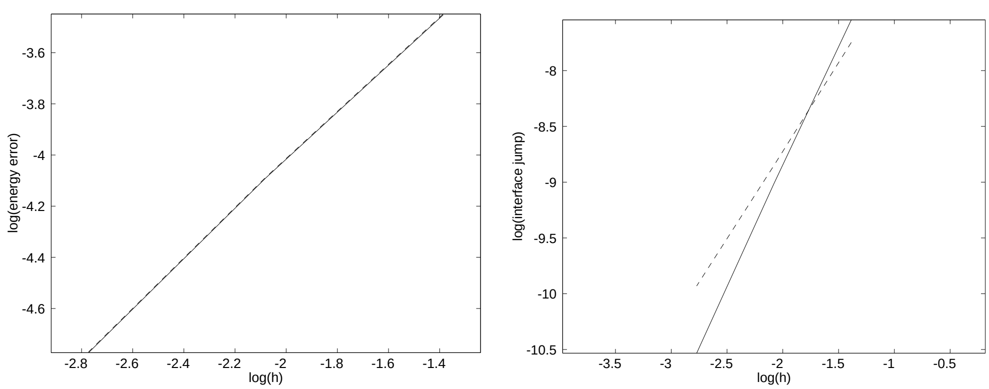

FIGURE 3. Convergence in energy and convergence of the jump term.

7.3. **Adaptive computations.** We present results of adaptive computations on the L-shaped domain

$$\Omega = (0,1) \times (0,1) \setminus (1/2,1) \times (0,1/2).$$

The problem is boundary driven ($f = 0$), with boundary data corresponding to the exact solution $u = r^{2/3} \sin(2\theta/3)$ in polar coordinates (with origin at $(1/2, 1/2)$). We let $\Omega_1 = (0,1/2) \times (0,1)$ and $\Omega_2 = (1/2,1) \times (1/2,1)$, and use a non-matching triangulation. The purpose of this example is not to obtain exact error control, but rather to show how the adaptive algorithm behaves with respect to the elements adjacent to the interface. In consequence, while we consider adaptive control of the maximum error, we have not approximately solved the dual problem (11). Instead we have simply tuned the interpolation constants to approximately match the maximum exact error.

In Figure 4 and 5 we show a sequence of adaptive meshes resulting from equilibrating the error distribution over the set of elements (for details, see [9, 6]). Note that the interface enforces a slight increased refinement as compared with a single-domain solution.

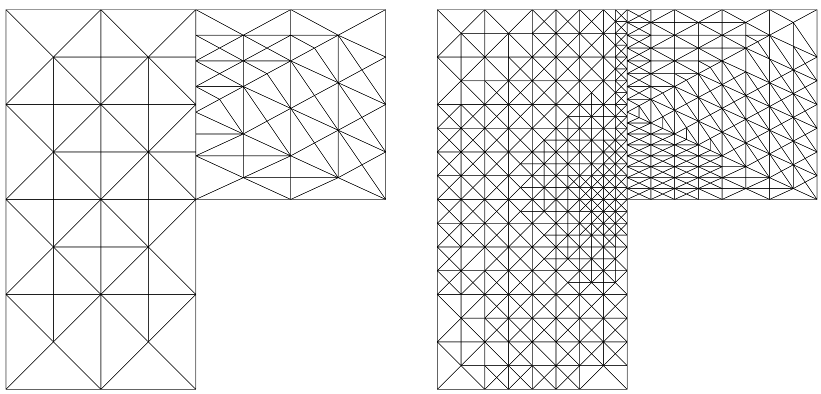

FIGURE 4. First mesh and first adapted mesh.

REFERENCES

[1] J.-P. Aubin, *Approximation of Elliptic Boundary-Value Problem* (Wiley, 1972).

[2] C. Baiocchi, F. Brezzi, and L. D. Marini, Stabilization of Galerkin methods and applications to domain decomposition, in: *Future Tendencies in Computer Science, Control and Applied Mathematics,* Springer Lecture Notes in Computer Science 653, A. Bensoussan and J.-P. Verjus (eds.), (Springer, 1992), 345–355.

[3] J. C. Barbosa and T. J. R. Hughes, Boundary Lagrange multipliers in finite element methods: error analysis in natural norms, Numer. Math. **62,** (1992) 1–15

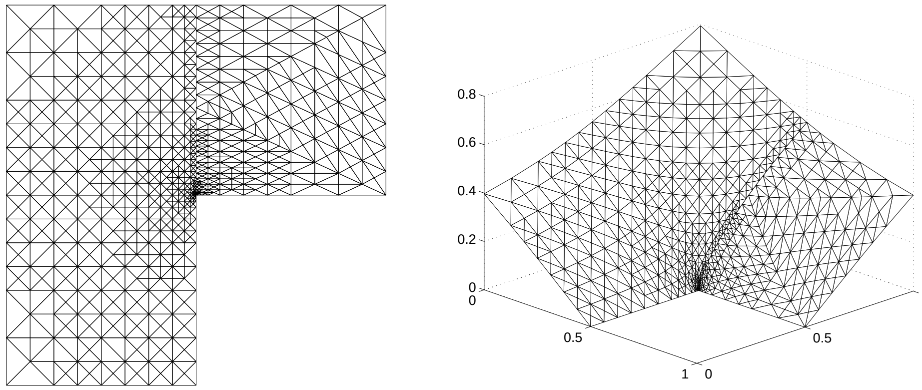

FIGURE 5. Final mesh and elevation of the solution.

[4] J. W. Barrett and C. M. Elliot, Finite element approximation of the Dirichlet problem using the boundary penalty method, Numer. Math. **49,** (1986) 343–366.

[5] R. Becker and P. Hansbo, Discontinuous Galerkin methods for convection-diffusion problems with arbitrary Péclet number, in preparation.

[6] R. Becker and R. Rannacher, A feed-back approach to error control in finite element methods: basic analysis and examples, East-West J. Numer. Math. **4** (1996) 237–264

[7] C. Bernadi, Y. Maday, and A. Patera, A new nonconforming approach to domain decomposition: the mortar element method, in: *Nonlinear Partial Differential Equations and Their Application,* H. Brezis and J. L. Lions (eds.), (Pitman, 1989).

[8] F. Brezzi, L.P. Franca, D. Marini, and A. Russo, Stabilization techniques for domain decomposition methods with non-matching grids, IAN-CNR Report N. 1037, Istituto di Analisi Numerica Pavia.

[9] C. Johnson and P. Hansbo, Adaptive finite element methods in computational mechanics, Comput. Methods Appl. Mech. Engrg. **101,** (1992) 143–181.

[10] P. Le Tallec and T. Sassi, Domain decomposition with nonmatching grids: augmented Lagrangian approach, Math. Comp. **64,** (1995) 1367–1396.

[11] P. L. Lions, On the Schwarz alternating method III: a variant for nonoverlapping subdomains, in: *Third International Symposium on Domain Decomposition Methods for Partial Differential Equations,* T. F. Chan, R. Glowinski, J. Periaux, and O. B. Widlund (eds.), (SIAM, 1989), 202–223.

[12] J. Nitsche, Über ein Variationsprinzip zur Lösung von Dirichlet-Problemen bei Verwendung von Teilräumen, die keinen Randbedingungen unterworfen sind, Abh. Math. Univ. Hamburg **36,** (1970) 9–15.

[13] R. Stenberg, On some techniques for approximating boundary conditions in the finite element method, J. Comput. Appl. Math. **63,** (1995) 139–148.

[14] V. Thomée, *Galerkin Finite Element Methods for Parabolic Problems*, (Springer, 1997)

[15] B. I. Wohlmuth, A residual based error estimator for mortar finite element discretizations, Preprint, Math. Institut, Universität Augsburg.

Unité de recherche INRIA Sophia Antipolis
2004, route des Lucioles - B.P. 93 - 06902 Sophia Antipolis Cedex (France)

Unité de recherche INRIA Lorraine : Technopôle de Nancy-Brabois - Campus scientifique
615, rue du Jardin Botanique - B.P. 101 - 54602 Villers lès Nancy Cedex (France)
Unité de recherche INRIA Rennes : IRISA, Campus universitaire de Beaulieu - 35042 Rennes Cedex (France)
Unité de recherche INRIA Rhône-Alpes : 655, avenue de l'Europe - 38330 Montbonnot St Martin (France)
Unité de recherche INRIA Rocquencourt : Domaine de Voluceau - Rocquencourt - B.P. 105 - 78153 Le Chesnay Cedex (France)

Éditeur
INRIA - Domaine de Voluceau - Rocquencourt, B.P. 105 - 78153 Le Chesnay Cedex (France)
http://www.inria.fr
ISSN 0249-6399