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Boumediène Chentouf, Jean-François Couchouron

INSTITUT NATIONAL DE RECHERCHE EN INFORMATIQUE ET EN AUTOMATIQUE

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Abstract: This paper deals with **nonlinear** feedback stabilization problem of a flexible beam clamped at a rigid body and free at the other end. We assume that there is no damping. The feedback law proposed here consists of a nonlinear control torque applied to the rigid body and either a nonlinear boundary control moment or a nonlinear boundary control force or both of them applied to the free end of the beam. This **nonlinear** feedback, which insures the exponential decay of the beam vibrations, extends the **linear** case studied by Laousy et al. to a more general class of controls. This new class of controls is in particular of the interest to be robust.

Key-words: Rotating body-beam, Nonlinear control, Exponential stability.

INRIA-Lorraine (projet Congé) & URA CNRS 399 (MMAS)
ISGMP, Bât. A, Université de Metz, Ile de Saulcy
57045 Metz cedex 01
tél. : 03 87 54 72 76 – fax : 03 87 54 72 77
e-mail : {chentouf,couchour }@loria.fr

Unité de recherche INRIA Lorraine
Technopôle de Nancy-Brabois, Campus scientifique,
615 rue de Jardin Botanique, BP 101, 54600 VILLERS LÈS NANCY (France)
Téléphone : (33) 83 59 30 30 – Télécopie : (33) 83 27 83 19
Antenne de Metz, technopôle de Metz 2000, 4 rue Marconi, 55070 METZ
Téléphone : (33) 87 20 35 00 – Télécopie : (33) 87 76 39 77

(Résumé : tsvp)

Stabilisation par feedback non linéaire d'un système disque-poutre en rotation sans frottements

Résumé : On aborde dans cet article un problème de stabilisation par feedback non linéaire d'un système constitué d'une poutre flexible fixée à l'une de ses extrémités à un corps rigide et libre à l'autre extrémité. On suppose qu'il n'y a pas de frottements. De plus, on propose une loi de commande composée d'une part d'un contrôle de couple exercée sur le disque et d'autre part d'un contrôle de force et/ou de moment appliqué simultanément à l'extrémité libre de la poutre. On montre ici que cette loi de commande assure un amortissement exponentiel des vibrations de la poutre. Cette conclusion est analogue à des résultats antérieurs obtenus avec des contrôles linéaires mais cette fois notre classe de contrôles est plus large et en particulier robuste.

Mots-clé : Disque-poutre en rotation, Contrôle non linéaire , Stabilité exponentielle.

1 Introduction

The purpose of this paper is to study the nonlinear feedback stabilization problem of the system presented in the figure 1. This system has been introduced by Baillieul and Levi [2]. It consists of a disk (D) with a beam (B) attached to its center and perpendicular to the disk plane. The disk (D) rotates freely around its axis and the motion of the beam (B) is confined to a plane perpendicular to the disk (see fig. 1).

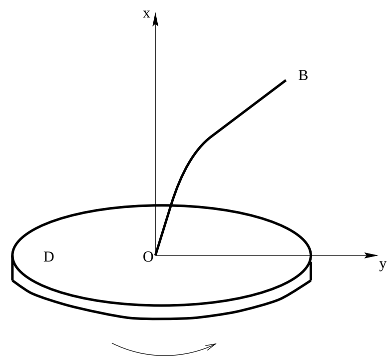

Figure 1: The body-beam system.

The body-beam system is governed by the following equations

$$\begin{cases} \rho y_{tt} + EI y_{xxxx} = \rho \omega^2(t) y, \\ y(0,t) = y_x(0,t) = 0, \\ y_{xx}(1,t) = \Gamma_1(t), \\ y_{xxx}(1,t) = \Gamma_2(t), \\ \dot{\omega}(t) = \dfrac{\Gamma_3(t) - 2\rho\omega(t) \displaystyle\int_0^1 y y_t \, dx}{I_d + \rho \displaystyle\int_0^1 y^2 \, dx}, \end{cases} \tag{1.1}$$

where the positive constants EI, ρ and I_d are respectively the flexural rigidity, the mass per unit length of the beam, and the disk's moment of inertia; $\omega(t)$ is the angular velocity of the disk at time t and the length of the beam is chosen as $L = 1$. Moreover, $\Gamma_1(t), \Gamma_2(t)$ and $\Gamma_3(t)$ are respectively the moment control, the force control and the torque control to be determined so that the solution's energy of the resulting closed-loop system decays to zero in some functional space.

In this paper, we take a nonlinear boundary control moment $\Gamma_1(t)$ and/or a nonlinear boundary control force $\Gamma_2(t)$ applied to the free end of the beam (B) while a nonlinear control torque $\Gamma_3(t)$ is exerted on the disk (D).
Then, as in the linear case [13], we will show that in our case the system (1.1) is still stabilizable at an equilibrium with exponential decay.

The stabilization problem of the body-beam system has been extensively studied in the literature [1], [2], [5], [13], [14], [15], [16], [20] and [21]. In [2], the authors showed that with structural damping and without control, the body-beam system has a finite number of rotating equilibrium states. Later, Bloch and Titi [5] showed that in the more difficult case of viscous damping, a linear inertial manifold exists for the body-beam system. By taking into account the effect of damping, and for any constant angular velocity smaller than a critical one, an exponentially stabilizing feedback torque control law has been given in [20]. In the same case, and by adding a boundary control force, the system is also stabilizable for any constant angular velocity [21]. The stabilization problem of similar systems has been studied in [14], [15] and [16]. Recently, for the body-beam system without damping, exponential stabilization was established in [13] as soon as at least one of two **linear** boundary controls (force or moment) is present at the free end of the beam with, in addition, a control torque of the disk. The last result on this subject has been obtained by d'Andéa-Novel and Coron [1]: without damping nor controls on the free boundary of the beam, the authors found a torque control which insured the strong stability of the system but not the exponential one.

The contribution of this paper consists in extending the class of controls proposed in [13]. In fact, our work is motivated by two arguments: First, the interest of such an extension is to highlight the robustness of controls. Actually, a minimum degree of robustness is needed in applications since neither sensors nor actuators can ensure an infinitely precise linear control as required in [13]. Second, a wide class of controls is provided. For instance, this helps us to avoid eventually some saturation phenomena in the actuators.

The new feedback control proposed here is the following:

$$\begin{cases} \Gamma_1(t) = -f(u_{tx}(1,t)), \;\; \Gamma_2(t) = g\,(u_t(1,t))\,; \\ \\ \Gamma_3(t) = -\gamma(\omega - \omega^*), \end{cases} \tag{1.2}$$

for each given $\omega^* \in \mathbb{R}$. Contrary to [13], these controls are nonlinear since f, g and γ are nonlinear real functions satisfying hypotheses F1), F2) and F3) given in Section 3.

The main result of our paper is then (Theorem 2 in Section 3):

Under Hypotheses F1), F2), F3) and the condition $\mid \omega^* \mid < \sqrt{9EI/\rho}$, *the solutions of the closed-loop system (1.1)-(1.2) decay exponentially to its equilibrium in a suitable state space.*

We forewarn the reader that as in [13], the decay rate although exponential is not uniform.

To obtain this result within a nonlinear framework, it was necessary to overcome new difficulties among which, well-posedness of the system (1.1)-(1.2), regularity of its solutions, density of the domain of the nonlinear unbounded operator governing the evolution problem, lack of exponential stability result in the perturbation theory of fully nonlinear systems, no Duhamel's formula...For instance, the linear argument of [13] to deduce the exponential decay of the solutions of (1.1)-(1.2) from the exponential stability of the associated homogenous system fails in our approach. In fact, Theorem 2 has been obtained by means of two essential tools: on the one hand, the Bénilan's integral inequalities (see [3] or [4]) and on the other hand, the multiplier method (see [12]).

All the systems considered in this paper (and in particular (1.1)-(1.2)) can be written in the following abstract form:

$$CP = \begin{cases} \dot{x}(t) + Ax(t) + B(t, x(t)) \ni 0, & \text{on } J, \\ \\ x(0) = x^0, \end{cases} \tag{1.3}$$

where J is any interval in $\mathbb{R}^+$ starting at zero; A is a (multivalued) **nonlinear m-accretive** unbounded operator on the Banach space $(X, \|\;\|)$; and x^0 belongs to the closure of the domain $\mathcal{D}(A)$ of A. The nonlinear **non accretive** operator B is in general (except in Section 2.3) locally Lipschitz from $J \times X$ to X. In fact, in our application to the body-beam system, A is single-valued, X is a Hilbert space and $\mathcal{D}(A)$ is dense in X. Furthermore, let us note that in this application, the nonlinearity of A unfortunately occurs in its domain and the density of $\mathcal{D}(A)$ has been obtained by non standard arguments (see Lemma 4). This paper is organised as follows.

Section 2 is devoted to preliminary results needed to prove global existence, regularity and exponential decay of the solutions of problem CP. In section 3, we state and show in several steps the exponential decay result of the rotating body-beam system (1.1)-(1.2).

2 Preliminaries on the abstract problem CP

2.1 The Cauchy problem CP

We consider the evolution problem CP introduced in (1.3). If necessary, we will precise some arguments in (CP) as follows: $CP = CP(x^0, J)$. Throughout this subsection, we assume the following hypotheses.

(HI) A is a multivalued unbounded nonlinear m-accretive operator with domain $\mathcal{D}(A)$.

(HII) B is locally Lipschitz on $J \times X$ and bounded on the bounded subsets of $J \times X$.

(HIII) For every $x^0 \in \overline{\mathcal{D}(A)}$ and $T \in J$, there is a constant $C = C(x^0, T)$ such that for all $T_0 \in [0, T]$, each solution $x(\cdot)$ of $CP(x^0, [0, T_0])$ satisfies:

$$||x(t)|| \leq C, \ \forall t \in [0, T_0].$$

We will precise below the meaning of solutions considered in this paper.

Definition 1 . *The continuous function x_* is a solution of CP if it is the mild solution of the quasi-autonomous problem*

$$Q = \begin{cases} \dot{x}(t) + Ax(t) \ni -B(t, x_*(t)), & t \in [0, T], \\ x(0) = x^0, & \end{cases}$$

for all $T \in J$.

We refer, for instance, the reader to [8] for the notion of mild solution for the quasi-autonomous case.

Proposition 1 below extends the well-known Picard-Lindeloff-Lipschitz Theorem.

Proposition 1 . *Assume that ((***HI***), (***HII***) and (***HIII***) hold. Then, for* $x^0 \in \overline{\mathcal{D}(A)}$, *the problem* $CP(x^0, J)$ *has a unique solution* $S(\cdot)x^0$ *and the map* $x^0 \mapsto S(t)x^0$ *is continuous on* $\overline{\mathcal{D}(A)}$, *for all* $t \in J$.

Proof of Proposition 1. The proof will be divided into four steps and without loss of generality, we will suppose $J = [0, T]$ for some $T > 0$.

Step 1. Due to M. Pierre result (pp. 126 of [18]), the problem CP has a unique local solution $x(\cdot)$, i.e, there are $T_0 \in]0, T[$ and $x(\cdot)$ solution of $CP(x^0, [0, T_0[)$.

Step 2. In order to show that CP has a global solution on J, we will prove that the local solution $x(\cdot)$ considered in step 1 can be extended into a solution of $CP(x^0, [0, T_1])$ for some $T_1 \in [T_0, T]$ with the property $T_1 > T_0$ if $T_0 < T$. From Definition 1 and the Bénilan's integral inequalities (see [3] or [4]), it follows

$$\begin{aligned} &||x(t+h) - x(t)|| \leq ||x(s+h) - x(s)|| \\ &+ \int_s^t [x(\tau+h) - x(\tau), -B(\tau+h, x(\tau+h)) + B(\tau, x(\tau))] \, d\tau, \end{aligned} \tag{2.1}$$

for $0 \leq s \leq t < t+h < T_0$; we have set as usual,

$$[y, z] = \lim_{\lambda \downarrow 0} \frac{||y + \lambda z|| - ||y||}{\lambda}, \quad \forall y, z \in X.$$

Assumptions **(HII)**-**(HIII)** and the inequality (2.1) provide then

$$||x(t+h) - x(t)|| \leq ||x(s+h) - x(s)|| + 2M(t-s), \tag{2.2}$$

with $M = \sup\{||B(t, \xi)||; t \in [0, T], \ ||\xi|| \leq C\}$.

From (2.2) and the continuity of x on $[0, T_0[$, it follows that $\lim_{t \to T_0^-} x(t)$ exists. Let

$$x(T_0) := \lim_{t \to T_0^-} x(t). \tag{2.3}$$

The inequality (2.3) extends x into a continuous function on $[0, T_0]$ which is necessarily an integral solution of the quasi-autonomous problem $Q(x^0, g)$ governed by A with the second member $g(t) = -B(t, x(t))$. Consequently, x is the unique mild solution of $Q(x^0, g)$ on $[0, T_0]$ and thus a solution of $CP(x^0, [0, T_0])$. Suppose now $T_0 < T$ (if $T_0 = T$, then the proof is already done). Then, the problem $CP(x(T_0), [T_0, T])$ has local solution (from step 1) and thus x can be extended in the required sense (see [4] or [8] for the continuation property of mild solutions).

Eventually, $CP(x^0, J)$ has a unique global solution $x(\cdot) = S(\cdot)x^0$.

Step 3. We will prove the following uniform estimation which does not require Assumption **(HIII)**.

Lemma 1 . *Let $r > 0$ and $x^0 \in \overline{\mathcal{D}(A)}$ be such that B is L-Lipschitz on $[0, r] \times \mathcal{B}(x^0, r)$. Let $x^1 \in \mathcal{B}(x^0, r/8) \cap \mathcal{D}(A)$; pick $\widehat{x^1} \in Ax^1$ and a real $T_1 \in]0, r[$ satisfying*

$$\left(\frac{r}{4} + T_1(||\widehat{x^1}|| + M_1)\right) e^{LT_1} < \frac{r}{2}, \tag{2.4}$$

with

$$M_1 = \sup\left\{||B(\tau, x^1)|| \,;\, \tau \in [0, r]\right\}.$$

Then we have $S(t)x_0 \in \mathcal{B}(x^0, r)$ for all $[0, T_1]$ and all $x_0 \in \mathcal{B}(x^0, r/8)$.

Proof of Lemma 1. Let $x_0 \in \mathcal{B}(x^0, r/8)$ and set $y(\cdot) = S(\cdot)x_0$. Define the set

$$\mathcal{E} = \{t \in [0, T_1]; ||y(\tau) - x_0|| < r; \text{ for all } \tau \in [0, t]\}.$$

Clearly, $\mathcal{E}$ is a non empty open subset of $[0, T_1]$. Let us prove that it is closed in order to conclude $\mathcal{E} = [0, T_1]$. To accomplish this aim, set

$$t_\infty = \sup \mathcal{E}.$$

The Bénilan's integral inequalities and the usual bracket properties yield (see [3] or [4]) for $t \in [0, t_\infty[$,

$$\begin{aligned}
||y(t)-x^1|| &\leq ||x_0-x^1||+\int_0^t [y(\tau)-x^1,-B(\tau,y(\tau))-\widehat{x^1}]d\tau \\
+ &\leq \frac{r}{4}+\int_0^t ([y(\tau)-x^1,-B(\tau,y(\tau))+B(\tau,x^1)] \\
&\qquad\qquad +[y(\tau)-x^1,-B(\tau,x^1)-\hat{x_1}])\; d\tau \\
&\leq \frac{r}{4}+\int_0^t \Big(L||y(\tau)-x^1||+||B(\tau,x^1)||+||\widehat{x^1}||\Big)\, d\tau.
\end{aligned}$$

By vertue of Gronwall's Lemma, the latter implies

$$||y(t)-x^1|| \leq \Big(\frac{r}{4}+t(||\widehat{x^1}||+M_1)\Big)\, e^{Lt},\ \forall t\in[0,t_\infty[.$$

According to the choice of T_1 in (2.4) and the relation $0\leq t_\infty \leq T_1$, it comes, since x is continuous on $[0,T]$

$$||y(t_\infty)-x^1|| \leq \frac{r}{2}. \tag{2.5}$$

Now by using the triangle inequality in (2.5), we conclude

$$||y(t_\infty)-x^0|| \leq \frac{r}{2}+||x^0-x^1|| \leq \frac{r}{2}+\frac{r}{8} < r. \tag{2.6}$$

From (2.6) and the definitions of $\mathcal{E}$ and t_∞, it results $t_\infty \in \mathcal{E}$ and thus $\mathcal{E}$ is closed. □

step 4. Now, let $(x_n^0)_{n\in\mathbb{N}^*}$ be a sequence converging in $\overline{\mathcal{D}(A)}$ to x^0. Define x_n and x as

$$x_n(t)=S(t)x_n^0;\ x(t)=S(t)x^0,\ \ t\in[0,T].$$

We will prove that the sequence $(x_n)_n$ converges uniformly to x on $[0,T]$. It suffices to establish that the set

$$\mathcal{K}=\{\theta\in]0,T];x_n\to x \ \text{ uniformly on } \quad [0,\theta]\}$$

is non empty, open and closed in $[0,T]$.

i) $\mathcal{K}$ is non empty:

The positive real T_1 introduced in Lemma 1 belongs to $\mathcal{K}$. Indeed, using again Bénilan's integral inequalities, we have

$$||x_n(t)-x(t)|| \leq ||x_n^0-x^0||+\int_0^t ||B(\tau,x_n(\tau))-B(\tau,x(\tau))||d\tau \tag{2.7}$$

for $t \in [0, T_1]$.

But Lemma 1 insures

$$x_n(t) \in \mathcal{B}(x^0, r), \quad \text{for } t \in [0, T_1] \text{ and } n \geq m, \tag{2.8}$$

where the integer m being sufficiently large. Since B is L-Lipschitz on $[0, T_1] \times \mathcal{B}(x^0, r)$, it follows from (2.7), (2.8) and the Gronwall's Lemma, that

$$||x_n(t) - x(t)|| \leq ||x_n^0 - x^0|| e^{Lt}$$

for $t \in [0, T_1]$ and $n \geq m$.

Hence $(x_n)_n$ converges uniformly to x on $[0, T_1]$.

ii) $\mathcal{K}$ is open:

Let $\theta_1 \in \mathcal{K}$ be such that $\theta_1 < T$. Then, in particular, $x_n(\theta_1) \to x(\theta_1)$. As in i), we can show that there exists $T_1^{'} \in [0, T - \theta_1[$ such that (x_n) converges uniformly to x on $[\theta_1, \theta_1 + T_1^{'}]$. Therefore, one has $[0, \theta_1 + T_1^{'}] \subset \mathcal{E}$ which proves the claim ii).

iii) $\mathcal{K}$ **is closed**:

Let $\theta_n \in \mathcal{E}, n \in \mathbb{N}^*$ be such that $\theta_n \uparrow \theta_\infty$. We have to prove that $(x_n)_n$ converges uniformly to x on $[0, \theta_\infty]$. We can find $r > 0$ then $T_2 \in]0, r[$ and then $m \in \mathbb{N}^*$ as follows:

a) B is L-Lipschitz on $[\theta_\infty - r, \theta_\infty] \times \mathcal{B}\left(x(\theta_\infty), r\right)$;

b) $||x(\theta_\infty - \tau) - x(\theta_\infty)|| \leq \frac{r}{8}$, for all τ in $[0, T_2]$;

c) $\left(\frac{r}{4} + T_2(||\widehat{x^1}|| + M_2)\right) e^{LT_2} < \frac{r}{2}$, with $x^1 \in \mathcal{B}(x^0, r/8) \cap \mathcal{D}(A)$ and $M_2 = \sup_{\tau \in [0,r]} ||B\left(x(\theta_\infty) - \tau, x^1\right)||$;

d) $||x_n(\theta_\infty - T_2) - x(\theta_\infty - T_2)|| \leq \frac{r}{8}$ for $n \geq m$.

Then, as in Lemma 1, one can show that the following relation holds

$$x_n((t) \in \mathcal{B}\left(x(\theta_\infty), r\right), \tag{2.9}$$

for $n \geq m$ and $t \in [\theta_\infty - T_2, \theta_\infty[$.

But the Bénilan's inequalities provide

$$||x_n((t) - x(t)|| \leq ||x_n(\theta_\infty - T_2) - x(\theta_\infty - T_2)|| + \int_{\theta_\infty - T_2}^{\theta_\infty} ||B\left(\tau, x_n(\tau)\right) - B\left(\tau, x(\tau)\right)|| d\tau,$$

for $t \in [\theta_\infty - T_2, \theta_\infty]$.

Then according to (2.9), the Lebesgue dominated convergence theorem used in the last inequality gives the uniform convergence of $(x_n)_n$ to x on $[\theta_\infty - T_2, \theta_\infty]$ and finally on $[0, \theta_\infty]$. Thus, as required at the begining of iii) we have $\theta_\infty \in \mathcal{K}$. The proof of Proposition 1 is now complete. $\square$

2.2 Two Lemmas without (HIII)

The two following technical lemmas which do not involve hypotheses **(HIII)** are used in Section 3 in order to check **(HIII)** for CP problem stemmed from the body-beam system.

Lemma 2 *Let $V : X \to \mathbb{R}$ be a continuous function, $x^0 \in \overline{\mathcal{D}(A)}$ and x be the solution of $CP(x^0, [0,T])$. Assume that* **(HI)** *and* **(HII)** *hold. Suppose in addition that each local solution y of $CP(x_0, [0,T])$ with $x_0 \in \mathcal{D}(A)$ satisfies:*

$$V(y(t)) \leq V(x_0),$$

on the domain of y. Then, we have

$$V(x(t)) \leq V(x^0),$$

for all $t \in [0,T]$. Consequently, the hypothesis **(HIII)** *holds if, for instance,*
$V(z) \geq K\|z\|$ for any $z \in X$ and for some positive constant K.

Proof of Lemma 2. The function x being the solution of $CP(x^0, [0,T])$ set

$$\mathcal{G} = \left\{ t \in [0,T]; V(x(\tau)) \leq V(x^0),\ \forall \tau \in [0,t] \right\}.$$

Clearly, $\mathcal{G}$ is closed in $[0,T]$.

Now, we want to prove that $\mathcal{G}$ is not reduced to $\{0\}$. Let $x_n^0 \to x^0$ with $x_n^0 \in \mathcal{D}(A)$ for all $n \in \mathbb{N}^*$. From Lemma 1 (which does not suppose **(HIII)**), there exists $T_1 \in]0,T]$ such that the solution $S(\cdot)x_n^0$ exist on $[0,T_1]$ for n sufficiently large and such that (see also step 4 i) in the proof of Proposition 1) $\left(S(\cdot)x_n^0\right)_n$ converges to $x(\cdot) = S(\cdot)x^0$ uniformly on $[0,T_1]$.

Therefore, we have

$$V(S(t)x_n^0) \leq V(x_n^0), \quad \text{for } t \in [0, T_1], \quad n \in \mathbb{N}^*.$$

Since V is continuous, the previous relation yields

$$V(S(t)x^0) \leq V(x^0), \quad \text{for } t \in [0, T_1].$$

and thus $[0, T_1] \subset \mathcal{G}$. Finally, repeating these arguments, one can easily show that $\mathcal{G}$ is open in $[0, T]$. Therefore, $\mathcal{G} = [0, T]$ and thus the proof of Lemma 2 is ended. □

In a Hilbert space and more generally in a Banach space X which has the Radon-Nykodym property, each Lipschitz continuous map from $[0, T]$ to X is a.e. differentiable on $[0, T]$.

Lemma 3 *Suppose that* **(HI)** *and* **(HII)** *hold and that the Banach space X has the Radon-Nykodym property. Then, each solution x of $CP(x^0, [0, T])$, with $x^0 \in \mathcal{D}(A)$, is a strong one.*

Proof of Lemma 3. **a)** Let $x^0 \in \mathcal{D}(A)$, $r > 0$ and T_1 chosen as in Lemma 1 with $x^1 = x^0$ for instance. For $t, t+h \in [0, T_1]$ with $h \geq 0$, the Bénilan's inequalities imply:

$$\begin{aligned} ||x(t+h) - x(t)|| &\leq ||x(h) - x^0|| + ||B(\tau + h, x(\tau + h)) - B(\tau, x(\tau))|| \\ &\leq ||x(h) - x^0|| + \int_0^t (Lh + L||x(\tau + h) - x(\tau)||)\, d\tau. \end{aligned} \tag{2.10}$$

But we have (see [3] or [4])

$$\left\{ \begin{array}{l} ||x(h) - x^0|| \leq h||\widehat{x^1}|| + M_1 h, \\ \widehat{x^1} \in Ax^0 \text{ and } M_1 = \sup_{\tau \in [0,T]} ||B(\tau, x(\tau))||. \end{array} \right. \tag{2.11}$$

Clearly, (2.11) and the Gronwall's Lemma applied in (2.10) show that x is Lipschitz on $[0, T]$. Because X has the Radon-Nykodym property, the map x is differentiable a.e. on $[0, T_1]$. Then, in this case (see [3] or [4]), x belongs to $W^{1,1}([0, T_1]; X)$ and satisfies

$$\left\{ \begin{array}{l} x(t) \in \mathcal{D}(A), \quad \text{a.e on} \quad [0, T_1] \\ \dot{x}(t) + Ax(t) + B(t, x(t)) \ni 0. \end{array} \right. \tag{2.12}$$

In other words, x is a strong solution of $CP(x^0, [0, T_1])$.

b) Now introduce the set

$$\mathcal{L} = \left\{ t \in [0, T];\, x\Big|_{[0,t]} \text{is a strong solution of } CP(x^0, [0, t]) \right\}.$$

This set is clearly closed and contains $[0, T_1]$. In order to have $\mathcal{L} = [0, T]$, it remains to prove that $\mathcal{L}$ is open.

Consider $t_0 \in \mathcal{L}$ with $t_0 < T$ and some open ball $\mathcal{B}(x(t_0), r)$ such that the operator B is L-Lipschitz on $[t_0 - r, t_0 + r] \times \overline{\mathcal{B}(x(t_0), r)}$. Then, from (2.12) and the continuity of x, there is $h \in]0, t_0]$ such that $x(t_0 - h) \in \mathcal{D}(A)$ and $x(\tau) \in \mathcal{B}(x(t_0), r)$ for $\tau \in [t_0 - h, t_0]$. But by vertue of part a) of this proof, x is a strong solution of $CP(x(t_0 - h), [0, T_1'])$ where T_1' is any real in $]t_0, T] \cap]t_0, t_0 + r[$ satisfying

$$t_0 - h \leq t \leq T_1' \;\Rightarrow\; x(t) \in \mathcal{B}(x(t_0), r)\,.$$

Of course, due to the continuity of x such a T_1' exists and thus we have $[0, T_1'[\subset \mathcal{L}$. The proof is now done. □

2.3 An exponential decay result

In this subsection, our hypotheses on A, B and CP are different from those of the previous developments.

Let $x^0 \in X$ and denote by **(HIV)** and **(HV)** the following hypotheses:

(HIV) The problem $CP(x^0, [0, +\infty[)$ has a unique solution x.

(HV) There is a locally integrable positive real valued function μ on $[0, +\infty[$ satisfying

$$||B(t, x(t))|| \leq \mu(t)||x(t)||, \text{ and } \lim_{\xi \to +\infty} \int_{\xi}^{\xi+\delta} \mu(\tau)d\tau = 0,\ \forall \delta > 0.$$

The result of this subsection is the following.

Theorem 1 . *Assume that the* **nonlinear** *semigroup e^{-At} generated by the operator $-A$ is exponentially (uniformly) stable. Then, under hypotheses* **(HI)**, **(HIV)** *and* **(HV)**, *there* *exist positive constants M and κ such that*

$$\|x(t)\| \leq M e^{-\kappa t}, \ \forall t \geq 0.$$

Remark 1 . *The decay rate obtained in Theorem 1, although exponential, is not uniform. In return, the* **nonlinear** *semigroup e^{-At} being uniformly stable, there exists constants $M_1, \ \omega_1 > 0$ satisfying $\|e^{-At}x^0\| \leq M_1 e^{-\omega_1 t}\|x^0\|$, for all $x^0 \in X$ and $t \geq 0$. Then, the constants M and κ of* **Theorem 1** *can be chosen as follows*

$$\kappa = \frac{\omega_0}{\delta}, \quad M = \max\left(M_1 e^{\omega_0(2+t_0/\delta)}, \ \sup_{0\leq t\leq t_0} \|x(t)\| e^{\kappa t_0} \right),$$

with

$$\omega_0 > \max(1, \omega_1), \ \delta > \frac{1}{\omega_1}(\ln M_1 + 2\omega_0),$$

and t_0 defined by

$$\int_{\xi}^{\xi+\delta} \mu(\tau) d\tau \leq \frac{\omega_0}{e^{\omega_0 \delta}}, \ \forall \xi \geq t_0.$$

Remark 2 . *Even when A is linear, such a result is not quite obvious. Nevertheless, if $\mu(t) \to 0$ as $t \to +\infty$, the result is clear when A is linear.*

Proof of Theorem 1. The **nonlinear** semigroup e^{-At} being uniformly stable, there exist constants $M_1, \ \omega_1 > 0$ such that $\|e^{-At}x^0\| \leq M_1 e^{-\omega_1 t}\|x^0\|$. Let $\delta > 0$ and $t_k = t_0 + k\delta$. Since the operator A is accretive, the Bénilan's inequalities give for any $t \in [t_{k-1}, t_k]$,

$$\|x(t) - e^{-(t-t_{k-1})A} x(t_{k-1})\| \leq \int_{t_{k-1}}^{t_k} \|B(\tau, x(\tau))\| d\tau.$$

Thus, for $t \in [t_{k-1}, t_k]$, we have

$$\begin{aligned} \|x(t)\| &\leq \|e^{-(t-t_{k-1})A} x(t_{k-1})\| + \int_{t_{k-1}}^{t_k} \|B(\tau, x(\tau))\| d\tau \\ &\leq M_1 e^{-\omega_1(t-t_{k-1})} \|x(t_{k-1})\| + \int_{t_{k-1}}^{t_k} \mu(\tau) \|x(\tau)\| d\tau. \end{aligned} \tag{2.13}$$

For $t \in [t_{k-1}, t_k]$, set

$$y_k(t) = e^{\omega_1(t-t_{k-1})} \|x(t)\|. \tag{2.14}$$

Then, for $t \in [t_{k-1}, t_k]$, (2.13) gives ,

$$y_k(t) \leq M_1 y_k(t_{k-1}) + e^{\omega_1 \delta} \int_{t_{k-1}}^{t} \mu(\tau) y_k(\tau) d\tau.$$

Applying the Gronwall's formula, we get

$$y_k(t) \leq M_1 y_k(t_{k-1}) \ \exp\left(e^{\omega_1 \delta} \int_{t_{k-1}}^{t} \mu(\tau) d\tau \right),\ t \in [t_{k-1}, t_k], \tag{2.15}$$

and in particular

$$y_k(t_k) \leq M_1 e^{\alpha_k} y_k(t_{k-1}), \tag{2.16}$$

where

$$\alpha_j = e^{\omega_1 \delta} \int_{t_{j-1}}^{t_j} \mu(\tau) d\tau, \ \ j \in \mathbb{N}^*.$$

This implies, together with (2.14),

$$\|x(t_k)\| \leq M_1^k \ \exp\left(-k\omega_1 \delta + \sum_{j=1}^{j=k} \alpha_j \right) \|x(t_0)\|. \tag{2.17}$$

Since we have

$$\sum_{j=1}^{j=k} \alpha_j = e^{\omega_1 \delta} \int_{t_0}^{t_k} \mu(\tau) d\tau,$$

the inequality (2.17) becomes

$$\|x(t_k)\| \leq \exp\left(k \ln M_1 - k\omega_1 \delta + e^{\omega_1 \delta} \int_{t_0}^{t_k} \mu(\tau) d\tau \right) \|x(t_0)\|. \tag{2.18}$$

Now, we take a positive constant ω_0 such that $\omega_0 > \max(1, \omega_1)$. Next, according to hypothesis **(HV)**, we choose δ and then t_0 as follows:

$$\left\{ \begin{array}{l} \delta > \dfrac{1}{\omega_1}(\ln M_1 + 2\omega_0), \\ \int_{\xi}^{\xi+\delta} \mu(\tau) d\tau \leq \dfrac{\omega_0}{e^{\omega_0 \delta}},\ \forall \xi \geq t_0. \end{array} \right. \tag{2.19}$$

It comes from (2.19)

$$\alpha_j \leq e^{\omega_1 \delta} \frac{\omega_0}{e^{\omega_0 \delta}} \leq \omega_0, \tag{2.20}$$

for $j \in \mathbb{N}^*$. According to (2.18), (2.19) and (2.20), we get

$$\|x(t_k)\| \leq \exp\left(-\omega_0\left(\frac{t_k - t_0}{\delta}\right)\right)\|x(t_0)\|. \tag{2.21}$$

One can show that (2.14) and (2.15) (with $k+1$ instead of k) give

$$\|x(t)\| \leq M_1 e^{-\omega_1(t-t_k)} e^{\alpha_{k+1}}\|x(t_k)\|,$$

for all $t \in [t_k, t_{k+1}]$. Then, combining this last inequality with (2.21), we obtain

$$\|x(t)\| \leq M_1 e^{-\omega_1(t-t_{k-1})+\alpha_{k+1}} \exp\left(-\omega_0\left(\frac{t_k - t_0}{\delta}\right)\right)\|x(t_0)\|,\ \forall t \in [t_k, t_{k+1}]. \tag{2.22}$$

We deduce from (2.20) and (2.22) that for any $t \in [t_k, t_{k+1}]$, we have

$$\begin{aligned}
\|x(t)\| &\leq M_1 e^{-\omega_1(t-t_k)} e^{\omega_0} \exp\left(-\omega_0 \frac{(t_k - t_0)}{\delta}\right)\|x(t_0)\| \\
&\leq M_1 e^{\omega_0} e^{\omega_0 \frac{t_0}{\delta}} e^{\frac{\omega_0(t-t_k)}{\delta}} e^{-\omega_0 \frac{t}{\delta}}\|x(t_0)\| \\
&\leq M_1 e^{-\omega_0 \frac{t}{\delta}} e^{\omega_0(2+\frac{t_0}{\delta})}\|x(t_0)\|.
\end{aligned}$$

Hence, the required estimation of Theorem 1 follows. □

3 Stability of a rotating body-beam system

This section deals with exponential stability of a body-beam system (1.1)-(1.2) under suitable conditions on the nonlinear controls f, g and γ.

3.1 Hypotheses, notations and main result

In the sequel, assume that the following hypotheses are verified:

F1) $f + g \not\equiv 0$ and $f, g \in \{h \in C^0(\mathbb{R}); h \text{ is increasing};\ h(0) = 0\}$.

F2) There exist nonnegative constants $L_1,\ L_2$ and L_3 such that $L_1 + L_2 > 0$ and

$$L_1 \mid x \mid \leq \mid f(x) \mid \leq L_3 \mid x \mid \text{ and } L_2 \mid x \mid \leq \mid g(x) \mid \leq L_3 \mid x \mid,\ \ \forall x \in \mathbb{R}.$$

F3) The function γ is Lipschitz on each bounded subset of $\mathbb{R}$ and for some $L_4 > 0$,

$$\gamma(x)x \geq 0, \quad \mid \gamma(x) \mid \geq L_4 \mid x \mid, \; \forall x \in \mathbb{R}.$$

The conditions F1)-F2) ensure in particular the presence of at least one control in the flexible beam.

Setting $z = y_t$, we can write the closed loop system (1.1)-(1.2) as follows:

$$\begin{cases} \rho y_{tt} + EIy_{xxxx} = \rho\omega^2(t)y, \\ y(0,t) = y_x(0,t) = 0, \\ y_{xx}(1,t) = -f(u_{tx}(1,t)), \\ y_{xxx}(1,t) = g\left(u_t(1,t)\right), \\ \dot{\omega}(t) = \dfrac{-\gamma(\omega-\omega^*) - 2\rho\omega(t)\displaystyle\int_0^1 y(t)z(t)dx}{I_d + \rho\displaystyle\int_0^1 y(t)^2 dx}. \end{cases} \tag{3.1}$$

For the phase space of the system, we take the real Hilbert space

$$X = \mathcal{H} \times \mathbb{R} = H_0^2 \times L^2(0,1) \times \mathbb{R},$$

equipped with the inner product

$$\langle (u,v,\xi), (\tilde{u},\tilde{v},\tilde{\xi}) \rangle_X = \langle (u,v), (\tilde{u},\tilde{v}) \rangle_{\mathcal{H}} + \xi\tilde{\xi},$$

where for $n \in \mathbb{N}$,

$$H_0^n = \{u \in H^n(0,1);\; u(0) = u_x(0) = 0\},$$

and the space $\mathcal{H}$ is endoweded with the inner product

$$\langle (u,v), (\tilde{u},\tilde{v}) \rangle_{\mathcal{H}} = \int_0^1 \left(EIu_{xx}\tilde{u}_{xx} - \rho{\omega^*}^2 u\tilde{u} + \rho v\tilde{v} \right) dx.$$

Note that the norm induced by this scalar product is equivalent to the usual one of the Hilbert space $H^2(0,1) \times L^2(0,1)$ provided that $\mid \omega^* \mid < \sqrt{9EI/\rho}$ (see [13] for details).

Set $\Phi = (y, z)$. Then, the system (3.1) can be written as follows

$$\begin{cases} \dot{\Phi}(t) + \tilde{A}\Phi(t) + \big(0, ({\omega^*}^2 - \omega^2(t))y(t)\big) = 0, \\ \\ \dot{\omega}(t) = \dfrac{-\gamma\,(\omega(t) - \omega^*) - 2\rho\,\omega(t) < y(t), z(t) >_{L^2(0,1)}}{I_d + \rho\|y(t)\|^2_{L^2(0,1)}}, \end{cases} \tag{3.2}$$

where the **nonlinear** operator $\tilde{A}$ is defined by

$$\mathcal{D}(\tilde{A}) = \left\{ (y, z) \in H_0^4 \times H_0^2;\ \begin{array}{l} y_{xx}(1) = -f(z_x(1)), \\ y_{xxx}(1) = g(z(1)) \end{array} \right\}, \tag{3.3}$$

and

$$\tilde{A}(y, z) = \left(-z, \frac{EI}{\rho} y_{xxxx} - {\omega^*}^2 y \right). \tag{3.4}$$

Then, it is easy to verify that the system (3.2) is equivalent to the following one

$$\left(\dot{\Phi}(t), \dot{\omega}(t)\right) + (A + B)\,(\Phi(t), \omega(t)) = 0, \tag{3.5}$$

where we have set,

$$\begin{cases} A(\Phi, \omega) = \left(\tilde{A}\Phi, 0\right), \text{ with } \mathcal{D}(A) = \mathcal{D}(\tilde{A}) \times \mathbb{R}, \\ \\ B(\Phi, \omega) = \left(0, ({\omega^*}^2 - \omega^2)y, \dfrac{-\gamma\,(\omega - \omega^*) - 2\rho\,\omega(t) < y, z >_{L^2(0,1)}}{I_d + \rho\|y\|^2_{L^2(0,1)}}\right), \end{cases} \tag{3.6}$$

We will show in the sequel that (3.5) is in fact a formulation of the closed loop system (1.1)-(1.2) in a CP form.

The main result of this section is:

Theorem 2 . *Suppose that* $|\,\omega^*\,| < \sqrt{9EI/\rho}$. *Then under Hypotheses F1), F2) and F3), for each initial data* $(\Phi_0, \omega_0) \in \mathcal{D}(\mathcal{A})$ *the solution* $(\Phi(t), \omega(t))$ *of the closed-loop system (3.5) tends exponentially to* $(0, \omega^*)$ *in* X *as* $t \to +\infty$.

First, we deal with the existence and uniqueness of the solution of the following subsystem

$$\begin{cases} \dot{\Phi}(t) + \tilde{A}\Phi(t) = 0, \\ \Phi(0) = \Phi_0, \end{cases} \tag{3.7}$$

where $\tilde{A}$ is the nonlinear operator defined by (3.3)-(3.4).

3.2 Wellposedness of the subsystem (3.7)

We have the following proposition

Proposition 2 . *Assume that* $\mid \omega^* \mid < \sqrt{9EI/\rho}$. *The operator* $\tilde{A}$ *defined by (3.3)-(3.4) is m-accretive in* $\mathcal{H}$ *with dense domain.*

Proof of Proposition 2. A straightforward computation shows that for all (y,z), $(\hat{y},\hat{z})$ in $\mathcal{D}(\tilde{A})$,

$$\begin{aligned} \langle\, \tilde{A}(y,z) - \tilde{A}(\hat{y},\hat{z}), (y,z) - (\hat{y},\hat{z}) \,\rangle_{\mathcal{H}} \;&=\; EI\,(z(1) - \hat{z}(1)\,)\left(\, g(z(1)) - g(\hat{z}(1))\,\right) \\ &+\; EI\,(z_x(1) - \hat{z}_x(1))\left(f(z_x(1)) - f(\hat{z}_x(1))\right). \end{aligned} \tag{3.8}$$

Using the hypothesis F1), we deduce: $\langle\, \tilde{A}(y,z) - \tilde{A}(\hat{y},\hat{z}), (y,z) - (\hat{y},\hat{z}) \,\rangle_{\mathcal{H}} \geq 0$; and thus the operator $\tilde{A}$ is accretive.

Now, we have to show the maximality of $\tilde{A}$, that amounts to saying that (see [6]), for any given $(u,v) \in \mathcal{H} = H_0^2 \times L^2(0,1)$, there exists $(y,z) \in \mathcal{D}(\tilde{A})$ such that $(I + \tilde{A})(y,z) = (u,v)$. Equivalently, we seek y and z satisfying

$$\begin{cases} y - z = u, \\ z - {\omega^*}^2 y + \dfrac{EI}{\rho} y_{xxxx} = v, \\ y_{xx}(1) + f(z_x(1)) = 0, \\ y_{xxx}(1) - g\,(v(1)) = 0, \\ y \in H_0^4, z \in H_0^2. \end{cases} \tag{3.9}$$

Eliminating the unknown z in (3.9), one obtains

$$\begin{cases} EIy_{xxxx} + \rho\left(1 - {\omega^*}^2\right) y = \rho\,(u+v)\,, \\ y_{xx}(1) + f\left(y_x(1) - u_x(1)\right) = 0, \\ y_{xxx}(1) - g\left(y(1) - u(1)\right) = 0, \\ y \in H_0^4. \end{cases} \tag{3.10}$$

Now let us define, as in [19], the function $J(\cdot)$ on H_0^2 by

$$J(\psi) = \frac{1}{2}\left\{\int_0^1 [EI\psi_{xx}^2 + \rho\left(1 - {\omega^*}^2\right)\psi^2]\;dx + \right\}$$
$$-\rho\int_0^1 (u+v)\;\psi\;dx + EI\left[F(\psi_x(1) - u_x(1)) + G(\psi(1) - u(1))\right],$$

where

$$F(x) = \int_0^x f(\xi)d\xi, \qquad G(x) = \int_0^x g(\xi)d\xi, \qquad \forall x \in \mathbb{R}.$$

From hypothesis F1), we deduce that F and G are continuously differentiable on $\mathbb{R}$, convex and $F(x) \geq 0,\ G(x) \geq 0$ for all $x \in \mathbb{R}$. Consequently, we can claim that J is convex, coercive and continuous in H_0^2. Hence, by a minimization theorem (Proposition 38.15, p. 155, [22]), there exists a function $y \in H_0^2$ such that $J(y) = \inf_{\psi \in H_0^2} J(\psi)$.

This implies that the function

$$\Theta : \lambda \longmapsto \Theta(\lambda) = J(y + \lambda\psi)$$

admits a minimum at $\lambda = 0$ and thus $\frac{d}{d\lambda}\left(J(y+\lambda\psi)\right)|_{\lambda=0} = 0,\ \forall \psi \in H_0^2$. This means, thanks to a direct computation, that for any $\psi \in H_0^2$, we have

$$\int_0^1 y_{xx}\psi_{xx}\;dx = \int_0^1 [\rho({\omega^*}^2 - 1)y + \rho(u+v)]\;\psi\;dx + g(y(1) - u(1))\;\psi(1)$$
$$+EIf(y_x(1) - u_x(1))\;\psi_x(1). \tag{3.11}$$

Integrating by parts in the last relation, we can prove after a careful computation that we have $y \in H_0^4$ and

$$\begin{cases} EIy_{xxxx} + \rho\left(1 - \omega^{*2}\right)y = \rho\left(u+v\right), \\ y_{xx}(1) + f(y_x(1) - u_x(1)) = 0, \\ y_{xxx}(1) - g\left(y(1) - u(1)\right) = 0. \end{cases} \tag{3.12}$$

We deduce that y is solution of the system (3.10).

Now define (y,z) by

$$\begin{cases} y \text{ solution of } (3.10) \\ z = y - u, \end{cases}$$

which satisfies clearly system (3.9) and thus the maximality of the operator $\tilde{A}$ is proved.

Finally, it remains to prove that the domain $\mathcal{D}(\tilde{A})$ is dense in $\mathcal{H}$.

Lemma 4 . *The domain $\mathcal{D}(\tilde{A})$ of $\tilde{A}$ is dense in $\mathcal{H}$.*

Proof of Lemma 4. Let $(u^0, v^0) \in \mathcal{H}$ and $(\lambda_n) \downarrow 0$. Because of $\tilde{A}$ is maximal, there exits $(u^n, v^n) \in \mathcal{D}(\tilde{A})$ such that

$$\left(I + \lambda_n \tilde{A}\right)(u^n, v^n) = (u^0, v^0). \tag{3.13}$$

We want to prove that

$$(u^n, v^n) \to (u^0, v^0) \quad \text{in } \mathcal{H}. \tag{3.14}$$

Let us define, for $\delta, \mu > 0$, a **linear** operator $A^{\delta,\mu}$ by

$$\mathcal{D}(A^{\delta,\mu}) = \left\{ (u,v) \in H_0^4 \times H_0^2;\ \begin{array}{l} u_{xxx}(1) - \delta v(1) = 0, \\ u_{xx}(1) + \mu v_x(1) = 0 \end{array} \right\},$$

and

$$A^{\delta,\mu}(u,v) = \left(-v, \frac{EI}{\rho}u_{xxxx} - \omega^{*2}u\right).$$

We note that $\overline{\mathcal{D}(A^{\delta,\mu})} = \mathcal{H}$ since $A^{\delta,\mu}$ is a linear m-accretive operator in the reflexive space $\mathcal{H}$ (see for instance [17]). We set

$$\delta_n = \begin{cases} \dfrac{g(v^n(1))}{v^n(1)}, & \text{if } v^n(1) \neq 0, \\ 0, & \text{if } v_n(1) = 0, \end{cases} \quad \text{and} \quad \mu_n = \begin{cases} \dfrac{f(v_x^n(1))}{v_x^n(1)}, & \text{if } v_x^n(1) \neq 0, \\ 0, & \text{if } v_x^n(1) = 0. \end{cases} \tag{3.15}$$

Clearly, $\tilde{A}(u^n, v^n) = A^{\delta_n,\mu_n}(u^n, v^n)$ and so, (3.13) becomes

$$\left(I + \lambda_n A^{\delta_n,\mu_n}\right)(u^n, v^n) = (u^0, v^0). \tag{3.16}$$

Moreover, thanks to hypothesis $F2$), the sequences (δ_n) and (μ_n) are bounded in $\mathbb{R}^+$ and thus the sequence $(\delta_n, \mu_n)_n$ has a cluster value $(\delta_\infty, \mu_\infty)$ in $\mathbb{R}^{+2}$, i.e, $\delta_\infty = \lim_{p\to\infty} \delta_{n_p} \in \mathbb{R}^+$ and $\mu_\infty = \lim_{p\to\infty} \mu_{n_p} \in \mathbb{R}^+$.

From the m-accretivity of the linear operator $A^{\delta_\infty,\mu_\infty}$, we have

$$\left(I + \lambda_{n_p} A^{\delta_\infty,\mu_\infty}\right)(u_\infty^{n_p}, v_\infty^{n_p}) = (u^0, v^0), \tag{3.17}$$

for some $(u_\infty^{n_p}, v_\infty^{n_p}) \in \mathcal{D}(A^{\delta,\mu})$ and

$$\lim_{p\to\infty} (u_\infty^{n_p}, v_\infty^{n_p}) = (u^0, v^0) \text{ in } \mathcal{H}. \tag{3.18}$$

The substraction of (3.16) and (3.17) leads us to

$$(u^{n_p} - u_\infty^{n_p}, v^{n_p} - v_\infty^{n_p}) = -\lambda_{n_p}\left(A^{\delta_{n_p},\mu_{n_p}}(u^{n_p}, v^{n_p}) - A^{\delta_\infty,\mu_\infty}(u_\infty^{n_p}, v_\infty^{n_p})\right). \tag{3.19}$$

Then, multiplying (3.19) for the inner product by $\left(u^{n_p} - u_\infty^{n_p}, v^{n_p} - v_\infty^{n_p}\right)$, we obtain

$$\begin{aligned} \| \left(u^{n_p} - u_\infty^{n_p}, v^{n_p} - v_\infty^{n_p}\right) \|_{\mathcal{H}}^2 = -\lambda_{n_p} EI \Big[&\Big(\delta_{n_p} v^{n_p}(1) - \delta_\infty v_\infty^{n_p}(1)\Big)\Big(v^{n_p}(1) - v_\infty^{n_p}(1)\Big) \\ &+ \Big(\mu_{n_p} v_x^{n_p}(1) - \mu_\infty (v_\infty^{n_p})_x(1)\Big)\Big(v_x^{n_p}(1) - (v_\infty^{n_p})_x(1)\Big)\Big]. \end{aligned} \tag{3.20}$$

On the other hand, getting the expression of $\left(u_\infty^{n_p} - u^0, v_\infty^{n_p} - v^0\right)$ from (3.17) and using the definition of $A^{\delta_\infty,\mu_\infty}$, it comes

$$\left\langle \left(u_\infty^{n_p}, v_\infty^{n_p}\right), \left(u_\infty^{n_p} - u^0, v_\infty^{n_p} - v^0\right)\right\rangle_{\mathcal{H}} = -\lambda_{n_p} EI \left[\delta_\infty \left(v_\infty^{n_p}(1)\right)^2 + \mu_\infty \left((v_\infty^{n_p})_x(1)\right)^2\right]. \tag{3.21}$$

According to assumption F2), we see that we have the following properties for some integer p_0:

$$\begin{aligned} \text{a) } \delta_\infty = 0 &\iff v^{n_p}(1) = 0 \text{ for } p \geq p_0, \\ &\iff v^{n_{p_q}}(1) = 0 \text{ for some subsequence } n_{p_q} \text{ of } n_p; \\ \text{b) } \mu_\infty = 0 &\iff v_x^{n_p}(1) = 0 \text{ for } p \geq p_0, \\ &\iff v_x^{n_{p_q}}(1) = 0 \text{ for some subsequence } n_{p_q} \text{ of } n_p. \end{aligned} \tag{3.22}$$

So we will examine the four cases, namely

$$\begin{array}{ll} c_1) \ \delta_\infty = 0 \text{ and } \mu_\infty = 0 & c_2) \ \delta_\infty \neq 0 \text{ and } \mu_\infty \neq 0, \\ c_3) \ \delta_\infty = 0 \text{ and } \mu_\infty \neq 0 & c_4) \ \delta_\infty \neq 0 \text{ and } \mu_\infty \neq 0 \end{array}$$

In case c_1), relation (3.20) and property (3.22) insure

$$u^{n_p} = u_\infty^{n_p} \text{ and } v^{n_p} = v_\infty^{n_p};$$

then from (3.18), the required claim (3.14) holds.
For case c_2), in view of (3.22) for p sufficiently large, Relation (3.20) can be rewritten in the following form:

$$\begin{aligned} &\| \left(u^{n_p} - u_\infty^{n_p}, v^{n_p} - v_\infty^{n_p} \right) \|_{\mathcal{H}}^2 \leq \\ &-EI\lambda_{n_p} \left[\delta_{n_p} \left(v^{n_p}(1) \right)^2 \left(-1 + \frac{\delta_\infty}{\delta_{n_p}} \frac{v_\infty^{n_p}(1)}{v^{n_p}(1)} \right) \left(-1 + \frac{v_\infty^{n_p}(1)}{v^{n_p}(1)} \right) \right. \\ &\left. + \mu_{n_p} \left((v^{n_p})_x (1) \right)^2 \left(-1 + \frac{\mu_\infty}{\mu_{n_p}} \frac{\left(v_\infty^{n_p} \right)_x (1)}{v_x^{n_p}(1)} \right) \left(-1 + \frac{\left(v_\infty^{n_p} \right)_x (1)}{v_x^{n_p}(1)} \right) \right]. \end{aligned} \tag{3.23}$$

Now since $\lim \frac{\delta_\infty}{\delta_{n_p}} = \frac{\mu_\infty}{\mu_{n_p}} = 1$, it is an exercise left to the reader to check that the unique cluster value in $\overline{\mathbb{R}^+}$ of the second member of (3.23) must be zero. Thus (3.14) is true again. Cases c_3) and c_4) are true with the same arguments as in case c_2), since according to (3.22), relation (3.20) leads to (3.23) where one term of the right hand side fails. Thus, (3.14) holds in all cases. This brings the proof of Lemma 4 to an end. $\square$

We have thus shown Proposition 2. $\square$

The following lemma is a consequence of Proposition 2 and the results of Brezis [6].

Lemma 5 . *i) For any initial data $\Phi_0 = (y_0, z_0) \in \mathcal{D}(\tilde{A})$, the system (3.7) admits a unique solution $\Phi(t) = (y(t), z(t)) \in \mathcal{D}(\tilde{A})$ such that*

$$(y, z) \in L^\infty(\mathbb{R}^+; \mathcal{H}), \quad \text{and} \quad \frac{d}{dt}(y, z) \in L^\infty(\mathbb{R}^+; \mathcal{H}).$$

The solution $\Phi = (y, z)$ is given by $\Phi(t) = e^{-\tilde{A}t}\Phi_0$, for all $t \geq 0$ where $e^{-\tilde{A}t}$ is the semigroup generated by $-\tilde{A}$ on $\overline{\mathcal{D}(\tilde{A})} = \mathcal{H}$. Moreover, the function $t \longmapsto ||\tilde{A}\Phi(t)||_{\mathcal{H}}$ is decreasing.
ii) For any initial data $\Phi_0 = (y_0, z_0) \in \overline{\mathcal{D}(\tilde{A})} = \mathcal{H}$, the equation (3.7) admits a unique mild solution $\Phi(t) = e^{-\tilde{A}t}\Phi_0$ which is bounded on $\mathbb{R}^+$ by $||\Phi_0||_{\mathcal{H}}$ and

$$\Phi = (y, z) \in C^0(\mathbb{R}^+; \mathcal{H}).$$

3.3 Asymptotic stability of $e^{-\tilde{A}t}$

In this subsection, using the LaSalle's invariance principle [11]-[9], we show that the nonlinear semigroup $e^{-\tilde{A}t}$ is asymptotically stable.

Proposition 3 . *Assume that $|\ \omega^*\ | < \sqrt{9EI/\rho}$. The semigroup $e^{-\tilde{A}t}$ is asymptotically stable in $\mathcal{H}$, i.e, for any initial data $\Phi_0 = (y_0, z_0) \in \overline{\mathcal{D}(\tilde{A})} = \mathcal{H}$, we have $e^{-\tilde{A}t}\Phi_0 \to 0$ as $t \to +\infty$.*

Proof of Proposition 3. Since the nonlinear semigroup $e^{-\tilde{A}t}$ is contractive and since $\mathcal{D}(\tilde{A})$ is dense in $\mathcal{H}$, it suffices to prove the Proposition for any initial data $\Phi_0 = (y_0, z_0) \in \mathcal{D}(\tilde{A})$. Let $\Phi_0 = (y_0, z_0) \in \mathcal{D}(\tilde{A})$. By virtue of Lemma 5, we know that the trajectory $\{e^{-\tilde{A}t}\Phi_0;\ t \geq 0\}$ is a bounded set for the graph norm. Furthermore, one can show directly that the injection

$$i : (\mathcal{D}(\tilde{A}), ||\ \ ||_{\mathcal{D}(\tilde{A})}) \longrightarrow \mathcal{H}$$

is compact. This implies that the considered trajectory is precompact in $\mathcal{H}$. Applying the LaSalles's invariance principle ([11], [9]), we deduce that the ω-limit set

$$\omega\left(\Phi_0\right)=\left\{\left(\psi_1,\psi_2\right)\in\mathcal{H};\left(\psi_1,\psi_2\right)=\lim_{n\to\infty}S(t_n)\left(\Phi_0\right)\quad\text{with}\;\; t_n\to\infty\;\;\text{as}\;\; n\to\infty\right\}$$

is non empty, compact, invariant under the semigroup $e^{-\tilde{A}t}$ and

$$e^{-\tilde{A}t}\Phi_0\longrightarrow\omega\left(\Phi_0\right)\qquad\quad\text{as}\;\; t\to+\infty.$$

Moreover, we deduce from the m-accretivity of $\tilde{A}$ (see [6])

$$\omega\left(\Phi_0\right)\subset\mathcal{D}(\tilde{A}).$$

In order to prove the asymptotic stability, it is sufficient to show that the ω-limit set $\omega\left(\Phi_0\right)$ reduces to $\{0\}$. For this, let $(\tilde{\phi}_1,\tilde{\phi}_2)\in\omega\left(\Phi_0\right)$ and

$$(\tilde{y},\tilde{z})(t)=e^{-\tilde{A}t}(\tilde{\phi}_1,\tilde{\phi}_2)\subset\omega(\Phi_0)\subset\mathcal{D}(\tilde{A}),\;\forall t\geq 0.$$

But, $\|(\tilde{y},\tilde{z})(t)\|_{\mathcal{H}}$ is constant on $\mathbb{R}^+$ (see [10]) and thus $\frac{d}{dt}\|(\tilde{y},\tilde{z})(t)\|^2_{\mathcal{H}}=0$, on $\mathbb{R}^+$. Since $(\tilde{y},\tilde{z})$ is a strong solution, this means

$$\langle\,\tilde{A}(\tilde{y},\tilde{z})(t),(\tilde{y},\tilde{z})(t)\,\rangle_{\mathcal{H}}=0,\;\;\text{a. e. } t\geq 0. \tag{3.24}$$

Now, from (3.8) and (3.24), it follows

$$\langle\,\tilde{A}(\tilde{y},\tilde{z})(t),(\tilde{y},\tilde{z})(t)\,\rangle_{\mathcal{H}}=EI\tilde{z}(1,t)g(\tilde{z}(1))+EI\tilde{z}_x(1)f(\tilde{z}_x(1))=0,\;\;\text{a. e. } t\geq 0. \tag{3.25}$$

We deduce from (3.25) and F1) that $\tilde{y}$ is the strong solution on $\mathbb{R}^+$ of the following linear system

$$\begin{cases}\rho y_{tt}+EIy_{xxxx}=\rho{\omega^*}^2y,\\ y(0,t)=y_x(0,t)=0,\\ y_{xx}(1,t)=0,\\ y_{xxx}(1,t)=y_t(1,t)=0,\\ (y(0),y_t(0))=(\tilde{\phi}_1,\tilde{\phi}_2)\in\omega\left(\phi_1,\phi_2\right).\end{cases} \tag{3.26}$$

However, it was shown in [13] that zero is the unique (mild) solution of (3.26). Of course, for such a linear system the notions of mild, strong or classical solutions coincide. Therefore, the proof of Proposition 3 is complete. $\square$

3.4 Exponential stability of $e^{-\tilde{A}t}$

In this subsection, by using the multiplier method (see [12]), we obtain an exponential stability result of the semigroup $e^{-\tilde{A}t}$ under hypotheses F1) and F2).

Proposition 4 . *Assume that* $\mid \omega^* \mid < \sqrt{9EI/\rho}$. *Let* $\Phi = (y,z)$ *be the solution of (3.7) stemmed from* $\Phi_0 = (y_0, z_0) \in \overline{\mathcal{D}(\tilde{A})} = \mathcal{H}$. *If hypotheses F1) and F2) are satisfied, then there exist (uniform) constants* $\tilde{M}$, $\tilde{\mu} > 0$ *such that*

$$||\Phi(t))||_{\mathcal{H}} \leq \tilde{M} e^{-\tilde{\mu}t} ||\Phi_0||_{\mathcal{H}}, \ \ \forall t \geq 0.$$

Proof of Proposition 4. As in the proof of Proposition 3, we may assume that $\Phi_0 = (y_0, z_0) \in \mathcal{D}(\tilde{A})$. Then, let us write $\Phi(t) = e^{-\tilde{A}t}\Phi_0 = (y, y_t)(t)$ and define the functional ξ as

$$\xi(t) = 2\int_0^1 x y_t y_x dx, \ \ t \geq 0. \tag{3.27}$$

Returning to the definition of mild solutions, we note that the function $t \mapsto \xi(t)$ is everywhere defined and continuous on $[0, +\infty[$ (even without assuming $(y_0, z_0) \in \mathcal{D}(\tilde{A})$). By means of Cauchy-Schwarz's inequality, we obtain

$$\mid \xi(t) \mid \leq ||(y(t), z(t))||^2_{\mathcal{H}}, \ \ \forall t \geq 0. \tag{3.28}$$

On the other hand, according to Lemma 5 and the equations of (3.1), a straightforward computation gives

$$\begin{aligned} \xi_t(t) = & -2\frac{EI}{\rho} y_x(1,t) \ f(y_{xt}(1,t)) - 2\frac{EI}{\rho} y_x(1,t) \ g\,(y_t(1,t)) \\ & + y_t^2(1,t) + \frac{EI}{\rho} f^2(y_{xt}(1,t)) + {\omega^*}^2 y^2(1,t) - \int_0^1 \left[y_t^2 + 3\frac{EI}{\rho} y_{xx}^2 + {\omega^*}^2 y^2 \right] dx, \end{aligned} \tag{3.29}$$

a.e. $t \geq 0$.

Moreover, using once again the Cauchy-Schwarz's inequality, we have

$$\left\{ \begin{array}{l} -2\dfrac{EI}{\rho}y_x(1,t)\; g\left(y_t(1,t)\right) \leq \dfrac{EI}{\rho\theta}y_x^2(1,t) + \theta\dfrac{EI}{\rho}g^2\left(y_t(1,t)\right), \\ -2\dfrac{EI}{\rho}y_x(1,t)\; f(y_{xt}(1,t)) \leq \dfrac{EI}{\rho\theta}y_x^2(1,t) + \theta\dfrac{EI}{\rho}f^2\left(y_{xt}(1,t)\right), \end{array} \right. \tag{3.30}$$

for any $\theta > 0$.

From $y(0) = y_x(0) = 0$, it easily follows

$$y_x^2(1,t) \leq \int_0^1 y_{xx}^2 dx, \qquad \text{and} \qquad y^2(1,t) \leq \frac{1}{3}\int_0^1 y_{xx}^2 dx.$$

This, together with (3.30), implies

$$\begin{array}{r} -2\dfrac{EI}{\rho}y_x(1,t)\; g\left(y_t(1,t)\right) - 2\dfrac{EI}{\rho}y_x(1,t)\; f(y_{xt}(1,t)) \leq \theta\dfrac{EI}{\rho}g^2\left(y_t(1,t)\right) \\ +\theta\dfrac{EI}{\rho}f^2\left(y_{xt}(1,t)\right) + 2\dfrac{EI}{\rho\theta}\displaystyle\int_0^1 y_{xx}^2 dx. \end{array} \tag{3.31}$$

Combining (3.29) and (3.31), we get

$$\begin{array}{l} \xi_t(t) \leq y_t^2(1,t) + \theta\dfrac{EI}{\rho}g^2\left(y_t(1,t)\right) + (1+\theta)\dfrac{EI}{\rho}f^2\left(y_{xt}(1,t)\right) \\ \qquad +\dfrac{1}{\rho}\displaystyle\int_0^1 \left[-\rho y_t^2 + \left(\frac{1}{2\theta} + \frac{\rho\omega^{*2}}{3EI} - 3\right)EIy_{xx}^2 - \rho\omega^{*2}y^2\right]dx, \end{array} \tag{3.32}$$

a.e. $t \geq 0$.

Taking into account the condition $\mid \omega^* \mid < \sqrt{9EI/\rho}$, we can choose θ in order to have

$$\frac{1}{2\theta} + \frac{\rho\omega^{*2}}{3EI} - 3 < 0.$$

Thus, we have proved the existence of a positive constant K_1 satisfying

$$\begin{array}{l} \xi_t(t) \leq y_t^2(1,t) + \theta\dfrac{EI}{\rho}g^2\left(y_t(1,t)\right) + (1+\theta)\dfrac{EI}{\rho}f^2\left(y_{xt}(1,t)\right) \\ \qquad -K_2||(y(t),z(t))||_{\mathcal{H}}^2, \end{array} \tag{3.33}$$

a.e. $t \geq 0$.

Given $\epsilon > 0$, we introduce (see [7] and [19]) the perturbed energy by

$$E^\epsilon(t) = E_1(t) + \epsilon\xi(t), \tag{3.34}$$

where

$$E_1(t) = \frac{1}{2}\|(y(t), z(t))\|^2_{\mathcal{H}},$$

for all $t \geq 0$.

The function $E_1(t)$ is a decreasing function: more precisely, we have

$$E_{1t}(t) = -EIy_t(1,t)\ g(y_t(1,t)) - EIy_{xt}\ f(y_{xt}(1,t)) \text{ a.e } \ t \geq 0. \tag{3.35}$$

But (3.28) and (3.34) imply that for all $t \geq 0$, we have

$$M^{-\frac{1}{2}} E^\epsilon(t) \leq E_1(t) \leq M^{\frac{1}{2}} E^\epsilon(t), \tag{3.36}$$

for any $M > 1$ and provided $\epsilon \leq \frac{1}{2}\left(1 - M^{-\frac{1}{2}}\right)$.

By using (3.34) and (3.32), we obtain

$$\begin{aligned} E^\epsilon_t(t) \leq E_{1t}(t) + \epsilon y_t^2(1,t) + \epsilon\,\theta \frac{EI}{\rho} g^2\left(y_t(1,t)\right) + \epsilon(1+\theta)\frac{EI}{\rho} f^2(y_{xt}(1,t)) \\ -2K_2 E_1(t), \end{aligned} \tag{3.37}$$

a.e. $t \geq 0$. In addition, hypothesis F2) leads us to

$$\begin{cases} y_t^2(1,t) + \theta \dfrac{EI}{\rho} g^2\left(y_t(1,t)\right) \leq \left(\dfrac{1}{L_2} + \theta L_3 \dfrac{EI}{\rho}\right) y_t(1,t)\ g(y_t(1,t)), \\ (1+\theta)\dfrac{EI}{\rho} f^2(y_{xt}(1,t)) \leq (1+\theta)\dfrac{EI}{\rho} L_3 y_{xt}(1,t)\ f(y_{xt}(1,t)). \end{cases}$$

a.e. $t \geq 0$.

Plugging these two last inequalities into (3.37) and using (3.35), we get

$$\begin{aligned} E^\epsilon_t(t) \leq -2K_2 E_1(t) + \left[\epsilon\left(\tfrac{1}{L_2} + \theta L_3 \tfrac{EI}{\rho}\right) - EI\right] y_t(1,t)\ g(y_t(1,t)) \\ + \left[\epsilon L_3(1+\theta)\frac{EI}{\rho} - EI\right] y_{xt}(1,t)\ f(y_{xt}(1,t)). \end{aligned}$$

This implies

$$E^\epsilon_t(t) \leq -2K_2 E_1(t), \tag{3.38}$$

a.e. $t \geq 0$, provided that, in addition, ϵ satisfies

$$\begin{cases} \epsilon\left(\tfrac{1}{L_2} + \theta L_3 \tfrac{EI}{\rho}\right) - EI \leq 0, \\ \epsilon L_3(1+\theta)\tfrac{EI}{\rho} - EI \leq 0. \end{cases}$$

Finally, we deduce from (3.36) and (3.38)

$$E_t^\epsilon(t) \leq -2M^{-\frac{1}{2}} K_2 E^\epsilon(t), \tag{3.39}$$

a.e. $t \geq 0$. Solving this differential inequality (remark that $E_t^\epsilon \in L^\infty([0, +\infty[, \mathbb{R})$) and using once again (3.36), we obtain the required exponential decay of $E_1(t) = ||\Phi(t)||_{\mathcal{H}}$ with $\tilde{\mu} = 2M^{-\frac{1}{2}} K_2$. □

3.5 The global system (3.5)

a) Cauchy problem.

By using Proposition 1, we will show that the global system (3.5) is well posed on $[0, \infty[$. Indeed, we recall that the global system (3.5) can be written as an evolution equation

$$\begin{pmatrix} \dot{\Phi}(t) \\ \dot{\omega}(t) \end{pmatrix} + (A + B) \begin{pmatrix} \Phi(t) \\ \omega(t) \end{pmatrix} = 0,$$

where the operators A and B are defined in (3.6).

First, we have shown in Proposition 4 that the operator $\tilde{A}$ defined by (3.3)-(3.4) is m-accretive in $\mathcal{H}$ with dense domain $\mathcal{D}(\tilde{A})$. We deduce thus, from (3.6), that the operator A is also m-accretive with dense domain $\mathcal{D}(A) = \mathcal{D}(\tilde{A}) \times \mathbb{R}$ in $X = \mathcal{H} \times \mathbb{R}$.

Second, as the reader can easily see, the operator B, defined in (3.6), is Lipschitz on bounded subsets of X and therefore B satisfies Hypothesis **(HII)** of Section 2. So (3.5) has a local solution $(\Phi, \omega)(t) = (y(t), z(t), \omega(t))$.

Consider now the function $V : X \to \mathbb{R}^+$ defined by

$$V(y_1, z_1, \omega_1) = \frac{1}{2} I_d \, (\omega_1 - \omega^*)^2 + \frac{1}{2}(\omega_1 - \omega^*)^2 \int_0^1 \rho y_1^2 dx + \frac{1}{2} \int_0^1 (\rho z_1^2 + EI y_{1xx}^2) dx \\ -\frac{1}{2} \omega^{*2} \int_0^1 \rho y_1^2 dx. \tag{3.40}$$

for $(y_1, z_1, \omega_1) \in X$. We claim that this function is a reasonable choice of Lyapunov function for (3.5). Indeed, on the one hand, it is easy to check

$$V(y_1, z_1, \omega_1) \geq K ||(y_1, z_1, \omega_1)||_X, \tag{3.41}$$

for some positive constant K and for all $(y_1, z_1, \omega_1) \in X$. On the other hand, from Lemma 3, we know that each local solution of (3.5), with initial data in $\mathcal{D}(A)$, is a strong one. Moreover, a straightforward computation shows that for any initial condition $(y_0, z_0, \omega_0) \in \mathcal{D}(A)$, the corresponding strong solution $(y, z, \omega)(t)$ of (3.5) satisfies

$$\begin{aligned} \frac{dV}{dt}(\Phi, \omega)(t) = -EI\left[y_{xt}(1,t)\, f\Big(y_{xt}(1,t)\Big) + y_t(1,t)\, g\Big(y_t(1,t)\Big)\right] \\ -(\omega(t) - \omega^*)\, \gamma\Big(\omega(t) - \omega^*\Big) \leq 0, \end{aligned} \tag{3.42}$$

almost everywhere on the domain $[0, T_{max})$ of this local solution $(y, z, \omega)(t)$. Thus, the function $t \longmapsto V(y(t), z(t), \omega(t))$ is non increasing on its domain. Then by Lemma 2 and (3.41), **(HIII)** holds for the system (3.5).

Now, from Proposition 1 we are able to claim that for any initial condition $(y_0, z_0, \omega_0) \in X$, the global system (3.5) has a unique solution $(\Phi, \omega)(t)$ on $[0, +\infty]$.

b) Proof of Theorem 2.

Initially, we will restrict ourselves to the system

$$\begin{cases} \dot{\Phi}(t) + \tilde{A}\Phi(t) + \tilde{B}(t, \Phi(t)), \quad \text{on} \quad [0, \infty[; \\ \Phi(0) = \Phi_0, \end{cases}$$

where $\Phi(t) = (y, z)(t)$, $\Phi_0 \in \mathcal{D}(\tilde{A})$ and

$$\tilde{B}(t, (u, v)) = (0, ({\omega^*}^2 - \omega^2(t))u); \tag{3.43}$$

$\omega(t)$ being such that $(\Phi(t), \omega(t))$ is the solution of the global system (3.5) with initial data $(\Phi_0, \omega_0) \in \mathcal{D}(A)$.

From (3.43), we have

$$||\tilde{B}(t, \Phi(t))||_{\mathcal{H}} \leq |{\omega^*}^2 - \omega^2(t)| ||\Phi(t)||_{\mathcal{H}}.$$

In addition, one can show that (3.40), (3.42) and F3) give

$$\omega^* - \omega(\cdot) \in L^2\left([0, \infty[; \mathbb{R}\right) \cap L^\infty\left([0, \infty[; \mathbb{R}\right).$$

Consequently, (HV) holds with $\mu(t) = |{\omega^*}^2 - \omega^2(t)|$. (Really, we have the stronger condition $\lim_{t \to +\infty} \omega(t) = \omega^*$). Therefore, all the assumptions of Theorem 1 are satisfied. Thus, there exist positive constants M and κ such that

$$\|\Phi(t)\|_{\mathcal{H}} \leq Me^{-\kappa t}, \; \forall t \geq 0.$$

Finally, returning to the second equation of (3.5) and using F3) we prove analogously to [13] that $\omega^* - \omega(\cdot)$ tends also exponentially to zero. The proof of Theorem 2 is now complete.

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Unit´e de recherche INRIA Lorraine, Technopˆole de Nancy-Brabois, Campus scientifique,
615 rue du Jardin Botanique, BP 101, 54600 VILLERS LÈS NANCY
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