Analyticity of Iterates of Random Non-Expansive Maps

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Abstract: This paper focuses on the analyticity of the limiting behavior of a class of dynamical systems defined by iteration of non expansive random operators. The analyticity is understood in function of the parameters which govern the law of the operators. The proofs are based on contraction with respect to certain projective semi norms. We first concentrate on the analyticity of Lyapunov exponents. Such exponents can be seen as functionals of a Bernoulli process whenever each operator is sampled independently between two possible values. Our main concern is then the domain of analyticity of these exponents seen as functions of the parameter of the Bernoulli law. Several examples are considered, including Lyapunov exponents associated with products of random matrices both in the conventional algebra, and in the $(\max, +)$ semi-field, and Lyapunov exponents associated with non–linear dynamical systems arising in stochastic control. For the class of reducible operators (defined in the paper), we also address the issue of analyticity of the expectation of functionals of the limiting behavior in function of the parameters of the law, and connect this with contraction properties with respect to the supremum norm. We give several applications to the analyticity of stationary response times in certain queueing networks in function of the intensity of the arrival process and the parameters of the law of the service times.

Key-words: contraction, non-expansiveness, analyticity, vectorial recurrence relation, Lyapunov exponents, asymptotic mean value. *AMS 1991 subject classifications* : Primary 47H09, 32D05, 60B99, 34D05; Secondary 26E05, 47H40, 34D08, 28A18.

(Résumé : tsvp)

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Analyticité des itérées de fonctions aléatoires non-expansives

Résumé : Cet article est consacré à l'analyticité du comportement asymptotique d'une classe de systèmes dynamiques obtenus par itération d'opérateurs aléatoires non-expansifs, la dépendance analytique étant entendue par rapport aux paramètres de la loi des opérateurs. Les preuves sont fondées sur des propriétés de contraction par rapport à certaines semi-normes projectives. Nous étudions d'abord l'analyticité des exposants de Lyapounov associés à ces systèmes. Ces exposants peuvent être considérés comme des fonctionnelles d'un processus de Bernoulli lorsque chaque opérateur prend deux valeurs possibles de manière indépendante. Notre but principal est alors de trouver le domaine d'analyticité de ces exposants vus comme des fonctions des paramètres de la loi de Bernoulli. Plusieurs exemples sont considérés : exposants de Lyapounov associés aux produits de matrices aléatoires dans l'algèbre conventionnelle et dans le semi-anneau $(\max, +)$, exposants associés à des systèmes dynamiques non–linéaires qui apparaissent dans les problèmes de contrôle stochastique etc. Pour la classe des opérateurs réductibles (définie dans cet article), nous obtenons des résultats sur l'analyticité de l'espérance de certaines fonctionnelles du comportement limite en fonction des paramètres de la loi: dans ce cas la contraction est établie par rapport à la norme L_∞. Nous donnons plusieurs applications sur l'analyticité de la moyenne du temps de réponse stationnaire dans certaines files d'attentes en fonction de l'intensité du processus d'arrivée et des paramètres de la loi des services.

Mots-clé : contraction, non-expansivité, analyticité, relation de récurrence vectorielle, exposant de Lyapounov, espérance asymptotique. *classification AMS 1991* : Primaire 47H09, 32D05, 60B99, 34D05; Secondaire 26E05, 47H40, 34D08, 28A18.

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1 Introduction

Contraction properties w.r.t. projective norms were used by D. Ruelle ([22]) for investigating properties of Lyapunov exponents in the conventional algebra. This line of thoughts was then developed by Y. Peres ([18], [19]) to find the domain of analyticity of the top Lyapunov exponent of random matrices of the conventional algebra with a Markov dependence.

We show that the approach of D. Ruelle and Y. Peres can be extended to address analyticity questions for iterates of certain classes of random non-expansive maps.

A first example is that of Lyapunov exponents associated with products of random matrices in the $(\max, +)$ semi-field. The results obtained by this approach allow us to give improved estimates on the radius of convergence of the Taylor series expansion which were recently established in [2].

We also consider the case of *reducible* non-expansive operators, which arise for instance in the modeling of open queueing networks, and for which differentiability and analyticity with respect to the parameters of the law are often open questions (see e.g. [5], [3]). In particular, it was shown lately by A. Jean-Marie [15] that even for isolated queues with Poisson input, the radius of convergence of the Taylor expansion of the mean asymptotic response times (seen as a function of the intensity λ of the Poisson process) could be strictly smaller than the maximal value of λ granting stability. The approach via contraction can also be used within this setting and leads to a better understanding of these questions.

2 Analyticity via contraction

2.1 Contraction, topical maps

In this section we give some basic definitions and technical lemmas which will be useful in the following.

Definition 1 (projective metric) *The projective metric $\mathcal{D}$ on $\mathbb{R}^d$ is defined by*

$$\begin{aligned}\mathcal{D}(X,Y) &= \max_{1\le i,j\le d}(X_i - Y_i - X_j + Y_j)\\ &= \max_{1\le i\le d}(X-Y)_i - \min_{1\le i\le d}(X-Y)_i.\end{aligned}$$

Remark 1 *$\mathcal{D}$ defines in fact a pseudo-norm on $\mathbb{R}^d$ and a norm on the projective space which results from $\mathcal{D}$.*

Remark 2 *If there exists a coordinate i_0 such that $X_{i_0} - Y_{i_0} \le X_j - Y_j$ for all j, then*

$$\mathcal{D}(X,Y) = ||V - W||_\infty,$$

where V and W are the vectors with coordinates $V_j = X_j - X_{i_0}$ and $W_j = Y_j - Y_{i_0}$, respectively.

Definition 2 (contraction coefficient) *For $G : \mathbb{R}^l \to \mathbb{R}^d$ let*

$$\mathcal{D}_G(Z,Z') = \mathcal{D}\left(G(Z), G(Z')\right).$$

1. *The pointwise contraction coefficient of the operator $\mathcal{A} : \mathbb{R}^l \to \mathbb{R}^l$ w.r.t. $\mathcal{D}_G$ is defined at point $(Z,Z') \in \mathbb{R}^l \times \mathbb{R}^l$ by*

$$\tau_G(\mathcal{A}, Z, Z') = \begin{cases} \frac{\mathcal{D}_G(\mathcal{A}(Z),\mathcal{A}(Z'))}{\mathcal{D}_G(Z,Z')} & \text{if } \mathcal{D}_G(Z,Z') \neq 0,\\ 0 & \text{if } \mathcal{D}_G(Z,Z') = 0.\end{cases}$$

2. *The uniform contraction coefficient of $\mathcal{A}$ w.r.t. $\mathcal{D}_G$ is defined by*

$$\tau_G(\mathcal{A}) = \sup_{Z,Z'}\left\{\tau_G(\mathcal{A}, Z, Z'), \mathcal{D}_G(Z,Z') \neq 0\right\}.$$

Definition 3 *Let $F, G : \mathbb{R}^l \to \mathbb{R}^d$ and let α be a positive real number.*

1. *F is α–Lipschitz w.r.t. $\mathcal{D}_G$ if*

$$||F(Z) - F(Z')||_\infty \le \alpha\mathcal{D}_G(Z,Z'), \ \ \forall (Z,Z') \in \mathbb{R}^l \times \mathbb{R}^l.$$

2. *An operator* $\mathcal{A} : \mathbb{R}^l \to \mathbb{R}^l$ *is non-expansive w.r.t.* $\mathcal{D}_G$ *if*

$$\mathcal{D}_G\left(\mathcal{A}(Z), \mathcal{A}(Z')\right) \leq \mathcal{D}_G(Z, Z').$$

3. *An operator* $\mathcal{A} : \mathbb{R}^l \to \mathbb{R}^l$ *is a* $\mathcal{D}_G$*-class function if* $\mathcal{D}_G(Z, Z') = 0$ *implies that* $\mathcal{D}_G\left(\mathcal{A}(Z), \mathcal{A}(Z')\right) = 0$.

Note that a non-expansive operator is a $\mathcal{D}_G$-class function.

Following J. Gunawardena [13], we will adopt the following terminology:

Definition 4 (Topical functions) $\mathcal{H} : \mathbb{R}^d \to \mathbb{R}^k$ *is called topical (resp. subtopical or supertopical) if it is monotone i.e.*

$$\mathcal{H}(X) \leq \mathcal{H}(Y) \quad \textit{if} \quad X \leq Y, \tag{1}$$

and 1–homogeneous i.e.

$$\mathcal{H}(X + a\mathbf{1}) = \mathcal{H}(X) + a\mathbf{1} \ \ \forall X \in \mathbb{R}^d \ \textit{and} \ \forall a > 0, \tag{2}$$

(resp. 1–sub–homogeneous or 1–super–homogeneous i.e.

$$\mathcal{H}(X + a\mathbf{1}) \leq \mathcal{H}(X) + a\mathbf{1} \ \ \textit{or} \ \mathcal{H}(X + a\mathbf{1}) \geq \mathcal{H}(X) + a\mathbf{1} \ \ \forall X \in \mathbb{R}^d \ \textit{and} \ \forall a > 0),$$

where $\mathbf{1}$ *is the vector with all its coordinates*[1] *equal to* 1.

It is easy to check that if (2) holds for all $a > 0$, then it holds for all $a \in \mathbb{R}$.

The non expansiveness of topical functions w.r.t. the supremum norm was first established by M. Crandall and L. Tartar in [10]. It is easily extended to subtopical functions as follows.

Lemma 1 *A subtopical function* $\mathcal{H}$ *is non-expansive w.r.t. the supremum norm. A topical function is also non-expansive w.r.t.* $\mathcal{D}$.

Proof
If $\mathcal{H}$ is topical, we have

$$\begin{aligned} \mathcal{H}(X) &= \mathcal{H}(Y + (X - Y)) \\ &\leq \mathcal{H}(Y + \mathbf{1}\max_i(X - Y)_i) \\ &\leq \mathcal{H}(Y) + \mathbf{1}\max_i(X - Y)_i. \end{aligned} \tag{3}$$

So

$$\mathcal{H}(X) - \mathcal{H}(Y) \leq \mathbf{1}\max_i(X - Y)_i$$

which implies that

$$\max_i(\mathcal{H}(X) - \mathcal{H}(Y))_i \leq \max_i(X - Y)_i. \tag{4}$$

[1]Note that its dimension is d on the L.H.S of (2), and k on its R.H.S.

This together with the symmetrical relation obtained by interchanging X and Y give that

$$\begin{aligned}||\mathcal{H}(X)-\mathcal{H}(Y)||_\infty &= \max\left(\max_i(\mathcal{H}(X)-\mathcal{H}(Y))_i, \max_i(\mathcal{H}(Y)-\mathcal{H}(X))_i\right)\\ &\leq \max\left(\max_i(X-Y)_i, \max_i(Y-X)_i\right) = ||X-Y||_\infty.\end{aligned}$$

Similarly,

$$\begin{aligned}\mathcal{D}(\mathcal{H}(X),\mathcal{H}(Y)) &= \max_i(\mathcal{H}(X)-\mathcal{H}(Y))_i + \max_i(\mathcal{H}(Y)-\mathcal{H}(X))_i\\ &\leq \max_i(X-Y)_i + \max_i(Y-X)_i = \mathcal{D}(X,Y).\end{aligned}$$

For a subtopical function (3) is not true when $\max_i(X-Y)_i$ is negative. However it verifies the following equation

$$\max_i(\mathcal{H}(X)-\mathcal{H}(Y))_i \leq \max_i(X-Y)_i \vee 0, \tag{5}$$

which is sufficient to establish the non-expansiveness w.r.t. the supremum norm. ♠

We will need the following lemma:

Lemma 2 *Let $\mathcal{H}:\mathbb{R}^d\to\mathbb{R}^d$ and $f:\mathbb{R}^d\to\mathbb{R}^k$ be topical functions, and let $G_1, G_2:\mathbb{R}^l\to\mathbb{R}^d$. Define the function*

$$F(Z) = f\circ\mathcal{H}(G_1(Z)) - f\circ G_2(Z), \tag{6}$$

1. *If $G_1 = G_2 = G$, then F is 1–Lipschitz w.r.t. $\mathcal{D}_G$.*
2. *If G_1 is subtopical and G_2 is supertopical then F is 1–Lipschitz w.r.t. $\mathcal{D}$.*

Proof
From (4), we get

$$\begin{aligned}(F(Z)-F(Z'))_j &= \left(f\circ\mathcal{H}\circ G_1(Z) - f\circ G_2(Z) - f\circ\mathcal{H}(G_1(Z')) + f\circ G_2(Z')\right)_j\\ &\leq \max_i(G_1(Z)-G_1(Z'))_i - \left(G_2(Z)-G_2(Z')\right)_j. \end{aligned}\tag{7}$$

If $G_1 = G_2 = G$, from (7) and interchanging Z and Z',

$$||F(Z)-F(Z')||_\infty \leq \mathcal{D}_G(Z,Z').$$

The second property also results from (7) and the fact that for the supertopical function G_2,

$$\min_i(Z-Z')_i \leq \min_i(G_2(Z)-G_2(Z'))_i.$$

♠

2.2 Analyticity of dynamical systems

2.2.1 The dynamical system and its parametric law

Consider the discrete time dynamical system:

$$Z_{n+1} = \mathcal{A}_n(Z_n), \tag{8}$$

where $\{\mathcal{A}_n\}$ is a sequence of independent and identically distributed random operators: $\mathcal{A}_{n,\omega} : \mathbb{R}^l \to \mathbb{R}^l$ and where $Z_0 \in \mathbb{R}^l$ is an independent initial condition.

We can see the variables Z_n as functionals of the sequence $\{\mathcal{A}_0, \mathcal{A}_1, \ldots\}$, and we will often consider the same variable but for the shifted sequence $\{\mathcal{A}_1, \mathcal{A}_2, \ldots\}$, which will be denoted $Z_n \circ \theta$ (θ being then the shift on the sequence of operators).

As explained above, the main focus of this paper is the analyticity of the limiting law of Z_n or more generally $F(Z_n)$ in the parameters of the law of the operators. In what follows, the law $\mathcal{P}_\nu$ of $\mathcal{A}_n$ will be assumed to be a probability measure on some measurable space $(S, \mathcal{S})$, and to depend on some parameter $\nu \in \mathbb{R}^k$.

We assume that for all sets $B \in \mathcal{S}$, $\mathcal{P}_\nu(B)$ can be continued to a complex valued set function $\mathcal{P}^*_\nu(B)$, when replacing ν initially in $\mathbb{R}^k$ by a complex number in a subset Δ_0 of $\mathbb{C}^k$, and more generally that the expectation operator $\mathbb{E}$ admits a complex continuation $\mathbb{E}^*$. By this we mean that for all integers n and for all measurable functions $f : S^n \to \mathbb{R}^d_+$, and for all ν in Δ_0, the operators,

$$\mathbb{E}^*\left[f(\mathcal{A}_0, \cdots, \mathcal{A}_{n-1})\right] \triangleq \int_{S^n} f(a_0, \ldots, a_{n-1}) \mathcal{P}^*_\nu(da_0) \cdots \mathcal{P}^*_\nu(da_{n-1}),$$

and

$$||\mathbb{E}^*||\left[f(\mathcal{A}_0, \cdots, \mathcal{A}_{n-1})\right] \triangleq \int_{S^n} ||f(a_0, \ldots, a_{n-1})||_\infty |\mathcal{P}^*_\nu(da_0)| \cdots |\mathcal{P}^*_\nu(da_{n-1})|,$$

where $|z|$ denotes the modulus of $z \in \mathbb{C}$ (bellow, for a vector $Z \in \mathbb{C}^k$, we will more generally denote $|Z|$ the vector of the modulus of coordinates), are well defined and satisfy the following set of properties:

1. $||\ |\mathbb{E}^*(f)|\ ||_\infty \leq ||\mathbb{E}^*||(f)$,
2. If $||f||_\infty \leq ||g||_\infty$ then $||\mathbb{E}^*||(f) \leq ||\mathbb{E}^*||(g)$,
3. $\mathbb{E}^*\left[f \circ \theta\right] = \mathbb{E}^*\left[f\right]$.
4. Whenever the random variables f and g are independent, $\mathbb{E}^*\left[fg\right] = \mathbb{E}^*\left[f\right] \mathbb{E}^*\left[g\right]$.

Two simple cases where such a continuation exists are when $\mathcal{P}^*_\nu$ is a point mass distribution, and when it admits a density w.r.t. a measure which does not depend on ν and this density can be analytically continued.

2.2.2 Main result

The main result of the present paper is the following theorem:

Theorem 1 *Assume that the sequence $\{\mathcal{A}_n\}$ is i.i.d. (i.e. made of independent and identically distributed variables) and that there exist $F, G : \mathbb{R}^l \to \mathbb{R}^d$ such that*

1. *The function F is α–Lipschitz w.r.t. $\mathcal{D}_G$.*
2. *For all n, the operator $\mathcal{A}_{n,\omega}$ is a $\mathcal{D}_G$-class function.*
3. *For all n, $F(Z_n)$ is integrable when ν belongs to the set $\Delta_0 \subset \mathbb{C}^k$ and $\mathbb{E}^*[F(Z_n)]$ is an analytic function of ν in this domain.*

Let Δ be a subset of Δ_0, on every compact of which the series

$$\sum_{n=1}^{\infty} ||\mathbb{E}^*||[\tau_G(\mathcal{A}_n, Z_n, Z_{n-1} \circ \theta) \cdots \tau_G(\mathcal{A}_1, Z_1, Z_0 \circ \theta)\mathcal{D}_G(Z_1, Z_0 \circ \theta)] \tag{9}$$

is uniformly convergent in ν.

Then for $\nu \in \Delta$, when n goes to ∞, $\mathbb{E}^*[F(Z_n)]$ converges to a function which is analytic in the variable ν.

In particular the domain defined by

$$||\mathbb{E}^*||[\tau_G(\mathcal{A}_0)] < 1$$

is included in Δ.

Proof

Assume in a first step that for all $n \geq 0$, $\mathcal{D}_G\left(Z_{n+1}, Z_n \circ \theta\right) \neq 0$. Then for all n,

$$\begin{aligned} ||F(Z_{n+1}) - F(Z_n) \circ \theta||_\infty &\leq \alpha.\mathcal{D}_G\left(Z_{n+1}, Z_n \circ \theta\right) \\ &= \alpha.\mathcal{D}_G\left(\mathcal{A}_n(Z_n), \mathcal{A}_n(Z_{n-1} \circ \theta)\right) \\ &= \alpha.\tau_G(\mathcal{A}_n, Z_n, Z_{n-1} \circ \theta)\mathcal{D}_G\left(Z_n, Z_{n-1} \circ \theta\right) \\ &\vdots \\ &= \alpha.\tau_G(\mathcal{A}_n, Z_n, Z_{n-1} \circ \theta) \times \cdots \\ &\qquad \times \tau_G(\mathcal{A}_1, Z_1, Z_0 \circ \theta)\mathcal{D}_G(Z_1, Z_0 \circ \theta)]. \end{aligned}$$

Then

$$\begin{aligned} &||\ \mathbb{E}^*[|F(Z_{n+1}) - F(Z_n)|]\ ||_\infty \\ &\quad \leq \ \alpha||\mathbb{E}^*||[\tau_G(\mathcal{A}_n, Z_n, Z_{n-1} \circ \theta) \cdots \tau_G(\mathcal{A}_1, Z_1, Z_0 \circ \theta)\mathcal{D}_G(Z_1, Z_0 \circ \theta)]. \end{aligned}$$

It follows that

$$\mathbb{E}^*[F(Z_n)] = \left(\sum_{k=1}^{n-1} \mathbb{E}^*[F(Z_{k+1}) - F(Z_k)]\right) + \mathbb{E}^*[F(Z_0)]$$

converges in the Cauchy sense in $\mathbb{R}^d$. The analyticity of the limit

$$\lim_{n \to \infty} \mathbb{E}^*[F(Z_n)]$$

follows from the uniform convergence.

In case $\mathcal{D}_G\left(Z_{n+1}, Z_n \circ \theta\right) = 0$ for some n, then $\mathcal{D}_G\left(Z_{m+1}, Z_m \circ \theta\right) = 0$, for all $m \geq n$ due to the assumption that $\mathcal{A}_n$ is a $\mathcal{D}_G$–class function. Using the same arguments as above, we get that $||F(Z_{m+1}) - F(Z_m) \circ \theta||_\infty = 0$ for all $m \geq n$, and this gives rise to the same condition as above in view of the convention on τ_G.

Finally, the uniform contraction region is obviously included in Δ.

♠

Remark 3 *Note that if $\nu \in \mathbb{R}^k \cap \Delta$ is such that $\mathcal{P}_\nu$ is a probability measure, then $\{U_n\}$, with $U_n = Z_n \circ \theta^{-n}$ is a Cauchy sequence w.r.t. $\mathcal{D}_G$ in the L_1 sense, i.e. for all ε, if n large is large enough, then*

$$\mathbb{E}\mathcal{D}_G(U_n, U_{n+m}) < \varepsilon$$

for all $m > 0$. Whenever $Im(G)$ is projectively closed, this implies the existence of a random variable U such that

$$\mathbb{E}\mathcal{D}_G(U_n, U) \rightarrow_{n \to \infty} 0.$$

To see this, consider the projection $\widetilde{V}_n$ of $V_n = G(U_n)$ on the hyper-plane $\sum_i V_i = 0$. Then

$$||\widetilde{V}_{n+1} - \widetilde{V}_n||_\infty \leq \mathcal{D}(\widetilde{V}_{n+1}, \widetilde{V}_n) \leq 2\ ||\widetilde{V}_{n+1} - \widetilde{V}_n||_\infty$$

which means that the supremum norm and projective metric are equivalent on this hyper-plane. Since the space of integrable $\mathbb{R}^d$–valued random variables endowed of the L_1 norm is complete (cf. p.50 [17]), for all $\nu \in \mathbb{R}^k \cap \Delta$, there exists V integrable such that $\mathbb{E}\mathcal{D}(G(U_n), V) \to 0$ when $n \to \infty$. Using the fact that V is the a.s. limit of a subsequence of $\widetilde{V}_n$, and the assumption that $Im(G)$ is a closed set, we see that there exists U with the above properties.

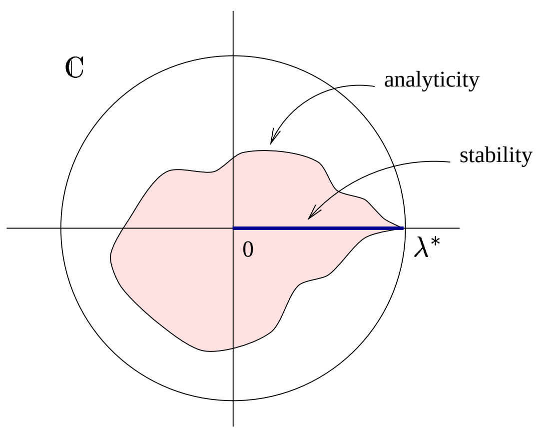

Figure 1: $k = 1$: stability and analyticity.

This type of property was studied by several authors including A. Borovkov [6], [7] and R.N. Bhattacharya and O. Lee [4], primarily for proving the existence of stationary and ergodic regimes for Markov chains and other stochastic systems defined by recurrence equations under contraction assumptions (see also the survey of P. Diaconis and D. Freedman [11]). These results can then be seen as special cases of the situation considered in Theorem 1, when the parameters are real–valued.

2.3 Classes of systems

2.3.1 Iterates of random topical maps

Consider a dynamical system in $\mathbb{R}^d$ defined through the recurrence relation

$$X_{n+1} = \mathcal{H}_n(X_n),$$

where $\{\mathcal{H}_n\}$ is a sequence of independent and identically distributed topical maps from $\mathbb{R}^d$ to $\mathbb{R}^d$. Under mild integrability assumptions, such a dynamical system admits both a top and a bottom Lyapunov exponent, defined by the relations

$$\gamma_{\text{top}} = \lim_{n\to+\infty} \frac{\mathbb{E}[||X_n||_\infty]}{n}$$

and

$$\gamma_{\text{bot}} = \lim_{n\to+\infty} \frac{\mathbb{E}[\inf_{i=1,\ldots,d} X_n^i]}{n}$$

respectively (see [24]). More generally, we will say that this dynamical system admits a ψ–Lyapunov exponent, $\Gamma_\psi \in \mathbb{R}^k$, for some function $\psi : \mathbb{R}^d \to \mathbb{R}^k$ if $\psi(X_n)$ is integrable for all n, and if

$$\lim_{n\to+\infty} \frac{\mathbb{E}[\psi(X_n)]}{n} = \Gamma_\psi.$$

A typical example is that where ψ is a coordinate map.

2.3.2 Augmented system

In the above setting, assume for instance that $\gamma_{\text{top}} > 0$. Then $\mathbb{E}||X_n||_\infty$ tends to ∞, which prevents us of using Theorem 1 directly to prove analyticity properties of γ_{top}.

One can then consider the associated *augmented* dynamical system, $Z_{n+1} = \mathcal{A}_n(Z_n)$ defined as follows:

$$Z_n = \begin{pmatrix} X_{n+1} \\ X_n \end{pmatrix} \quad \text{and} \quad \mathcal{A}_n(Z_n) = \begin{pmatrix} \mathcal{H}_{n+1}(X_{n+1}) \\ X_{n+1} \end{pmatrix}.$$

It is then easy to check that for all n, $\mathcal{A}_n$ is also topical and that the mapping

$$F(Z_n) = ||X_{n+1}||_\infty - ||X_n||_\infty = f \circ \mathcal{H}_n \circ G(Z_n) - f \circ G(Z_n), \tag{10}$$

is 1–Lipschitz w.r.t. $\mathcal{D}_G$ (for this last property, we apply Lemma 2; G denotes the projection of $\mathbb{R}^{2d}$ on the last d coordinates and f denotes the supremum norm). So we can use Theorem 1 to show that the limit (when it exists) of $\mathbb{E}[||X_{n+1}||_\infty - ||X_n||_\infty]$, which is equal to γ_{top}, is analytic in ν in a certain domain.

2.3.3 Reducible system

Consider a dynamical system in $\mathbb{R}^d$ defined through the recurrence relation

$$Z_{n+1} = \mathcal{A}_n(Z_n),$$

where $\{\mathcal{A}_n\}$ is a sequence of topical maps from $\mathbb{R}^d$ to $\mathbb{R}^d$. This dynamical system is reducible if there exists a partition of the set of coordinates $\{1,\ldots,d\}$ into two non-empty subsets S, of cardinality k and R of cardinality $d-k$, such that $X_n = (Z_n^i, \in R)$ and $Y_n = (Z_n^i, i \in S)$ satisfy the relation

$$\begin{aligned} Y_{n+1} &= g_n(Y_n) \\ X_{n+1} &= h_n(Z_n), \end{aligned}$$

where g_n is a topical map from $\mathbb{R}^k$ to $\mathbb{R}^k$ and h_n a topical map from $\mathbb{R}^d$ to $\mathbb{R}^{d-k}$. A topical map for which no such decomposition exists is said to be irreducible.

Remark 4 *In the reducible case, contraction w.r.t. the projective norm is closely related to certain contraction properties w.r.t. the supremum norm (see §3.3.2, §3.3.3 and §4.2 below).*

This connection is based on the following observation, which concerns a reducible system where $k=1$. It is then easy to check that $g_n : \mathbb{R} \to \mathbb{R}$ is necessarily of the form

$$g_n(y) = y + a_n,$$

where a_n is some real number, and then that for all n,

$$Y_{n+1} - Y_n = (Y_n - Y_{n-1}) \circ \theta = a_n.$$

Let W_n denote the vector with coordinates $X_n^i - Y_n$, $i \in R$. If the inequality

$$W_1 \geq W_0 \circ \theta$$

holds, then an immediate induction shows that for all n and for all $i \in R$,

$$X_n^i - X_{n-1}^i \circ \theta \geq Y_n - Y_{n-1} \circ \theta,$$

so that (see Remark 2), for all n,

$$\mathcal{D}(Z_n, Z_{n-1} \circ \theta) = ||W_n - W_{n-1} \circ \theta||_\infty.$$

Therefore in this special case, the key property used for proving Theorem 1 is actually the non-expansiveness of W_n w.r.t. the supremum norm.

3 Applications to linear systems

3.1 Top Lyapunov exponent in the conventional algebra

In this section, we give a brief account of the analyticity properties of the top Lyapunov exponent in the conventional algebra established by Y. Peres [19], together with a slightly simplified proof of this result, based on Theorem 1.

In this section, we denote $\ln X_n$ the vector with its i-th coordinate equal to $\ln(X_n)_i$ when $X_n > 0$. In the same way, e^{X_n} will denote the vector with i-th coordinate equal to $e^{(X_n)_i}$.

The reference algebra is the conventional algebra, $\{A_n\}$ is a sequence of random matrices in $\mathbb{R}_+^{d\times d}$ with non-negative entries and $\{Y_n\}$ a sequence of random vectors in $\mathbb{R}_+^d$, also with non-negative entries, satisfying the recurrence relation

$$Y_{n+1} \quad = \quad A_n Y_n, \tag{11}$$

By definition, the top Lyapunov exponent is the following limit, when it exists:

$$\gamma_{\text{top}} = \lim_{n\to+\infty} \frac{\mathbb{E}[||\ln Y_n||_\infty]}{n}.$$

Consider the case when the random matrices are sampled independently, within a finite set, i.e. :

- $\{A_n\}$ is an i.i.d. sequence of random matrices in $\mathbb{R}_+^{d\times d}$;
- A_n can take a finite number of different values, say:

$$A_n = \begin{cases} A_0' & \text{with probability } p_0 \\ \vdots & \qquad \vdots \\ A_m' & \text{with probability } p_m. \end{cases}$$

One of the main results of [19] can then be stated as follows:

Result 1 (Y. Peres) *Assume $\mathcal{L} = \{A_0', \cdots, A_m'\}$ is a set of non-negative matrices such that each row has at least one strictly positive entry. Then the top Lyapunov exponent exists and is analytic in $P = (p_0, \ldots, p_m)$ in the domain $\Delta(\mathcal{L})$ defined by:*

$$\Delta(\mathcal{L}) = \left\{ P \in \mathbb{C}^{m+1} \text{ s.t. } \sum_{j=0}^{m} \tau^B(A_j')|p_j| < 1 \right\}, \tag{12}$$

where $\tau^B(A)$ is Birkhoff's contraction coefficient of A.

Let $\mathcal{D}^H$ be Hilbert's projective metric which is defined by

$$\mathcal{D}^H(X,Y) = \ln \max_{1\le i,j\le d} \frac{x_i y_j}{x_j y_i}.$$

Birkhoff's contraction coefficient of a real matrix A is defined by:

$$\begin{aligned} \tau^B(A) &= \sup\left\{ \frac{\mathcal{D}^H(AX,AY)}{\mathcal{D}^H(X,Y)},\ \mathcal{D}^H(X,Y) \neq 0, X>0, Y>0 \right\} \\ &= \frac{1-\Psi(A)^{\frac{1}{2}}}{1+\Psi(A)^{\frac{1}{2}}}, \end{aligned}$$

with

$$\Psi(A) = \min_{i,j,k,l} \frac{A_{ik}A_{jl}}{A_{il}A_{jk}}.$$

Proof of Result 1: The proof is based on the augmented system (cf. §2.3.2) associated with the sequence of topical operators $\mathcal{H}_n : \mathbb{R}^d \to \mathbb{R}^d$ defined by

$$X \to \mathcal{H}_n(X) = \ln A_n e^X.$$

Let us now study the convergence of the series of Theorem 1 for this augmented system associated to the sequence $\{X_n = \ln Y_n\}$. The operator $\mathcal{A}_n$ has a uniform contraction coefficient w.r.t. G given by

$$\tau_G(\mathcal{A}_n) \quad = \quad \tau^B(A_n),$$

(because the supremum defining τ_G is computed on vectors of the form $Z = (A_nX, X)$. and $Z' = (A_nX', X')$). This together with the independence assumption imply that

$$\begin{aligned}
& ||\mathbb{E}^*||\left[\tau_G(\mathcal{A}_n, Z_n, Z_{n-1}\circ\theta)\cdots\tau_G(\mathcal{A}_1, Z_1, Z_0\circ\theta)\mathcal{D}_G(Z_1, Z_0\circ\theta)\right] \\
\leq\ & ||\mathbb{E}^*||\left[\tau^B(A_n)\cdots\tau^B(A_1)\mathcal{D}(A_0X_0, X_0)\right] \\
=\ & ||\mathbb{E}^*||\left[\mathcal{D}(A_0X_0, X_0)\right]\Big(||\mathbb{E}^*||[\tau^B(A_0)]\Big)^n.
\end{aligned}$$

Therefore, if

$$||\mathbb{E}^*||[\tau^B(A_0)] = \sum_{i=0}^{d} |p_i|\tau^B(A_i^{'})$$

is smaller than 1, then Theorem 1 together with a Cesaro average argument show that the top Lyapunov exponent,

$$\gamma = \gamma(P) = \lim_{n\to\infty} \mathbb{E}^* F(Z_n),$$

with F defined as in (10), is analytic in P on $\Delta(\mathcal{L})$. ♠

Remark 5 *Note that if we restrict our domain to the set of positive and real vectors P such that $\sum_{j=1}^m p_j = 1$, then the uniform convergence is guaranteed as soon as one of the matrices in $\mathcal{L}$ has its contraction coefficient smaller than one.*

Remark 6 *In case the law of $\tau^B(A_0)$ is a general probability measure on $\mathbb{R}_+$, say $\mathcal{P}$, we then obtain the very same conclusion whenever $\mathcal{P}(\{1\}) < 1$.*

3.2 Top Lyapunov exponent in the $(\max, +)$ semi-field

We now consider a linear dynamical system but this time in the $(\max, +)$–semi-field ([1]), i.e. $\{X_n\}$ is the sequence of random vectors in $\mathbb{R}^d_{\max}$ defined by some independent and integrable initial condition X_0 and by the linear recurrence equation:

$$X_{n+1} = A_n \otimes X_n,$$

where $\{A_n\}$ is an i.i.d. sequence of matrices in $\mathbb{R}^{d\times d}_{\max}$, taking their values in some finite set $\mathcal{L} = \left\{A_0^{'}, \ldots, A_m^{'}\right\}$, according to the probability vector $P = (p_0, \ldots, p_m)$. Under these assumptions, if there exists a finite constant C such that $\mathbb{E}||A_n \otimes \cdots \otimes A_1||_\infty \geq C.n$, for all n, then the top Lyapunov exponent

$$\gamma_{\text{top}} = \lim_{n\to\infty} \frac{\mathbb{E}||X_n||_\infty}{n}$$

exists (as shown by a direct sub-additivity argument [9], [1]).

Theorem 2 *Assume that each matrix A_i' has at least one entry different from ε on each row. Then the Lyapunov exponent $\gamma_{top} = \gamma(P)$ is an analytic function of P on the domain*

$$\Delta(\mathcal{L}) = \left\{ P \in \mathbb{C}^{m+1} \text{ s.t. } \sum_{j=0}^{m} \tau_{\max}(A_j')|p_j| < 1 \right\}, \tag{13}$$

where $\tau_{\max}(A)$ denotes the uniform contraction coefficient of $A \in \mathbb{R}_{\max}^{d,d}$ w.r.t. $\mathcal{D}$.

Proof
The proof is again based on the augmented system associated with the sequence of topical maps

$$\mathcal{H}_n(X) = A_n \otimes X.$$

Defining F as in (10), we get from Theorem 1 that

$$|\mathbb{E}^* [F(Z_{n+1}) - F(Z_n) \circ \theta]| \le ||\mathbb{E}^*|| [\mathcal{D}(A_0 \otimes X_0, X_0)] ||\mathbb{E}^*|| [\tau_{\max}(A_0)]^n. \tag{14}$$

♠

Consider now the following set of assumptions:

(H1) : *The matrix $A = A_0'$ is irreducible,*

(H2) : *The matrix A is scs1–cyc1,* (a condition which ensures that A has a unique eigenvector – see [2])

(H3) : *Each matrix A_i' has at least one entry different from ε on each row,*

Under (H1)–(H3), γ_{top}, seen as a function of $Q = (p_1, \dots, p_m)$, admits a Taylor series expansion at point $(0, \dots, 0)$, the coefficients of which are given in [2]. It was also shown there that the radius of convergence of this expansion is at least $\frac{1}{2cm}$, where c is the first *coupling time* of A (see [2]). The domain of analyticity is in fact much larger as shown by Corollary 1 below.

Remark 7 *Under assumptions (H1)–(H3),*

$$\lim_{n \to \infty} \frac{\mathbb{E} X_n^i}{n} = \gamma$$

for all $i = 1, \dots, d$ (see [2]).

Corollary 1 *The radius of convergence of the analytic expansion of $\gamma_{top} = \gamma(Q)$ at point 0 is larger than or equal to $2^{\frac{1}{c}} - 1$ for the L_1 norm. In particular, if $c = 1$, $\gamma(Q)$ is analytic if $p_0 \neq 0$.*

Proof
First note that the domains of analyticity w.r.t. Q or P are the same in view of the linear relationship between them. If $c = 1$, $\tau_{\max}(A) = 0$ and the contraction region is

$$\sum_{j=1}^{m} \tau(A_j')|p_j| < 1,$$

and so, the region

$$\sum_{j=1}^{m} |p_j| < 1$$

is included in the contraction region. Therefore $\gamma(Q)$ is analytic w.r.t. Q if $p_0 \neq 0$.
If $c > 1$, consider the system $\widetilde{X}_n = X_{nc}$. The associated matrices are $\widetilde{A}'_0 = {A'_0}^c$ and $\widetilde{A}'_i$, $i = 1, \ldots, (m+1)^c - 1$, where $\{\widetilde{A}'_i\}$, $i \neq 0$ is the set of matrices which are the product of c matrices $\mathcal{L}$, with at least one of them different from A'_0. The same argument as above shows that the contraction region includes the domain

$$(\sum_{j=0}^{m} |p_j|)^c - |p_0|^c < 1.$$

This last relation is equivalent to

$$\sum_{j=1}^{m} |p_j| < e^{\frac{\ln(1+|p_0|^c)}{c}} - |p_0|.$$

This last function of p_0 is decreasing on $[0,1]$, so that it is minimal at 1, where it is equal to $2^{\frac{1}{c}} - 1$. ♠

Remark 8 *For the supremum norm, the radius of convergence of $\gamma(Q)$ is larger than or equal to $\frac{1}{m} \times \left(2^{\frac{1}{c}} - 1\right)$. This result is sharper than the one obtained by direct bounds on the coefficients of the analytic expansion, which leads to the conclusion that the radius of convergence of $\gamma(Q)$ is larger than or equal to $\frac{1}{2cm}$ for the supremum norm [2]. Since $2^{\frac{1}{c}} - 1$ is larger than $\frac{1}{2c}$, we obtain a larger domain by the contraction argument indeed.*

Remark 9 *It is interesting to remark that for all $A \in \mathbb{R}_{\max}^{d,d}$, the contraction coefficient $\tau_{\max}(A)$ is either 0 or 1. To see this, observe that for all vector X, if we denote X' the vector obtained from X by varying one of its component by a small value α, then if the rank of A is larger than one, one can make $\mathcal{D}(A \otimes X, A \otimes X')$ equal to α.*

3.3 Linear systems: increments

The linear systems considered in these sections will be first in the $(\max, +)$ setting, where the increments defined here receive a nice physical interpretation, particularly in the reducible case. Similar results also hold in the conventional algebra.

3.3.1 $(\max, +)$–Increments: irreducible case

The assumptions are those of §3.2. Here and below, we will call (i,j) increment of order n, where $1 \leq i, j \leq d$, the difference $X^j_{n+1} - X^i_n$. Assume that its asymptotic mean value exists (this can be proved under assumptions (H1)–(H3) as in [2] 6.1.1, 6.1.2):

$$\exists \lim_{n \to \infty} \mathbb{E}[X^j_{n+1} - X^i_n] = \phi^1_{i,j}(P).$$

Note that in view of Remark 7 $\phi^1_{i,i}(P) = \gamma(P)$ for all i, so that the functions $\phi^1_{i,j}(P)$ can be seen as generalizations of the Lyapunov exponent.

The analyticity of $\phi^1_{i,j}(P)$ can be addressed applying Theorem 1 to the augmented system and when selecting the following functions in Lemma 2:

- G_1 is the projection on the last d coordinates;
- $G_2 = G_1 \circ \pi$, where π is a permutation of the last d coordinates such that $\pi(i) = j$;
- f is the i-th coordinate mapping.

The analyticity of

$$\phi_{i,j}^{k}(P) = \lim_{n\to\infty} \mathbb{E}[X_{n+k}^{j} - X_{n}^{i}]$$

can be assessed in the same way.

3.3.2 A simple two dimensional reducible case: the $G/G/1$ queue

In this section and the next one, we investigate the case of reducible (max, +)–matrices, where pointwise contraction coefficients have to be used.

Let $\{a_n\}$ and $\{b_n\}$ be two independent sequences of non-negative random variables. Each sequence is assumed to be made of i.i.d. random variables. Consider the two–dimensional linear recurrence $Z_{n+1} = \mathcal{A}_n \otimes Z_n$ with

$$Z_n = \begin{pmatrix} x_n \\ y_n \end{pmatrix} \quad \text{and} \quad \mathcal{A}_n = \begin{pmatrix} a_n & b_n \\ \varepsilon & b_n \end{pmatrix}, \tag{15}$$

that is

$$\begin{cases} x_{n+1} &= a_n \otimes x_n \oplus b_n \otimes y_n, \\ y_{n+1} &= b_n \otimes y_n. \end{cases}$$

The function $F(Z_n) = W_n = x_n - y_n$ is 1–Lipschitz w.r.t. $\mathcal{D}$ by a direct application of Lemma 2, with $G_1(Z_n) = x_n$, $G_2(Z_n) = y_n$, $\mathcal{H}(Z) = Z$ and $f = Id$.
Here the contraction properties will be verified w.r.t. the supremum norm (see Remark 4).

Remark 10 *This recurrence receives an interpretation in queueing theory: if customer n arrives at time y_n to a single server queue and has a service time a_n, then the time when it begins its service is precisely x_n, so that W_n is the time it has to wait to get served. Here b_n represents the inter-arrival time between customers n and $n+1$.*

Theorem 3 *Let $\mathcal{P}_\lambda$ and $\mathcal{P}_\mu$, be the laws of a_0 and b_0, which are assumed to depend on some real valued parameters λ and μ respectively. If these distributions on $\mathbb{R}^+$ admit exponential moments, then*

$$\exists \lim_{n\to\infty} \mathbb{E}[W_n] = \phi_{2,1}^{1}(\lambda, \mu)$$

and the last function is analytic in (λ, μ) in the open domain

$$\{(\lambda, \mu) : \exists \zeta \ s.t. \ |\Lambda_\lambda|(-\zeta)|\Lambda_\mu|(\zeta) < 1\}, \tag{16}$$

where

$$|\Lambda_\mu|(\zeta) = \int e^{\zeta x} |P_\mu(dx)| \quad and \quad |\Lambda_\lambda|(-\zeta) = \int e^{-\zeta x} |P_\lambda(dx)|.$$

Proof
We first prove that if $x_0 = y_0$, then for all n, the pointwise contraction coefficient of $\mathcal{A}_n$ w.r.t. $\mathcal{D}$ is such that

$$\tau(\mathcal{A}_n, Z_n, Z_{n-1} \circ \theta) \le \mathbb{1}_{b_n < a_n + W_n}. \tag{17}$$

Under the above assumption, $W_0 = 0$ and $\{W_n\}$ satisfies the recurrence equation:

$$W_{n+1} \quad = \quad (W_n \otimes a_n - b_n) \oplus 0. \tag{18}$$

We show by induction that

$$\forall n \geq 0,\ W_{n+1} \geq W_n \circ \theta \tag{19}$$

and that for $n > 0$,

$$W_{n+1} - W_n \circ \theta \quad = \quad \tau(\mathcal{A}_n, Z_n, Z_{n-1} \circ \theta) \left\{ W_n - W_{n-1} \circ \theta \right\}$$

and (17) is immediate from (18) and (19). So,

$$\begin{aligned} W_{n+1} - W_n \circ \theta &\leq \mathbb{1}_{b_n < a_n + W_n} \left\{ W_n - W_{n-1} \circ \theta \right\} \\ &\vdots \\ &\leq \left(\mathbb{1}_{b_n < a_n + W_n} \cdots \mathbb{1}_{b_1 < a_1 + W_1} \mathbb{1}_{b_0 < a_0} \right) a_0 \\ &= \left(\mathbb{1}_{W_{n+1} > 0} \cdots \mathbb{1}_{W_2 > 0} \mathbb{1}_{W_1 > 0} \right) a_0 \\ &= \mathbb{1}_{\Upsilon > n+1} a_0, \end{aligned} \tag{20}$$

where

$$\Upsilon = \inf \left\{ n > 0 \text{ s.t } W_n = 0 \right\}.$$

But for all $n < \Upsilon$,

$$W_n = \sum_{i=0}^{n-1} a_i - b_i,$$

so that

$$\mathbb{1}_{\Upsilon > n+1} \leq \mathbb{1}_{\sum_{i=0}^{n} b_i \leq \sum_{i=0}^{n} a_i}. \tag{21}$$

It then follows from the Cauchy-Schwartz inequality that

$$\mathbb{E}\left[W_{n+1} - W_n \circ \theta \right] \leq \mathbb{P}\left(\sum_{i=0}^{n} b_i \leq \sum_{i=0}^{n} a_i \right) \sqrt{\mathbb{E}\left[a_0^2 \right]}.$$

By the exponential inequality:

$$\begin{aligned} \mathbb{P}\left(\sum_{i=0}^{n} b_i \leq \sum_{i=0}^{n} a_i \right) &\leq \mathbb{E}\left[e^{-\zeta(\sum_{i=0}^{n} b_i - a_i)} \right] \\ &\leq \mathbb{E}\left[e^{\zeta(a_0 - b_0)} \right]^{n+1},\ \forall \zeta \geq 0. \end{aligned} \tag{22}$$

We can then use the ideas of Theorem 1 with

$$\mathcal{P}_\nu(dt_n ds_n) = \mathcal{P}_\lambda(dt_n)\mathcal{P}_\mu(ds_n),$$

where $\nu = (\lambda, \mu)$. Let $\lambda = \lambda_1 + i\lambda_2 = |\lambda| e^{i\phi}$ with $\lambda_1 = \Re(\lambda)$ and $\lambda_2 = \Im(\lambda)$.

Then,

$$
\begin{aligned}
&|\mathbb{E}^* [W_{n+1} - W_n]| \\
&= |\mathbb{E}^* [W_{n+1} - W_n \circ \theta]| \\
&\leq ||\mathbb{E}^*|| \left(W_{n+1} - W_n \circ \theta\right) \\
&\leq ||\mathbb{E}^*|| \left(1\!\!1_{\sum_{i=0}^n b_i \leq \sum_{i=0}^n a_i} a_0\right) \\
&\leq \sqrt{\mathbb{E}[a_0^2]} \int_{(\mathbb{R}^+)^{n+1}} \int_{(\mathbb{R}^+)^{n+1}} e^{-\zeta(\sum_{i=0}^n t_i)} e^{\zeta(\sum_{i=0}^n s_i)} \\
&\qquad |\mathcal{P}_\lambda(dt_0) \cdots \mathcal{P}_\lambda(dt_n)||\mathcal{P}_\mu(ds_0) \cdots \mathcal{P}_\mu(ds_n)| \\
&= \sqrt{\mathbb{E}[a_0^2]} \int_{(\mathbb{R}^+)^{n+1}} e^{-\zeta(\sum_{i=0}^n t_i)} |\mathcal{P}_\lambda(dt_0) \cdots \mathcal{P}_\lambda(dt_n)| \\
&\qquad \int_{(\mathbb{R}^+)^{n+1}} e^{\zeta(\sum_{i=0}^n s_i)} |\mathcal{P}_\mu(ds_0) \cdots \mathcal{P}_\mu(ds_n)|,
\end{aligned}
$$

so that

$$
|\mathbb{E}^* [W_{n+1} - W_n]| \leq \sqrt{\mathbb{E}[a_0^2]} \left(\left\{ \int_{(\mathbb{R}^+)} e^{-\zeta t} |\mathcal{P}_\lambda(dt)| \right\} \left\{ \int_{(\mathbb{R}^+)} e^{\zeta s} |\mathcal{P}_\mu(ds)| \right\} \right)^{n+1}. \tag{23}
$$

We conclude from this that $\mathbb{E}W$ is analytic in (λ, μ) in the region where (16) is satisfied. ♠

Let us now look at a few applications of this theorem, where we mix both well known and new results.

Relaxation times of the M/M/1 queue Consider the case when the two sequences $\{a_n\}$ and $\{b_n\}$ are independent and both a_0 and b_0 follow an exponential distribution with respective parameters μ and λ, which corresponds to the M/M/1 queue. Then

$$
\mathbb{E}\left[e^{\zeta(a_0 - b_0)}\right] = \frac{\lambda}{\lambda + \zeta} \frac{\mu}{\mu - \zeta}.
$$

This last function has its minimum at $]-\lambda, \mu[$ at $\frac{\mu - \lambda}{2}$, and therefore

$$
\mathbb{E}[W_{n+1} - W_n] \leq \sqrt{\mathbb{E}\left[a_0^2\right]} \left(\frac{4\lambda\mu}{(\lambda + \mu)^2} \right)^{n+1}. \tag{24}
$$

This implies that the convergence of the expectation of the workload sequence to its stationary regime is geometrically bounded and uniform on every compact of the domain of stability, a result which is well known (this constant is the so called relaxation time of the M/M/1 queue – see e.g. [23]).

Analyticity of the waiting times of the M/GI/1 queue in the intensity of the input process The random variables $\{b_n\}$ are still assumed to be i.i.d and exponentially distributed, whereas the sequence $\{a_n\}$ is now assumed to be i.i.d and to have a general law $\mathcal{P}$, with exponential moments. This corresponds to the $M/GI/1$ queue. Let us assume for a moment that we ignore the Pollaczek-Khinchine formula (see e.g. [23]) and that we wish to address the issue of analyticity of the mean waiting times in this queue using our approach. In this case, we have:

$$|\mathbb{E}^* [W_{n+1} - W_n]| \quad \leq \quad \sqrt{\mathbb{E}[a_0^2]} \left(\frac{|\lambda|}{\zeta + \Re(\lambda)} \Lambda_a(\zeta) \right)^{n+1}, \tag{25}$$

where

$$\Lambda_a(\zeta) = \mathbb{E}\left[e^{\zeta a_0}\right],$$

and where $|\lambda|$ and $\Re(\lambda)$ respectively denote the modulus and the real part of λ.

The stability condition for this queue is $\lambda \mathbb{E}[a_0] < 1$, where λ denotes the intensity of the input Poisson process. Using the above bound, it is easy to check that under this condition, the series $\mathbb{E}[W_{n+1} - W_n]$ converges geometrically towards the expectation of the stationary limit $\mathbb{E}W$.
We can also address the analyticity of $\mathbb{E}W$ as above. It is easy to see that when λ tends to zero, there exists $\zeta > 0$ for which

$$\left(\frac{|\lambda|}{\zeta + \Re(\lambda)} \Lambda_a(\zeta) \right) \quad < \quad 1. \tag{26}$$

This is the usual light traffic analyticity result (see e.g. [20]).

To get the analyticity region more precisely, let us first find ζ which minimizes

$$h(\lambda_1, \zeta) = \ln \left(\frac{\lambda_1}{\zeta + \lambda_1} \Lambda_a(\zeta) \right),$$

where λ_1 is a positive real number which satisfies the stability condition. Since this function is convex w.r.t. ζ, and with negative derivative at zero, there is a unique minimum which is the zero of its derivative w.r.t. ζ. In the interior of the definition domain of $\Lambda_a(\zeta)$, we have:

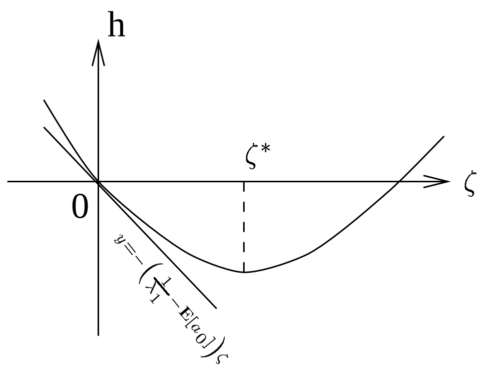

Figure 2: The function $h(\lambda_1, \zeta)$.

$$\frac{\partial h(\lambda_1, \zeta)}{\partial \zeta} = e^{-h(\lambda_1, \zeta)} \left(\frac{\lambda_1}{\zeta + \lambda_1} \right) \left(\mathbb{E}\left[a_0 e^{\zeta a_0}\right] - \frac{1}{\zeta + \lambda_1} \Lambda_a(\zeta) \right)$$

which is zero at $\zeta^*(\lambda_1)$ such that:

$$h(\lambda_1,\zeta^*) = \ln\left(\lambda_1 \mathbb{E}\left[a_0 e^{\zeta^* a_0}\right]\right) < 0.$$

So we have contraction in the region:

$$h(\Re(\lambda),\zeta^*(\Re(\lambda))) < \ln\left(\frac{\Re(\lambda)}{|\lambda|}\right). \tag{27}$$

Let us give qualitative properties of this region. We can rewrite (26) as follows:

$$\mathbb{E}\left[e^{\zeta a_0}\right] \quad < \quad \frac{\zeta+\Re(\lambda)}{|\lambda|} = \frac{\zeta}{|\lambda|} + \cos(\phi). \tag{28}$$

For a given ϕ, let $\rho(\phi)$ be the supremum value of $|\lambda|$ such that (28) holds true for some $\zeta \geq 0$.

Since $\mathbb{E}\left[e^{\zeta a_0}\right]$ is a convex function, it is easy to see that $\rho(\phi)$ is an even function on $]-\pi,\pi]$, where it decreases when ϕ varies from 0 to π (see Figure 3; $\frac{1}{\rho(\phi)}$ is the slope of the tangent to $\mathbb{E}\left[e^{\zeta a_0}\right]$ passing through point $(0,\cos(\phi))$). So for all ϕ, $\rho(\phi) > \rho(\pi)$, which implies that

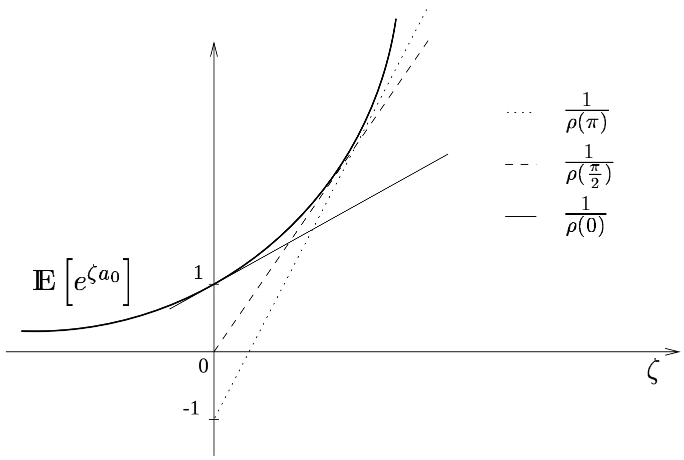

Figure 3: The function $\rho(\phi)$.

the figure we obtain below (cf. Figure 4) shows a typical shape for the region defined by (27) and that the first violation of Condition (28) is located on the negative half-line $\mathbb{R}^-$.

The following proposition, proved in the appendix, indicates how the contraction region depends on the variance of a_0.

Proposition 1 *The assumptions are those of the $GI/GI/1$ queue in Theorem (3). Let*

$$\rho(\phi) = \sup\left\{|\lambda| \mid \ \exists\zeta \geq 0 \ s.t. \ ||\mathbb{E}^*|| \left\{e^{-\zeta t_0}\right\} \mathbb{E}\left\{e^{\zeta a_0}\right\} < 1\right\}. \tag{29}$$

Then $\inf_{x\in]-\pi,\pi]} \rho(\phi)$ is a decreasing function of the variance of a_0, when this variance is small.
For the $M/GI/1$ queue we have:

$$\frac{\overline{\rho}_0(\pi)}{\overline{\rho}(\pi)} \ \leq \ 1 + \frac{|\ln \overline{\rho}_0|}{\mathbb{E}[a_0]^2}\left(1+\frac{|\ln\overline{\rho}_0|}{2}\right) Var(a_0)$$

where $\overline{\rho}(\pi) = \rho(\pi)\mathbb{E}[a_0]$ *and* $\overline{\rho}_0(\pi)$ *is the function* $\overline{\rho}(\pi)$ *in the special case* $Var(a_0) = 0$.

Analyticity of the waiting times in queues with alternating service The following example was studied in [15] where it was first shown that the mean stationary workload is not always analytic in its region of stability (see [14], [15]).
The $M/D_2/1$ queue is characterized by Poisson arrival of parameter λ and two service times a and b which alternate periodically from a_0 which equal to a or b with probability $\frac{1}{2}$:

$$\text{Conditionally to} \quad \begin{aligned} \sigma_0 = a: \ \sigma_1 = b, \sigma_2 = a, \sigma_3 = b \text{ etc.} \\ \sigma_0 = b: \ \sigma_1 = a, \sigma_2 = b, \sigma_3 = a \text{ etc.} \end{aligned}$$

This case can be approached by the above considerations. Let $\sigma = \frac{a+b}{2}$. Then, we obtain directly from (22) that $|\mathbb{E}^* [W_{n+1} - W_n]|$ is bounded from above by $\sigma\left(\frac{|\lambda|}{\zeta+\lambda_1} e^{\zeta\sigma}\right)^{n+1}$, which

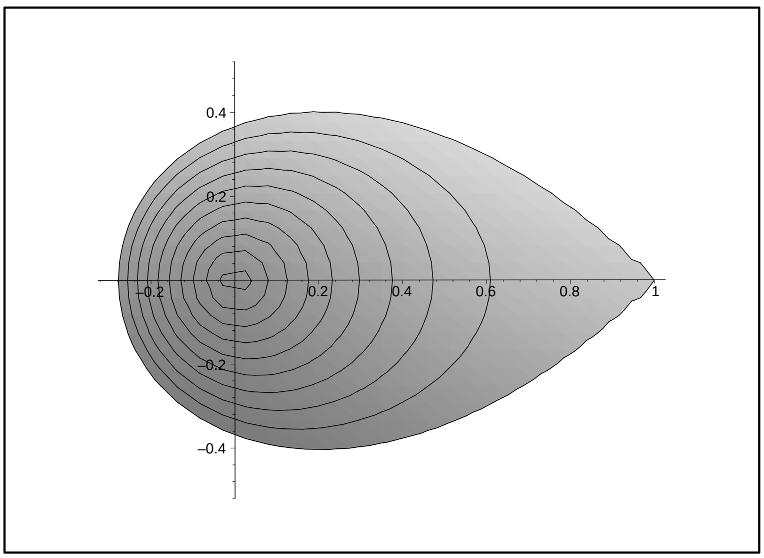

Figure 4: Contraction region of $\mathbb{E}W$: $x = \Re(\lambda).\sigma$, $y = \Im(\lambda).\sigma$.

is minimal at $\zeta = \frac{1}{\sigma} - \lambda_1$, so that:

$$|\mathbb{E}^* [W_{n+1} - W_n]| \leq \sigma\left(\sigma|\lambda|e^{1-\sigma\lambda_1}\right)^n.$$

Thus, $\mathbb{E}W$ is analytic in the region where $\sigma|\lambda|e^{1-\sigma\lambda_1} < 1$. The singularity with minimal modulus is the negative real point $\sigma\lambda = -0.2784645428$. We find back the singularity obtained by a study of Lambert's function in [15].
In this case, the region of convergence given by Equation (27) is exactly the radius of convergence of the Taylor expansion of $\mathbb{E}W$ in λ at point 0.

Remark 11 *Here Equation (27) gives an exact radius of convergence. For the* $M/M/1$ *queue, we find a first violation of the contraction condition at* -0.1715728752, *whereas the radius of convergence is in fact 1, as we know from the Pollaczek-Khinchine formula. This is explained by the roughness of the bound (21). This bound is better when the service times have a small variance as seen in Proposition 1.*

3.3.3 Higher dimensional reducible $(\max,+)$-linear systems

Let $\{A_n\}$ be a sequence of i.i.d. random matrices in $\mathbb{R}_{\max}^{d,d}$, for instance generated as in §3.2. Let $\{B_n\}$ and $\{b_n\}$ be two sequences of i.i.d. matrices, independent of $\{A_n\}$, with $B_n \in \mathbb{R}_+{}^{d,1}$ and $b_n \in \mathbb{R}_+$. Consider the following linear recurrence relation in $\mathbb{R}_{\max}^{d+1}$, which generalizes that considered in §3.3.2:

$$Z_{n+1} = \mathcal{A}_n \otimes Z_n, \quad \text{with} \quad Z_n = \begin{pmatrix} X_n \\ y_n \end{pmatrix} \quad \text{and} \quad \mathcal{A}_n = \begin{pmatrix} A_n & B_n \\ \mathcal{E} & b_n \end{pmatrix},$$

where $\mathcal{E}$ here denotes the matrix of $\mathbb{R}_{\max}^{1,d}$ with all its entries equal to $-\infty$, and where $X_n \in \mathbb{R}_{\max}^{d}$ and $y_n \in \mathbb{R}_+$.

Such systems were studied in [1], and more recently in [3], where it was shown that if the random variables b_n are exponentially distributed with parameter λ, then, under certain stability and integrability conditions, the random variables

$$W_n^i = X_n^i - y_n, \quad i = 1, \ldots, d,$$

are such that when n tends to ∞, $\mathbb{E}W_n^i$ converges to a function of λ which is infinitely differentiable in λ.

We show below how the analyticity question can be approached here too using Theorem 1, or more precisely its translations into contraction properties w.r.t the supremum norm (see Remark 4).

We have

$$W_{n+1} = A_n \otimes C(b_n) \otimes W_n \oplus B_n \otimes C(b_n), \tag{30}$$

where $C(x)$ is the diagonal matrix of $\mathbb{R}_{\max}^{d\times d}$ with all its diagonal entries equal to $-x$.

First approach From (30), we easily verify by induction that if $W_0 = 0$ and $B_0 \otimes C(b_0)$ is a positive random variable, then

$$\forall n \geq 0, \; W_{n+1} \geq W_n \circ \theta$$

and

$$\begin{aligned}
&||W_{n+1} - W_n \circ \theta||_\infty \\
&\quad = \; ||A_n \otimes C(b_n) \otimes W_n \oplus B_n \otimes C(b_n) - A_n \otimes C(b_n) \otimes W_{n-1} \circ \theta \oplus B_n \otimes C(b_n)||_\infty \\
&\quad \leq \; ||A_n \otimes C(b_n) \otimes W_n \oplus 0 - A_n \otimes C(b_n) \otimes W_{n-1} \circ \theta \oplus 0||_\infty \\
&\quad \leq \; \mathbb{1}_{b_n < \alpha_n + ||W_n||_\infty} ||W_n - W_{n-1} \circ \theta||_\infty \\
&\quad \vdots \\
&\quad \leq \; \left(\mathbb{1}_{b_n < \alpha_n + ||W_n||_\infty} \cdots \mathbb{1}_{b_1 < \alpha_1 + ||W_1||_\infty} \mathbb{1}_{b_0 < \alpha_0}\right) \alpha_0,
\end{aligned}$$

where

$$\alpha_n = \max_{(A_n)_{ij} \neq \epsilon} \{|(A_n)_{ij}|\}.$$

We denote $\{\overline{Z}_n\}$ the two–dimensional sequence defined like in (15), but with associated sequence $\{\alpha_n, b_n\}$. It is immediate that

$$\forall n, \quad ||W_n||_\infty \leq \overline{W}_n.$$

In case $\mathbb{E}[\alpha_0] < \mathbb{E}[b_0]$, a condition which is stronger than the stability condition, we can then use Theorem 3 to evaluate the analyticity region.

A better bound Here we only assume that $\gamma < \mathbb{E}[b_0]$, where

$$\gamma = \lim_{n\to\infty} \frac{\mathbb{E}\left[||A_n \otimes \cdots \otimes A_1||_\infty\right]}{n},$$

is the (top) Lyapunov exponent of the sequence $\{A_n\}$. This condition gives the exact stability region for $\{W_n\}$, i.e. the region for which this sequence admits a finite limit, for instance in the sense of the weak convergence (see [3]).

By induction, $\forall K \geq 1$,

$$\begin{aligned} W_{K(n+1)} - W_{Kn} \circ \theta^K &= \widetilde{A}_n \otimes C(\widetilde{b}_n) \otimes W_{Kn} \oplus \widetilde{B}_n \otimes C(\widetilde{b}_n) \\ &\quad -\widetilde{A}_n \otimes C(\widetilde{b}_n) \otimes W_{K(n-1)} \circ \theta^K \oplus \widetilde{B}_n \otimes C(\widetilde{b}_n), \end{aligned}$$

with

$$\begin{cases} \widetilde{b}_n &= (t_{K(n+1)} - t_{Kn}) \\ \widetilde{A}_n &= A_{K(n+1)-1} \otimes \cdots \otimes A_{Kn} \\ \widetilde{B}_n &= B_{K(n+1)} \bigoplus_{i=1}^{K-1} A_{K(n+1)-1} \otimes C(b_{K(n+1)-1}) \otimes \cdots \\ & \qquad \cdots \otimes A_{K(n+1)-i} \otimes C(b_{K(n+1)-i}) \otimes B_{K(n+1)-i-1}. \end{cases}$$

For $\epsilon > 0$, let $K = K(\epsilon)$ be such that

$$\forall n \geq K(\epsilon), \left| \frac{\mathbb{E}||A_n \otimes \cdots \otimes A_1||_\infty}{n} - \gamma \right| \leq \epsilon.$$

Let

$$\widetilde{\alpha}_n = ||\widetilde{A}_n||_\infty.$$

Since $\mathbb{E}\widetilde{\alpha}_o < \mathbb{E}\widetilde{b}_0$ for $\epsilon < \mathbb{E}[b_0] - \gamma$, we can apply the results of the first approach and prove that

$$||\mathbb{E}^*[W_{K(n+1)} - W_{Kn}]||_\infty \leq D \left[\left(\int_{(\mathbb{R}^+)} e^{-\zeta t} |P(dt)| \right)^K \Lambda_{\widetilde{\alpha}_o}(\zeta) \right]^{n+1},$$

where $\mathcal{P}$ is the law of b_0 and where D is a constant. When the arrival is a homogeneous Poisson process then:

$$||\mathbb{E}^*[W_{K(n+1)} - W_{Kn}]||_\infty \leq D \left[\left(\frac{|\lambda|}{\zeta + \Re(\lambda)} \right)^K \Lambda_{\widetilde{\alpha}_o}(\zeta) \right]^{n+1},$$

and a sufficient condition for analyticity is

$$e^{\left(\frac{\ln \mathbf{E}[e^{\zeta \widetilde{\alpha}_o}]}{K}\right)} < \frac{\zeta}{|\lambda|} + \cos(\phi). \tag{31}$$

For $K = 1$ we find back Condition (28). This last equation gives a larger region of analyticity if the function

$$h_K(\zeta) = e^{\left(\ln \mathbf{E}[e^{\zeta \widetilde{\alpha}_o}]/K\right)}$$

is a decreasing function of K on $\mathbb{R}^+$ (this is not generally not true as it can be seen in the deterministic case). However we can verify that this function is decreasing w.r.t. n when we choose $K = 2^n$.

Example Consider a four dimensional stochastic network introduced in [2] §3.2.2. One of the possible states of this model of network with breakdowns is depicted by the Petri net of Figure 5, where the initial model of [2] is completed by a homogeneous Poisson arrival process.

The evolution of this system can be described by the recurrence relation

$$X_{n+1} = A(n) \otimes X_n \oplus B(n) \otimes T_n,$$

where X_n^i denote the n-th epochs when a token leaves transition i, $B(n) = (\tau(n), \tau(n), \tau(n), \tau(n))^t$ describes the Poisson arrival epoch, T_n is the epoch of n–th external arrival, $\tau(n) = T_{n+1} - T_n$ is the n-th inter-arrival time, and $A(n)$ is equal to A or A' with probability $1-p$ and p. (for more details, see [2]).
The corresponding topical system is

$$Z_{n+1} = \begin{pmatrix} X_{n+1} \\ T_{n+1} \end{pmatrix} = \begin{pmatrix} & & & & \tau(n) \\ & & A(n) & & \tau(n) \\ & & & & \tau(n) \\ & & & & \tau(n) \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \tau(n) \end{pmatrix} \otimes Z_n.$$

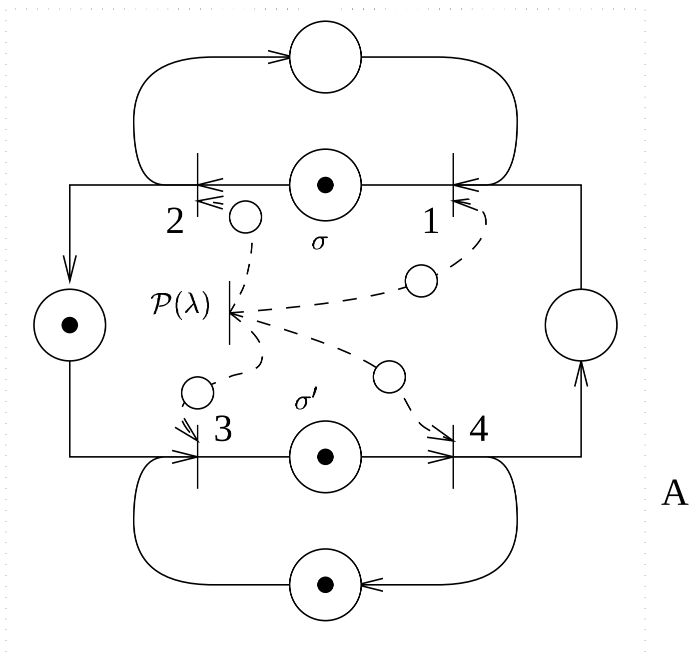

Figure 5: A network with 3 servers corresponding to matrix A with Poisson arrival.

We take $\sigma = 1$ and $\sigma' = 2$, so that

$$A = \begin{pmatrix} 1 & \varepsilon & 2 & \varepsilon \\ 1 & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & 0 & \varepsilon & 0 \\ \varepsilon & \varepsilon & 2 & \varepsilon \end{pmatrix},$$

$$A' = \begin{pmatrix} 1 & \varepsilon & 2 & \varepsilon \\ 1 & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & 0 & 2 & \varepsilon \\ \varepsilon & \varepsilon & 2 & \varepsilon \end{pmatrix},$$

and $c = 4$, $\gamma(0) = 1$ and

$$\gamma(p) = 1 + p - p^2 + 2p^3 - 3p^4 + 5p^5 - 8p^6 + \cdots$$

For $p = 0.1$, $\gamma(0.1) \sim 1.091742$, so that the stability region is given by

$$\lambda < \gamma(0.1)^{-1} \sim 0.915967.$$

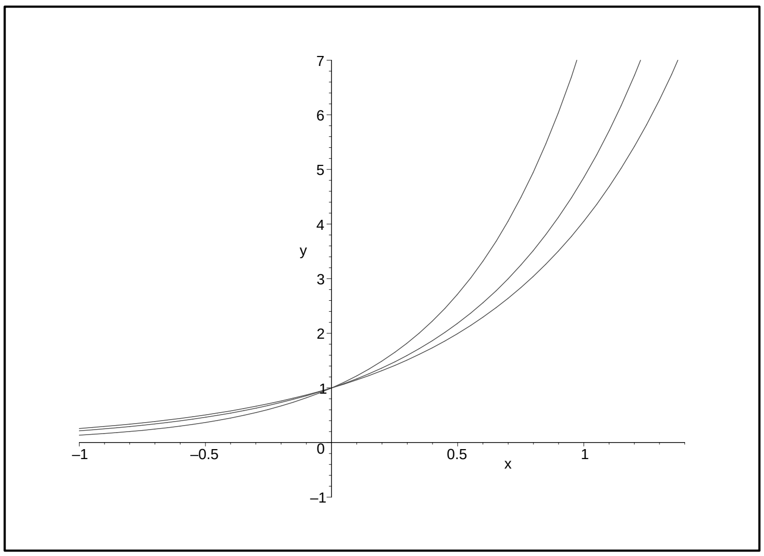

Figure 6: The function $h_K(\zeta) = e^{\left(\ln \mathbf{E}[e^{\zeta \widetilde{\alpha_0}}]/K\right)}$: $K = 1, 2, 3$.

$K = 1$ gives the bound of Condition (16). In this case, $\zeta^*(\pi) = \frac{1}{2}W(e^{-1}) + \frac{1}{2} = 0.6392322714$, and the first violation of the condition for contraction is at

$$\rho(\pi) = \frac{1}{2}W(e^{-1}) = 0.1392322713.$$

Here, W denotes the principal branch of Lambert's function, which is the solution of the functional equation

$$W(x)e^{W(x)} = x,$$

with x complex.

For $K = 2$, we get $\zeta^*(\pi) = 0.8023503752$, which implies that $\rho(\pi) = 0.1770828307$. For $K = 3$, $\zeta^*(\pi) = 0.8989503482$ so that $\rho(\pi) = 0.1995441968$.

3.3.4 An example of affine equation in the conventional algebra

Consider the recurrence relation in $\mathbb{R}_+^d$

$$X_{n+1} = P_n.X_n + Q_n,$$

where P_n are i.i.d. stochastic matrices and Q_n i.i.d. positive matrices.
Put

$$\mathcal{H}_n(X) = P_n.X + Q_n.$$

The contraction coefficient which is involved here is that of some Markov transition matrix P w.r.t. projective metric. This can be explicitly given by the formula:

$$\tau(P) = \max_{i,j} \sum_{k \in I_{ij}^+} (P_{ik} - P_{jk}),$$

where

$$I_{ij}^+ \quad = \quad \{1 \leq k \leq d \mid P_{ik} \geq P_{jk}\}.$$

To prove this, choose two rows i and j of P and introduce I_{ij}^+ to maximize the contraction rate.

4 Applications to non–linear systems

It is beyond the scope of the present paper to give a complete review the domains of applications of topical functions where Theorem 1 can be applied. The reader interested in a structured description of these domains of applications should read the comprehensive paper by S. Gaubert and J. Gunawardena [12]. We will limit ourselves to two non-linear classes of systems arising in stochastic control and in multi-server queues.

4.1 Stochastic control equation

Consider the equations which arise in stochastic control (cf. [21]):

$$V_{n+1}(i) = \max_a [R(i,a) + \sum_j P_{ij}(a) V_n(j)].$$

Here V_n is the maximal expected return of a gambler in state i and who can still play n times. If he chooses the action a, the return is $R(i,a)$ and his next state is j with probability $P_{ij}(a)$. If we assume that there are d states, this equation can be rewritten as

$$V_{n+1} = \max_a [R(a) + P(a).V_n],$$

where $V_n = (V_n(1), .., V_n(d))^t$, $R(a) = (R(1,a), .., R(d,a))^t$ and $P(a) = (P_{ij}(a))_{1 \leq i,j \leq d}$.
Let us now consider the case when the transition probability P and the return function R are random, say i.i.d., with respective values P_n and R_n at time n. Then the evolution of V_n becomes

$$V_{n+1} = \max_a [R_n(a) + P_n(a).V_n].$$

Put

$$\mathcal{H}_n(X) = \max_a [R_n(a) + P_n(a).X].$$

Then $\mathcal{H}_n$ is a topical function on which we can use Lemma 2 and Theorem 1 to derive analyticity properties of the asymptotic return rate γ_{top}.

4.2 The Kiefer and Wolfowitz representation of $GI/GI/s$ queues

The $G/G/s$ queue is characterized by $s \geq 1$ servers where the new arriving customer is assigned to the server with the smallest workload and is then served at unit speed until completion with FIFO policy. Such a system is usually described introducing the ordering function $\mathcal{R} : \mathbb{R}^s \to \mathbb{R}^s$ which arranges vectors of $\mathbb{R}^s$ in increasing order: $\mathcal{R}(x) = (x_{\sigma(1)}, \ldots, x_{\sigma(s)})$, where σ is a permutation of the coordinates such that

$$x_{\sigma(1)} \leq x_{\sigma(2)} \leq \cdots \leq x_{\sigma(s)}.$$

The evolution of such a queue can then be captured by the following equations, where $T_n \in \mathbb{R}^1$ and $X_n \in \mathbb{R}^s$:

$$\begin{aligned} T_{n+1} &= T_n + \tau_n \\ X_{n+1} &= \mathcal{R}(X_n + \sigma \mathbf{e}) \vee ((T_n + \tau_n)\mathbf{i}), \end{aligned}$$

where e and $\imath$ are the following vectors of $\mathbb{R}^s$: $\mathbf{e} = (1, 0, \ldots, 0)^t$ and $\mathbf{i} = (1, \ldots, 1)^t$.
The above equation can be rewritten as

$$(T_{n+1}, X_{n+1}) = \mathcal{H}_n(T_n, X_n),$$

where $\{\mathcal{H}_n\}$ is a sequence of topical maps: $\mathbb{R}^{s+1} \to \mathbb{R}^{s+1}$. This sequence is i.i.d. if the arrival process is a renewal process and if the service times are i.i.d., which will be assumed in what follows.

This dynamical system is reducible, and we can reduce the analysis of its contraction properties to the supremum norm contraction properties of the recurrence relation

$$W_{n+1} = \mathcal{R}\left\{(W_n + \sigma_n \mathbf{e} - \tau_n \mathbf{i})^+\right\} \tag{32}$$

Indeed, the non-negativity of $W_{n+1} - W_n \circ \theta$ still holds whenever $W_0 = 0$, and one can easily verify that if $X \geq Y$,

$$||\mathcal{R}(X) - \mathcal{R}(Y)||_\infty \leq ||\mathcal{R}(X - Y)||_\infty = ||(X - Y)||_\infty.$$

The method of §3.3.3 can then be applied. Let β denote the busy period, namely the first index n such that $W_n^s = 0$. It follows from the relations

$$\begin{aligned} ||W_{n+1} - W_n \circ \theta||_\infty &\leq \left(\mathbb{1}_{\tau_n < (\sigma_n + W_n^1) \vee W_n^s} \cdots \mathbb{1}_{\tau_1 < \sigma_1 \vee W_1^s} \mathbb{1}_{\tau_0 < \sigma_0}\right) \sigma_0 \\ &\leq \mathbb{1}_{\{\beta \geq t_{n+1}\}} && (33) \\ &\leq \mathbb{1}_{\{\sum_{i=1}^n \tau_i \leq \sum_{i=1}^n \sigma_i\}} \sigma_0 && (34) \end{aligned}$$

that some domain of analyticity can be derived using (23).

Remark 12 *The last upper bound is quite rough. One can actually get a better bound when using more general renovating event (see [7]) than the busy period β. Consider the event*

$$\Omega_n = \{W_n^1 = 0, W_n^2 \leq \tau_n, W_n^3 \leq \tau_n + \tau_{n+1}, \cdots, W_n^s \leq \tau_n + \cdots + \tau_{n+s-2}\}.$$

It is well known that Ω_n is a renovating event of order $s-1$, which implies that for $n \geq s-2$, $W_{n+1} - W_n \circ \theta$ is equal to zero on

$$\Omega_0 \cup \Omega_1 \cup \cdots \cup \Omega_{n-s+2}.$$

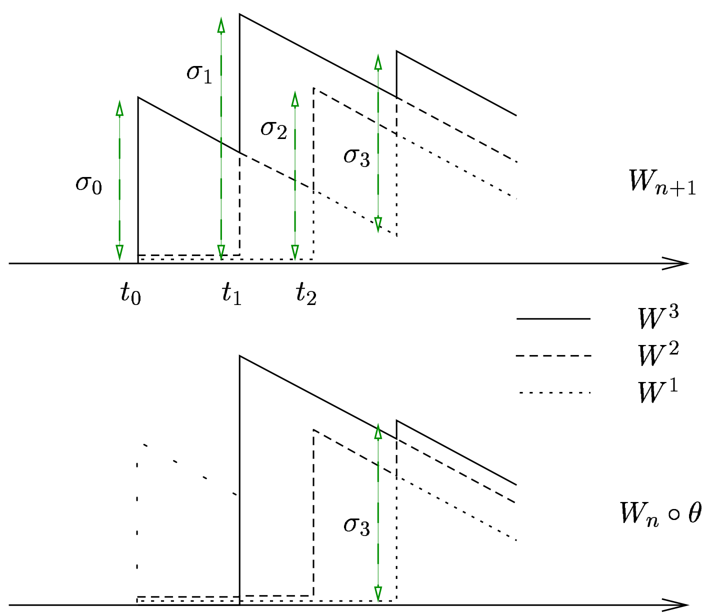

Figure 7: Non-negativity of $W_{n+1} - W_n \circ \theta$.

Therefore we have

$$||W_{n+1} - W_n \circ \theta||_\infty \quad \leq \quad (1 - \mathbb{1}_{\Omega_0 \cup \Omega_1 \cup \cdots \cup \Omega_{n-s+2}}) \, .$$

This upper bound could be used to derive a larger domain of analyticity of $\mathbb{E}W$.

5 Appendix

We give here a proof of Proposition (1). We have

$$\begin{aligned}
\mathbb{E}\left\{e^{\zeta a_0}\right\} &= e^{\zeta \mathbb{E}[a_0]}\mathbb{E}\left\{e^{\zeta(a_0 - \mathbb{E}[a_0])}\right\} \\
&= e^{\zeta \mathbb{E}[a_0]}\mathbb{E}\left\{1 + (a_0 - \mathbb{E}[a_0])\zeta + (a_0 - \mathbb{E}[a_0])^2\frac{\zeta^2}{2} + o((a_0 - \mathbb{E}[a_0])^2)\right\} \\
&= e^{\zeta \mathbb{E}[a_0]}\left\{1 + Var(a_0)\frac{\zeta^2}{2} + o(Var(a_0))\right\}
\end{aligned}$$

Since this function is increasing w.r.t. $Var(a_0)$, it follows that $\rho(\phi)$ is a decreasing function of $Var(a_0)$, at least when this quantity is small.
For the M/GI/1 queue, we put:

$$f(\zeta) = e^{\zeta\sigma}\left\{1 + Var(a_0)\frac{\zeta^2}{2}\right\},$$

with $\sigma = \mathbb{E}[a_0]$. We have

$$f'(\zeta) = e^{\zeta\sigma}\left\{\sigma\left(1 + Var(a_0)\frac{\zeta^2}{2}\right) + Var(a_0)\zeta\right\}.$$

Then

$$\overline{\rho}(\pi) \sim \frac{\sigma}{f'(\zeta^*)}$$

where ζ^* satisfies

$$f(\zeta^*) \quad = \quad f^{'}(\zeta^*)\zeta^* - 1. \tag{35}$$

So

$$\begin{aligned}\frac{1}{\overline{\rho}(\pi)} &\sim \frac{1+f(\zeta^*)}{\sigma\zeta^*} \\ &= \frac{1+e^{\zeta^*\sigma}\left(1+Var(a_0)\frac{\zeta^{*2}}{2}\right)}{\sigma\zeta^*}.\end{aligned}$$

and

$$\frac{e^{-\zeta^*\sigma}}{\overline{\rho}(\pi)} \sim \frac{e^{-\zeta^*\sigma}+1+Var(a_0)\frac{\zeta^{*2}}{2}}{\sigma\zeta^*}.$$

Now from (35) we deduce that

$$1+e^{-\zeta^*\sigma} = \sigma\zeta^* + (1+\sigma\zeta^*)Var(a_0)\frac{\zeta^{*2}}{2}.$$

Therefore

$$\begin{aligned}\frac{e^{-\zeta^*\sigma}}{\overline{\rho}(\pi)} &= 1+(1+\frac{\sigma\zeta^*}{2})\frac{\zeta^{*2}Var(a_0)}{\sigma\zeta^*} \\ &= 1+(1+\frac{\sigma\zeta^*}{2})\frac{(\zeta^*\sigma)}{\sigma^2}Var(a_0).\end{aligned}$$

Since $\overline{\rho}_0(\pi) = \frac{1}{\sigma e^{\sigma\zeta_0^*}}$ and $\zeta_0^* > \zeta^*$, the proposition is proved. ♠

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