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Homogenization of elliptic difference operators

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Abstract: We develop some aspects of general homogenization theory for second order elliptic difference operators and consider several models of homogenization problems for random discrete elliptic operators with rapidly oscillating coefficients. More precisely, we study the asymptotic behavior of effective coefficients for a family of random difference schemes whose coefficients can be obtained by the discretization of random high-contrast checkerboard structures. Then we compare, for various discretization methods, the effective coefficients obtained with the homogenized coefficients for corresponding differential operators.

Key-words: Random media, homogenization, G–convergence, difference operator, percolation, random walk

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Homogénéisation d'opérateurs elliptiques aux différences

Résumé : Nous étudions quelques aspects de la théorie de l'homogénéisation dans le cas d'opérateurs elliptiques aux différences (variables discrètes) à coefficients à variations rapides. Nous nous intéressons ensuite à des problèmes plus précis : les coefficients de l'opérateur correspondent à la discrétisation d'un milieu de type damier, à deux composants homogènes, fortement contrasté. Nous étudions le comportement asymptotique du coefficient effectif d'une marche aléatoire sur ce milieu.

Mots-clés : Milieu aléatoire, homogénéisation, G–convergence, opérateur aux différences, percolation, marche aléatoire

Contents

1 Introduction

We develop some aspects of general G–convergence and homogenization theory for second order elliptic difference operators and consider several homogenization problems for random discrete elliptic operators with rapidly oscillating coefficients. More precisely, we study the asymptotic behavior of effective coefficients for a family of random difference schemes whose coefficients can be obtained by the discretization of random high-contrast checker-board structures. Then we compare, for various discretization methods, the effective coefficients obtained with the homogenized coefficients for corresponding differential operators.

Many results can also be formulated in terms of the Central Limit Theorem for random walks in random statistically homogeneous media.

In the recent years, a significant progress has been achieved in the homogenization theory of random differential operators. We refer to Jikov et al [9] and to the bibliography of this report. In particular, in case of random high-contrast checker-board structures, the asymptotic of effective diffusion have been constructed by Jikov et al [9]. Berlyand and Golden [3] have improved this result in a special case.

In contrast with differential operators, the homogenization theory of difference operators is not so well-developed. There are only few mathematical works on this subject, among them Kozlov [11], [12], Künnemann [14], Krasnianski [13]. Künnemann [14] proved that the Central Limit Theorem holds for symmetric random walk in random ergodic statistically homogeneous media. Then, many interesting results for various kinds of random walks in random media were obtained in Kozlov [11]. First homogenization results were formulated and proved in Kozlov [12]. Perhaps, the difference operators with rapidly oscillating coefficients did not attract the attention of mathematicians because these operators did not appear in the classical difference schemes approximation approach: the fast oscillation of coefficients of difference schemes would contradict the regularity and even the measurability of coefficients of the initial differential equations.

Moreover, many modern practical and numerical applications involve various homogenization problems for discrete operators. For instance, when discretizing micro-inhomogeneous media, due to the natural restrictions, it is not possible to keep the size of the numerical grid much smaller than the typical size of inhomogeneity (the microscopic length scale) of the medium. This leads to the appearance of difference operators with rapidly oscillating coefficients. The most important question here is: how far could the effective coefficients of a difference scheme diverge from ones of corresponding differential operators ? The first successful attempt to answer this question was done in Avellaneda et al [2] where it was shown that, in multidimensional case, the finite difference approach does not provide the right homogenized coefficients unless the ratio of the size of a discretization mesh to the microscopic length scale goes to 0.

In the present work we show that the effective coefficients of the difference schemes depend essentially on the discretization method.

The paper is divided in two sections. In section 2, we expose the G- convergence and the homogenization of difference operators, including basic results like the convergence of energies and the convergence of arbitrary solutions. In section 3, we discretize high–contrast two–dimensional checkerboard structures and describe the asymptotic behavior of the effective diffusion. The main point is to prove that different discretizations methods lead to different asymptotics. Some technical tools are put in appendix.

1.1 Difference elliptic operators

Let $Q \subset \mathbb{R}^n$ be a smooth bounded domain and let $Q_\varepsilon = Q \cap \varepsilon\mathbb{Z}^n$, where $\mathbb{Z}^n$ is the standard integer lattice in $\mathbb{R}^n$ and $\varepsilon > 0$. We consider the discrete Dirichlet problem in Q_ε:

$$A_\varepsilon u^\varepsilon(x) = \sum_{z,z' \in \Lambda \setminus \{0\}} \partial^\varepsilon_{-z} \left(a^\varepsilon_{zz'}(x) \partial^\varepsilon_{z'} u^\varepsilon(x) \right) = f^\varepsilon(x) \qquad \text{in } Q_\varepsilon \,, \tag{1}$$

$$u^\varepsilon(x) = 0 \qquad \text{on } \partial Q^\Lambda_\varepsilon \,. \tag{2}$$

Here Λ is a fixed finite subset of $\mathbb{Z}^n$ symmetric with respect to 0, the matrix $\mathcal{A}^\varepsilon = \{a^\varepsilon_{zz'}\}$ is symmetric, $\partial Q^\Lambda_\varepsilon$ is the boundary of Q_ε defined by:

$$\partial Q^\Lambda_\varepsilon \triangleq (Q_\varepsilon + \varepsilon\Lambda) \setminus Q_\varepsilon = \{x + \varepsilon z \,|\, x \in Q_\varepsilon,\, z \in \Lambda\} \setminus Q_\varepsilon \,,$$

and ∂^ε_z is the standard difference operator:

$$(\partial^\varepsilon_z v)(x) \triangleq \frac{1}{\varepsilon} \left(v(x + \varepsilon\, z) - v(x) \right) \,.$$

For functions defined on Q_ε, we introduce the following norm (the $L^2(Q_\varepsilon)-$ norm):

$$\forall v^\varepsilon : Q_\varepsilon \mapsto \mathbb{R}\ ,\ \|v^\varepsilon\|^2_{L^2(Q_\varepsilon)} \stackrel{\triangle}{=} \varepsilon^n \sum_{x\in Q_\varepsilon} |v^\varepsilon(x)|^2 \,.$$

We say that a function v^ε defined on $\varepsilon\mathbb{Z}^n$, belongs to the space $W_0^{1,2}(Q_\varepsilon)$ if $v(x)=0$ for $x\notin Q_\varepsilon$. We define the norm on the space $W_0^{1,2}$ as follows:

$$\|v^\varepsilon\|^2_{W_0^{1,2}(Q_\varepsilon)} = \varepsilon^n \sum_{x\in\overline{Q_\varepsilon}} \sum_{i=1}^n |\partial^\varepsilon_{e_i} v^\varepsilon(x)|^2 \,,$$

where $\{e_i\}_{i=1,\dots,n}$ is the standard basis in $\mathbb{R}^n$, and

$$\overline{Q_\varepsilon} \stackrel{\triangle}{=} Q_\varepsilon + \varepsilon\Lambda = Q_\varepsilon \cup \partial Q_\varepsilon^\Lambda \,.$$

$W^{-1,2}(Q_\varepsilon)$ is the dual space to $W_0^{1,2}(Q_\varepsilon)$.

Definition 1.1 *We say that the family of problems (1)-(2) (or, simply, problem (1)-(2)) is* uniformly elliptic *if there are c_1, $c_2>0$ and $\varepsilon_0>0$ such that*

$$|a^\varepsilon_{zz'}(x)| \le c_1 \tag{3}$$

$$c_2\|v^\varepsilon\|^2_{W_0^{1,2}(Q_\varepsilon)} \le \sum_{x\in\overline{Q_\varepsilon}} \sum_{z,z'\in\Lambda\setminus\{0\}} a^\varepsilon_{zz'}(x)\partial_{z'}v^\varepsilon(x)\partial_z v^\varepsilon(x) \tag{4}$$

for any $v^\varepsilon\in W_0^{1,2}(Q_\varepsilon)$ and any $\varepsilon<\varepsilon_0$.

Remark 1.2 *The uniform boundness of the matrix $\mathcal{A}^\varepsilon$ implies the following upper bound:*

$$\sum_{x\in\overline{Q_\varepsilon}} \sum_{z,z'\in\Lambda\setminus\{0\}} a^\varepsilon_{zz'}(x)\partial_{z'}v^\varepsilon(x)\partial_z v^\varepsilon(x) \le\ c(\Lambda)\|v^\varepsilon\|^2_{W_0^{1,2}(Q_\varepsilon)} \,.$$

Indeed, it suffices to represent $\partial^\varepsilon_z v^\varepsilon(x)$ in the form $\sum\limits_{|z^k|=1} \partial^\varepsilon_{z^k} v^\varepsilon(x+z^1+\dots+z^{k-1})$ and take into account the finiteness of Λ.

It should be noted that, in general, the uniform ellipticity does not require the positiveness of the matrix $\{a^\varepsilon_{zz'}(x)\}$ and does not follow from the estimate

$$c_3|\xi|^2 \le \sum_{z,z'\in\Lambda\setminus\{0\}} a^\varepsilon_{zz'}(x)(\xi,z)(\xi,z') \le c_4|\xi|^2, \quad \xi\in\mathbb{R}^n,\ c_3>0\,,$$

where $(\cdot,\cdot)$ is the scalar product in $\mathbb{R}^n$.

Clearly, the uniform ellipticity implies the coerciveness of problem (1)-(2) and we have the following statement.

Proposition 1.3 *Let problem (1)-(2) be uniformly elliptic and $f^\varepsilon \in L^2(Q_\varepsilon)$. Then there exists a unique solution $u^\varepsilon \in W^{1,2}_0(Q_\varepsilon)$ and the estimate*

$$\|u^\varepsilon\|_{W^{1,2}_0(Q_\varepsilon)} \le c\|f^\varepsilon\|_{L^2(Q_\varepsilon)}$$

holds uniformly in ε.

Henceforth we usually suppose that $f^\varepsilon(\cdot)$ is a discretization of a given function $f\in L^2(Q)$.

Let us consider an important particular case of elliptic equations. Suppose we are given a family of functions $p^\varepsilon_z(x)$, $x\in Q_\varepsilon$, $z\in\Lambda$, possessing the following properties :

(i) positiveness: $p^\varepsilon_z(x)\ge 0,\ p_{\pm e_i}(x)\ge\delta>0,\ i=1,\dots,n\,,$

(ii) $\sum_{z\in\Lambda} p^\varepsilon_z(x)=1$ for each $x\in Q_\varepsilon$,

(iii) symmetry: $p^\varepsilon_z(x)=p^\varepsilon_{-z}(x+\varepsilon z)$.

Then, the family of problems

$$u^\varepsilon(x)=\sum_{z\in\Lambda} p^\varepsilon_z(x)\,u^\varepsilon(x+\varepsilon z)+\varepsilon^2 f^\varepsilon(x) \qquad \text{in } Q_\varepsilon\,, \tag{5}$$

$$u^\varepsilon(x)=0 \qquad \text{on } \partial Q^\Lambda_\varepsilon\,, \tag{6}$$

can be easily rewritten in the form (1)-(2) with

$$a^\varepsilon_{zz'}(x)=\begin{cases} p^\varepsilon_z(x) & \text{if } z=z',\ z\ne 0\\ 0 & \text{otherwise.}\end{cases} \tag{7}$$

Proposition 1.4 *Let $\{p^\varepsilon_z(x)\}$ possess the properties (i), (ii) and (iii) above. Then Problem (5)-(6) is uniformly elliptic.*

Proof Summing by parts, one can show after simple calculations that

$$\begin{aligned}\delta \sum_{x\in\overline{Q_\varepsilon}} \sum_{i=1}^{n} |\partial^\varepsilon_{e_i} v^\varepsilon|^2 &\le \sum_{x\in\overline{Q_\varepsilon}} \sum_{z,z'\in\Lambda\setminus\{0\}} a^\varepsilon_{zz'}(x)\partial_{z'}v^\varepsilon(x)\partial_z v^\varepsilon(x) \\ &\le C \sum_{x\in\overline{Q_\varepsilon}} \sum_{z\in\Lambda} |\partial^\varepsilon_z v^\varepsilon|^2 \le c(\lambda)\varepsilon^{-n}\|v^\varepsilon\|^2_{W^{1,2}_0(Q_\varepsilon)}\end{aligned}$$

uniformly in ε. Combining the latter estimate with the Friedrichs inequality (see Appendix C) we obtain the desired result. □

We also define the norm on the space $W^{1,2}(Q_\varepsilon)$ by:

$$\|v^\varepsilon\|^2_{W^{1,2}(Q_\varepsilon)} = \varepsilon^n \sum_{x\in\overline{Q_\varepsilon}} \sum_{i=1}^{n} |\bar\partial^\varepsilon_{e_i} v^\varepsilon(x)|^2 + \|v^\varepsilon\|^2_{L^2(Q^\varepsilon)},$$

where we use the notation:

$$\bar\partial^\varepsilon_z \varphi(x) = \begin{cases} \partial^\varepsilon_z\varphi(x) & \text{if } x+\varepsilon z \in \overline{Q_\varepsilon}, \\ 0 & \text{otherwise.}\end{cases}$$

1.2 Description of the random environment

In this section we introduce random difference elliptic operators with statistically homogeneous rapidly oscillating coefficients.

Let $(\Omega, \mathcal{F}, \mu)$ be a standard probability space, where $\mathcal{F}$ is a σ–algebra of subsets of Ω and μ a probability measure. Let $\{T_x : \Omega \mapsto \Omega;\ x \in \mathbb{Z}^n\}$ be a group of $\mathcal{F}$-measurable transformations which preserve the measure μ:

(*i*) $T_x : \Omega \mapsto \Omega$ is $\mathcal{F}$–measurable for all $x \in \mathbb{Z}^n$,

(*ii*) $\mu(T_x\mathcal{B}) = \mu(\mathcal{B})$, for any $\mathcal{B} \in \mathcal{F}$ and $x \in \mathbb{Z}^n$,

(*iii*) $T_0 = I\,,\ T_x \circ T_y = T_{x+y}$.

In what follows we assume that the group T_x is ergodic. That is: any $f \in L^1(\Omega)$ such that $f(T_x\omega) = f(\omega)$ μ-a.s for each $x \in \mathbb{Z}^n$, is equal to a constant μ–a.s.

Given a family of random variables $\{q(\omega, z),\ z \in \mathbb{Z}^n\}$ such that μ–a.s,

(i) $\sum_{z\in\mathbb{Z}^n} q(\omega,z)=1\,,$

(ii) $q(T_x\omega,z)=q(T_{x+z}\omega,-z)\,,$

(iii) $q(\omega,z)\geq 0,\quad q(\omega,\pm e_i)\geq\delta>0,\ i=1,\dots,n$ (ellipticity condition),

we define the family of transition probabilities as follows:

$$p_z(x)=q(T_x\omega,z)$$

where the argument ω, treated as a realization of the medium, is omitted. The important characteristic of a family of transition probabilities is the structure of its support

$$\Lambda=\big\{z\in\mathbb{Z}^n\,|\ \operatorname*{ess\,sup}_{\Omega} p_z(x)\ \neq\ 0\big\}.$$

In all the models considered below, the set Λ is finite.

Now, if we denote $p_z^\varepsilon(x)=p_z(\varepsilon^{-1}x),\ x\in Q_\varepsilon,\ z\in\Lambda$, then due to the assumptions on $q(\omega,x)$, Problem (5)-(6) is uniformly elliptic.

It is convenient to define the following "divergence" operators: for any vector-function $q\in\left(L^2(Q_\varepsilon)\right)^{|\Lambda|}$,

$$\operatorname{div}^\varepsilon_\Lambda q(x)\triangleq\sum_{z\in\Lambda}\partial^\varepsilon_{-z}q_z(x)\,,$$

for any random variable $v\in L^2(\Omega)$,

$$\operatorname{div}_\omega v(\omega)\triangleq\sum_{z\in\Lambda}v(T_{-z}\omega)-v(\omega)\,.$$

The discrete Dirichlet problem (5)-(6) can now be rewritten in the form:

$$A_\varepsilon\,u^\varepsilon=f^\varepsilon\,,\ u^\varepsilon\in W_0^{1,2}(Q_\varepsilon)\,, \tag{8}$$

with

$$A_\varepsilon\,u^\varepsilon=\operatorname{div}^\varepsilon_\Lambda\left(\sum_{z'\in\Lambda}a^\varepsilon_{\cdot z'}\,\partial^\varepsilon_{z'}u^\varepsilon\right)\ . \tag{9}$$

2 Tools for discrete operators analysis

In the chapter *Mathematical Approaches and Methods* of [8], Grégoire Allaire presents a good survey of the main methods and approaches in the mathematical theory of homogenization.

2.1 Compensated compactness lemma

One of the main tools in the homogenization of differential operators is the so–called compensated compactness lemma (see Tartar [16]), which gives a sufficient condition for passing to the limit in the inner product of two weakly converging sequences of vector-functions. Following an equivalent process, we prove a version of this lemma for functions defined on a grid. This is done in a rather straightforward way, but up to our knowledge it was not done yet.

Lemma 2.1 *Let q^ε and v^ε be sequences of vector-functions from $\left(L^2(Q_\varepsilon)\right)^{|\Lambda|}$ such that:*

$$q^\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} q^0 \text{ weakly in } \left(L^2(Q)\right)^{|\Lambda|}, \qquad \mathrm{div}^\varepsilon_\Lambda q^\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} f^0 \text{ in } W^{-1,2}(Q)\,,$$
$$v^\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} v^0 \text{ weakly in } \left(L^2(Q)\right)^{|\Lambda|}, \qquad v^\varepsilon_z(x) = \partial^\varepsilon_z u^\varepsilon(x) \text{ where } u^\varepsilon \in W^{1,2}(Q_\varepsilon)\,.$$

Then, the sequence $(q^\varepsilon\, v^\varepsilon)$ converges $\star$–weakly to $q^0\, v^0$:

$$q^\varepsilon\, v^\varepsilon \underset{\varepsilon\to 0}{\overset{\star}{\longrightarrow}} q^0\, v^0\,.$$

Proof According to Kozlov [12], the weak convergence of q^ε in $(L^2(Q))^{|\Lambda|}$ implies that $\mathrm{div}^\varepsilon_\Lambda q^\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} \sum_{z\in\Lambda} z\dfrac{\partial q^0}{\partial z}$ weakly in $W^{-1,2}(Q_\varepsilon)$. Thus, $\sum_{z\in\Lambda} z\dfrac{\partial q^0}{\partial z} = f^0$, and we have

$$\lim_{\varepsilon\to 0} \|\mathrm{div}^\varepsilon_\Lambda (q^\varepsilon - q^0)\|_{W^{-1,2}(Q)} = 0\,.$$

From now on, the notation like q^0 or v^0 is used both for the functions of continuous argument and for their discretization (see Appendix A). Using the representation

$$q^\varepsilon\, v^\varepsilon = (q^\varepsilon - q^0)\, v^\varepsilon + q^0\, v^\varepsilon$$

and taking into account the $\star$–weak convergence of q^0, v^ε to $q^0\, v^0$, one can assume, without loss of generality, that $q^0 = 0$. Also, under a proper choice of an arbitrary additive constant, $\sum_{x\in Q_\varepsilon} u^\varepsilon(x) = 0$. Then, by the Poincaré inequality, the sequence u^ε is uniformly bounded in the $W^{1,2}$-norm. For any $\varphi \in \mathcal{C}_0^\infty(Q)$ we get:

$$\sum_{x\in Q_\varepsilon} q^\varepsilon(x)\, v^\varepsilon(x)\, \varphi(x) =$$

$$= \sum_{x\in Q_\varepsilon} \sum_{z\in\Lambda} q_z^\varepsilon(x)\, \partial_z^\varepsilon u^\varepsilon(x)\, \varphi(x)$$

$$= \sum_{x\in Q_\varepsilon} \sum_{z\in\Lambda} \{q_z^\varepsilon(x)\, \partial_z^\varepsilon(u^\varepsilon(x)\, \varphi(x)) - q_z^\varepsilon(x)\, u^\varepsilon(x)\, \partial_z^\varepsilon \varphi(x)\} + \tau(\varepsilon)$$

with $\lim\limits_{\varepsilon\to 0} \tau(\varepsilon) = 0$ (see Appendix B). Summing up by parts in the latter expression leads to:

$$\sum_{x\in Q_\varepsilon} q^\varepsilon(x)\, v^\varepsilon(x)\, \varphi(x) =$$

$$= \sum_{x\in Q_\varepsilon} \sum_{z\in\Lambda} \left\{\partial_{-z}^\varepsilon q_z^\varepsilon(x)\, u^\varepsilon(x)\, \varphi(x) - q_z^\varepsilon(x)\, u^\varepsilon(x)\, \partial_z^\varepsilon \varphi(x)\right\} + \tau(\varepsilon)$$

$$= \sum_{x\in Q_\varepsilon} (\mathrm{div}_\Lambda^\varepsilon\, q^\varepsilon(x)\ , u^\varepsilon\, \varphi) - \sum_{x\in Q_\varepsilon} \sum_{z\in\Lambda} q_z^\varepsilon(x)\, u^\varepsilon(x)\, \partial_z^\varepsilon \varphi(x) + \tau(\varepsilon)\,.$$

Since u^ε is uniformly bounded in $W^{1,2}(Q_\varepsilon)$ and $\mathrm{div}_\Lambda^\varepsilon\, q^\varepsilon$ converges to 0 in $W^{-1,2}$-norm, the first term in the right hand side goes to 0 as $\varepsilon \to 0$. The second term goes to 0 because $q_z^\varepsilon\, \partial_z^\varepsilon \varphi$ converges to 0 in $L^2(Q)$ weakly. Finally, for any $\varphi \in \mathcal{C}_0^\infty(Q)$,

$$\lim_{\varepsilon\to 0} \sum_{x\in Q_\varepsilon} \sum_{z\in\Lambda} q_z^\varepsilon(x)\, v_z^\varepsilon(x)\, \varphi(x) = 0\,.$$

□

2.2 G–convergence and homogenization

The notion of G–convergence has been introduced by Spagnolo [15]. It concerns the sequence of symmetric, second order and elliptic operators.

In this section, we give the definitions of the G-convergence and the homogenization of discrete operators and then study the main properties of this convergence.

Consider a family of uniformly elliptic discrete Dirichlet problems:

$$A_\varepsilon u^\varepsilon = \mathrm{div}^\varepsilon_\Lambda \left(\sum_{z' \in \Lambda} a^\varepsilon_{\cdot z'}\, \partial^\varepsilon_{z'} u^\varepsilon \right) = f^\varepsilon\,, \quad u^\varepsilon \in W^{1,2}_0(Q_\varepsilon)\,,$$

and denote by $\mathcal{A}^\varepsilon(x)$ the matrices of the coefficients $\{a^\varepsilon_{zz'}(x)\}$. Let $\mathcal{A}(x) = \{a_{zz'}(x)\}$, $x \in Q$, be a $|\Lambda| \times |\Lambda|$ matrix.

Definition 2.2 (G–convergence) *We say that the matrix $\mathcal{A}^\varepsilon$ G*–converges *to $\mathcal{A}$ ($\mathcal{A}^\varepsilon \xrightarrow[\varepsilon\to 0]{G} \mathcal{A}$) if, for any sequence $f^\varepsilon \in W^{-1,2}(Q_\varepsilon)$ such that $f^\varepsilon \xrightarrow[\varepsilon\to 0]{} f$ in $W^{-1,2}(Q)$, we have:*

$$\begin{array}{rcll} u^\varepsilon & \xrightarrow[\varepsilon\to 0]{} & u^0 & \textit{weakly in } W^{1,2}_0(Q)\,, \\ p^\varepsilon = \displaystyle\sum_{z'\in\Lambda} a^\varepsilon_{zz'}\, \partial^\varepsilon_{z'} u^\varepsilon & \xrightarrow[\varepsilon\to 0]{} & p^0 = \displaystyle\sum_{z\in\Lambda} a_{zz'} \frac{\partial}{\partial z} u^0 & \textit{weakly in } L^2(Q)\,, \end{array}$$

where u^0 is the solution of the limit Dirichlet problem:

$$\sum_{z,z'\in\Lambda} -\frac{\partial}{\partial z} \left(a_{zz'}(x) \frac{\partial}{\partial z'} u^0 \right) = f, \quad u^0 \in W^{1,2}_0(Q).$$

The homogenization is a particular case of G-convergence. Given a matrix-valued function $\mathcal{A}^1(x) = \{a^1_{zz'}(x)\}$, z, $z' \in \Lambda$, $x \in \mathbb{Z}^n$, we define the sequence $\mathcal{A}^\varepsilon$ as follows:

$$\mathcal{A}^\varepsilon(x) = \mathcal{A}^1(x/\varepsilon), \quad x \in Q_\varepsilon.$$

Suppose that the corresponding family of operators A_ε (defined in (9)) is uniformly elliptic.

Definition 2.3 *The constant matrix $\mathcal{A}$ is the* homogenized matrix *for the family $\mathcal{A}^\varepsilon(x) = \{a^\varepsilon_{zz'}(x)\}$ if, for any sequence $f^\varepsilon \in W^{-1,2}(Q_\varepsilon)$ such that: $f^\varepsilon \xrightarrow[\varepsilon\to 0]{} f$ in $W^{-1,2}(Q)$, the solutions u^ε of the Dirichlet problems:*

$$\mathrm{div}^\varepsilon_\Lambda \left(\sum_{z'\in\Lambda} a^\varepsilon_{\cdot z'}\, \partial^\varepsilon_{z'} u^\varepsilon \right) = f^\varepsilon,\ \ u^\varepsilon \in W^{1,2}_0(Q_\varepsilon)\,,$$

converge to the solution u^0 of the limit Dirichlet problem:

$$- \sum_{z,z'\in\Lambda} \frac{\partial}{\partial z} a_{zz'} \frac{\partial}{\partial z'} u^0 = f\,,\ \ u^0 \in W_0^{1,2}(Q) \tag{10}$$

in the following sense:

$$\begin{aligned} u^\varepsilon \ &\xrightarrow[\varepsilon\to 0]{}\ u^0\,, \qquad \text{weakly in } W_0^{1,2}(Q)\,,\\ \sum_{z'\in\Lambda} a^\varepsilon_{zz'}\,\partial^\varepsilon_{z'} u^\varepsilon \ &\xrightarrow[\varepsilon\to 0]{}\ \mathcal{A}\,\nabla u^0\,, \ \ \text{weakly in } L^2(Q)\,. \end{aligned}$$

Remark 2.4 *The dimension of the difference gradient of functions defined on Q_ε is equal to $|\Lambda|$ and does not coincide with the dimension of the standard gradient of functions defined on Q. This is the reason we write the limit equation in the definitions above in a nonstandard form. This allows to define the convergence of streams. Of course, one can easily transform the limiting equation to the standard form*

$$\sum_{z,z'\in\Lambda} \frac{\partial}{\partial z} a_{zz'} \frac{\partial}{\partial z'} = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} a_{ij} \frac{\partial}{\partial z_j}\,, \quad a_{ij}(x) = \sum_{z,z'\in\Lambda} (z,i) a_{zz'}(x)(z',j)\,.$$

The following condition is important when analyzing a sequence of uniformly elliptic difference operators.

Definition 2.5 (Condition (N)) *We say that the sequence of operators A_ε defined in (9) satisfies the* condition (N) *in Q if there exists a sequence of mesh functions $N^\varepsilon_z \in W^{1,2}(Q_\varepsilon)$ such that the following limit relations hold:*

(P1) $N^\varepsilon_z \xrightarrow[\varepsilon\to 0]{} 0$ *weakly in* $W^{1,2}(Q)$,

(P2) $\overline{a}^\varepsilon_{zz'} \stackrel{\triangle}{=} a^\varepsilon_{zz'} + \sum_{z''} a^\varepsilon_{zz''}\,\partial^\varepsilon_{z''} N^\varepsilon_{z'} \longrightarrow l_{zz'}$ *weakly in* $L^2(Q)$,

(P3) $\displaystyle \left\| \sum_{z\in\Lambda} \partial^\varepsilon_{-z} \overline{a}^\varepsilon_{zz'} + \sum_{z\in\Lambda} \frac{\partial}{\partial z} l_{zz'} \right\|_{W^{-1,2}(Q_\varepsilon)} \xrightarrow[\varepsilon\to 0]{} 0\,.$

In the remaining part of the section we prove and quote a number of general results on the G-convergence and the homogenization of difference operators.

Proposition 2.6 *(see* Kozlov *[12], §2) Any uniformly elliptic sequence of problem defined in (8) contains a G-convergent subsequence. The limit problem involves a second order uniformly elliptic operator in divergence form:*

$$A\,u = -\sum_{z,z'\in\Lambda} \frac{\partial}{\partial z}\left(a_{zz'}(x)\,\frac{\partial}{\partial z'}\,u\right) = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\,\frac{\partial}{\partial x_j}u\right)\,.$$

The next statement provides a necessary and sufficient condition of $G-$ convergence in terms of condition (N).

Proposition 2.7 *A sequence of uniformly elliptic operators $\{A^\varepsilon\}$ G-converges as $\varepsilon \to 0$ to the operator A if and only if $\{A^\varepsilon\}$ satisfies* condition (N) *with $a_{zz'} = l_{zz'}$.*

Proof The fact that the condition (N) is sufficient for G-convergence has been proved in Kozlov ([12], §1).

In order to prove that the condition (N) is necessary for G-convergence, let us assume that $A^\varepsilon \xrightarrow[\varepsilon\to 0]{G} A$ and denote by b_z^ε the unique solution of the problem

$$A_\varepsilon b_z^\varepsilon = \sum_{z'\in\Lambda} \partial^\varepsilon_{-z'}\, a_{zz'}(x)\,, \quad b_z^\varepsilon \in W_0^{1,2}(Q_\varepsilon)\,;$$

the last equation is equivalent to the following one:

$$\sum_{z',z''\in\Lambda} \partial^\varepsilon_{-z'}\, a_{z'z''}(x)\,(\partial^\varepsilon_{z''} b_z^\varepsilon - \delta_{zz''}) = 0\,, \quad b_z^\varepsilon \in W_0^{1,2}(Q_\varepsilon)\,.$$

According to the uniform ellipticity condition, b_z^ε is uniformly bounded in $W_0^{1,2}(Q_\varepsilon)$. Hence, due to Theorem 2.9 below, b_z^ε converges in $W_0^{1,2}(Q)$ weakly, as $\varepsilon \to 0$, to the solution of the equation

$$\sum_{z',z''\in\Lambda} \frac{\partial}{\partial z'}\left(a_{z'z''}\frac{\partial}{\partial z''}b_z - \delta_{zz''}\right) = 0\,, \quad b_z \in W_0^{1,2}(Q)\,.$$

Denote $F_z = A\,b_z$, and put:

$$N_z^\varepsilon(x) = A_\varepsilon^{-1}\left(F_z(x) - \sum_{z'\in\Lambda} \partial^\varepsilon_{-z'}\, a^\varepsilon_{zz'}(x)\right)\,,\ x \in Q_\varepsilon\,.$$

We should verify that these functions satisfy all the properties *(P1)*, *(P2)*, *(P3)* of Definition 2.5.

Proof of (P1)

Denote $v_z^\varepsilon = A_\varepsilon^{-1} F_z$. By the definition of G-convergence, v_z^ε converges to b_z weakly in $W_0^{1,2}(Q)$ as $\varepsilon \to 0$ and we get:

$$N_z^\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} 0 \text{ weakly in } W_0^{1,2}(Q)\,.$$

Proof of (P2)

Since $\partial_{z'}^\varepsilon N_z^\varepsilon$ and $a_{zz'}^\varepsilon$ are uniformly bounded in $L^2(Q_\varepsilon)$, the functions $\overline{a}_{zz'}^\varepsilon(x)$ converge weakly in $L^2(Q_\varepsilon)$ if we take a proper subsequence. We denote this limit by $l_{zz'}(x)$.

Proof of (P3)

Since $\overline{a}_{zz'}^\varepsilon$ converges to $l_{zz'}$ weakly in $L^2(Q)$, we have:

$$\sum_{z\in\Lambda} \partial_{-z}^\varepsilon \overline{a}_{zz'}^\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} -\sum_{z\in\Lambda} \frac{\partial}{\partial z} l_{zz'} \qquad \text{weakly in } W^{-1,2}(Q)$$

(see Kozlov [12], §1, Prop. 5). Moreover, $\lim\limits_{\varepsilon\to 0} \sum\limits_{z\in\Lambda} \partial_{-z}^\varepsilon \overline{a}_{zz'}^\varepsilon = F_{z'}$. Indeed,

$$\begin{aligned}
\sum_{z\in\Lambda} \partial_{-z}^\varepsilon \overline{a}_{zz'}^\varepsilon &= \sum_{z\in\Lambda} \partial_{-z}^\varepsilon a_{zz'}^\varepsilon + \sum_{z\in\Lambda} \partial_{-z}^\varepsilon \left(\sum_{z''\in\Lambda} a_{zz''}^\varepsilon \partial_{z''}^\varepsilon N_{z'}^\varepsilon \right) \\
&= \sum_{z\in\Lambda} \partial_{-z}^\varepsilon a_{zz'}^\varepsilon + F_{z'} - \sum_{z''\in\Lambda} \partial_{-z''}^\varepsilon a_{z'z''}^\varepsilon + \tau_z^\varepsilon \\
&= F_{z'} + \tau_z^\varepsilon\,,
\end{aligned}$$

where $\|\tau_z^\varepsilon\|_{W^{-1,2}(Q_\varepsilon)}$ goes to 0 as $\varepsilon \to 0$. Passing to the limit, as $\varepsilon \to 0$, in the last relation gives:

$$\sum_{z\in\Lambda} \frac{\partial}{\partial z} l_{zz'} = F_{z'}\,.$$

Therefore,

$$\lim_{\varepsilon\to 0} \left\| \sum_{z\in\Lambda} \partial_{-z}^\varepsilon \overline{a}_{zz'}^\varepsilon + \sum_{z\in\Lambda} \frac{\partial}{\partial z} l_{zz'} \right\|_{W^{-1,2}(Q)} = \lim_{\varepsilon\to 0} \left\| \sum_{z\in\Lambda} \tau_z^\varepsilon \right\|_{W^{-1,2}(Q)} = 0\,.$$

Now, according to the first part of the proposition, $l_{zz'}$ coincide with the coefficients $a_{zz'}$ of the limit operator and, thus, are defined uniquely. Consequently, the whole sequence $\bar{a}_{zz'}$ converges. □

Remark 2.8 *The choice of functions N_z^ε is obviously not unique. In this proof we present one of the possibilities to construct such functions.*

2.2.1 Convergence of arbitrary solutions

One of the significant properties of G-convergence is the fact that the G-limit operator depends only on the original sequence of operators and but neither on the type of boundary conditions nor on the domain. In a general form, this can be formulated as follows:

Théorème 2.9 (Convergence of arbitrary solutions)
Let a sequence of uniformly elliptic operators A_ε G-converge in a domain Q to the limit operator A and let a sequence of functions $w^\varepsilon \in W^{1,2}(Q_\varepsilon)$ satisfy the conditions:

$$\begin{aligned} & w^\varepsilon \underset{\varepsilon \to 0}{\longrightarrow} w^0 \quad \textit{weakly in } W^{1,2}(Q)\,, \\ & \operatorname{div}^\varepsilon_\Lambda \left(\sum_{z' \in \Lambda} a^\varepsilon_{zz'} \; (g_{z'} + \partial^\varepsilon_{z'} w^\varepsilon) \right) = f\,, \end{aligned} \tag{11}$$

where $g \in (L^2(Q))^{|\Lambda|}$ and $f \in W^{-1,2}(Q)$ do not depend on ε. Then, w^0 satisfies the following homogenized equation:

$$- \sum_{z,z' \in \Lambda} \frac{\partial}{\partial z} [a_{zz'} \left(g_{z'} + \frac{\partial}{\partial z'} w^0 \right)] = f\,,$$

and the streams converge in $L^2(Q)$ weakly:

$$\sum_{z' \in \Lambda} a^\varepsilon_{zz'} \; (g_{z'} + \partial^\varepsilon_{z'} w^\varepsilon) \underset{\varepsilon \to 0}{\longrightarrow} \sum_{z' \in \Lambda} a_{zz'} \left(g_{z'} + \frac{\partial}{\partial z'} w^0 \right) .$$

Proof Under the conditions of the theorem, the streams are uniformly bounded in $L^2(Q_\varepsilon)$. Thus, taking a proper subsequence, we have:

$$\sum_{z'\in\Lambda} a^\varepsilon_{zz'}\,(g_{z'}+\partial^\varepsilon_{z'}w^\varepsilon) \underset{\varepsilon\to 0}{\longrightarrow} \xi_z \quad \text{weakly in } L^2(Q)\,.$$

Passing to the limit in Equation (11), one can easily check that

$$-\sum_{z\in\Lambda}\frac{\partial}{\partial z}\xi_z = f\,.$$

We have to prove the relation:

$$\xi_z = \sum_{z'\in\Lambda} a_{zz'}\left(g_{z'}+\frac{\partial}{\partial z'}w^0\right)\,.$$

Let u^0 be an arbitrary function from $W^{1,2}_0(Q)$. Denote by u^ε the solution of the Dirichlet problem:

$$\mathrm{div}^\varepsilon_\Lambda\left(\sum_{z'\in\Lambda} a^\varepsilon_{zz'}\,\partial^\varepsilon_{z'}u^\varepsilon\right) = \sum_{z,z'\in\Lambda}\frac{\partial}{\partial z}\left(a_{zz'}\,\frac{\partial}{\partial z'}\,u^0\right)\,,$$

and consider the following identity:

$$\sum_{z\in\Lambda}(g_z+\partial^\varepsilon_z w^\varepsilon)\sum_{z'\in\Lambda} a^\varepsilon_{zz'}\,\partial^\varepsilon_{z'}u^\varepsilon = \sum_{z\in\Lambda}\partial^\varepsilon_z u^\varepsilon\sum_{z'\in\Lambda} a^\varepsilon_{zz'}\,(g_{z'}+\partial^\varepsilon_{z'}w^\varepsilon)\,. \tag{12}$$

By the definition of G-convergence, we have:

$$\sum_{z'\in\Lambda} a^\varepsilon_{zz'}\,\partial^\varepsilon_{z'}u^\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} \sum_{z'\in\Lambda} a_{zz'}\,\frac{\partial}{\partial z'}u^0\,, \quad \text{weakly in } L^2(Q)\,,$$

while the limiting relation

$$\sum_{z\in\Lambda}(g_z+\partial^\varepsilon_z w^\varepsilon) \underset{\varepsilon\to 0}{\longrightarrow} \sum_{z\in\Lambda}\left(g_z+\frac{\partial}{\partial z}w^0\right)\,, \quad \text{weakly in } L^2(Q)$$

is an evident consequence of the weak convergence of w^ε . Now, passing to the limit on the left-hand side of (12), with the help of Lemma 2.1 and the fact that g_z does not depend on ε, we obtain:

$$\sum_{z\in\Lambda}(g_z+\partial_z^\varepsilon w^\varepsilon)\sum_{z'\in\Lambda}a_{zz'}^\varepsilon\,\partial_{z'}^\varepsilon u^\varepsilon \xrightarrow[\varepsilon\to 0]{\star} \sum_{z\in\Lambda}(g_z+\frac{\partial}{\partial z}w^0)\sum_{z'\in\Lambda}a_{zz'}\,\frac{\partial}{\partial z'}u^0\,.$$

Similarly, passing to the limit on the right-hand side of (12) gives:

$$\sum_{z\in\Lambda}\partial_z^\varepsilon u^\varepsilon\sum_{z'\in\Lambda}a_{zz'}^\varepsilon\,(g_{z'}+\partial_{z'}^\varepsilon w^\varepsilon) \xrightarrow[\varepsilon\to 0]{\star} \sum_{z\in\Lambda}\frac{\partial}{\partial z}u^0\,\xi_z\,.$$

Finally, considering the fact that u^0 is arbitrary function from $W_0^{1,2}(Q)$, we deduce:

$$\xi_z=\sum_{z'\in\Lambda}a_{zz'}^\varepsilon\,\left(g_{z'}+\frac{\partial}{\partial z'}w^0\right)\,.$$

□

Corollary 2.10 (Local property of G-convergence)
If $A_\varepsilon \xrightarrow[\varepsilon\to 0]{G} A$ in a domain Q, then $A_\varepsilon \xrightarrow[\varepsilon\to 0]{G} A$ in any sub-domain $Q_1\subset Q$.

2.2.2 Convergence of energies

In this section, we address a family of Dirichlet problems with non homogeneous boundary conditions:

$$\mathrm{div}_\Lambda^\varepsilon\left(\sum_{z'\in\Lambda}a_{zz'}^\varepsilon\,\partial_{z'}^\varepsilon u^\varepsilon\right)=f\,,\;\; u^\varepsilon-u^0\in W_0^{1,2}(Q_\varepsilon)\,, \tag{13}$$

where $u^0\in W^{1,2}(\mathbb{R}^n)$ and $f\in W^{-1,2}(Q)$ are fixed given functions.

We suppose that the family $\{A_\varepsilon\}$ is uniformly elliptic and G-converges to the limit operator A. Then, one can assume without loss of generality that the function u^0 satisfies the equation $A\,u^0=f$ in the domain Q.

Substituting the function $(u^\varepsilon-u^0)$ in Equation (13) and summing by parts, we can easily see that the family $\{u^\varepsilon\}$ is uniformly bounded in $W^{1,2}(\overline{Q_\varepsilon})$.

By Theorem 2.9 (Convergence of arbitrary solution), any weak limiting point of the sequence $\{u^\varepsilon\}$ coincides with u^0 in Q. Hence, the whole family $\{u^\varepsilon\}$ converges to u^0 in $W^{1,2}(\overline{Q})$ weakly.

Proposition 2.11 (convergence of energies) *Let $A_\varepsilon \xrightarrow[\varepsilon\to 0]{G} A$ and let u^ε be the solution of problem (13). Then the following limit relation holds true:*

$$\sum_{x\in\overline{Q_\varepsilon}}\sum_{z,z'\in\Lambda} a^\varepsilon_{zz'}(x)\,\partial^\varepsilon_z u^\varepsilon(x)\,\partial^\varepsilon_{z'}u^\varepsilon(x) \underset{\varepsilon\to 0}{\longrightarrow} \int_Q \sum_{z,z'\in\Lambda} a_{zz'}(x)\,\frac{\partial}{\partial z}u^0(x)\,\frac{\partial}{\partial z'}u^0(x)\,dx\,.$$

Proof By (13) we have:

$$\begin{aligned}
&\sum_{x\in\overline{Q_\varepsilon}}\sum_{z,z'\in\Lambda} a^\varepsilon_{zz'}(x)\,\partial^\varepsilon_z(u^\varepsilon-u^0)(x)\,\partial^\varepsilon_{z'}(u^\varepsilon-u^0)(x) = \\
&\quad = \sum_{x\in\overline{Q_\varepsilon}}\sum_{z,z'\in\Lambda} a^\varepsilon_{zz'}(x)\,\partial^\varepsilon_z u^\varepsilon(x)\,\partial^\varepsilon_{z'}(u^\varepsilon-u^0)(x) - \\
&\qquad \sum_{x\in\overline{Q_\varepsilon}}\sum_{z,z'\in\Lambda} a^\varepsilon_{zz'}(x)\,\frac{\partial}{\partial z}u^0(x)\,\partial^\varepsilon_{z'}(u^\varepsilon-u^0)(x) + \tau(\varepsilon) \\
&\quad = \sum_{x\in Q_\varepsilon} f(x)\,(u^\varepsilon-u^0)(x) - \sum_{x\in\overline{Q_\varepsilon}}\sum_{z,z'\in\Lambda} a^\varepsilon_{zz'}(x)\,\frac{\partial}{\partial z}u^0(x)\,\partial^\varepsilon_{z'}u^\varepsilon(x) + \\
&\qquad \sum_{x\in\overline{Q_\varepsilon}}\sum_{z,z'\in\Lambda} a^\varepsilon_{zz'}(x)\,\frac{\partial}{\partial z}u^0(x)\,\frac{\partial}{\partial z'}u^0(x)\; + \;\tau(\varepsilon)\,,
\end{aligned}$$

where $\tau(\varepsilon)\to 0$ as $\varepsilon\to 0$. On the other hand,

$$\begin{aligned}
&\sum_{x\in\overline{Q_\varepsilon}}\sum_{z,z'\in\Lambda} a^\varepsilon_{zz'}(x)\,\partial^\varepsilon_z(u^\varepsilon-u^0)(x)\,\partial^\varepsilon_{z'}(u^\varepsilon-u^0)(x) = \\
&\quad = \sum_{x\in\overline{Q_\varepsilon}}\sum_{z,z'\in\Lambda} a^\varepsilon_{zz'}(x)\,\partial^\varepsilon_z u^\varepsilon(x)\,\partial^\varepsilon_{z'}u^\varepsilon(x) \\
&\qquad -2\sum_{x\in\overline{Q_\varepsilon}}\sum_{z,z'\in\Lambda} a^\varepsilon_{zz'}(x)\,\frac{\partial}{\partial z}u^0(x)\,\partial^\varepsilon_{z'}u^\varepsilon(x) \\
&\qquad +\sum_{x\in\overline{Q_\varepsilon}}\sum_{z,z'\in\Lambda} a^\varepsilon_{zz'}(x)\,\frac{\partial}{\partial z}u^0(x)\,\frac{\partial}{\partial z'}u^0(x)\,.
\end{aligned}$$

After subtraction we find:

$$\sum_{x\in\overline{Q_\varepsilon}}\sum_{z,z'\in\Lambda} a^\varepsilon_{zz'}(x)\,\partial^\varepsilon_z u^\varepsilon(x)\,\partial^\varepsilon_{z'}u^\varepsilon(x) - \sum_{x\in\overline{Q_\varepsilon}}\sum_{z,z'\in\Lambda} a^\varepsilon_{zz'}(x)\,\frac{\partial}{\partial z}u^0(x)\,\partial^\varepsilon_{z'}u^\varepsilon(x)$$
$$-\sum_{x\in Q_\varepsilon} f(x)\,(u^\varepsilon - u^0)(x) + \tau(\varepsilon) = 0\,. \tag{14}$$

Passing to the limit in the last relation, and taking into account the weak convergence of $u^\varepsilon - u^0$ to 0 in $W^{1,2}_0(Q)$ and the weak convergence of the streams $a^\varepsilon_{zz'}\,\partial^\varepsilon_{z'}u^\varepsilon$ in $L^2(Q)$, we obtain:

$$\sum_{x\in\overline{Q_\varepsilon}}\sum_{z,z'\in\Lambda} a^\varepsilon_{zz'}(x)\,\partial^\varepsilon_z u^\varepsilon(x)\,\partial^\varepsilon_{z'}u^\varepsilon(x) \underset{\varepsilon\to 0}{\longrightarrow} \int_Q \sum_{z,z'\in\Lambda} a_{zz'}(x)\,\frac{\partial}{\partial z}u^0(x)\,\frac{\partial}{\partial z'}u^0(x)\,dx\,.$$

□

2.2.3 Neumann problem

The notion of the G-limit operator has been expressed in terms of the operators of the corresponding Dirichlet problems. But, as was already mentioned in the previous section, we can consider as well other boundary value problems. In this section, the Neumann problem is investigated.

Definition 2.12 *Let $f \in (L^2(Q))^{|\Lambda|}$. We say that $u^\varepsilon \in W^{1,2}(Q_\varepsilon)$ is a solution of the Neumann problem for the equation*

$$\mathrm{div}^\varepsilon_\Lambda\left(\sum_{z'\in\Lambda} a^\varepsilon_{\cdot z'}\,\partial^\varepsilon_{z'}u^\varepsilon\right) = \sum_{z\in\Lambda}\partial^\varepsilon_{-z}f^\varepsilon_z$$

if the relation

$$\sum_{x\in\overline{Q_\varepsilon}}\sum_{z,z'\in\Lambda} a^\varepsilon_{zz'}(x)\,\bar\partial^\varepsilon_z\varphi^\varepsilon(x)\,\bar\partial^\varepsilon_{z'}u^\varepsilon(x) = \sum_{x\in\overline{Q_\varepsilon}}\sum_{z\in\Lambda} f^\varepsilon_z(x)\,\bar\partial^\varepsilon_z\varphi^\varepsilon(x) \tag{15}$$

holds true for any $\varphi \in W^{1,2}(Q)$; here we use the notation

$$\bar\partial^\varepsilon_z\varphi = \begin{cases} \partial^\varepsilon_z\varphi & \textit{if}\ \ x+\varepsilon z \in \overline{Q_\varepsilon} \\ 0 & \text{otherwise.}\end{cases}$$

Clearly, the functions u^ε are defined up to an additive constant. To fix the choice of the constant, we assume that:

$$\sum_{x\in\overline{Q_\varepsilon}} u^\varepsilon(x) = 0\,.$$

In order to study the Neumann problem, we should modify the definition of ellipticity and impose a more strong condition because the Definition 1.1 above does not ensure the coerciveness of Problem (15).

Definition 2.13 *We say that the family of operators* $\{A_\varepsilon\}$ *is* N-elliptic *in a domain* Q_ε *if the inequality*

$$\sum_{x\in\overline{Q_\varepsilon}}\sum_{z,z'\in\Lambda} a^\varepsilon_{zz'}(x)\,\bar\partial^\varepsilon_z\varphi(x)\bar\partial^\varepsilon_{z'}\varphi(x) \geq \sum_{x\in\overline{Q_\varepsilon}}\sum_{i=1}^{n}\left(\bar\partial^\varepsilon_{\pm e_i}\varphi(x)\right)^2 \tag{16}$$

holds for any φ.

It should be noted that N-ellipticity implies the uniform ellipticity in the same domain Q and that, under the condition of Proposition 1.4, the family of operators is always N-elliptic.

Proposition 2.14 *Suppose that a family of* N*-elliptic operators* $\{A_\varepsilon\}$ $G-$ *converges to the operator* A *in the domain* Q*. Then the solutions* u^ε *of Problem (15) converge as* $\varepsilon\to 0$ *to the solution of the limit Neumann problem: for any* $\varphi\in W^{1,2}(Q)$,

$$\int_Q\left(\sum_{z,z'\in\Lambda} a_{zz'}(x)\frac{\partial}{\partial z}\varphi(x)\,\frac{\partial}{\partial z'}u^0(x)\right)\,dx = \int_Q\left(\sum_{z\in\Lambda} f_z(x)\frac{\partial}{\partial z}\varphi(x)\right)\,dx\,.$$

Moreover, the streams do also converge.

Proof Using the Poincaré inequality, we derive from the N-ellipticity the uniform coerciveness of Problem (15). Thus, the family u^ε is uniformly bounded in $W^{1,2}(Q_\varepsilon)$. By Theorem 2.9, any limit point w^0 of the family u^ε satisfies the G-limit equation and

$$\sum_{z'\in\Lambda} a^\varepsilon_{zz'}\,\partial^\varepsilon_{z'}u^\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} \sum_{z'\in\Lambda} a_{zz'}\,\frac{\partial}{\partial z'}w^0 \quad \text{weakly in}\ \ L^2(Q)\,.$$

So, for any $\varphi \in W^{1,2}(Q)$, passing to the limit in (15), we get:

$$\int_Q \left(\sum_{z,z' \in \Lambda} a_{zz'}(x) \frac{\partial}{\partial z}\varphi(x) \, \frac{\partial}{\partial z'} w^0(x) \right) dx = \int_Q \left(\sum_{z \in \Lambda} f_z(x) \frac{\partial}{\partial z}\varphi(x) \right) dx \, .$$

Moreover, $\int_Q w^0(x)\, dx = 0$.

□

2.2.4 Γ–convergence

The Γ–convergence is a notion of functional convergence which has been introduced by De Giorgi [4], [5].

In this section, we exhibit the relation between the G-convergence of operators the Γ-convergence, which is a special kind of convergence of corresponding quadratic forms.

Proposition 2.15 *Let A_ε be a N-elliptic family of operators in a domain Q. Then, $A_\varepsilon \xrightarrow[\varepsilon\to 0]{G} A$ in Q if and only if the following conditions are satisfied:*

(*i*) *For any $u^0 \in W^{1,2}(Q)$ and for any sequence $w^\varepsilon \in W^{1,2}(Q_\varepsilon)$ such that $w^\varepsilon \xrightarrow[\varepsilon\to 0]{} u^0$ weakly in $W^{1,2}(Q)$, the following inequality holds:*

$$\liminf_{\varepsilon\to 0} \sum_{x \in \overline{Q_\varepsilon}} \sum_{z,z' \in \Lambda} a^\varepsilon_{zz'}(x)\, \bar\partial^\varepsilon_z w^\varepsilon(x)\, \bar\partial^\varepsilon_{z'} w^\varepsilon(x) \geq \int_Q \sum_{z,z' \in \Lambda} a_{zz'}(x)\, \frac{\partial}{\partial z} u^0(x)\, \frac{\partial}{\partial z'} u^0(x)\, dx \, .$$

(*ii*) *For any $u^0 \in W^{1,2}(Q)$, there exists a sequence $u^\varepsilon \in W^{1,2}(Q_\varepsilon)$ such that $u^\varepsilon \xrightarrow[\varepsilon\to 0]{} u^0$ weakly in $W^{1,2}(Q)$, $u^\varepsilon - u^0 \in W^{1,2}_0(Q)$, and:*

$$\lim_{\varepsilon\to 0} \sum_{x \in \overline{Q_\varepsilon}} \sum_{z,z' \in \Lambda} a^\varepsilon_{zz'}(x)\, \bar\partial^\varepsilon_z u^\varepsilon(x)\, \bar\partial^\varepsilon_{z'} u^\varepsilon(x) = \int_Q \sum_{z,z' \in \Lambda} a_{zz'}(x)\, \frac{\partial}{\partial z} u^0(x)\, \frac{\partial}{\partial z'} u^0(x)\, dx \, .$$

Proof Suppose that $A_\varepsilon \xrightarrow[\varepsilon\to 0]{G} A$.

(i) Consider the Neumann problem (15), with $f_z = \left(\sum_{z'\in\Lambda} a_{zz'}\frac{\partial u^0}{\partial z'}\right)_{z\in\Lambda}$, where u^0 is the solution of the G-limit Neumann problem. The solution u^ε provides the minimum in the following variational problem:

$$E = \inf_{v\in W^{1,2}(Q_\varepsilon)} J^\varepsilon(v)\,,$$

where

$$J^\varepsilon(v) = \sum_{x\in\overline{Q_\varepsilon}}\sum_{z,z'\in\Lambda}\left[a^\varepsilon_{zz'}(x)\,\bar\partial^\varepsilon_z v(x)\,\bar\partial^\varepsilon_{z'}v(x) - 2a_{zz'}(x)\,\bar\partial^\varepsilon_z v(x)\,\frac{\partial}{\partial z'}u^0(x)\right].$$

For any sequence $\{w^\varepsilon\}$ such that $w^\varepsilon \to u^0$ weakly in $W^{1,2}(Q)$, we have

$$J^\varepsilon(w^\varepsilon) \ge J^\varepsilon(u^\varepsilon)\,. \tag{17}$$

Then,

$$\partial^\varepsilon_z u^\varepsilon \xrightarrow[\varepsilon\to 0]{} \frac{\partial}{\partial z}u^0 \quad \text{weakly in } L^2(Q)$$

and, therefore,

$$\begin{aligned}\sum_{x\in\overline{Q_\varepsilon}}\sum_{z,z'\in\Lambda} a^\varepsilon_{zz'}(x)\,\bar\partial^\varepsilon_z u^\varepsilon(x)\,\bar\partial^\varepsilon_{z'}u^\varepsilon(x) = \\ = \sum_{x\in\overline{Q_\varepsilon}}\sum_{z,z'\in\Lambda} a_{zz'}(x)\,\bar\partial^\varepsilon_z u^\varepsilon(x)\,\frac{\partial}{\partial z'}u^0(x) \\ \xrightarrow[\varepsilon\to 0]{} \int_Q \sum_{z,z'\in\Lambda} a_{zz'}(x)\,\frac{\partial}{\partial z}u^0(x)\,\frac{\partial}{\partial z'}u^0(x)\,dx\,.\end{aligned}$$

Now, taking the infimum limit in the both sides of (17), we obtain the required inequality.

(ii) It is the statement of Proposition 2.11.

The remaining part of the proposition is an easy consequence of the uniqueness of G-limit. □

Remark 2.16 *The statements of Propositions 2.11 and 2.15 remain valid if we replace the sums over $x \in \overline{Q_\varepsilon}$ by the sums over $x \in Q_\varepsilon$.*

2.3 Homogenization of random operators

This section is devoted to the operators with random statistically homogeneous coefficients. Following the construction of Section 1.2, we introduce the matrix-valued $\mathcal{F}$-measurable function $\{a_{zz'}(\omega)\}$, $z, z' \in \Lambda$ and, then, define the matrices $\mathcal{A}^\varepsilon(x)$ as follows:

$$a^\varepsilon_{zz'}(x) = a_{zz'}(T_{x/\varepsilon}\omega), \quad x \in \varepsilon\mathbb{Z}^n,\ z, z' \in \Lambda\,. \tag{18}$$

Also, we suppose here that $\pm e_i \in \Lambda$, $i = 1, \dots, n$, and that

$$\sum_{z,z' \in \Lambda} a_{zz'}(\omega)\eta_z\eta_{z'} \geq c \sum_{i=1}^{n} |\eta_{\pm e_i}|^2\,, \quad \eta \in \mathbb{R}^{|\Lambda|}\,. \tag{19}$$

It is easy to see that the latter inequality implies the N-ellipticity and the uniform ellipticity of the family A_ε in any regular domain Q.

2.3.1 Auxiliary problem

Let us define the following subspaces of $\left(L^2(\Omega)\right)^{|\Lambda|}$ (see Kozlov [12]):

$L^2_{pot}(\Omega,\, \Lambda)$ is the closure of the set:
$\left\{v \in (L^2(\Omega))^{|\Lambda|}\,;\ v_z(\omega) = u(T_z\omega) - u(\omega) \text{ for some } u \in L^\infty(\Omega)\right\}$,
$L^2_{sol}(\Omega,\, \Lambda)$ is the closure of the set: $\left\{v \in (L^2(\Omega))^{|\Lambda|}\,;\ \mathrm{div}_\omega\, v = 0\right\}$

For $\lambda \in \mathbb{R}^{|\Lambda|}$ we denote by $\mathcal{V}^2_{pot,\lambda}(\Omega,\, \Lambda)$ the closed set $\left\{v + \lambda\,;\, v \in L^2_{pot}(\Omega,\, \Lambda)\right\}$.

Consider the following auxiliary problem:
given $\lambda \in \mathbb{R}^{|\Lambda|}$, find $v \in \mathcal{V}^2_{pot,\lambda}(\Omega,\, \Lambda)$ such that

$$\mathrm{div}_\omega \left(\sum_{z' \in \Lambda} a_{zz'}(\omega)\, v_{z'}(\omega)\right) = 0\,. \tag{20}$$

In order to prove the existence and uniqueness of the solution of this problem we introduce the operator

$$\begin{array}{rcl} A_{\mathrm{pot}} \ : \ L^2_{pot}(\Omega,\,\Lambda) & \mapsto & L^2_{pot}(\Omega,\,\Lambda) \\ (v_z)_{z\in\Lambda} & \mapsto & \Pi_{pot}\left(\displaystyle\sum_{z'\in\Lambda} a_{zz'}(\omega)\, v_{z'}(\omega)\right) , \end{array}$$

where Π_{pot} is the orthogonal projection onto the subspace $L^2_{pot}(\Omega,\,\Lambda)$.

In view of the Weyl decomposition (see Kozlov [12]): $\forall \lambda \in \mathbb{R}^{|\Lambda|}$,

$$L^2(\Omega) \;=\; \mathcal{V}^2_{pot,\lambda}(\Omega) \oplus L^2_{sol}(\Omega),$$

we can rewrite the problem (20) in the following form: given $\lambda \in \mathbb{R}^{|\Lambda|}$, find $v \in \mathcal{V}^2_{pot}(\Omega,\,\Lambda)$ such that

$$A_{\mathrm{pot}} v = \Pi_{pot}\left(\sum_{z'\in\Lambda} a_{zz'}(\omega)\,\lambda_{z'}\right) .$$

The operator A_{pot} is coercive. Indeed, for any $v \in L^2_{pot}(\Omega,\,\Lambda)$, we have

$$\begin{aligned} (A_{\mathrm{pot}} v, v) &= \sum_{z\in\Lambda}\left(\Pi_{pot}\left(\sum_{z'\in\Lambda} a_{zz'}(\omega)\, v_{z'}(\omega)\right),\; v_z(\omega)\right)_{L^2(\Omega)} \\ &= \sum_{z\in\Lambda}\left(\sum_{z'\in\Lambda} a_{zz'}(\omega)\, v_{z'}(\omega),\; \Pi_{pot}\left(v_z(\omega)\right)\right)_{L^2(\Omega)} \\ &= \sum_{z\in\Lambda}\left(\sum_{z'\in\Lambda} a_{zz'}(\omega)\, v_{z'}(\omega),\; v_z(\omega)\right)_{L^2(\Omega)} . \end{aligned}$$

According to Hypothesis (19), we get:

$$(A_{\mathrm{pot}} v, v) \geq c\, E\left[\sum_{i=1}^{n} |v_{\pm e_i}|^2\right] .$$

where E is the expectation with respect to the measure μ. On the other hand,

$$\begin{aligned}
\|v\|^2_{(L^2(\Omega))^{|\Lambda|}} &= E\,[\sum_{z\in\Lambda} |v_z(\omega)|^2] \\
&= E\,[\sum_{z\in\Lambda} |u(T_z\omega) - u(\omega)|^2] \\
&= E\,[\sum_{z\in\Lambda} \left| \sum_{i=0}^{N(z)-1} u(T_{\zeta_{i+1}}\omega) - u(T_{\zeta_i}\omega) \right|^2]
\end{aligned}$$

where $\zeta_0 = 0$, $\zeta_{N(z)} = z$, $|\zeta_{i+1} - \zeta_i| = 1$ and $N(z) \leq n\,\mathrm{diam}(\Lambda)$. Therefore,

$$\begin{aligned}
\|v\|^2_{(L^2(\Omega))^{|\Lambda|}} &\leq E\,[\,n(\mathrm{diam}(\Lambda))^2|\Lambda| \left(\sum_{i=1}^{n} v_{\pm e_i}(\omega) \right)^2] \\
&\leq c_1(n,\Lambda)\, E\,[\sum_{i=1}^{n} (v_{\pm e_i}(\omega))^2]\,.
\end{aligned}$$

Thus,

$$(A_{\mathrm{pot}} v, v) \geq c_2\, \|v\|^2_{(L^2(\Omega))^{|\Lambda|}}\,,$$

and the desired existence and uniqueness follow from the Lax-Milgram lemma. □

2.3.2 Homogenization

In this section, we study the family of random operators $\{A_\varepsilon\}$

$$A_\varepsilon\, u^\varepsilon = \mathrm{div}^\varepsilon_\Lambda \left(\sum_{z'\in\Lambda} a^\varepsilon_{\cdot z'}\, \partial^\varepsilon_{z'} u^\varepsilon \right)\,,$$

with statistically homogeneous coefficients $\{a^\varepsilon_{zz'}\}$ given by (18). The main result here is the following theorem:

Théorème 2.17 *Let $\mathcal{A}^\varepsilon = \mathcal{A}^\varepsilon(\omega)$ be a matrix defined on the probability space Ω and let the condition of ellipticity (19) be fulfilled. Then, almost surely, $\mathcal{A}^\varepsilon$ admits homogenization and the homogenized matrix $\mathcal{A}$ does not depend on ω.*

Proof For a fixed $f \in W^{-1,2}(Q)$, consider the following Dirichlet problems:

$$\operatorname{div}_\Lambda^\varepsilon \left(\sum_{z' \in \Lambda} a^\varepsilon_{zz'} \partial^\varepsilon_{z'} u^\varepsilon \right) = f \, , \quad u^\varepsilon \in W_0^{1,2}(Q_\varepsilon) \, .$$

Since (u^ε) are uniformly bounded in $W^{1,2}(Q_\varepsilon)$ and $(a^\varepsilon_{zz'} \, \partial^\varepsilon_{z'} u^\varepsilon)$ are uniformly bounded in $L^2(Q_\varepsilon)$, we have for a proper subsequence:

$$u^\varepsilon \xrightarrow[\varepsilon \to 0]{} u^0 \text{ in } W_0^{1,2}(Q) \, ,$$

$$p^\varepsilon \xrightarrow[\varepsilon \to 0]{} p^0 \text{ in } \left(L^2(Q)\right)^{|\Lambda|} \, ,$$

where we denote $p^\varepsilon_z = \sum\limits_{z' \in \Lambda} a^\varepsilon_{zz'} \, \partial^\varepsilon_{z'} u^\varepsilon$.

Let $v_z(\omega)$ solve the auxiliary problem (20).

If we denote : $v^\varepsilon(x) \stackrel{\triangle}{=} v\left(T_{x/\varepsilon}\, \omega\right)$

$q^\varepsilon \stackrel{\triangle}{=} v^\varepsilon \mathcal{A}^\varepsilon$, i.e: $\forall z \in \Lambda$, $q^\varepsilon_z = \sum_{z' \in \Lambda} v^\varepsilon_{z'} \, a^\varepsilon_{zz'}$,

then, the identity

$$\sum_{x \in Q_\varepsilon} \sum_{z \in \Lambda} p^\varepsilon_z(x) \, v^\varepsilon_z(x) = \sum_{x \in Q_\varepsilon} \sum_{z \in \Lambda} q^\varepsilon_z(x) \, \partial^\varepsilon_z u^\varepsilon(x) \tag{21}$$

obviously holds. By the Birkhoff ergodic theorem, we have

$$v^\varepsilon \xrightarrow[\varepsilon \to 0]{} E(v^\varepsilon) = \lambda \quad \text{weakly in } L^2(Q) \text{ a.s.}$$

$$q^\varepsilon \xrightarrow[\varepsilon \to 0]{} E(q^\varepsilon) \stackrel{\triangle}{=} \lambda \, \mathcal{A} \quad \text{weakly in } L^2(Q) \text{ a.s.}$$

The last relation here follows from the fact that q^ε is a linear functional of λ. Thus, the matrix $\mathcal{A}$ is well defined. By Lemma 2.1,

$$\sum_{z \in \Lambda} p^\varepsilon_z \, v^\varepsilon_z \xrightarrow[\varepsilon \to 0]{\star} \sum_{z \in \Lambda} p^0_z \, \lambda_z \, ,$$

and

$$\sum_{z \in \Lambda} q^\varepsilon_z \, \partial^\varepsilon_z u^\varepsilon \xrightarrow[\varepsilon \to 0]{\star} \sum_{z, z' \in \Lambda} \lambda_z \, a_{zz'} \, \frac{\partial}{\partial z'} u^0 \, ,$$

or, equivalently,

$$\sum_{z,z'\in\Lambda} v^\varepsilon_{z'}\, a^\varepsilon_{zz'}\, \partial^\varepsilon_z u^\varepsilon \xrightarrow[\varepsilon\to 0]{\star} \sum_{z,z'\in\Lambda} \lambda_z\, a_{zz'}\, \frac{\partial}{\partial z'} u^0 \,.$$

Hence, passing to the limit in (21) and bearing in mind that λ is an arbitrary vector, we find:

$$p^0_z = \sum_{z'\in\Lambda} a_{zz'}\, \frac{\partial}{\partial z'} u^0 \,.$$

Since $\sum_{z\in\Lambda} \frac{\partial}{\partial z} p^0_z = f$, the function u^0 is the solution of the homogenized problem, and $\mathcal{A}$ is the limit matrix.

□

3 Asymptotic behavior of the effective coefficient

In this second part, we consider the difference operators obtained by discretizing a random two-dimensional high-contrast checker-board structure, as various discretization methods are applied.

To define the random media, we split the plane $\mathbb{R}^2$ into regular squares $\{[-\frac{1}{2},\frac{1}{2}]^2 + j\}$, $j \in \mathbb{Z}^2$, and assign a value of permeability, independently at each square, as follows:

$$\kappa(y) \triangleq \begin{cases} \delta & \text{with probability} \quad p \\ 1 & \text{with probability} \quad 1-p \end{cases}, \quad y \in \left[-\frac{1}{2},\frac{1}{2}\right]^2 + j,\ j\in\mathbb{Z}^2,$$

where δ is a small strictly positive parameter (see Figure 1). Then, we consider the grid $\mathbb{Z}^2$, fix a finite set $\Lambda \subset \mathbb{Z}^2$ and define the transition probabilities $\{p_z(x);\, x \in \mathbb{Z}^2,\, z \in \Lambda\}$ to be a function of $\{\kappa(x+z)\}$, $z \in \Lambda$. Finally, we define the coefficients of operator A_ε in terms of $\{p_z(x)\}$ in the usual way described by (7).

Henceforth, we suppose that the properties (*i*), (*ii*), (*iii*) on page 8 are satisfied. It is also clear that the family $\{p_z(x)\}$ is ergodic. Now, the following assertion is a direct consequence of Theorem 2.17 (see also Kozlov [12], §2).