Propositional Circumscriptions

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Abstract: Circumscription formalizes in terms of classical logic various aspects of common sense reasoning: exceptions are made as rare as possible from the given knowledge. We give the main properties of the reasoning thanks to circumscriptions. We provide various counter-examples to prove some counter-properties. We restrict our study to propositional circumscriptions, and we show that even in this case, things are not that easy. We evoke the predicate calculus case, showing that the propositional case suffices to get a good understanding of the logical properties of circumscriptions in any case. As we stick to traditional circumscriptions, we have adapted significantly the literature on "preferential models". We study carefully the more general "formula circumscription", examining when sets of formulas give rise to the same circumscription, and providing the first known characterization result. Finally, an intuitive presentation of some of these (counter-)properties is given. Examples of real circumscriptions illustrate the utility of this study for translating a common sense situation in terms of circumscription. This part contains arguments against some classical methods and proposes new methods instead, which have a better behavior and use simpler circumscriptions than the classical methods. Our study can be considered as giving the first steps towards an automatic way of translating sets of rules in terms of circumscription, which should allow to use seriously circumscription for what it has been proposed till the beginning. Also, as we describe syntactically all the sets of formulas which give rise to some given propositional circumscription, this study should help the automatization.

Key-words: Circumscription, common sense reasoning, knowledge representation, minimal models logic, non monotonic reasoning, preferential entailment, sceptical default inference.

(Résumé : tsvp)

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Circonscriptions propositionnelles

Résumé : La circonscription est un formalisme logique bien adapté au raisonnement de sens commun, et en particulier aux règles avec exceptions: les exceptions sont rendues aussi rares que possible en tenant compte des données. Nous donnons les principales propriétés du raisonnement par circonscription connues de nous à ce jour. Nous fournissons aussi des contre-exemples afin de préciser les limites précises des propriétés satisfaites. Nous nous restreignons au cas de la logique propositionnelle. Cependant, cette étude montre que ce seul cas est déjà suffisamment complexe pour permettre de bien cerner les propriétés essentielles de la circonscription en général. D'ailleurs, afin de justifier cette affirmation, nous donnons à chaque fois que c'est possible l'équivalent en terme de circonscription de prédicats des propriétés et contre-propriétés citées. Comme notre but est la circonscription, nous nous en tenons ici à l'inférence préférentielle la plus naturelle, qui suffit dans ce cas, ce qui s'écarte parfois de la littérature sur une "inférence préférentielle" plus complexe. Nous étudions aussi en détail la notion plus générale de circonscription de formules, en particulier nous précisons quand deux ensembles de formules donnent naissance à la même circonscription, et nous donnons un résultat de caractérisation des circonscriptions de formules, même dans le cas infini. Enfin, une partie importante est réservée à la signification intuitive, en termes de raisonnement, de quelques unes des propriétés et contre-propriétés présentées. Plusieurs exemples illustrent l'utilité de cette étude quand il s'agit de traduire précisément une situation donnée par des règles de sens commun en terme de circonscriptions. Cette partie contient une critique de méthodes classiques et présente des méthodes inédites. Ces nouvelles méthodes se contentent des circonscriptions les plus simples, là où les méthodes classiques nécessitaient parfois des versions plus exotiques.

En conclusion, nous montrons comment cette étude fournit les bases d'une méthode systématique de traduction d'un ensemble de règles informelles en termes de circoncriptions. Cela permettrait enfin d'utiliser la sérieusement la circonscription pour ce pourquoi elle a été conçue dès l'origine.

Mots-clé : Circonscription, inférence préférentielle, inférence prudente par défauts, logique des modèles minimaux, raisonnement de sens commun, raisonnement non monotone, représentation des connaissances.

Contents

1 Introduction

Circumscription has been introduced by McCarthy at the end of the seventies as a way of formalizing important aspects of common sense reasoning. It uses classical logic for representing knowledge with implicit information or rules with exceptions, allowing to reason non monotonically. Non monotony is a "negative feature", and the study of "positive properties" has been initiated by [Gab85] in the middle of the eighties and studied thoroughly in a context of general preferential entailment since by [KLM90, Mak94] and other authors.

Strangely enough, this important study has never been applied seriously to the traditional circumscriptions. As a first step towards this study, we examine propositional circumscriptions, in the infinite case. For the sake of comparison, we evoke also very concisely the largely unexplored case of the predicate circumscription: indeed the properties concerned are the same ones, only the conditions of applicability are more complicated and not detailed here.

We examine the main properties of propositional circumscriptions, and we precise which variant of each property holds. To our knowledge, the great majority of these results are new, and references are given for the exceptions. The properties studied here are simple (except a few "auxiliary" ones) and they have easy and important interpretations in terms of knowledge representation. For example, the well known "case reasoning", or (CR), means that, if we know that *birds fly*, and also that *bats fly*, then, if all we know about *Tweety* is that it is *a "bird or a bat"*, we get the expected conclusion that *Tweety flies.* This gives the main motivation for this study: it is necessary to know which properties hold in order to know whether circumscription is adapted to what we want or not.

We give also two examples showing how the study of these intuitively simple and natural properties may have some important technical consequences. One example uses an infinite variant of (CR): we establish that some way of expressing finite circumscriptions only thanks to their "inaccessible formulas", does not extend to the infinite case. Precisely, it extends only in the cases when this extension is easy, and in particular it does not extend when varying propositions are present. This result was unexpected for us. Another example uses a property called (rather strangely) rational monotony. We show how, from the known fact that any non trivial circumscription falsifies rational monotony, we may deduce that a technical result which was believed to be true cannot be true in reality.

These examples let us think that further studies on the subject, not only are necessary in order to improve our comprehension of the way circumscription "reasons", but could also give various more technical results as a bonus.

Hopefully also, some of the results given in this text could be used in order to help the automatization of circumscription.

As the examples of practical use of circumscription show, it is generally much more natural and simple to use the more general notion called formula circumscription. Thus we study also this version in great details. One of our results answers a problem stated as open by Makinson in 1994: the characterization of formula circumscription.

We begin (section 3) by a reminder about propositional preferential entailments, as this is now relatively well-known, we do not give again the proofs of the results given here (see [MR99] for precise references and proofs[1]). Notice however that we depart significantly from the intricate (and these complications are useless for studying circumscriptions, even in the predicate calculus) "preferential models" or related formalisms used in [KLM90] and its followers. Then (section 4), we deal in great details with propositional circumscriptions. We give the precise variants of the properties which are satisfied. We give all the proofs, including all the counter-examples, in order to make precise the border line between the properties satisfied by circumscriptions and related properties which are not satisfied. Then (section 5), we examine formula circumscription, which is very useful in applications of circumscriptions, providing three equivalent definitions. We examine when two sets of formulas are "equivalent" with respect to formula circumscription, and we show why in fact two different kinds of equivalence must be considered. Also, we show that in the finite case the notion of formula circumscription is equivalent to the apparently more general notion of cumulative and consistency preserving preferential entailment. This last result was probably ready to get out, because we have discovered it in two recent apparently unrelated papers at the time of printing! Elaborating from a known result, we have shown how the elimination of the fixed propositions in a formula circumscription makes precise the equivalence between formula circumscription and sceptical Poole's defaults without constraint. The more technical results about formula circumscription (and also a few related results about ordinary propositional circumscription) are examined in section 6. There, the problem of the equivalences of sets of formulas with respect to formula circumscription is developed, and it is shown how this gives naturally rise to the introduction of various sets of "positive formulas". It is known till the introduction of the notion of circumscription that the notion of "positive formulas" plays a central role. However, and rather strangely, what this notion denotes exactly in the case of formula circumscription, or even in the case of ordinary (propositional or predicate) circumscription with varying objects, had never been studied. We show that various notions of "positive formulas" are necessary. In this more technical section also, the problem of the characterization of the notion of formula circumscription in the infinite case is solved: to our knowledge, it is the first time that such a characterization result appears in the literature.

Finally (section 7) comes a rather detailed intuitive explanation of the main properties, and counter properties, as some of them are not very well known. We explain why we think that the set of the already known properties of circumscription makes that circumscription is a good candidate for translating sets of rules with exceptions. In order to confirm this opinion, examples of real circumscriptions used for translating some rules are given. One

[1] [MR99] was begun much earlier than the present report (before 1994). It deals with great details with circumscriptions and preferential entailments, in the predicate calculs and in the propositional calculus. As the subject is rather complex, in particular the limits of the interesting properties are not always easy to be precised exactly, we have decided to extract the results for propositional circumscriptions, which has given the present report. The "big" report is almost achieved, only it needs polishing, and this polishing may take some time, but it will be published in a near future. Thus, for results about general preferential entailments, even when these results are ours, we do not provide the proofs here, in order to keep the present report short enough, and we refer to our "big" future report. However, all the results concerning specifically circumscriptions are proved in this present report, except a very small number of easy or well known results.

of these examples shows how this study can be applied in order to decide which circumscription is better to translate some given set of rules. Another example shows that we cannot expect too much from this study alone: the real situation to be translated must be studied carefully in order to obtain an appropriate translation. Also, this example shows that in some cases when the tradition uses unions of classical circumscriptions, in fact a unique formula circumscription gives a better solution. In all these examples, we show how a pertinent combination of the sets of formulas associated to individual rules may give a good answer for translating the "rule" corresponding to the set of these "individual rules".

We have taken great care in attributing any result about circumscription to the first text (or at least to a more elaborated text by the same author[s]) where it appears, to the best of our knowledge, even in cases when the relation with our own result was not obvious. We have tried to make as many connections as possible with equivalent formalisms. Certainly, there are still omissions and wrong attributions, and we welcome any comment on the subject. This report has taken some times (the situation is much worse for the "big report" evoked in note 1...), partly because the characterization of formula circumscription in the infinite case was not so obvious, and we wanted to avoid publishing this report without such a result. Thus, sometimes we have discovered in the recent literature a result that we had got before its publication, thinking it was new. Naturally, even in these cases, we attribute the concerned result to (whom we think is) the right person.

2 Notations

$\boldsymbol{\mathcal{L}}$ is the propositional logic considered, which contains propositional symbols noted $P, Q, Z, \cdots$. $\mathcal{V}(\boldsymbol{\mathcal{L}})$ is the set of all the propositional symbols. As usual, $\boldsymbol{\mathcal{L}}$ denotes also the set of all the formulas existing in this logic. As we are in a propositional logic, our logic is compact and complete. Also, for preferential entailments, a relation between models may be assimilated to a relation between complete theories.

The set of *theories* is $\mathcal{J} = \{\mathcal{T}/\mathcal{T} \subseteq \boldsymbol{\mathcal{L}},\ \mathcal{T} = Th(\mathcal{T})\}$ where $Th(\mathcal{T}) = \{\varphi/\mathcal{T} \models \varphi\}$.

Letters φ and ψ will denote formulas in $\boldsymbol{\mathcal{L}}$. $\perp$ denotes the false formula, and $\top$ the true formula.

$\mathcal{C}$ denotes the set of all the *complete* theories $\mathcal{T}$ in $\boldsymbol{\mathcal{L}}$: $\varphi \in \mathcal{T}$ exclusive or $\neg\varphi \in \mathcal{T}$ for any φ.

The letter $\mathcal{T}$ (with possible subscript or superscript) will denote subsets of $\boldsymbol{\mathcal{L}}$, when $\mathcal{T}$ is a theory ($\mathcal{T} \in \mathcal{J}$), this will be precised, and $\mathcal{T}'$ (with possible subscript) will be reserved to complete theories ($\mathcal{T}' \in \mathcal{C}$). For any theories $\mathcal{T}_1$ and $\mathcal{T}_2$ in $\boldsymbol{\mathcal{L}}$ we have $\mathcal{T}_1 \models \mathcal{T}_2$ iff $\mathcal{T}_1 \supseteq \mathcal{T}_2$. Notice that letters $\Phi, \Psi, \cdots$ and X will also denote sets of formulas, but only in cases when considering the logical closure by Th of these sets is not intended.

For any subset $\mathcal{T}$ of $\mathcal{L}$, $\mathcal{V}(\mathcal{T})$ denotes the set of the propositional symbols appearing in $\mathcal{T}$ and $\mathcal{L}(\mathcal{T})$ is the language such that $\mathcal{V}(\mathcal{L}(\mathcal{T})) = \mathcal{V}(\mathcal{T})$. If $\mathbf{P} \subseteq \mathcal{L}$, then we note $\mathbf{P}(\mathcal{T})$ for the set $\mathbf{P} \cap \mathcal{V}(\mathcal{T})$ and $\mathbf{P}^*(\mathcal{T})$ for $\mathbf{P} - \mathbf{P}(\mathcal{T})$.

When we consider *partitions* of $\mathcal{V}(\mathcal{L})$, e.g. $(\mathbf{P}, \mathbf{Q}, \mathbf{Z})$, we mean as usual $\mathbf{P} \cup \mathbf{Q} \cup \mathbf{Z} = \mathcal{V}(\mathcal{L})$ and the pairwise intersections are empty, however, notice that we allow empty sets among $\mathbf{P}, \mathbf{Q}, \mathbf{Z}$.

We use the notations $\mathcal{T}_1 \sqcup \mathcal{T}_2 = Th(\mathcal{T}_1 \cup \mathcal{T}_2)$, $\bigsqcup_{i \in I} \mathcal{T}_i = Th(\bigcup_{i \in I} \mathcal{T}_i)$ and $\mathcal{T} \sqcup \varphi = \mathcal{T} \sqcup \{\varphi\}$.

For any $\mathcal{T} \in \mathcal{J}$, $\Gamma(\mathcal{T})$ denotes the set of all the complete theories which entail (i.e. contain) $\mathcal{T}$. We know that $\mathcal{T} = \bigcap_{\mathcal{T}' \in \Gamma(\mathcal{T})} \mathcal{T}'$.

Letters μ and ν denote interpretations for $\mathcal{L}$, μ may be assimilated to its corresponding complete theory $Th(\mu) = \{\varphi \ / \ \varphi \in \mathcal{L}, \mu \models \varphi\}$ ($Th(\mu) \in \mathcal{C}$) (this ambiguous meaning of $\models$ and of Th is classical in logic and should not provoke confusion). We will denote the interpretations by the subset of $\mathcal{V}(\mathcal{L})$ that they satisfy: e.g. if $\mathcal{V}(\mathcal{L}) = \{P, Q, Z\}$ and $\mu = \{P, Z\}$, then $Th(\mu) = Th(P \wedge \neg Q \wedge Z)$.

$\mathcal{M}$ denotes the set of all the interpretations for $\mathcal{L}$, i.e the set of all the subsets of $\mathcal{V}(\mathcal{L})$. For any subset $\mathcal{M}'$ of $\mathcal{M}$, we note $Th(\mathcal{M}')$ for the set $\bigcap_{\mu \in \mathcal{M}'} Th(\mu) = \{\varphi \ / \ \varphi \in \mathcal{L}, \mu \models \varphi$ for any $\mu \in \mathcal{M}'\}$.

An expression such as "for any φ" will mean "for any formula φ in $\mathcal{L}$", similarly "for any $\mathcal{T}$" will mean "for any subset of $\mathcal{L}$" and "for any μ" will mean "for any interpretation μ for $\mathcal{L}$", and similarly when "for any" is replaced by "there exists some".

$\mathcal{M}(\mathcal{T})$ denotes the set of all the models of $\mathcal{T}$: $\mathcal{M}(\mathcal{T}) = \{\mu \in \mathcal{M} \ / \ \mu \models \mathcal{T}\}$. Thus we have $\mu \in \mathcal{M}(\mathcal{T})$ iff $Th(\mu) \in \Gamma(\mathcal{T})$.

We note $\Gamma(\varphi)$ instead of $\Gamma(\{\varphi\})$ and $\mathcal{M}(\varphi)$ instead of $\mathcal{M}(\{\varphi\})$, and also $\mathbf{P}(\varphi)$ for $\mathbf{P}(\{\varphi\})$, $\mathcal{V}(\varphi)$ for $\mathcal{V}(\{\varphi\})$.

As we work in propositional logic, there exists an obvious correspondence between $\mathcal{M}$ and $\mathcal{C}$, $\mathcal{M}(\mathcal{T})$ and $\Gamma(\mathcal{T})$, μ and $Th(\mu)$ and so on. However, we like better to keep the two kinds of notations, as sometimes it is easier to consider the interpretations, and sometimes it may be easier to consider the complete theories.

$TC(\cdots)$ denotes the classical topological closure: for any set $S \subseteq \mathcal{C}$, we note $TC(S)$ for the closure of the set S, i.e. $TC(S) = \Gamma(\bigcap_{\mathcal{T}' \in S} \mathcal{T}')$. Thanks to the correspondence between $\mathcal{C}$ and $\mathcal{M}$, for any set $\mathcal{M}' \subseteq \mathcal{M}$, we will also denote $TC(\mathcal{M}')$ for the closure of $\mathcal{M}'$, i.e. $TC(\mathcal{M}') = \mathcal{M}(Th(\mathcal{M}'))$.

$\mathcal{A}$ is the set of the *finitely axiomatizable theories* of $\mathcal{L}$: $\mathcal{T} \in \mathcal{A}$ iff there exists some φ such that $\mathcal{T} = Th(\varphi)$. If $\mathcal{T} \notin \mathcal{J}$, we will as usual call $\mathcal{T}$ *finitely axiomatizable* iff $Th(\mathcal{T}) \in \mathcal{A}$.

$\equiv$ (without any subscript) is a meta symbol meaning "equivalent to".

The term *formula* will generally be used in fact to denote the quotient of this notion by logical equivalence: we will note $\varphi = \psi$ for $\varphi \equiv \psi$, i.e. $\Gamma(\varphi) = \Gamma(\psi)$. This means that we often assimilate the set of all the formulas $\mathcal{L}$ to the quotient of $\mathcal{L}$ by logical equivalence (Lindenbaum algebra). However, it is not always possible to make this assimilation: precisely, when we consider $\mathcal{V}(\varphi)$ (or $\mathcal{V}(\mathcal{T})$), we cannot assimilate φ to its equivalence class by logical equivalence. The context should make clear what is intended by a *formula*, and this abusive partial assimilation, which simplifies considerably the expression and proof of some results, should not provoke confusion.

3 Propositional preferential entailment

3.1 Pre-circumscriptions and a menagerie of logical properties

Definition 3.1 A *pre-circumscription* f (in $\mathcal{L}$) is an extensive (i.e. $f(\mathcal{T}) \supseteq \mathcal{T}$ for any $\mathcal{T}$) mapping from $\mathcal{J}$ to $\mathcal{J}$. For any subset $\mathcal{T}$ of $\mathcal{L}$ not in $\mathcal{J}$, we use the abbreviation $f(\mathcal{T}) = f(Th(\mathcal{T}))$[2]. We use also the notation $f(\varphi)$ for $f(\{\varphi\}) = f(Th(\varphi))$.

A *pre-circumscriptions restricted to* $\mathcal{U} \subseteq \mathcal{J}$ is the restriction to $\mathcal{U}$ of a pre-circumscription f.

The *union* $f_1 \sqcup f_2$ and the *intersection* $f_1 \cap f_2$ of two pre-circumscriptions f_1 and f_2 are the pre-circumscriptions defined by $(f_1 \sqcup f_2)(\mathcal{T}) = f_1(\mathcal{T}) \sqcup f_2(\mathcal{T})$ and $(f_1 \cap f_2)(\mathcal{T}) = f_1(\mathcal{T}) \cap f_2(\mathcal{T})$ respectively. Unions and intersections of pre-circumscriptions restricted to $\mathcal{U}$ are pre-circumscriptions restricted to $\mathcal{U}$. $\square$

Definitions 3.2 Here are various properties a pre-circumscription may possess. $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_i$ are in $\mathcal{J}$:

[2] Thus, for a reader familiar with the terminology used in [KLM90], a pre-circumscription is an "inference operation" satisfying the full (or theory) versions of reflexivity, "LLE", "RW" and "AND".

Idempotence :	$f(f(\mathcal{T})) = f(\mathcal{T})$	**(Idem)**
Reverse monotony :	if $\mathcal{T} \subseteq \mathcal{T}''$, then $f(\mathcal{T}'') \subseteq f(\mathcal{T}) \sqcup \mathcal{T}''$	**(RM)**
Case reasoning :	$f(\mathcal{T}_1) \cap f(\mathcal{T}_2) \subseteq f(\mathcal{T}_1 \cap \mathcal{T}_2)$	**(CR)**
id. (infinite version) :	$\bigcap_{i\in I} f(\mathcal{T}_i) \subseteq f(\bigcap_{i\in I} \mathcal{T}_i)$	**(CR∞)**
Conjunctive coherence :	$f(\mathcal{T} \sqcup \mathcal{T}'') \subseteq f(\mathcal{T}) \sqcup f(\mathcal{T}'')$	**(CC)**
id. (infinite version) :	$f(\bigsqcup_{i\in I} \mathcal{T}_i) \subseteq \bigsqcup_{i\in I} f(\mathcal{T}_i) \quad (I \neq \emptyset)$	**(CC∞)**
Restricted identity :	if $f(\mathcal{T}_1) \subseteq \mathcal{T}_2$ then $f(\mathcal{T}_2) = \mathcal{T}_2$	**(RI)**
Disjunctive coherence :	$f(\mathcal{T}_1 \cap \mathcal{T}_2) \subseteq f(\mathcal{T}_1) \sqcup f(\mathcal{T}_2)$	**(DC)**
Disjunctive rationality :	$f(\mathcal{T}_1 \cap \mathcal{T}_2) \subseteq f(\mathcal{T}_1) \cup f(\mathcal{T}_2)$	**(DR)**
Monotony :	$f(\mathcal{T}) \subseteq f(\mathcal{T} \sqcup \mathcal{T}'')$	**(MON)**
Cumulative monotony :	if $\mathcal{T} \subseteq \mathcal{T}'' \subseteq f(\mathcal{T})$ then $f(\mathcal{T}) \subseteq f(\mathcal{T}'')$	**(CM)**
Cumulative transitivity :	if $\mathcal{T} \subseteq \mathcal{T}'' \subseteq f(\mathcal{T})$ then $f(\mathcal{T}'') \subseteq f(\mathcal{T})$	**(CT)**
Cumulativity :	if $\mathcal{T} \subseteq \mathcal{T}'' \subseteq f(\mathcal{T})$ then $f(\mathcal{T}) = f(\mathcal{T}'')$	**(CUMU)**
Preservation of consistency :	if $f(\mathcal{T}_1) = Th(\perp) = \boldsymbol{\mathcal{L}}$ then $\mathcal{T}_1 = \boldsymbol{\mathcal{L}}$	**(PC)**
Coherent non monotony :	if $\perp \notin f(\mathcal{T}) \sqcup \mathcal{T}''$ then $\perp \notin f(\mathcal{T}) \sqcup f(\mathcal{T} \sqcup \mathcal{T}'')$	**(CNM)**
Rational monotony :	if $\perp \notin f(\mathcal{T}) \sqcup \mathcal{T}''$ then $f(\mathcal{T}) \subseteq f(\mathcal{T} \sqcup \mathcal{T}'')$	**(RatM)**

P′: if $f(\mathcal{T}) \sqcup \varphi \neq \boldsymbol{\mathcal{L}}$ then $\Gamma(f(\mathcal{T})) \cap \Gamma(\varphi) \cap \bigcup_{\mathcal{T}'\in\Gamma(\mathcal{T})} \bigcap_{\mathcal{T}''\in\Gamma(\mathcal{T})} \Gamma(f(\mathcal{T}' \cap \mathcal{T}'')) \neq \emptyset$ **(P′)**
□

These properties come from various texts and are well-know except (CC), (CC∞), (CR∞) and (RI), which are ours. Our names come from the literature, except when some conflict existed between the notations used in various texts, or when no specific name has been given before, to our knowledge. (PC) appears in a context very close to circumscription in [BS85]. (CT) and (CM) appear for the first time in the context of non monotonic reasoning in the pioneer [Gab85] (which has initiated the systematic study of such properties in this context) under the respective names of cut and restricted monotonicity, the names we have given in this text being rather common now. (CR), often called also OR, as in [KLM90], or distributivity as in [Sch92], is also one of the "oldest" properties of this kind which has been considered. (RM) is called by various names: its formula-only version (called (RM0) below) is called deduction principle in [Sho88] where it makes it first apparition in a context of preferential entailment and circumscription; among the various names given to the full version (RM) we may cite [infinite] conditionalization in [Sch92, Mak94] and, in the lines of Shoham, deductivity in [FL93]. (DC) has no name in [Sat90] where it makes its first apparition to our knowledge (formula-only version, called (DC0) below, which, as remarked by Satoh who was mainly concerned by circumscription, "corresponds [in another context] to the property called (R8) in [KM91]"). The formula versions of (DR) and (RatM) (called (DR0) and (RatM0) below) appear in [KLM90]. (P′) appears without name at all ((P′) itself is not a real name either...) in lemma 3.4.9 in [Mak94] which uses it differently than we do (Makinson does not give the characterization result that we give in proposition 3.8-2 below). The formula-only version of (CNM) appears without other name than (13) in [LM92]. Beware that a few texts denote rational monotony by (RM), but the name "rational monotony" is misleading: see below in subsection 7.5 why we consider (RatM) as not "rational" at all. Moreover (RatM) is not a property of circumscriptions, while some variant of

(RM) ((RM1) given below) is a fundamental property of circumscriptions, thus it is much better in texts about circumscriptions to use a short name for (RM) and a longer name for (RatM), which is rather exotic in this context.

For an intuitive meaning of most of these properties, see section 7 below.

In order to study circumscriptions, we need also some weaker versions of the properties given above:

Definitions 3.3 Formula versions Formula-only versions

	Formula versions		Formula-only versions
(RM1)	$f(\mathcal{T} \sqcup \varphi) \subseteq f(\mathcal{T}) \sqcup \{\varphi\}$	**(RM0)**	$f(\psi \wedge \varphi) \subseteq f(\psi) \sqcup \varphi$
(CR1)	$f(\mathcal{T} \sqcup \varphi) \cap f(\mathcal{T} \sqcup \psi) \subseteq f(\mathcal{T} \sqcup \varphi \vee \psi)$	**(CR0)**	$f(\varphi) \cap f(\psi) \subseteq f(\varphi \vee \psi)$
(DC1)	$f(\mathcal{T} \sqcup \varphi \vee \psi) \subseteq f(\mathcal{T} \sqcup \varphi) \sqcup f(\mathcal{T} \sqcup \psi)$	**(DC0)**	$f(\varphi \vee \psi) \subseteq f(\varphi) \sqcup f(\psi)$
(DR1)	$f(\mathcal{T} \sqcup \varphi \vee \psi) \subseteq f(\mathcal{T} \sqcup \varphi) \cup f(\mathcal{T} \sqcup \psi)$	**(DR0)**	$f(\varphi \vee \psi) \subseteq f(\varphi) \cup f(\psi)$
(CC1)	$f(\mathcal{T} \sqcup \varphi \wedge \psi) \subseteq f(\mathcal{T} \sqcup \varphi) \sqcup f(\mathcal{T} \sqcup \psi)$	**(CC0)**	$f(\varphi \wedge \psi) \subseteq f(\varphi) \sqcup f(\psi)$
(CT1)	if $\varphi \in f(\mathcal{T})$ then $f(\mathcal{T} \sqcup \varphi) \subseteq f(\mathcal{T})$	**(CT0)**	if $\varphi \in f(\psi)$ then $f(\varphi \wedge \psi) \subseteq f(\psi)$
(CM1)	if $\varphi \in f(\mathcal{T})$ then $f(\mathcal{T}) \subseteq f(\mathcal{T} \sqcup \varphi)$	**(CM0)**	if $\varphi \in f(\psi)$ then $f(\psi) \subseteq f(\varphi \wedge \psi)$
(CNM1)	if $\neg\varphi \notin f(\mathcal{T})$ and $\psi \in f(\mathcal{T})$ then $\neg\psi \notin f(\mathcal{T} \sqcup \varphi)$	**(CNM0)**	if $\neg\varphi \notin f(\varphi')$, $\psi \in f(\varphi')$ then $\neg\psi \notin f(\varphi \wedge \varphi')$
(RatM1)	if $\neg\varphi \notin f(\mathcal{T})$ then $f(\mathcal{T}) \subseteq f(\mathcal{T} \sqcup \varphi)$	**(RatM0)**	if $\neg\varphi \notin f(\varphi')$ then $f(\varphi') \subseteq f(\varphi \wedge \varphi')$
(RI1)	if $f(\mathcal{T}) \subseteq \mathcal{T} \sqcup \varphi$ then $f(\mathcal{T} \sqcup \varphi) = \mathcal{T} \sqcup \varphi$.	**(RI0)**	if $f(\psi) \subseteq Th(\varphi \wedge \psi)$ then $f(\varphi \wedge \psi) = Th(\varphi \wedge \psi)$
(MON1)	$f(\mathcal{T}) \subseteq f(\mathcal{T} \sqcup \varphi)$	**(MON0)**	$f(\varphi) \subseteq f(\varphi \wedge \psi)$.

□

Remark 3.4 The properties of pre-circumscriptions can be restricted to $\mathcal{U}$, ($\mathcal{U} \subseteq \mathscr{T}$), their name being subscripted by $_{\mathcal{U}}$, e.g. $(\text{RM})_{\mathcal{U}}$ is: if $\mathcal{T}_1 \in \mathcal{U}, \mathcal{T}_2 \in \mathcal{U}, \mathcal{T}_1 \subseteq \mathcal{T}_2$ then $f(\mathcal{T}_2) \subseteq f(\mathcal{T}_1) \sqcup \mathcal{T}_2$. Thus, (RM0) is $(\text{RM})_{\mathscr{A}}$,... Depending of the property, $\mathcal{U}$ must be stable for intersections and/or unions. □

We use freely results (referenced as "known") about preferential entailments coming from [Sho88, KLM90, Sat90, Sch92, LM92, FL93, Mak94, MR94a, Sch97, MR99]. All these results are referenced, completed and precised in [MR99]. As already indicated in note 1 page 5, this text concentrates its attention to the traditional (propositional) circumscriptions, and does not give again the proofs for general (propositional) pre-circumscriptions or preferential entailments.

Proposition 3.5 (known or obvious) For any pre-circumscription:
(RM1) and (CR1) are equivalent, as are (RM0) and (CR0).
(RM1) implies (CC1), (CNM1) and (RI1).
(RM) implies (CT), (CR), (CNM), (RI), (P′) and (CC∞), (CT) implies (Idem).
(DR) implies (DC), ((RatM)+(PC)) implies (DR) and (CNM).

Any full version implies its corresponding formula version, any formula version implies its corresponding formula-only version: (CR) implies (CR1) which in turn implies (CR0). Also, an infinite version implies its corresponding standard full version: (CR∞) implies (CR) (notice that formula versions of (CR∞) could be defined, but we think that our text gives already enough properties...) and (CC∞) implies (CC). □

3.2 Preferential entailments and their logical properties

Definitions 3.6 A *preference relation* in $\mathcal{L}$ is any binary relation $\prec$ over $\mathcal{C}$. We note $\Gamma_{\prec}(\mathcal{T})$ for the set of the elements $\mathcal{T}'$ of $\Gamma(\mathcal{T})$ *minimal for* $(\mathcal{T}, \prec)$: $\mathcal{T}' \in \Gamma(\mathcal{T})$ and no $\mathcal{T}'' \in \Gamma(\mathcal{T})$ is such that $\mathcal{T}'' \prec \mathcal{T}'$.

Using the correspondence between $\mathcal{C}$ and $\mathcal{M}$, we may also consider $\prec$ as a relation over $\mathcal{M}$. We will indifferently note $Th(\mu) \prec Th(\nu)$ or $\mu \prec \nu$. $\mathcal{M}_{\prec}(\mathcal{T})$ denotes the set of the models of $\mathcal{T}$ minimal for $\prec$, and corresponds to $\Gamma_{\prec}(\mathcal{T})$:
$\Gamma_{\prec}(\mathcal{T}) = \{Th(\mu) \ / \ \mu \in \mathcal{M}_{\prec}(\mathcal{T})\}$ and $\mathcal{M}_{\prec}(\mathcal{T}) = \{\mu \in \mathcal{M} \ / \ Th(\mu) \in \Gamma_{\prec}(\mathcal{T})\}$.

The *preferential entailment* $f = f_{\prec}$ is the pre-circumscription in $\mathcal{L}$ defined by

$$f_{\prec}(\mathcal{T}) = \bigcap_{\mathcal{T}' \in \Gamma_{\prec}(\mathcal{T})} \mathcal{T}' = Th(\mathcal{M}_{\prec}(\mathcal{T})). \ \Box$$

Remark 3.7 We have $\Gamma(f_{\prec}(\mathcal{T})) = TC(\Gamma_{\prec}(\mathcal{T}))$ or equivalently, $\mathcal{M}(f_{\prec}(\mathcal{T})) = TC(\mathcal{M}_{\prec}(\mathcal{T}))$. (Remind that $TC(\cdots)$ denotes the classical topological closure.)

Our definition is the classical definition of preferential entailments in the propositional case, contrarily to the definitions in e.g. [KLM90] which allows states (which, as noted in [Sch97], is equivalent to allow copies of models), a complication useless for studying circumscriptions (propositional or predicate). □

Proposition 3.8 (known, the results involving (DCC), (CC), (CC∞) and (RI) are from [MR99], including the characterization result in point 2 and most of the characterization result in point 3)

1. Any preferential entailment satisfies (CT), (P′) and (CR), thus (RM1), (CC1), (CNM1) and (RI1).

   A preferential entailment may falsify (RI), (CNM), (RM) or (CR∞).

   For preferential entailments, (RM) and (CC∞) (and (CC) when $\mathcal{L}$ is enumerable) are equivalent.

   Moreover, as we deal with propositional preferential entailments, we get 2 and 3 below:

2. Any preferential entailment satisfies: $f(\mathcal{T}) \subseteq \bigcap_{\mathcal{T}' \in \Gamma(\mathcal{T})} \bigsqcup_{\mathcal{T}'' \in \Gamma(\mathcal{T})} f(\mathcal{T}' \cap \mathcal{T}'')$ **(DCC).**

   Any preferential entailment satisfying (RM) satisfies also (DC).

   A pre-circumscription satisfies (P′) and (DCC) iff it is a preferential entailment.

3. If $\mathcal{V}(\boldsymbol{\mathcal{L}})$ is finite, then a pre-circumscription satisfies (CR) and (DCC) iff it satisfies (CR) and (DC) iff it is a preferential entailment. Remind that in this case (RM) is (RM0), thus also (CR0) or (CR), and similarly (DC) is (DC0). □

As announced above, we try to give as far as possible a few indications about the corresponding results in the predicate calculus case.

Remark 3.9 The predicate calculus case:

0. In the predicate calculus case, any complete theory has as many models as we want (and even more: we must go outside the notion of set!). Thus we must split the notion of preferential entailments as given in definition 3.6 in two different notions: it is not the same thing to start from a relation among $\mathcal{C}$ and from a relation among $\mathcal{M}$. We get two characterization results.

1. For studying predicate circumscriptions, we must start from a preference relation $\prec$ defined among $\mathcal{M}$. In the predicate calculus, a preference relation $\prec$ is any binary relation among the class of all the interpretations $\mathcal{M}$, $\mathcal{M}_\prec(\mathcal{T})$, $\Gamma_\prec(\mathcal{T})$ and $f_\prec(\mathcal{T})$ are defined exactly as in definition 3.6, except that $\Gamma_\prec(\mathcal{T})$ must be defined from $\mathcal{M}_\prec(\mathcal{T})$ as given there, and $f_\prec$ is called a *preferential entailment*.

   A pre-circumscription f is a preferential entailment iff f satisfies the property of *common points*:

   for any formula formula φ, and any set of formulas $\mathcal{T}$, if $\Gamma(f(\mathcal{T})) \cap \Gamma(\varphi) \neq \emptyset$ then there exists $\mathcal{T}' \in \Gamma(f(\mathcal{T})) \cap \Gamma(\varphi)$, such that $\mathcal{T}' \in \bigcap_{\mathcal{T}' \in \Gamma(\mathcal{T}'') \subseteq \Gamma(\mathcal{T})} \Gamma(f(\mathcal{T}''))$. **(CP)**

2. Here is what corresponds to the preferential entailment in the propositional calculus: In the predicate calculus, a pre-circumscription f is a *regular preferential entailment* iff $f = f_\prec$ where $\prec$ is any binary relation among the set of the complete theories $\mathcal{C}$ and $f_\prec$ is defined as in definition 3.6, i.e. $\Gamma(f_\prec(\mathcal{T})) = TC(\Gamma_\prec(\mathcal{T}))$.

   A pre-circumscription f is a regular preferential entailment iff f satisfies (DCC) and (P′).

   As generally predicate circumscriptions (first order or *mixed*, i.e. "second order" in a first order frame) falsify (DCC), they are not regular preferential entailments, thus, the characterization result for preferential entailments which is useful for studying predicate circumscriptions is the result given in point 1 above.

3. If $\prec$ is *compatible with elementary equivalence* (i.e. $\mu' \prec \nu'$ whenever $\mu \prec \nu, Th(\mu) = Th(\mu')$ and $Th(\nu) = Th(\nu')$) then the preferential entailment $f_{\prec}$ is regular.

   From a more algebraic perspective, we get that a preferential entailment $f = f_{\prec}$ satisfies (DCC) iff it is regular, iff it satisfies the following equality, for any set of formulas $\mathcal{T}$:

   $$\{\mathcal{T}' \ / \ \mathcal{T}' \in \Gamma(\mathcal{T}),\ \forall \mathcal{T}'' \in \Gamma(\mathcal{T}),\ \mathcal{T}' \in \Gamma(f(\mathcal{T}' \cap \mathcal{T}''))\} = \Gamma_{\prec}(\mathcal{T}). \qquad \textbf{(Reg)}$$

4. All the results of proposition 3.8-1 apply for the preferential entailments in the predicate calculus, while the results of 3.8-2 apply only for the regular preferential entailments.

   Again, we refer to [MR99] for details and much more about the predicate calculus case.

□

Here is an obvious but useful result:

Proposition 3.10 The larger is the preference relation $\prec$, the stronger is the preferential entailment $f_{\prec}$. Formally: If $\mu \prec \nu$ implies $\mu \prec' \nu$, then for any $\mathcal{T}$, $f_{\prec}(\mathcal{T}) \subseteq f_{\prec'}(\mathcal{T})$. □

Indeed, when the relation increases, there are less minimal models.

Definition 3.11 Some particular preferential entailments have properties useful for circumscriptions:

1. A preference relation $\prec$ satisfies the *closure property* (**(cl)** for short) iff for any $\mathcal{T} \in \mathcal{J}$ we have $\Gamma_{\prec}(\mathcal{T}) = \Gamma(f_{\prec}(\mathcal{T}))$ (i.e. $\mathcal{M}_{\prec}(\mathcal{T}) = \mathcal{M}(f_{\prec}(\mathcal{T}))$).

2. A preference relation $\prec$ is *safely founded* (**(sf)** for short) iff, for any $\mathcal{T}' \in \Gamma(\mathcal{T})$, if $\mathcal{T}' \notin \Gamma_{\prec}(\mathcal{T})$, then there exists $\mathcal{T}'' \in \Gamma_{\prec}(\mathcal{T})$ such that $\mathcal{T}'' \prec \mathcal{T}'$.

3. $\prec$ is *well founded* (**(wf)** for short) iff it is transitive, irreflexive, and there exists no infinitely decreasing chain ($\mathcal{T}'_{i+1} \prec \mathcal{T}'_i$ for any $i \in \mathbb{N}$).

These properties may be restricted to subsets $\mathcal{U}$ of $\mathcal{L}$, and are then noted as in remark 3.4, and, as in definitions 3.3 we use the notations (cl0) and (sf0) for respectively $(\mathrm{cl})_{\mathcal{A}}$ and $(\mathrm{sf})_{\mathcal{A}}$. □

(cl) means that $\Gamma_{\prec}(\mathcal{T})$ is closed for the classical topology, thus its name (it is definability preserving of [Sch92, Sch97] and faithful of [Mak94]). (sf) appears in a context very close to circumscription as minimally modelable in [BS85], then it was confusingly called well-founded in e.g. [EMR85], and a closely related notion is smooth in [KLM90] and stoppered in [Mak94][3]. (wf) is a well known mathematical notion, which implies clearly (sf) while the converse is false (see below, e.g. proposition 4.5).

[3] As it is used in [KLM90] or [Mak94] for their preferential models (see page 47 below), the notion of smoothness or stopperedness implies only (CUMU) and not (PC), while in the "history" of circumscription

Proposition 3.12 (known)

1. If $\prec$ and $\prec'$ are two preference relations, $\prec$ being irreflexive, and if $f_{\prec} = f_{\prec'}$, then $\prec = \prec'$.
2. If $\prec$ satisfies (sf), then $\prec$ is transitive and irreflexive.
3. If $\mathcal{V}(\mathcal{L})$ is finite, then a preference relation satisfies (sf) iff it is transitive and irreflexive. $\square$

Proposition 3.13 (known or [MR99]) If $\prec$ satisfies (sf), $f_{\prec}$ satisfies (PC) and (CUMU). $\square$

There exist various partial converse results from which we extract the following ones, useful for circumscriptions, and which are easy to state if not to prove (see [MR99] for the proofs and also for more results of this kind).

Proposition 3.14 [MR99]

1. A preferential entailment $f_{\prec}$ satisfies (RM), (CUMU) and (PC) iff $\prec$ satisfies (sf) and (cl).
2. If $\mathcal{V}(\mathcal{L})$ is enumerable, then a preferential entailment $f_{\prec}$ satisfies (CUMU) and (PC) iff $\prec$ satisfies (sf). $\square$

Remind that from proposition 3.12 there is only one preference relation $\prec$ possible in these two cases.

Let us complete our characterization results already given above, in propositions 3.8 (last sentence of point 2, and point 3.) and 3.14. As the result we want now is easier to express when the pre-circumscription f satisfies (PC), we will restrict our attention to this case, which suffices when studying circumscriptions.

Definition 3.15 Let f be a pre-circumscription. We associate to f the preference relation $\prec_f$ defined as follows: $\mu \prec_f \nu$ iff $\mu \neq \nu$ and $Th(\mu) \subseteq f(Th(\{\mu, \nu\}))$.

Then we have the following results (as for any result in this section, see [MR99] for more details):

Proposition 3.16 1. If a pre-circumscription f satisfies (DCC), (P′) and (PC), then $f = f_{\prec_f}$, i.e. f is the preferential entailment associated to the preference relation $\prec_f$.

the first motivation for introducing such a property was to get (PC) (see [BS85, EMR85, Sho88]). The analog of smoothness or stopperedness in terms of preferential entailment is what we call (qsf): A preference relation $\prec$ is ***quasi safely founded***, or ***satisfies* (qsf)**, iff for any $\mathcal{T}$ and any $\mathcal{T}' \in \Gamma(\mathcal{T}) - \Gamma_{\prec}(\mathcal{T})$ such that $\mathcal{T}' \not\prec \mathcal{T}'$, there exists $\mathcal{T}'_1 \in \Gamma_{\prec}(\mathcal{T})$ such that $\mathcal{T}'_1 \prec \mathcal{T}'$. Clearly, (sf) implies (qsf), and there is equivalence if $\prec$ is irreflexive. As (qsf) is more complicated than (sf), and is of no use for studying circumscriptions, we refer to [MR99] for more about (qsf). Let us give one small result here: If $\mathcal{V}(\mathcal{L})$ is finite, $\prec$ satisfies (qsf) iff there exists $\prec'$ which is transitive and with *isolated loops* (if $\mathcal{T}' \prec' \mathcal{T}'$ then for any $\mathcal{T}'_1 \neq \mathcal{T}'$ we have $\mathcal{T}' \not\prec' \mathcal{T}'_1$ and $\mathcal{T}'_1 \not\prec' \mathcal{T}'$) such that $f_{\prec} = f_{\prec'}$ [MR98].

2. If $\mathcal{V}(\boldsymbol{\mathcal{L}})$ is finite, we may start from more "natural" properties: if a pre-circumscription f satisfies (CR0), (DC0) and (PC), then $f = f_{\prec_f}$.

   Also in this finite case we have an easy way to express a "constructive characterization result": a pre-circumscription f satisfies (CR0), (DC0), (CM0) and (PC) iff $\prec_f$ is transitive and $f = f_{\prec_f}$.

□

Remind that, as $\prec_f$ is irreflexive by definition, any preferential entailment $f_{\prec_f}$ may be defined by one preference relation only.

Proposition 3.17 (known) $\mathcal{U} \subseteq \boldsymbol{\mathcal{L}}$ is stable for finite unions and intersections. If $\prec$ satisfies $(\mathrm{cl})_{\mathcal{U}}$, then $f_\prec$ satisfies $(\mathrm{RM})_{\mathcal{U}}$ and $(\mathrm{DC})_{\mathcal{U}}$.
Conversely, if a preferential entailment satisfies (RM), then there exists some preference relation $\prec$ such that $f = f_\prec$ and $\prec$ satisfies (cl). Thus (cf proposition 3.12-1) if $\prec$ is irreflexive, then $f_\prec$ satisfies (RM) iff $\prec$ satisfies (cl) (again, remind that we are in the propositional case). □

We have given various properties of preferential entailments. Notice that we may also mix some of these properties together, getting many more "new properties". Cumulativity is a good candidate for getting interesting combinations of properties, let us give here two examples useful for circumscriptions:

Definitions 3.18 A pre-circumscription f satisfies *super case reasoning* iff:
for any $\mathcal{T}_1,\ \mathcal{T}_2$ in $\mathfrak{T}$, if $\mathcal{T}_2 \subseteq f(\mathcal{T}_1)$ then $f(\mathcal{T}_2) = f(\mathcal{T}_1 \cap \mathcal{T}_2)$; i.e.,
for any $\mathcal{T}_1,\ \mathcal{T}_2$ in $\mathfrak{T}$, we have $f(\mathcal{T}_1 \cap \mathcal{T}_2) = f(f(\mathcal{T}_1) \cap \mathcal{T}_2)$. **(SCR)**

A pre-circumscription satisfies *super reverse monotony* iff:
for any $\mathcal{T}_1, \mathcal{T}_2 \in \mathfrak{T}$, if $\mathcal{T}_1 \subseteq f(\mathcal{T}_2)$ then $f(\mathcal{T}_2) \subseteq f(\mathcal{T}_1) \sqcup \mathcal{T}_2$; i.e.
for any $\mathcal{T}_1, \mathcal{T}_2 \in \mathfrak{T}$, if $\mathcal{T}_1 \subseteq f(\mathcal{T}_1 \cap \mathcal{T}_2)$ then $f(\mathcal{T}_2) \subseteq f(\mathcal{T}_1) \sqcup \mathcal{T}_2$. **(SRM)**

Here are the formula versions (respectively "unary version" and "formula-only" version):
For any $\mathcal{T} \in \mathfrak{T}$, $\varphi_1, \varphi_2 \in \boldsymbol{\mathcal{L}}$, if $\varphi_1 \in f(\mathcal{T}, \varphi_2)$ then $f(\mathcal{T}, \varphi_2) \subseteq f(\mathcal{T}, \varphi_1) \sqcup \varphi_2$. **(SRM1)**
For any $\varphi_1, \varphi_2 \in \boldsymbol{\mathcal{L}}$, if $\varphi_1 \in f(\varphi_2)$ then $f(\varphi_2) \subseteq f(\varphi_1) \sqcup \varphi_2$. **(SRM0)** □

Proposition 3.19 • A pre-circumscription satisfies (CUMU) and (CR) iff it satisfies (SCR).
Thus, a preferential entailment satisfies (CUMU) iff it satisfies (SCR).

• A pre-circumscription satisfies (RM) and (CUMU) iff it satisfies (SRM).
Thus, a preferential entailment satisfying (CUMU) satisfies (RM) iff it satisfies (SRM) [FL93].

• (RM1) + (CUMU1) is equivalent to (SRM1) while (RM0) + (CUMU0) is equivalent to (SRM0).
Thus, a preferential entailment satisfies (CUMU1) iff it satisfies (SRM1) and it satisfies (CUMU0) iff it satisfies (SRM0). □

Again, we refer to [MR99] for the proofs of the equivalences appearing in definitions 3.18 and in proposition 3.19 and for precise references[4]. As (CR1) is equivalent to (RM1), and (CR0) to (RM0), (SRM1) and (SRM0) may be considered as the formula and formula-only versions of (SCR) also.

Here is another property of the preference relation which may be useful ("negatively", see below) for studying circumscriptions:

Definition 3.20 [LM92] A preference relation $\prec$ is *ranked* (**(rk)** for short) iff it is a strict order and: if $\mu_2 \not\prec \mu_1$ and $\mu_3 \prec \mu_1$ then $\mu_3 \prec \mu_2$. □

Proposition 3.21 [LM92] A strict order relation $\prec$ on a set E is ranked iff there exists some linear strict order relation $<$ on a set X and a mapping r from E to X such that $x \prec y$ iff $r(x) < r(y)$. $r(x)$ is the *rank* of x. □

Proposition 3.22

1. [LM92] If $\prec$ satisfies (rk), $f_\prec$ satisfies (RatM1) and (DR).
2. [MR99] If $\prec$ satisfies (rk) and (cl), or is a strict linear order, $f_\prec$ satisfies (RatM).
3. [MR99] Any preferential entailment satisfying (RatM) and (PC) satisfies (DR), and also, as noted already in [LM92], any preferential entailment satisfying (RatM1) and (PC) satisfies (DR1).
4. [MR99] If $\prec$ is irreflexive, then $f_\prec$ satisfies (RM)+(RatM) iff $\prec$ satisfies (cl)+(rk). □

4 Propositional circumscription

Predicate circumscription (see e.g. [McC86, PM86, Lif94]) has already been simplified into propositional circumscription (e.g. [Sat90, EG93]). As a first step towards the study of predicate circumscription, we think it is important to study thoroughly the logical properties of propositional circumscription, which, strangely enough, have not been studied a lot so far. We will see that even in the propositional case, the situation is sufficiently complicated to be significative. We indicate, for each property for which it presents some interest, what is the situation in the predicate calculus case (if this situation is known to us, see [MR99] in preparation for more about the predicate calculus case).

[4] Here are a few indications, to the best of our knowledge. These equivalences are new, but the first formulation of (SCR) and the two formulations of (SRM) appear (without name) with a few partial results in [FL93, Mak94] (generally these texts are concerned only by preferential entailments satisfying already (CUMU), which could explain why they were not interested in the equivalences given here). The formula versions are ours.

4.1 Definitions and first properties

Definition 4.1 $(\mathbf{P}, \mathbf{Q}, \mathbf{Z})$ is a partition of $\mathcal{V}(\boldsymbol{\mathcal{L}})$. $\mathbf{P}$ is the set of the *circumscribed propositional symbols*, $\mathbf{Z}$ of the *variable* ones, the remaining propositional symbols, in $\mathbf{Q}$, being *fixed*.

A circumscription is a preferential entailment $CIRC(\mathbf{P}, \mathbf{Q}, \mathbf{Z}) = f_{\prec}$ where
$\mathcal{T}'_1 \prec \mathcal{T}'_2$ iff $\{P \in \mathbf{P} \,/\, \mathcal{T}'_1 \models P\} \subset \{P \in \mathbf{P} \,/\, \mathcal{T}'_2 \models P\}$ ($\subset$ strict) and
$\{Q \in \mathbf{Q} \,/\, \mathcal{T}'_1 \models Q\} = \{Q \in \mathbf{Q} \,/\, \mathcal{T}'_2 \models Q\}$.

If $\mathbf{P} = \emptyset$, $f_{\prec}$ is the identity (we call it the *trivial circumscription*). □

As soon as $\mathbf{P}$ is not empty, there exist $\mathcal{T}$ and φ such that $CIRC(\mathbf{P}, \mathbf{Q}, \mathbf{Z})(\mathcal{T}) \models \varphi$ and $\mathcal{T} \not\models \varphi$:
if $P \in \mathbf{P}$, $\mathcal{T} = Th(\emptyset)$ and $\varphi = \neg P$, we get $\varphi \in CIRC(\mathbf{P}, \mathbf{Q}, \mathbf{Z})(\mathcal{T}) = Th(\{\neg P' \;/\; P' \in \mathbf{P}\})$ and $\varphi \notin \mathcal{T}$.

Remark 4.2 Considered as a relation over $\mathcal{M}$, $\prec$ is defined by:
$\mu \prec \nu$ iff $\mathbf{P} \cap \mu \subset \mathbf{P} \cap \nu$ and $\mathbf{Q} \cap \mu = \mathbf{Q} \cap \nu$. □

It is useful to remind here an easy consequence of this definition: The more propositional symbols are circumscribed and/or allowed to vary, the stronger is the circumscription:

Proposition 4.3 $\mathbf{P}_1, \mathbf{P}_2, \mathbf{Q}, \mathbf{Z}_1, \mathbf{Z}_2$ being a partition of $\mathcal{V}(\boldsymbol{\mathcal{L}})$, we get:

$$CIRC(\mathbf{P}_1 \cup \mathbf{P}_2,\ \mathbf{Q},\ \mathbf{Z}_1 \cup \mathbf{Z}_2)(\mathcal{T}) \models CIRC(\mathbf{P}_1,\ \mathbf{Q} \cup \mathbf{P}_2 \cup \mathbf{Z}_2,\ \mathbf{Z}_1)(\mathcal{T}). \ \square$$

Proof: Use proposition 3.10. □

In definition 4.1, we could suppose only $\mathbf{P} \cup \mathbf{Q} \cup \mathbf{Z} = \mathcal{V}(\boldsymbol{\mathcal{L}})$ without any requirement about disjoint sets (remind anyway our non classical definition of a "partition of $\mathcal{V}(\boldsymbol{\mathcal{L}})$" given in section 2). Indeed, if we use definition 4.1 without this requirement, it is obvious that we get $CIRC(\mathbf{P}, \mathbf{Q}, \mathbf{Z}) = CIRC(\mathbf{P} - \mathbf{Q}, \mathbf{Q}, \mathbf{Z} - \mathbf{P} - \mathbf{Q})$. However, we will try to avoid using this extended definition of $CIRC(\mathbf{P}, \mathbf{Q}, \mathbf{Z})$, sticking to definition 4.1 as it stands, even if this may sometimes complicate the notations (see e.g. corollary 4.23 below).

Remark 4.4 As a circumscription is a preferential entailment, we get that it satisfies (CR), (CT), (Idem), (RM1), (CC1), (CNM1), (DCC), (P′), ...

As we will see, circumscriptions may falsify full (RM), (CNM) and also (CR∞).

From remark 3.9, we know that the "positive results" (first sentence) extend to predicate propositional circumscriptions (first order or mixed or even full second order, as all these are preferential entailments), except for (DCC). The negative results extend to predicate circumscriptions also [MR99]. □

In order to obtain other results, we must refine our study:

Proposition 4.5 1. The preference relation $\prec$ associated to propositional circumscriptions in definition 4.1 satisfies (sf) (thus it is transitive and irreflexive).

2. Also the opposite $\succ$ of $\prec$ satisfies (sf).

3. If $\mathbf{P}$ is infinite, $\prec$ falsifies (wf). $\square$

A (rather technical) bibliographical comment is in order here. To our knowledge, points 1 and 2 applied to $CIRC(\mathbf{P},\mathbf{Q},\mathbf{Z})$ are new (point 3 is folklore), and the proof we give below is thus new. However, the proof for point 1 is closely related, even if it is different, to the proof about predicate mixed[5] circumscription (defined in remark 3.9-2 page 12): the associated $\prec$ satisfies (sf) for the universal theories, provided that only predicates (no function) are allowed to vary (see [BS85, EMR85, Lif94][6]). Moreover, for what concerns the propositional case, using the following three correpondances, we get that point 1 is a consequence of Observation 3.4.11 in [Mak94] (the other part of this Observation concerns precisely the relations evoked in 1) below, which do not concern us directly here). Let us enunciate already these three correpondances, which will come later in this text. 1) The equivalence between "sceptical Poole's defaults without constraint" and formula circumscription reminded in subsection 5.2. 2) The expression of any circumscription $CIRC(\mathbf{P},\mathbf{Q},\mathbf{Z})$ in terms of a formula circumscription without fixed propositions reminded in proposition 5.9-2, which includes a result from [dKK89]. 3) The expression of any formula circumscription in terms of an ordinary circumscription given in definition 5.1, together with the expression of any formula circumscription in terms of a preferential entailment given in proposition 5.5. See also our bibliographical comment for corollary 5.7-2.

Proof: 1. Transitivity and irreflexivity are obvious. Thus we already get (sf) restricted to the theories $\mathcal{T}$ which have a finite set of models. The remaining cases concern theories with an (eventual) infinitely decreasing (for $\prec$) chain of models. We give the proof for (sf) in these cases.

$\prec$ denotes the preference relation associated to $CIRC(\mathbf{P},\mathbf{Q},\mathbf{Z})$ as in definition 4.1. We suppose $\mu \in \mathcal{M}(\mathcal{T}) - \mathcal{M}_{\prec}(\mathcal{T})$, and also, for any $\nu \in \mathcal{M}(\mathcal{T})$, if $\nu \prec \mu$, then $\nu \notin \mathcal{M}_{\prec}(\mathcal{T})$. From Zorn lemma, we know that there exists $(\nu_\xi)_{\xi<\lambda}$, a maximal sequence such that $\nu_0 = \mu$, and for any $\xi < \xi' < \lambda$, $\nu_\xi \in \mathcal{M}(\mathcal{T})$ and $\nu_{\xi'} \prec \nu_\xi$. Thus, λ is a limit ordinal (because $\nu_\xi \notin \mathcal{M}_{\prec}(\mathcal{T})$). We define the set $X = \bigcap_{\xi<\lambda}(\nu_\xi \cap \mathbf{P})$, as the sequence $(\nu_\xi \cap \mathbf{P})_{\xi<\lambda}$ is strictly decreasing, we get $X \subset \nu_\xi \cap \mathbf{P}$ for any $\xi < \lambda$. We define $\mathcal{A} = \mathcal{T} \cup \{Q \;/\; Q \in \mu \cap \mathbf{Q}\} \cup \{\neg Q \;/\; Q \in \mathbf{Q} - \mu\} \cup \{\neg P \;/\; P \in \mathbf{P} - X\}$. If $\nu \models \mathcal{A}$, then $\nu \models \mathcal{T}$ and for any $\xi < \lambda$, $\nu \cap \mathbf{P} \subseteq X \subset \nu_\xi \cap \mathbf{P}$ thus $\nu \prec \nu_\xi$, a contradiction with the maximality of $(\nu_\xi)_{\xi<\lambda}$. Thus $\mathcal{A}$ is inconsistent. By compactness there exists an inconsistent set $\mathcal{A}' = \mathcal{T} \cup \{Q \;/\; Q \in \mu \cap \mathbf{Q}\} \cup \{\neg Q \;/\; Q \in$

[5] The preference relation $\prec$ associated to the first order circumscription falsifies (sf), even without varying predicate and for universal finitely axiomatizable theories, and in fact this circumscription falsifies (CM0) (ex. 5.2 in [MR94b]).

[6] The original proof, where all the predicates are circumscribed, is in [BS85]. [EMR85] and finally Lifschitz have shown that this proof extends to the cases where fixed predicates and finally also varying predicates are allowed.

$\mathbf{Q} - \mu\} \cup \{\neg P_1, \cdots \neg P_n\}$ where $P_i \in \mathbf{P} - X$ for any i such that $1 \leq i \leq n \in \mathbb{N}$. As $P_i \notin X$, there exists $\xi_i < \lambda$ such that $P_i \notin \nu_{\xi_i} \cap \mathbf{P}$. Let $\xi = \sup_{1 \leq i \leq n} \xi_i$, ν_ξ is a model of $\mathcal{A}'$, a contradiction with the inconsistency of $\mathcal{A}'$. Thus, our initial assumption about μ is false. Notice that an easier proof can be given when $\mathcal{V}(\boldsymbol{\mathcal{L}})$ is enumerable.

2. The proof for the opposite $\succ$ of $\prec$ is similar (see also corollary 5.7-4 below and its comment).

3. See example 4.6 below. This last point, showing that the relation $\prec$ is not well founded in the traditional mathematical meaning of the expression, is given here in order to remind that we certainly cannot restrict our attention to relations satisfying (wf) (as was proposed in the pioneer [Sho88]) when studying circumscriptions, even in the propositional case. This remark is folklore now, but from times to times it seems to have been forgotten. □

Example 4.6 [folklore] $\mathcal{V}(\boldsymbol{\mathcal{L}}) = \mathbf{P} = \{P_i\}_{i \in \mathbb{N}^*}$. $\prec$ is the relation associated to $CIRC(\mathbf{P}, \emptyset, \emptyset)$. We define $\mu_i = \mathcal{V}(\boldsymbol{\mathcal{L}}) - \{P_j\}_{0 \leq j \leq i}$. We have $\mu_1 \prec \mu_0$, $\mu_2 \prec \mu_1, \cdots, \mu_{i+1} \prec \mu_i, \cdots$. □

Proposition 4.7 Any (propositional) circumscription satisfies (PC) and (CUMU), thus also (SCR). □

Proof: Use propositions 3.13 and 4.5 (and 3.19). □

This is to be related to the similar result about mixed predicate circumscription without varying function, for universal theories: it satisfies (PC) (thus the weaker first order circumscription also) and (CUMU) (this result does not extend to the first order version [MR94b]).

4.2 When $\mathcal{T}$ is finite: the circumscription axiom

We have given a semantical definition of propositional circumscription, a syntactical definition is also well known, and we give a few words about it now.

Remark 4.8 In definition 4.1, we may intersect the sets $\mathbf{P}, \mathbf{Q}, \mathbf{Z}$ with $\mathcal{V}(\mathcal{T})$.

Precisely, for any formula $\varphi \in \boldsymbol{\mathcal{L}}$ we have (remind the notations given in section 2):

$CIRC(\mathbf{P}, \mathbf{Q}, \mathbf{Z})(\mathcal{T}) \models \varphi$ iff
$CIRC(\mathbf{P}(\mathcal{T}), \mathbf{Q}(\mathcal{T}), \mathbf{Z}(\mathcal{T}))(\mathcal{T}) \cup \{\neg P \ / \ P \in \mathbf{P}^*(\mathcal{T})\} \models \varphi$.

Notice that the second $CIRC$ is defined in the sublanguage $\boldsymbol{\mathcal{L}}(\mathcal{T})$. □

The proof is straightforward and left to the reader.

Notations 4.9 If $\mathbf{p} = \{\varphi_i\}_{i \in I}$ is a set of formulas in $\boldsymbol{\mathcal{L}}$ (or of elements in $\mathcal{V}(\boldsymbol{\mathcal{L}})$, assimilated here to their corresponding formula), we will sometimes assimilate this set to the sequence $(\varphi_i)_{i \in I}$ (element of $\boldsymbol{\mathcal{L}}^I$). If $\mathbf{p} = (\varphi_i)_{i \in I}$ and $\mathbf{p}' = (\varphi'_i)_{i \in I}$ are two sequences, we note $\mathbf{p} \Rightarrow \mathbf{p}'$ for the sequence of implications $(\varphi_i \Rightarrow \varphi'_i)_{i \in I}$. At the end we are only interested in the set $\{\varphi_i \Rightarrow \varphi'_i\}_{i\in I}$, thus this abusive assimilation of sets to sequences, which simplifies the exposition, should not provoke confusion.

We note here $\mathbf{P} = \{P_i\}_{i\in I}$, $\mathbf{Z} = \{Z_j\}_{j\in J}$, $\mathbf{Q} = \{Q_k\}_{k\in K}$, with $\mathcal{V}(\boldsymbol{\mathcal{L}}) = \mathbf{P}\cup\mathbf{Z}\cup\mathbf{Q}$. For any set I, we note $\mathcal{F}(I)$ for the set of all the sequences $(\varphi_i)_{i\in I}$, where each φ_i is either $\bot$ or $\top$. We define the subsets (finite if $\mathcal{T}$ is finite) $I(\mathcal{T})$ and $J(\mathcal{T})$ of I and J respectively by $\mathbf{P}(\mathcal{T}) = \{P_i\}_{i\in I(\mathcal{T})}$, $\mathbf{Z}(\mathcal{T}) = \{Z_j\}_{j\in J(\mathcal{T})}$. $\mathbf{p} = (\varphi_i)_{i\in I(\mathcal{T})}$ and $\mathbf{z} = (\varphi'_j)_{j\in J(\mathcal{T})}$ are sequences of formulas in $\boldsymbol{\mathcal{L}}$. As usual in the literature about circumscription, $\mathcal{T}[\mathbf{p},\mathbf{z},\mathbf{Q}]$ denotes $\mathcal{T}$ in which any occurrence of P_i and Z_j is replaced respectively by φ_i and φ'_j (as $\mathbf{Q}$ is fixed, we keep the Q_k's unchanged, and we replace only the P_i's and the Z_j's).

If $\mathcal{T}$ is finite, it is equivalent to a single formula and so is $\mathcal{T}[\mathbf{p},\mathbf{z},\mathbf{Q}]$, thus we use (abusively, assimilating this last set to a formula) the notation $\mathcal{T}[\mathbf{p},\mathbf{z},\mathbf{Q}] \Rightarrow \psi$ where ψ is a formula. □

Proposition 4.10 We use the preceding notations. If $\mathcal{T}$ is finite, we have:

$CIRC(\mathbf{P},\mathbf{Q},\mathbf{Z})(\mathcal{T}) = \mathcal{T} \sqcup CIRCAX(\mathbf{P},\mathbf{Q},\mathbf{Z})(\mathcal{T})$ where

$$CIRCAX(\mathbf{P},\mathbf{Q},\mathbf{Z})(\mathcal{T}) =$$
$$\{\mathcal{T}[\mathbf{p},\mathbf{z},\mathbf{Q}] \Rightarrow ((\mathbf{p}\Rightarrow\mathbf{P}(\mathcal{T}))\Rightarrow(\mathbf{P}(\mathcal{T})\Rightarrow\mathbf{p}))) \,/\, \mathbf{p}\in\mathcal{F}(I(\mathcal{T})),\ \mathbf{z}\in\mathcal{F}(J(\mathcal{T}))\} \,\cup$$
$$\{\neg P \,/\, P\in\mathbf{P}^*(\mathcal{T})\}.$$

The set of formulas $CIRCAX(\mathbf{P},\mathbf{Q},\mathbf{Z})(\mathcal{T})$ is the *circumscription axiom schema*, known since [McC80, PM86], with two adaptations to the propositional case. The first one is a simplification: restricting our attention to the formulas φ_i, φ'_i which are $\top$ or $\bot$. The other one is a complication: we cannot exclude the fact that $\mathbf{P}$ is infinite as soon as we want to allow an infinite number of "individuals", which may provoke the existence of infinitely many P_i's in $\mathbf{P}$, thus we must split the set $\mathbf{P}$ into the finite part $\mathbf{P}(\mathcal{T})$ (or any finite superset of $\mathbf{P}(\mathcal{T})$ as in the proof of proposition 4.11 below) and its complement $\mathbf{P}^*(\mathcal{T})$ in $\mathbf{P}$.

When $\mathbf{P}$ is finite, the set $CIRCAX(\mathbf{P},\mathbf{Q},\mathbf{Z})(\mathcal{T})$ is finite thus it is equivalent to a single formula. □

This result is well known in the circumscription literature, being a simplifying adaptation of the corresponding result for predicate circumscription [McC80, PM86].

Proposition 4.11 $CIRCAX$ satisfies the *disjunctive equation*:

For any finite sets $\mathcal{T}_1$ and $\mathcal{T}_2$, let us assimilate here $\mathcal{T}_1$ and $\mathcal{T}_2$ to the conjunctions of their respective elements, and note $\mathcal{T}_1 \vee \mathcal{T}_2$ for the set having as only formula the disjunction of these two conjunctions (thus $Th(\mathcal{T}_1\vee\mathcal{T}_2) = Th(\mathcal{T}_1)\cap Th(\mathcal{T}_2)$).

$$Th(CIRCAX(\mathbf{P},\mathbf{Q},\mathbf{Z})(\mathcal{T}_1\vee\mathcal{T}_2)) =$$
$$CIRCAX(\mathbf{P},\mathbf{Q},\mathbf{Z})(\mathcal{T}_1) \sqcup CIRCAX(\mathbf{P},\mathbf{Q},\mathbf{Z})(\mathcal{T}_2) \qquad \textbf{(DE0)}\ \square$$

The "0" in (DE0) indicates that this property is restricted here to finite theories (assimilated to formulas).

Proof: We have $\mathcal{T}_1[\mathbf{p},\mathbf{z},\mathbf{Q}] \vee \mathcal{T}_2[\mathbf{p},\mathbf{z},\mathbf{Q}] \equiv (\mathcal{T}_1\vee\mathcal{T}_2)[\mathbf{p},\mathbf{z},\mathbf{Q}]$. Let us note here $(\mathcal{T}_1,\mathcal{T}_2)$ for $(\mathcal{T}_1\cup\mathcal{T}_2)$. We use the notations 4.9 and we denote the formula $(\mathbf{p} \Rightarrow \mathbf{P}(\mathcal{T}_1,\mathcal{T}_2)) \Rightarrow$

$(\mathbf{P}(\mathcal{T}_1, \mathcal{T}_2) \Rightarrow \mathbf{p})$ by $\phi[\mathbf{p}]$.
$CIRCAX(\mathbf{P}, \mathbf{Q}, \mathbf{Z})(\mathcal{T}_1)$ is equivalent to
$\{\mathcal{T}_1[\mathbf{p}, \mathbf{z}, \mathbf{Q}] \Rightarrow \phi[\mathbf{p}] \;/\; \mathbf{p} \in \mathcal{F}(I(\mathcal{T}_1, \mathcal{T}_2)),\ \mathbf{z} \in \mathcal{F}(J(\mathcal{T}_1, \mathcal{T}_2))\} \cup \{\neg P \;/\; P \in \mathbf{P}^*(\mathcal{T}_1, \mathcal{T}_2)\}$,
and similarly for $CIRCAX(\mathbf{P}, \mathbf{Q}, \mathbf{Z})(\mathcal{T}_2)$.
Also, $CIRCAX(\mathbf{P}, \mathbf{Q}, \mathbf{Z})(\mathcal{T}_1 \vee \mathcal{T}_2)$ is equivalent to
$\{(\mathcal{T}_1[\mathbf{p}, \mathbf{z}, \mathbf{Q}] \vee \mathcal{T}_2[\mathbf{p}, \mathbf{z}, \mathbf{Q}]) \Rightarrow \phi[\mathbf{p}] \;/\; \mathbf{p} \in \mathcal{F}(I(\mathcal{T}_1, \mathcal{T}_2)),\ \mathbf{z} \in \mathcal{F}(J(\mathcal{T}_1, \mathcal{T}_2))\} \cup \{\neg P \;/\; P \in \mathbf{P}^*(\mathcal{T}_1, \mathcal{T}_2)\}$,
thus, from the equivalence between $(\varphi \vee \varphi') \Rightarrow \psi$ and $\{\varphi \Rightarrow \psi,\ \varphi' \Rightarrow \psi\}$, it is also equivalent to
$\{\mathcal{T}_1[\mathbf{p}, \mathbf{z}, \mathbf{Q}] \Rightarrow \phi[\mathbf{p}]; / \mathbf{p} \in \mathcal{F}(I(\mathcal{T}_1, \mathcal{T}_2)),\ \mathbf{z} \in \mathcal{F}(J(\mathcal{T}_1, \mathcal{T}_2))\} \cup \{\neg P \;/\; P \in \mathbf{P}^*(\mathcal{T}_1, \mathcal{T}_2)\} \cup$
$\{\mathcal{T}_2[\mathbf{p}, \mathbf{z}, \mathbf{Q}] \Rightarrow \phi[\mathbf{p}] \;/\; \mathbf{p} \in \mathcal{F}(I(\mathcal{T}_1, \mathcal{T}_2)),\ \mathbf{z} \in \mathcal{F}(J(\mathcal{T}_1, \mathcal{T}_2))\} \cup \{\neg P \;/\; P \in \mathbf{P}^*(\mathcal{T}_1, \mathcal{T}_2)\}$. □

Corollary 4.12 Any circumscription satisfies (RM0) and (DC0). □

Proof: Again, we assimilate the finite sets of formulas to their conjunction.

- (RM0) uses the $\supseteq$ part of the equality (DE0): $\mathfrak{A}$ is stable for finite intersections, also for any subsets $\mathcal{T}, \mathcal{T}''$ of $\boldsymbol{\mathcal{L}}$, $\mathcal{T} \sqcup \mathcal{T}'' = Th(\mathcal{T}) \sqcup Th(\mathcal{T}'')$ and for any $\mathcal{T}, \mathcal{T}'', \mathcal{T}'''$ in $\mathscr{T}$, we have $\mathcal{T} \sqcup \mathcal{T}'' \subseteq \mathcal{T}'''$ iff ($\mathcal{T} \subseteq \mathcal{T}'''$ and $\mathcal{T}'' \subseteq \mathcal{T}'''$). Thus the side $\supseteq$ of the equality in (DE0) means that $Th(CIRCAX(\mathbf{P}, \mathbf{Q}, \mathbf{Z}))$ is a decreasing mapping from $\mathfrak{A}$ to $\mathscr{T}$: if $Th(\mathcal{T}_1) \subseteq Th(\mathcal{T}_2)$, with $Th(\mathcal{T}_1) \in \mathfrak{A}, Th(\mathcal{T}_2) \in \mathfrak{A}$, we get $Th(CIRCAX(\mathbf{P}, \mathbf{Q}, \mathbf{Z})(\mathcal{T}_2)) \subseteq Th(CIRCAX(\mathbf{P}, \mathbf{Q}, \mathbf{Z})(\mathcal{T}_1))$. Thus, this side $\supseteq$ of (DE0) means that, over $\mathfrak{A}$, circumscriptions are "antitonic" as defined in [KL95], which shows that they respect $(\mathrm{RM})_{\mathfrak{A}}$=(RM0). The proof of this last affirmation appears in [KL95] and is immediate: As $Th(\psi) \subseteq Th(\varphi \wedge \psi)$, we get:
$CIRC(\mathbf{P}, \mathbf{Q}, \mathbf{Z})(\psi \wedge \varphi) = Th(\psi \wedge \varphi) \sqcup CIRCAX(\mathbf{P}, \mathbf{Q}, \mathbf{Z})(\psi \wedge \varphi) = Th(\psi \wedge \varphi) \sqcup Th(CIRCAX(\mathbf{P}, \mathbf{Q}, \mathbf{Z})(\psi \wedge \varphi)) \subseteq Th(\psi \wedge \varphi) \sqcup Th(CIRCAX(\mathbf{P}, \mathbf{Q}, \mathbf{Z})(\psi)) = Th(\psi) \sqcup Th(CIRCAX(\mathbf{P}, \mathbf{Q}, \mathbf{Z})(\psi)) \sqcup Th(\varphi) = CIRC(\mathbf{P}, \mathbf{Q}, \mathbf{Z})(\psi) \sqcup \varphi$ (RM0).

- (DC0) uses the $\subseteq$ part of the equality (DE0): As we have $Th(\varphi \vee \psi) \subseteq Th(\varphi \wedge \psi) = Th(\varphi) \sqcup Th(\psi)$ we get: $CIRC(\mathbf{P}, \mathbf{Q}, \mathbf{Z})(\varphi \vee \psi) = Th(\varphi \vee \psi) \sqcup CIRCAX(\mathbf{P}, \mathbf{Q}, \mathbf{Z})(\varphi \vee \psi) \subseteq Th(\varphi \vee \psi) \sqcup (CIRCAX(\mathbf{P}, \mathbf{Q}, \mathbf{Z})(\varphi) \sqcup CIRCAX(\mathbf{P}, \mathbf{Q}, \mathbf{Z})(\psi)) \subseteq Th(\varphi) \sqcup Th(\psi) \sqcup CIRCAX(\mathbf{P}, \mathbf{Q}, \mathbf{Z})(\varphi) \sqcup CIRCAX(\mathbf{P}, \mathbf{Q}, \mathbf{Z})(\psi) = CIRC(\mathbf{P}, \mathbf{Q}, \mathbf{Z})(\varphi) \sqcup CIRC(\mathbf{P}, \mathbf{Q}, \mathbf{Z})(\psi)$ (DC0). □

4.3 Refining our results about the logical properties

The preceding result did not even use the properties of preferential entailments, being an immediate consequence of the writing of the circumscription axiom schema. The drawback is that this result is limited to finite sets (it can obviously be extended, with some care concerning the vocabulary, to finitely axiomatizable theories). Thanks to the study of preferential entailments, we may overcome slightly this limitation, for reverse monotony only. We know from proposition 3.8-1 that any circumscription satisfies (RM1), which is already better than (RM0). In a few particular cases, we get full (RM):

Proposition 4.13 1. If $\mathbf{P} \cup \mathbf{Q}$ is finite, then the circumscription satisfies (RM). (RM) is falsified as soon as $\mathbf{P} \cup \mathbf{Q}$ is infinite (and $\mathbf{P} \neq \emptyset$).

2. A circumscription satisfies (RM) iff its preference relation $\prec$ satisfies (cl). $\square$

Again, the situation is similar in predicate calculus: first order and mixed predicate circumscriptions satisfy (RM1), but generally they falsify (RM) [MR99].

Proof:
1. $\mathbf{P} \cup \mathbf{Q}$ is finite. Let μ be in $TC(\mathcal{M}_{\prec}(\mathcal{T}))$ and ψ be the conjunction of all the literals satisfied by μ made from $\mathbf{P} \cup \mathbf{Q}$. $\mu \in \mathcal{M}(\psi)$ and $\mathcal{M}(\psi)$ is an open set (ψ is a formula) thus there exists $\nu \in \mathcal{M}_{\prec}(\mathcal{T})$ such that $\nu \in \mathcal{M}(\psi)$. If $\mu \notin \mathcal{M}_{\prec}(\mathcal{T})$, let $\mu' \in \mathcal{M}(\mathcal{T})$ be such that $\mu' \prec \mu$, we get $\mu \cap \mathbf{P} = \nu \cap \mathbf{P}$ and $\mu \cap \mathbf{Q} = \nu \cap \mathbf{Q}$ thus $\mu' \prec \nu$, a contradiction. This shows that $\mu \in \mathcal{M}_{\prec}(\mathcal{T})$: $\prec$ satisfies (cl). Use proposition 3.17: f satisfies (RM).
The following two counter-examples, which may easily be generalized, complete the proof.

2. "if": Use propositions 3.14-1 and 4.7. "only if": Use proposition 3.17. $\square$

Again, a small technical bibliographical comment is in order. We will see in proposition 5.19 that a result related to the positive part (the first sentence) of proposition 4.13-1 is equivalent to Observation 3.3.4 in [Mak94], if we take into account the correspondances evoked in our comment about proposition 4.5. Even if [Mak94] evokes the negative part (second sentence of proposition 4.13-1) in a comment following Observation 3.4.11, no indication for this part is given there (see example 6.26 below).

Example 4.14 $\mathbf{P} = \{P\}$, $\mathbf{Z} = \emptyset$, $\mathbf{Q} = \{Q_i\}_{i \in \mathbb{N}}$.
We define the interpretations $\mu_n = \{P\} \cup \{Q_i\}_{i \geq n}$, $\mu_\omega = \{P\}$, $\nu = \emptyset$ and the theory $\mathcal{T} = \bigcap_{n \in \mathbb{N}} Th(\mu_n) \cap Th(\nu)$. We get $\Gamma(\mathcal{T}) = \{Th(\mu_n)\}_{n \in \mathbb{N}} \cup \{Th(\mu_\omega), Th(\nu)\}$ and $\Gamma_{\prec}(\mathcal{T}) = \Gamma(\mathcal{T}) - \{Th(\mu_\omega)\}$ (indeed, $Th(\nu) \prec Th(\mu_\omega)$).
Thus $\Gamma_{\prec}(\mathcal{T})$ is not a closed set, and $f = f_{\prec}$ falsifies (RM). We give here an explicit counter-example: $\mathcal{T}_1 = \mathcal{T}$, $\mathcal{T}_2 = Th(\mu_\omega) \cap Th(\nu)$, thus $\mathcal{T}_1 \subseteq \mathcal{T}_2$, $f(\mathcal{T}_1) = \mathcal{T}_1$ and $f(\mathcal{T}_2) = Th(\nu) \not\subseteq f(\mathcal{T}_1) \sqcup \mathcal{T}_2 = \mathcal{T}_2$. We cannot choose a theory $\mathcal{T}_2$ which is finitely axiomatizable (even "with respect to $\mathcal{T}_1$") as we know that (RM1) is satisfied. $\square$

Example 4.15 $\mathbf{P} = \{P_i\}_{i \in \mathbb{N}}$, $\mathbf{Z} = \mathbf{Q} = \emptyset$.
We define the interpretations $\mu_n = \{P_i\}_{i \neq n}$, $\mu_\omega = \mathbf{P}$, and the theory $\mathcal{T} = Th(\{P_i \vee P_j\}_{i \neq j})$. We get thus $Th(\mu_n) \in \Gamma_{\prec}(\mathcal{T})$ for any $n \in \mathbb{N}$ and $Th(\mu_\omega) \notin \Gamma_{\prec}(\mathcal{T})$ ($Th(\mu_n) \prec Th(\mu_\omega)$): again $\Gamma_{\prec}(\mathcal{T})$ is not a closed set, and $f = f_{\prec}$ falsifies (RM). $\square$

Proposition 4.16 A circumscription satisfies (CNM) iff $\mathbf{P}$ and $\mathbf{Q}$ are finite or $\mathbf{P}$ has less than two elements. $\square$

Once again, the situation is similar in predicate calculus: first order and mixed predicate circumscriptions satisfy (CNM1), but generally they falsify (CNM) [MR99].

Notice that from this result we get that example 4.14 satisfies (CNM), which shows that (CNM) does not imply (RM) for circumscriptions, and that example 4.15 falsifies (CNM) (see example 4.18 below).

<u>Proof:</u> If $\mathbf{P}$ and $\mathbf{Q}$ are finite, f satisfies (RM) (proposition 4.13-1) thus also (CNM) (proposition 3.5).

If $\mathbf{P} = \{P\}$, let us suppose that $\mathcal{T}_1 \subseteq \mathcal{T}_2$, $Th(\mu) \in \Gamma(f(T1)) \cap \Gamma(\mathcal{T}_2)$ and $\perp \in f(\mathcal{T}_1) \sqcup f(\mathcal{T}_2)$. As $Th(\mu) \notin \Gamma(f(\mathcal{T}_2))$, then $Th(\mu) \notin \Gamma_\prec(\mathcal{T}_2)$ and there exists $\mu' \in \mathcal{M}(\mathcal{T}_2)$ such that $\mu' \prec \mu$. We must have $\mu' \cap \mathbf{P} = \emptyset$ from definition 4.1, thus $\mu' \in \mathcal{M}_\prec(\mathcal{T}_1) \cap M_\prec(\mathcal{T}_2)$ and $Th(\mu') \in \Gamma(f(\mathcal{T}_1)) \cap \Gamma(f(\mathcal{T}_2)) = \Gamma(f(\mathcal{T}_1) \sqcup f(\mathcal{T}_2)) = \emptyset$, a contradiction. Thus, f satisfies (CNM).

The following two counter-examples, in which $CIRC(\mathbf{P}, \mathbf{Q}, \mathbf{Z})$ falsifies (CNM) and which may easily be generalized, complete the proof. □

Example 4.17 $\{P_1, P_2\} \subseteq \mathbf{P}$, $\mathbf{Q} = \{Q_i\}_{i \in \mathbb{N}}$.

We define the interpretations $\mu_n = \{P_1, P_2\} \cup \{Q_i\}_{i \geq n}$, $\mu = \{P_1, P_2\}$, $\mu' = \{P_1\}$ and $\mu'' = \emptyset$ and the theories $\mathcal{T}_1 = (\bigcap_{n \in \mathbb{N}} Th(\mu_n)) \cap Th(\mu') \cap Th(\mu'')$ and $\mathcal{T}_2 = Th(\mu) \cap Th(\mu')$. We get $T_1 \subseteq \mathcal{T}_2$ and $Th(\mu)$ is in $\Gamma(f(\mathcal{T}_1))$ because $Th(\mu_n) \in \Gamma_\prec(\mathcal{T}_1)$ for any $n \in \mathbb{N}$. Thus μ is a model of $f(\mathcal{T}_1) \sqcup \mathcal{T}_2$. $\mu' \prec \mu$ thus $\Gamma(f(\mathcal{T}_2)) = \{Th(\mu')\}$ and $Th(\mu')$ is isolated in $\Gamma(\mathcal{T}_1)$ because μ' satisfies $P_1 \wedge \neg P_2$. Moreover $\mu'' \prec \mu'$ thus $Th(\mu') \notin \Gamma(f(\mathcal{T}_1))$. Thus, $\perp \in f(\mathcal{T}_1) \sqcup f(\mathcal{T}_2)$. □

Example 4.18 (example 4.15, continued).

$\{P_i\}_{i \in \mathbb{N}} = \mathbf{P}$, $\mathbf{Z} = \mathbf{Q} = \emptyset$.

We define the interpretations $\mu_n = \mathbf{P} - \{P_n\}$, $\mu = \mathbf{P}$, $\mu' = \{P_{2i}\}_{i \in \mathbb{N}}$ and $\mu'' = \{P_{4i}\}_{i \in \mathbb{N}}$ and the theories $\mathcal{T}_1 = (\bigcap_{n \in \mathbb{N}} Th(\mu_n)) \cap Th(\mu') \cap Th(\mu'')$ and $\mathcal{T}_2 = Th(\mu) \cap Th(\mu')$. If $n = 4i$, $Th(\mu_n) \in \Gamma_\prec(\mathcal{T}_1)$ because $P_{4i} \in \mu'$, $P_{4i} \in \mu''$, and $P_{4i} \notin \mu_{4i}$. As $Th(\mu)$ is limit of the $Th(\mu_n)$'s, we get $Th(\mu) \in \Gamma(f(\mathcal{T}_1))$. Thus, $Th(\mu) \in \Gamma(f(\mathcal{T}_1)) \cap \Gamma(\mathcal{T}_2) = \Gamma(f(\mathcal{T}_1) \sqcup \mathcal{T}_2)$. Now, $\Gamma(f(\mathcal{T}_2)) = \Gamma_\prec(\mathcal{T}_2) = \{Th(\mu')\}$. As $\mu'' \prec \mu'$, $Th(\mu') \notin \Gamma_\prec(\mathcal{T}_1)$. Also, $Th(\mu')$ is isolated in $\Gamma(\mathcal{T}_1)$, thus $Th(\mu') \notin \Gamma(f(\mathcal{T}_1))$. Thus $\Gamma(f(\mathcal{T}_1) \sqcup f(\mathcal{T}_2)) = \emptyset$. □

Here is a simple property true for circumscriptions without varying proposition and false with varying propositions (and an infinite $\mathbf{P}$).

Proposition 4.19 A circumscription satisfies (CR∞) iff $\mathbf{P}$ is finite or $\mathbf{Z} = \emptyset$. □

In the predicate calculus also, circumscriptions satisfy (CR) but generally they falsify (CR∞), even if we restrict our attention to universal theories [MR99].

<u>Proof:</u> We provide here a proof which needs an enumerable $\mathcal{V}(\boldsymbol{\mathcal{L}})$, which should suffice for the not too mathematically oriented reader. Notice that below we will give a proof of this result without any restriction on $\mathcal{V}(\boldsymbol{\mathcal{L}})$ (see theorem 6.40).

$f = f_\prec$, $\mathcal{V}(\boldsymbol{\mathcal{L}})$ is enumerable.

Let $\mathcal{T} = \bigcap_{i\in I} \mathcal{T}_i$ and $\mathcal{T}' \in \Gamma_\prec(\mathcal{T}) - \Gamma(\bigcap_{i\in I} f(\mathcal{T}_i))$. Then, $\mathcal{T}' \in TC(\bigcup_{i\in I} \Gamma(\mathcal{T}_i)) - TC(\bigcup_{i\in I} \Gamma(f(\mathcal{T}_i)))$, thus $\mathcal{T}' \in TC(\bigcup_{i\in I}(\Gamma(\mathcal{T}_i) - \Gamma_\prec(\mathcal{T}_i)))$. Thus, there exists a sequence $(\mathcal{T}'_n)_{n\in \mathbb{N}}$ of elements all distinct such that $\mathcal{T}'_n \in \Gamma(\mathcal{T}_{i_n}) - \Gamma_\prec(\mathcal{T}_{i_n})$ and the limit of $\mathcal{T}'_n$ $(n \to \infty)$ is $\mathcal{T}'$. Let $\mathcal{T}''_n \in \Gamma(\mathcal{T}_{i_n})$ such that $\mathcal{T}''_n \prec \mathcal{T}'_n$. For any $\mathcal{T}_1 \in \mathcal{C}$, there exists one μ such that $\mathcal{T}_1 = Th(\mu)$ and we note $\mathcal{T}_1 \cap \mathbf{P}$ for $\mu \cap \mathbf{P}$. $\mathcal{T}''_n \cap \mathbf{Q} = \mathcal{T}'_n \cap \mathbf{Q}$ and the limit of $\mathcal{T}'_n \cap \mathbf{Q}$ is $\mathcal{T}' \cap \mathbf{Q}$. Let $\mathcal{T}''$ be an accumulation point of $\{\mathcal{T}''_n\}_{n\in \mathbb{N}}$ if this set is infinite (otherwise $\mathcal{T}''$ is one of the $\mathcal{T}''_n$ which corresponds to an infinite number of n's and such that $\mathcal{T}'' \cap \mathbf{Q} = \mathcal{T}' \cap \mathbf{Q}$).

• $\mathbf{P}$ is finite: Then, $\mathcal{T}''\cap\mathbf{Q} = \mathcal{T}'\cap\mathbf{Q}$ and $(\mathcal{T}'_n\cap\mathbf{P})_{n\in \mathbb{N}}$ is stationary, thus we may suppose $\mathcal{T}'_n \cap \mathbf{P} = \mathcal{T}' \cap \mathbf{P}$ for any n. Thus, $\mathcal{T}'' \cap \mathbf{P} \subset \mathcal{T}' \cap \mathbf{P}$ because $\mathcal{T}''_n \cap \mathbf{P} \subset \mathcal{T}'_n \cap \mathbf{P} = \mathcal{T}' \cap \mathbf{P}$ and $(\mathcal{T}''_n \cap \mathbf{P})_{n\in \mathbb{N}}$ is stationary. We get $\mathcal{T}'' \prec \mathcal{T}'$, which contradicts $\mathcal{T}' \in \Gamma_\prec(\mathcal{T})$ as $\mathcal{T}'' \in \Gamma(\mathcal{T})$.

• $\mathbf{Z} = \emptyset$: Same beginning as above, except that we choose $\mathcal{T}''_n \in \Gamma_\prec(\mathcal{T}_{i_n})$, which can be done because $\prec$ is (sf). $\mathcal{T}''$ is the limit of an infinite subsequence of $(\mathcal{T}''_n)_{n\in \mathbb{N}}$. We may in fact suppose that $\mathcal{T}'' = \lim_{n\to\infty} \mathcal{T}''_n$, it suffices to modify, if necessary, the $\mathcal{T}'n$ from which we started. For any $n \in \mathbb{N}$, $\mathcal{T}''_n \cap \mathbf{Q} = \mathcal{T}'_n \cap \mathbf{Q}$ and the limit of $\mathcal{T}'_n \cap \mathbf{Q}$ is $\mathcal{T}' \cap \mathbf{Q}$. Also, the limit of $\mathcal{T}''_n \cap \mathbf{Q}$ is $\mathcal{T}'' \cap \mathbf{Q}$, thus $\mathcal{T}' \cap \mathbf{Q} = \mathcal{T}'' \cap \mathbf{Q}$. If $P \in \mathcal{T}'' \cap \mathbf{P}$, there exists $N \in \mathbb{N}$ such that for any $n \geq N$ we have $P \in \mathcal{T}''_n \cap \mathbf{P}$, thus $P \in \mathcal{T}'_n \cap \mathbf{P}$, thus $P \in \mathcal{T}' \cap \mathbf{P}$. Thus $\mathcal{T}'' \cap \mathbf{P} \subseteq \mathcal{T}' \cap \mathbf{P}$ and, as $\mathcal{T}' \in \Gamma_\prec(\mathcal{T})$ and $\mathcal{T}'' \in \Gamma(\mathcal{T})$, we get also $\mathcal{T}'' \cap \mathbf{P} = \mathcal{T}' \cap \mathbf{P}$ and, as $\mathbf{Z} = \emptyset$, $\mathcal{T}' = \mathcal{T}''$. Thus $\mathcal{T}' \in \Gamma(\bigcap_{i\in I} f(\mathcal{T}_i))$, because for any $n \in \mathbb{N}$, there exists $i \in I$ such that $\mathcal{T}''_n \in \Gamma_\prec(\mathcal{T}_i)$. This contradicts our initial assumption.

We have proved that in these two cases, we have $\Gamma_\prec(\mathcal{T}) \subseteq \Gamma(\bigcap_{i\in I} f(\mathcal{T}_i))$, i.e. $TC(\Gamma_\prec(\mathcal{T})) \subseteq \Gamma(\bigcap_{i\in I} f(\mathcal{T}_i))$, i.e. $\Gamma(f(\mathcal{T})) \subseteq \Gamma(\bigcap_{i\in I} f(\mathcal{T}_i))$, i.e. f satisfies (CR∞).

The following counter-example proves that if $\mathbf{P}$ is infinite and $\mathbf{Z} \neq \emptyset$, then $CIRC(\mathbf{P}, \mathbf{Q}, \mathbf{Z})$ falsifies (CR∞). □

Example 4.20 $\mathbf{P} = \{P_i\}_{i\in \mathbb{N}}$, $\mathbf{Z} = \{Z\}$, $\mathbf{Q} = \emptyset$. We define $\mu_n = \{P_i\}_{i\geq n} \cup \{Z\}$, $\mu'_n = \{P_i\}_{i>n}$, $\mu_\omega = \{Z\}$, $\mu'_\omega = \emptyset$ and $\mathcal{T}_n = Th(\mu_n) \cap Th(\mu'_n)$, $\mathcal{T} = \bigcap_{n\in \mathbb{N}} \mathcal{T}_n$. We get $f_\prec(\mathcal{T}_n) = Th(\mu'_n)$ (because $Th(\mu'_n) \prec Th(\mu_n)$) and $Th(\mu_\omega) \in \Gamma(\mathcal{T})$ (because any $Th(\mu_n)$ is in $\Gamma(\mathcal{T})$). As $\mu_\omega \cap \mathbf{P} = \emptyset$, $Th(\mu_\omega) \in \Gamma_\prec(\mathcal{T}) \subseteq \Gamma(f_\prec(\mathcal{T}))$. $\Gamma(\bigcap_{n\in \mathbb{N}} f(\mathcal{T}_n)) = \Gamma(\bigcap_{n\in \mathbb{N}} Th(\mu'_n)) = \{Th(\mu'_n)\}_{n\in \mathbb{N}} \cup \{Th(\mu'_\omega)\}$ thus $Th(\mu_\omega) \notin \Gamma(\bigcap_{n\in \mathbb{N}} f_\prec(\mathcal{T}_n))$ thus $\bigcap_{n\in \mathbb{N}} f_\prec(\mathcal{T}_n) \not\subseteq f_\prec(\mathcal{T})$: (CR$\infty$) is falsified. We could have chosen any non empty $\mathbf{Z}$, any $\mathbf{Q}$ and any infinite $\mathbf{P}$. □

As already noticed in [MR98], this counter-property has an important consequence: a pre-circumscription falsifying (CR∞) cannot be expressed thanks to its "inaccessible formulas" by the natural way suggested in [Suc93] for predicate circumscriptions without varying predicates, and reminded and made more explicit for propositional circumscriptions, still without varying objects, in [SF96]. Inaccessible formulas are defined precisely in definition 5.28-1 below, and we postpone the complete treatment of propositional circumscription to that respect to subsection 6.3.

This counter-result sheds new lights on the behavior of circumscriptions: it was known that varying objects are necessary in terms of knowledge representation[7]. The falsification

[7] See the kangaroo example in [PM86]: in the presence of formulas such as $Dead_i \Leftrightarrow \neg Alive_i$, no circumscription without varying object can formalize the rule with exceptions: *an individual i is considered as*

of (CR∞) when varying propositions are present (and $\mathbf{P}$ infinite) shows that the price of introducing varying propositions is high, as it provokes the failure of a powerful property.

We have already showned that circumscriptions satisfy (DC0) (see corollary 4.12). We give now another proof of this fact, using the preferential entailment nature of the circumscription.

Before giving this other proof, let us notice that in the predicate calculus, it is known that the first order predicate circumscription satisfies (DC0). See [MR99] for this result: as in the propositional case, the easiest proof is similar to corollary 4.12, and a proof similar to the following one exists also. However a proof similar to the following one does not work for the "mixed" predicate circumscription (defined in the bibliographical comment about proposition 4.5) because it falsifies (cl0): see [MR99] which uses an example appearing as example 3.1 in [MR90] and already actualized a first time as example 6.3 in [MR94a].

Our proof makes use of an easy lemma:

Lemma 4.21 1) If $\mu \cap \mathcal{V}(\varphi) = \nu \cap \mathcal{V}(\varphi)$ then then $\mu \in \mathcal{M}(\varphi)$ iff $\nu \in \mathcal{M}(\varphi)$.

2) If $\mu \in \mathcal{M}_{\prec}(\varphi)$ then $\mu \cap \mathbf{P} \subseteq \mathbf{P}(\varphi)$.

3) If $\mu \cap \mathbf{P} \subseteq \mathbf{P}(\varphi)$, $\nu \cap \mathbf{P} \subseteq \mathbf{P}(\varphi)$ and $\mu \cap \mathcal{V}(\varphi) = \mu \cap \mathcal{V}(\varphi)$, then $\mu \in \mathcal{M}_{\prec}(\varphi)$ iff $\nu \in \mathcal{M}_{\prec}(\varphi)$. $\square$

Proof: 1) Obvious.

2) Let us suppose $\mu \cap \mathbf{P} \not\subseteq \mathbf{P}(\varphi)$, then let us define $\mu' = \mu - \mu \cap (\mathbf{P} - \mathbf{P}(\varphi))$. We have $\mu' \cap \mathbf{P} \subset \mu \cap \mathbf{P}$ and $\mu' \cap \mathbf{Q} = \mu \cap \mathbf{Q}$ thus $\mu' \prec \mu$. We have also $\mu' \cap \mathcal{V}(\varphi) = \mu \cap \mathcal{V}(\varphi)$ thus $\mu \notin \mathcal{M}_{\prec}(\varphi)$ from 1).

3) Let us suppose $\mu \cap \mathbf{P} \subseteq \mathbf{P}(\varphi)$, $\mu \cap \mathcal{V}(\varphi) = \mu \cap \mathcal{V}(\varphi)$ (thus $\nu \cap \mathbf{P} \subseteq \mathbf{P}(\varphi)$) and $\mu \in \mathcal{M}(\varphi) - \mathcal{M}_{\prec}(\varphi)$. There exists $\mu' \in \mathcal{M}(\varphi)$ such that $\mu' \prec \mu$, thus we have: $\mu' \cap \mathbf{P} \subset \mu \cap \mathbf{P} = \mu \cap \mathbf{P}(\varphi) = \nu \cap \mathbf{P}(\varphi)$. Let us define $\nu' = (\mu' \cap (\mathbf{P} \cup \mathbf{Z})) \cup (\nu \cap \mathbf{Q})$. We have $\nu' \cap \mathbf{P}(\varphi) = \mu' \cap \mathbf{P} = \mu' \cap \mathbf{P}(\varphi)$ thus $\mu' \cap \mathcal{V}(\varphi) = \nu' \cap \mathcal{V}(\varphi)$ and $\nu' \in \mathcal{M}(\varphi)$.

We have $\nu' \cap \mathbf{P} = \mu' \cap \mathbf{P} \subset \nu \cap \mathbf{P}(\varphi) \subseteq \nu \cap \mathbf{P}$ and $\nu' \cap \mathbf{Q} = \nu \cap \mathbf{Q}$ thus $\nu' \prec \nu$ and $\nu \notin \mathcal{M}_{\prec}(\varphi)$. As the hypothesis in point 3) are symmetrical, we can also prove that if $\nu \notin \mathcal{M}_{\prec}(\varphi)$ then $\mu \notin \mathcal{M}_{\prec}(\varphi)$. $\square$

Proposition 4.22
The preference relation $\prec$ associated to a circumscription $CIRC(\mathbf{P}, \mathbf{Q}, \mathbf{Z})$ (see definition 4.1) satisfies (cl0). $\square$

Thus, using proposition 3.17, we get that any circumscription satisfies (RM0) (which is subsumed by the fact that it satisfies (RM1) from proposition 3.8-1) and also (DC0) (which, as (RM0), is already given in corollary 4.12).

alive, except if we have good reasons to think i is dead. With varying predicates (propositions here) this is possible by $CIRC(\{Dead_i\}_{i \in I}, \mathbf{Q}, \{Alive_i\}_{i \in I} \cup \mathbf{Z}')$.

Proof: Let $\varphi \in \boldsymbol{\mathcal{L}}$ and $Th(\mu) \in \Gamma(\varphi) - \Gamma_{\prec}(\varphi)$.

• If $\mu \cap \mathbf{P} \not\subseteq \mathbf{P}(\varphi)$ there exists $P \in (\mu \cap \mathbf{P}) - \mathbf{P}(\varphi)$ such that $Th(\mu) \in \Gamma(\varphi \wedge P)$. For any $Th(\mu') \in \Gamma(\varphi \wedge P)$, $P \in \mu'$ thus $\mu' \cap \mathbf{P} \not\subseteq \mathbf{P}(\varphi)$, thus $Th(\mu') \notin \Gamma_{\prec}(\varphi)$. Thus $\Gamma(\varphi \wedge P) \cap \Gamma_{\prec}(\varphi) = \emptyset$.

• Otherwise, $\mu \cap \mathbf{P} \subseteq \mathbf{P}(\varphi)$ and we define ψ as the conjunction of all the literals built in $\mathcal{V}(\varphi)$ satisfied by μ. We have $Th(\mu) \in \Gamma(\varphi \wedge \psi)$. For any μ' such that $Th(\mu') \in \Gamma(\varphi \wedge \psi)$, we have $\mu' \cap \mathcal{V}(\varphi) = \mu \cap \mathcal{V}(\varphi)$ because $\psi \in Th(\mu')$. From point 3) in lemma 4.21, we get $Th(\mu') \notin \Gamma_{\prec}(\varphi)$ thus $\Gamma(\varphi \wedge \psi) \cap \Gamma_{\prec}(\varphi) = \emptyset$. Thus $\Gamma(\varphi) - \Gamma_{\prec}(\varphi)$ is an open set, thus $\Gamma_{\prec}(\varphi)$ is a closed set: $\prec$ satisfies (cl0). □

Here is a corollary of this result (and of propositions 4.3 and 4.13).

Corollary 4.23 $(\mathbf{P}, \mathbf{P}_1, \mathbf{P}_2, \mathbf{Q}, \mathbf{Z})$ is a partition of $\mathcal{V}(\boldsymbol{\mathcal{L}})$.

1. If $\mathcal{T}$ is finite, or if $\mathbf{P} \cup \mathbf{P}_1 \cup \mathbf{P}_2 \cup \mathbf{Q}$ is finite, we get:
   $CIRC(\mathbf{P} \cup \mathbf{P}_1, \mathbf{Q}, \mathbf{Z} \cup \mathbf{P}_2)(\mathcal{T}) \cup CIRC(\mathbf{P} \cup \mathbf{P}_2, \mathbf{Q}, \mathbf{Z} \cup \mathbf{P}_1)(\mathcal{T}) \models$
   $CIRC(\mathbf{P} \cup \mathbf{P}_1 \cup \mathbf{P}_2, \mathbf{Q}, \mathbf{Z})(\mathcal{T})$.

2. In any case we get:
   $CIRC(\mathbf{P} \cup \mathbf{P}_1 \cup \mathbf{P}_2, \mathbf{Q}, \mathbf{Z})(\mathcal{T}) \models$
   $CIRC(\mathbf{P} \cup \mathbf{P}_1, \mathbf{Q} \cup \mathbf{P}_2, \mathbf{Z})(\mathcal{T}) \cup CIRC(\mathbf{P} \cup \mathbf{P}_2, \mathbf{Q} \cup \mathbf{P}_1, \mathbf{Z})(\mathcal{T})$. □

These results may be enunciated as:

1. "When the set of the circumscribed and fixed propositions appearing in $\mathcal{T}$ is finite, a union of circumscriptions is as least as strong as the corresponding parallel circumscription, provided that in the union the propositional symbols concerned are either circumscribed or varying."

2. "A parallel circumscription is as least as strong as the corresponding union of the circumscriptions, provided that in the union the propositional symbols concerned are either circumscribed or fixed."

Proof:

1. Let us call $\prec_1$, $\prec_2$ and $\prec$ the preference relations associated respectively to $CIRC(\mathbf{P} \cup \mathbf{P}_1, \mathbf{Q}, \mathbf{Z} \cup \mathbf{P}_2)$, $CIRC(\mathbf{P} \cup \mathbf{P}_2, \mathbf{Q}, \mathbf{Z} \cup \mathbf{P}_1)$ and to $CIRC(\mathbf{P} \cup \mathbf{P}_1 \cup \mathbf{P}_2, \mathbf{Q}, \mathbf{Z})$. From definition 4.1 we get easily: If $\mu \prec \nu$ then $\mu \prec_1 \nu$ or $\mu \prec_2 \nu$. Thus we get $\mathcal{M}_{\prec_1}(\mathcal{T}) \cap \mathcal{M}_{\prec_2}(\mathcal{T}) \subseteq \mathcal{M}_{\prec}(\mathcal{T})$, i.e., from proposition 4.13 (case $\mathbf{P} \cup \mathbf{P}_1 \cup \mathbf{P}_2 \cup \mathbf{Q}$ finite) or 4.22 (case $\mathcal{T}$ finite), $TC(\mathcal{M}_{\prec_1}(\mathcal{T})) \cap TC(\mathcal{M}_{\prec_2}(\mathcal{T})) \subseteq TC(\mathcal{M}_{\prec}(\mathcal{T}))$, i.e. $\mathcal{M}(f_{\prec_1}(\mathcal{T}) \cup f_{\prec_2}(\mathcal{T})) \subseteq \mathcal{M}(f_{\prec}(\mathcal{T}))$, i.e. $f_{\prec}(\mathcal{T}) \subseteq f_{\prec_1}(\mathcal{T}) \sqcup f_{\prec_2}(\mathcal{T})$.

Notice that alternatively we could prove this result, when $\mathcal{T}$ is finite, by using proposition 4.10.

2. It is a mere rewriting of proposition 4.3, in the case where $\mathbf{Z}_2 = \emptyset$, and written twice (exchanging $\mathbf{P}_1$ and $\mathbf{P}_2$), given here only for the sake of comparison with the preceding result. □

Proposition 4.24 If $\mathbf{P} \cup \mathbf{Q}$ is finite, the circumscription satisfies (DC).

In almost all the other cases, circumscriptions falsify (DC1).

Precisely, if $\mathbf{P} \cup \mathbf{Q}$ is infinite and $\mathbf{P} \cup \mathbf{Z}$ as at least two elements (and naturally $\mathbf{P}$ is not empty) then the circumscription falsifies (DC1).

Finally, if $\mathbf{P}$ is a singleton and $\mathbf{Z} = \emptyset$, (DC1) is satisfied, but we still do not have full (DC) if moreover $\mathbf{Q}$ is infinite. □

Once again, the situation is similar in the predicate calculus: first order and mixed predicate circumscriptions generally falsify (DC1), while first order predicate circumscriptions satisfy (DC0) [MR99].

Proof: We know that if $\mathbf{P} \cup \mathbf{Q}$ is finite, the preference relation $\prec$ satisfies (cl) from proposition 4.13, thus $f_\prec$ satisfies (DC) from proposition 3.17 (or we could use proposition 3.8-2).

The next four counter-examples complete the proof. □

Example 4.25 $\mathbf{P} = \{P_i\}_{i \in \mathbb{N}}$, $\mathbf{Z} = \mathbf{Q} = \emptyset$.

Let us define $\mu_i = \mathbf{P} - \{P_{2i+1}\}$, $\nu_i = \mathbf{P} - \{P_{2i}\}$, $\mu_\omega = \mathbf{P}$, $\mu = \{P_{2i}\}_{i \in \mathbb{N}}$ and $\nu = \{P_{2i+1}\}_{i \in \mathbb{N}}$.

We have $TC(\{Th(\mu_i)\}_{i \in \mathbb{N}}) = \{Th(\mu_i)\}_{i \in \mathbb{N}} \cup \{Th(\mu_\omega)\}$ and $TC(\{Th(\nu_i)\}_{i \in \mathbb{N}}) = \{Th(\nu_i)\}_{i \in \mathbb{N}} \cup \{Th(\mu_\omega)\}$, thus there exists $\mathcal{T} \in \mathcal{J}$ such that $\Gamma(\mathcal{T}) = \{Th(\mu_i), Th(\nu_i)\}_{i \in \mathbb{N}} \cup \{Th(\mu_\omega), Th(\mu), Th(\nu)\}$. As $\mu \prec \mu_n$ and $\nu \prec \nu_n$ for any $n \in \mathbb{N}$ and also $\mu \prec \mu_\omega$ and $\nu \prec \mu_\omega$, we get $\Gamma_\prec(\mathcal{T}) = \{Th(\mu), Th(\nu)\}$. $Th(\mu)$ and $Th(\nu)$ being isolated in $\Gamma(\mathcal{T})$, we may find two formulas ψ_1, ψ_2 such that $\Gamma(\mathcal{T} \sqcup \psi_1) = \Gamma(\mathcal{T}) - \{Th(\mu)\}$, $\Gamma(\mathcal{T} \sqcup \psi_2) = \Gamma(\mathcal{T}) - \{Th(\nu)\}$ and $\Gamma(f(\mathcal{T} \sqcup (\psi_1 \vee \psi_2))) = \Gamma(f(\mathcal{T})) = \{Th(\mu), Th(\nu)\}$. For any $i \in \mathbb{N}$ we have $\nu \not\prec \mu_i$ and $\mu \not\prec \nu_i$ thus $Th(\mu_i) \in \Gamma_\prec(\mathcal{T} \sqcup \psi_1)$ and $Th(\nu_i) \in \Gamma_\prec(\mathcal{T} \sqcup \psi_2)$. Thus, $Th(\mu_\omega) \in \Gamma(f(\mathcal{T} \sqcup \psi_1)) \cap \Gamma(f(\mathcal{T} \sqcup \psi_2)) = \Gamma(f(\mathcal{T} \sqcup \psi_1) \sqcup f(\mathcal{T} \sqcup \psi_2))$ while $Th(\mu_\omega) \notin \Gamma(f(\mathcal{T}))$, which contradicts (DC1). The proof works as soon as $\mathbf{P}$ is infinite, for any $\mathbf{Z}$ and $\mathbf{Q}$. □

Example 4.26 $\mathbf{P} = \{P\}$, $\mathbf{Z} = \{Z\}$, $\mathbf{Q} = \{Q_i\}_{i \in \mathbb{N}}$.

We define $\mu_n = \{P\} \cup (\mathbf{Q} - \{Q_{2n}\})$, $\nu_n = \{P\} \cup (\mathbf{Q} - \{Q_{2n+1}\})$, $\mu_\omega = \{P\} \cup \mathbf{Q}$, and also $\mu'_n = \mathbf{Q} - \{Q_{2n}\})$, $\mu'_\omega = \mathbf{Q}$, $\nu'_n = \{Z\} \cup (\mathbf{Q} - \{Q_{2n+1}\})$, $\nu'_\omega = \{Z\} \cup \mathbf{Q}$.

We get $TC(\{Th(\mu_n)\}_{n \in \mathbb{N}}) = \{Th(\mu_n)\}_{n \in \mathbb{N}} \cup \{Th(\mu_\omega)\}$ and also $TC(\{Th(\nu_n)\}_{n \in \mathbb{N}}) = \{Th(\nu_n)\}_{n \in \mathbb{N}} \cup \{Th(\mu_\omega)\}$. Similarly, we get $TC(\{Th(\mu'_n)\}_{n \in \mathbb{N}}) = \{Th(\mu'_n)\}_{n \in \mathbb{N}} \cup \{Th(\mu'_\omega)\}$ and also $TC(\{Th(\nu'_n)\}_{n \in \mathbb{N}}) = \{Th(\nu'_n)\}_{n \in \mathbb{N}} \cup \{Th(\nu'_\omega)\}$.

Thus, the set $\{Th(\mu_n), Th(\mu'_n), Th(\nu_n), Th(\nu'_n)\}_{n \in \mathbb{N}} \cup \{Th(\mu_\omega), Th(\mu'_\omega), Th(\nu'_\omega)\}$ is a closed set and there exists $\mathcal{T} \in \mathcal{J}$ such that $\Gamma(\mathcal{T})$ is this set.

We define also the formulas $\psi_1 = P \vee Z$ and $\psi_2 = \neg Z$ and the theories $\mathcal{T}_1 = \mathcal{T} \sqcup \psi_1$ and $\mathcal{T}_2 = \mathcal{T} \sqcup \psi_2$.

We get $\Gamma(\mathcal{T}_1) = \Gamma(\mathcal{T}) - (\{Th(\mu'_n)\}_{n \in \mathbb{N}} \cup \{\mu'_\omega\})$ and $\Gamma(\mathcal{T}_2) = \Gamma(\mathcal{T}) - (\{Th(\nu'_n)\}_{n \in \mathbb{N}} \cup \{\nu'_\omega\})$.

For any $n \in \mathbb{N}$, $\mu_n \in \mathcal{M}_\prec(\mathcal{T}_1)$ and $\nu_n \in \mathcal{M}_\prec(\mathcal{T}_2)$, thus $Th(\mu_\omega) \in \Gamma(f(\mathcal{T}_1)) \cap \Gamma(f(\mathcal{T}_2)) = \Gamma(f(\mathcal{T}_1) \sqcup f(\mathcal{T}_2))$.

For any $n \in \mathbb{N}$, $\mu'_n \prec \mu_n$ and $\nu'_n \prec \nu_n$ thus $\neg P \in f(\mathcal{T}_1 \sqcap \mathcal{T}_2)$ and $Th(\mu_\omega) \notin \Gamma(f(\mathcal{T}_1 \sqcap \mathcal{T}_2))$. f falsifies (DC1). The proof works for any infinite $\mathbf{Q}$, as soon as $\mathbf{P}$ and $\mathbf{Z}$ are not empty. $\square$

Example 4.27 $\mathbf{P} = \{P_1, P_2\}$, $\mathbf{Z} = \emptyset$, $\mathbf{Q} = \{Q_i\}_{i \in \mathbb{N}}$.

We define $\mu_n = \{P_1, P_2\} \cup (\mathbf{Q} - \{Q_{2n}\})$, $\nu_n = \{P_1, P_2\} \cup (\mathbf{Q} - \{Q_{2n+1}\})$, $\mu_\omega = \{P_1, P_2\} \cup \mathbf{Q}$, and also $\mu'_n = \mu_n - \{P_1\}$, $\mu'_\omega = \mu_\omega - \{P_1\}$, $\nu'_n = \nu_n - \{P_2\}$, $\nu'_\omega = \nu_\omega - \{P_2\}$.

We get, as in example 4.26, $TC(\{Th(\mu_n)\}_{n \in \mathbb{N}}) = \{Th(\mu_n)\}_{n \in \mathbb{N}} \cup \{Th(\mu_\omega)\}$ and also $TC(\{Th(\nu_n)\}_{n \in \mathbb{N}}) = \{Th(\nu_n)\}_{n \in \mathbb{N}} \cup \{Th(\mu_\omega)\}$.

As in example 4.26 again, we define the theory $\mathcal{T}$ by
$\Gamma(\mathcal{T}) = \{Th(\mu_n), Th(\mu'_n), Th(\nu_n), Th(\nu'_n)\}_{n \in \mathbb{N}} \cup \{Th(\mu_\omega), Th(\mu'_\omega), Th(\nu'_\omega)\}$.

We define also the formulas $\psi_1 = P_1$ and $\psi_2 = P_2$ and the theories $\mathcal{T}_1 = \mathcal{T} \sqcup \psi_1$ and $\mathcal{T}_2 = \mathcal{T} \sqcup \psi_2$.

As in example 4.26, we get $\mu'_n \notin \mathcal{M}(\mathcal{T}_1)$ and $\nu'_n \notin \mathcal{M}(\mathcal{T}_2)$ for any $n \in \mathbb{N}$, thus $Th(\mu_\omega) \in \Gamma(f(\mathcal{T}_1)) \cap \Gamma(f(\mathcal{T}_2)) = \Gamma(f(\mathcal{T}_1) \sqcup f(\mathcal{T}_2))$. We have also $\neg P_1 \vee \neg P_2 \in f(\mathcal{T}_1 \sqcap \mathcal{T}_2)$ thus $Th(\mu_\omega) \notin \Gamma(f(\mathcal{T}_1 \sqcap \mathcal{T}_2))$. f falsifies (DC1). The proof works for any $\mathbf{Z}$, any $\mathbf{P}$ with at least two elements, and any infinite $\mathbf{Q}$. $\square$

Example 4.28 $\mathbf{P} = \{P\}$, $\mathbf{Z} = \emptyset$ and $\mathbf{Q} = \{Q_i\}_{i \in \mathbb{N}}$.

• Let us suppose that (DC1) is falsified: there exist $\mathcal{T}$ and ψ_1, ψ_2 such that, for $\mathcal{T}_1 = \mathcal{T} \sqcup \psi_1$ and $\mathcal{T}_2 = \mathcal{T} \sqcup \psi_2$, there exists an interpretation μ such that $\mu \in \mathcal{M}(f(\mathcal{T}_1)) \cap \mathcal{M}(f(\mathcal{T}_2)) - \mathcal{M}(f(\mathcal{T}_1 \sqcap \mathcal{T}_2))$.

Then, we may find sequences of interpretations $(\mu_n)_{n \in \mathbb{N}}$ and $(\nu_n)_{n \in \mathbb{N}}$ such that $Th(\mu) = \lim_{n \to \infty} Th(\mu_n) = \lim_{n \to \infty} Th(\nu_n)$ and, for any $n \in \mathbb{N}$, $\mu_n \in \mathcal{M}_\prec(\mathcal{T}_1)$, $\nu_n \in \mathcal{M}_\prec(\mathcal{T}_2)$, $\mu_n \notin \mathcal{M}_\prec(\mathcal{T}_1 \sqcap \mathcal{T}_2)$ and $\nu_n \notin \mathcal{M}_\prec(\mathcal{T}_1 \sqcap \mathcal{T}_2)$.

Thus, there exist sequences $(\mu'_n)_{n \in \mathbb{N}}$ and $(\nu'_n)_{n \in \mathbb{N}}$ such that $\mu'_n \prec \mu_n$, $\nu'_n \prec \nu_n$, $\mu'_n \in \mathcal{M}(\mathcal{T}_2)$ and $\nu'_n \in \mathcal{M}(\mathcal{T}_1)$.

By compactness, we know that these sequences possess subsequences which have a limit, thus we may consider without loss of generality that these sequences themselves have limits, which we will call μ' and ν' respectively (limits in the meaning of the topology of $\mathcal{C}$, assimilated to $\mathcal{M}$).

From $\mu'_n \prec \mu_n$ we get $\mu'_n \cap \mathbf{Q} = \mu_n \cap \mathbf{Q}$ thus, as $\mathbf{Z} = \emptyset$ and $\mathbf{P}$ is a singleton, $\mu'_n = \mu_n - \{P\}$ and similarly $\nu'_n = \nu_n - \{P\}$.

$\lim_{n \to \infty}(\mu'_n \cap \mathbf{Q}) = \lim_{n \to \infty}(\mu_n \cap \mathbf{Q}) = \mu \cap \mathbf{Q}$ thus $\mu' \cap \mathbf{Q} = \mu \cap \mathbf{Q}$. Similarly $\nu' \cap \mathbf{Q} = \nu \cap \mathbf{Q}$.

As $\mu' \cap \mathbf{P} = \nu' \cap \mathbf{P} = \emptyset$, we get in fact $\mu' = \nu' \in \mathcal{M}(\mathcal{T}_1) \cap \mathcal{M}(\mathcal{T}_2)$, thus $\mu' \in \mathcal{M}(\psi_1 \vee \psi_2)$. Thus, there exist infinitely many $n \in \mathbb{N}$ such that $\mu'_n \in \mathcal{M}(\psi_1)$, which contradicts $\mu_n \in \mathcal{M}_\prec(\mathcal{T}_1)$: f satisfies (DC1), and the proof applies a fortiori with any finite set $\mathbf{Q}$.

• We define $\mu_n = \{P\} \cup (\mathbf{Q} - \{Q_{2n}\})$, $\nu_n = \{P\} \cup (\mathbf{Q} - \{Q_{2n+1}\})$, $\mu_\omega = \{P\} \cup \mathbf{Q}$, and also $\mu'_n = \mu_n - \{P\}$, $\mu'_\omega = \mu_\omega - \{P\}$, $\nu'_n = \nu_n - \{P\}$.

We get $\lim_{n \to \infty} \mu_n = \lim_{n \to \infty} \nu_n = \mu_\omega$ and $\lim_{n \to \infty} \mu'_n = \lim_{n \to \infty} \nu'_n = \mu'_\omega$.

We define the theories $\mathcal{T}_1$ and $\mathcal{T}_2$ by: $\Gamma(\mathcal{T}_1) = \{Th(\mu_n), Th(\nu'_n)\}_{n \in \mathbb{N}} \cup \{Th(\mu_\omega), Th(\mu'_\omega)\}$ and $\Gamma(\mathcal{T}_2) = \{Th(\nu_n), Th(\mu'_n)\}_{n \in \mathbb{N}} \cup \{Th(\mu_\omega), Th(\mu'_\omega)\}$. We have

$Th(\mu_n) \in \Gamma_{\prec}(\mathcal{T}_1)$ and $Th(\nu_n) \in \Gamma_{\prec}(\mathcal{T}_2)$ thus $Th(\mu_\omega) \in \Gamma(f(\mathcal{T}_1)) \cap \Gamma(f(\mathcal{T}_2)) = \Gamma(f(\mathcal{T}_1) \sqcup f(\mathcal{T}_2))$. Now, as $\neg P \notin f(\mathcal{T}_1 \cap \mathcal{T}_2)$, $Th(\mu_\omega) \notin \Gamma(f(\mathcal{T}_1 \cap \mathcal{T}_2))$: f falsifies (DC). Notice that here the proof requires that $\mathbf{Q}$ is infinite. □

4.4 Two counter-properties, not so "rational"

We end our study of the properties of reasoning by circumscription by an easy application of the study of such properties.

As a consequence of a careful study of propositions 3.21 and 3.22 and obviously related results, it has been noticed already by Satoh in [Sat90] that a propositional circumscription falsifies (RatM) as soon as $\mathbf{P}$ contains at least three elements: indeed, the set inclusion falsifies (rk) as soon as there are at least three elements. However, the falsification of rational monotony is even more radical (this also has been remarked by Satoh): a soon as fixed propositions are present, two propositional symbols are enough to provoke the falsification of (rk).

We make these results precise now:

Proposition 4.29 (slight extension of [Sat90, theorem 3]) • If $\mathbf{Q} = \emptyset$ and $\mathbf{P}$ as at most two elements, then $CIRC(\mathbf{P}, \mathbf{Q}, \mathbf{Z})$ satisfies (RatM) and (DR).

Otherwise, any non trivial $CIRC(\mathbf{P}, \mathbf{Q}, \mathbf{Z})$ falsifies (RatM0) and (DR0). This means precisely: if $\mathbf{Q} \neq \emptyset$ (and $\mathbf{P} \neq \emptyset$) or if $\mathbf{Q} = \emptyset$ and $\mathbf{P}$ has at least three elements, (thus in most of the cases), then the circumscription falsifies (RatM0) and (DR0). □

Once again, the situation is the same in the predicate calculus: first order and mixed predicate circumscriptions generally falsify (RatM0) and (DR0).

Proof: As we extend slightly the result of Satoh (who was concerned only by the formula-only versions (RatM0) and (DR0)), and as [Sat90] is (unjustifiably) ignored by many people, even in the circumscription community[8], we provide our own complete proof of this result. If $\mathbf{Q} = \emptyset$, and if $\mathbf{P}$ has at most two elements, the relation $\prec$ associated to $CIRC(\mathbf{P}, \mathbf{Q}, \mathbf{Z})$ is obviously ranked, indeed, we have $\mu \prec \nu$ iff $\mu \cap \mathbf{P} \subset \nu \cap \mathbf{P}$, thus $\prec$ shares the rankedness property of the inclusion relation over a set of two elements or less. Also, in this case, from proposition 4.13, we know that $\prec$ satisfies (cl), thus we get (DR) and (RatM) from proposition 3.22-1 and -2.

The following two counter-examples complete the proof. □

Example 4.30 (a similar example is given in [Sat90, proof of theorem 3])

$\mathbf{P} = \{P \cup \{P_i\}_{i \in I}\}$, $\mathbf{Q} = \{Q \cup \{Q_k\}_{k \in K}\}$.

Choosing $\varphi = P$ and $\varphi' = P \vee Q$, we get $f(\varphi') = Th(\{(\neg P \wedge Q) \vee (P \wedge \neg Q)\} \cup \{\neg P_i\}_{i \in I}) = Th(\{\neg(P \Leftrightarrow Q)\} \cup \{\neg P_i\}_{i \in I})$, $\varphi \wedge \varphi' \equiv \varphi$ and $f(\varphi \wedge \varphi') = Th(\{P\} \cup \{\neg P_i\}_{i \in I})$. Now $\neg\varphi = \neg P \notin f(\varphi')$. Let $\psi = \neg(P \Leftrightarrow Q)$, we have $\psi \in f(\varphi')$ and $\psi \notin f(\varphi \wedge \varphi')$: (RatM0), is falsified.

[8] In justifications of this comment, see below: 1) our comments about [LS97], page 30, and 2) our bibliographical comments between proposition 5.24 and example 5.25 pages 46 and 47.

This corresponds to the fact that $\prec$ falsifies (rk) (cf proposition 3.22): Defining $\mu_1 = \{P, Q\}$, $\mu_2 = \{P\}$ and $\mu_3 = \{Q\}$, we have $\mu_2 \not\prec \mu_1$, $\mu_3 \prec \mu_1$ but $\mu_3 \not\prec \mu_2$. The fixed proposition Q is clearly the cause of the failure of (rk) here, because if Q was not in $\mathbf{Q}$, we would have $\mu_3 \prec \mu_2$.

(DR0) also is falsified: we choose now $\varphi = P \Leftrightarrow Q$ and $\psi = \neg P \Leftrightarrow Q$ (equivalent to $\neg(P \Leftrightarrow Q)$). $\varphi \vee \psi \equiv \top$, thus $f(\varphi \vee \psi) = Th(\{\neg P\} \cup \{\neg P_i\}_{i\in I})$, $f(\varphi) = Th(\{P \Leftrightarrow Q\}\{\neg P_i\}_{i\in I})$ and $f(\psi) = Th(\{\neg P \Leftrightarrow Q\}\{\neg P_i\}_{i\in I})$. We get $\neg P \in f(\varphi \vee \psi)$, $\neg P \notin f(\varphi) \cup f(\psi)$: (DR0) is falsified. $\Box$

Example 4.31 (again, a similar example is given in [Sat90]) $\mathbf{P} = \{A, B, C\} \cup \{P_i\}_{i\in I}\}, \mathbf{Q} = \emptyset$.

Choosing $\varphi = B \wedge (A \Leftrightarrow \neg C)$ and $\varphi' = (A \wedge \neg C) \vee (\neg A \wedge B \wedge C)$, we get $f(\varphi') = Th(\{(A \wedge \neg B \wedge \neg C) \vee (\neg A \wedge B \wedge C)\} \cup \{\neg P_i\}_{i\in I})$, $\varphi \wedge \varphi' \equiv \varphi$ and $f(\varphi \wedge \varphi') = Th(\{\varphi\} \cup \{\neg P_i\}_{i\in I})$. Now $\neg\varphi$ is equivalent to $\neg B \vee (A \Leftrightarrow \neg C)$, thus $\neg\varphi \notin f(\varphi')$. Let $\psi = B \Leftrightarrow \neg C$, we have $\psi \in f(\varphi')$ and $\psi \notin f(\varphi \wedge \varphi')$: (RatM0), is falsified.

This corresponds (again, cf proposition 3.22) to the fact that $\prec$ falsifies (rk): Defining $\mu_1 = \{A, B\}$, $\mu_2 = \{B, C\}$ and $\mu_3 = \{A\}$, we have $\mu_2 \not\prec \mu_1$, $\mu_3 \prec \mu_1$ but $\mu_3 \not\prec \mu_2$. The failure of (rk) is provoked here by the fact that $\subseteq$ falsifies (rk) as soon as it is considered in all the subsets of a set with at least three elements.

(DR0) also is falsified: we choose now $\varphi = \neg(A \Leftrightarrow (B \wedge C))$ and $\psi = \neg(C \Leftrightarrow (A \wedge B))$. $\varphi \vee \psi$ is equivalent to $(A \vee C) \wedge (\neg A \vee \neg B \vee \neg C)$, thus $f(\varphi \vee \psi) = Th(\{(A \wedge \neg B \wedge \neg C) \vee (\neg A \wedge \neg B \wedge C)\} \cup \{\neg P_i\}_{i\in I})$, $f(\varphi) = Th(\{(A \wedge \neg B \wedge \neg C) \vee (\neg A \wedge B \wedge C)\} \cup \{\neg P_i\}_{i\in I})$ and $f(\psi) = Th(\{(\neg A \wedge \neg B \wedge C) \vee (A \wedge B \wedge \neg C)\} \cup \{\neg P_i\}_{i\in I})$. We get $\neg B \in f(\varphi \vee \psi)$, $\neg B \notin f(\varphi) \cup f(\psi)$: (DR0) is falsified. $\Box$

It could seem that it is useless to write explicitly so obvious (and old) things, but this is not the case, because it has interesting consequences. The consequences in terms of knowledge representation of this falsification of (DR) and (RatM) are examined in section 7. There, we show why these properties are not very "rational" in fact, despite their names. Indeed, we show in subsections 7.4 and 7.5 that it is precisely because these properties are falsified by circumscriptions that circumscriptions are well adapted for translating rules with exceptions. Thus, we do not share the point of view defended in [Fre98], where it is claimed that it is a good thing to sudy the very particular circumscriptions which satisfy (DR0) and (RatM0).

Let us give now one immediate technical consequence of this falsification of (RatM) by circumscriptions. [LS97] has defined a *cardinality-based* circumscription $NCIRC(\mathbf{P}, \mathbf{Q}, \mathbf{Z})$ defined as CIRC of definition 4.1 except that the inclusion about $\mathbf{P}$ is replaced by $card(\{B \in \mathbf{P}/\mathcal{T}'_i \models B\}) < card(\{B \in \mathbf{P}/\mathcal{T}'_j \models B\})$. Obviously (see e.g. proposition 3.21), if $\mathbf{Q} = \emptyset$, the relation $\prec_N$ so defined satisfies (rk), thus $NCIRC$ satisfies (RatM1) when no fixed proposition appears. However, $NCIRC$ also falls prey to example 4.30 (it is equivalent to CIRC in this case, because $\mathbf{P}$ has no more than two elements), and it is easy to prove that as soon as neither $\mathbf{P}$ nor $\mathbf{Q}$ is empty, $\prec_N$ falsifies (rk) and $NCIRC$ falsifies (RatM1). This proves that not only corollary 9 in [LS97] is false, but no such result can exist: when a given cardinality-based circumscription $NCIRC$ contains fixed propositions, $NCIRC(\mathcal{T})$ cannot be equated to $NCIRC'(\mathcal{T} \sqcup \mathcal{T}_1)$ where $\mathcal{T}_1$ is some fixed theory (not depending of $\mathcal{T}$) and where $NCIRC'$ is another cardinality-based circumscription, without fixed propositions[9].

5 Formula circumscription

5.1 Definition, first results and motivations

Instead of using a propositional circumscription, it is generally better to use another version, called formula circumscription. Formula circumscription has been introduced as early as in [McC86], but it has not been used a lot since, even if in many situations it is much more natural and easier than ordinary circumscription of definition 4.1 (see examples in section 7). We give here the propositional version of the original definition.

Definition 5.1 $\boldsymbol{\mathcal{L}}$ is a language such that $\mathbf{Z}, \mathbf{Q}$ is a partition of $\mathcal{V}(\boldsymbol{\mathcal{L}})$. $\Phi = \{\varphi_i\}_{i\in I}$ is a set of formulas in $\boldsymbol{\mathcal{L}}$ and $\mathcal{T}$ is another subset of $\boldsymbol{\mathcal{L}}$. The *formula circumscription* $CIRCF$ of the formulas of Φ, with the propositions in $\mathbf{Z}$ allowed to vary and those in $\mathbf{Q}$ fixed, is as follows: We introduce P_i's as new (not appearing in the language $\boldsymbol{\mathcal{L}}$) and distinct propositional symbols, with the set $\mathbf{P} = \{P_i\}_{i\in I}$.

$CIRCF(\Phi; \mathbf{Q}, \mathbf{Z})(\mathcal{T}) \equiv CIRC(\mathbf{P}, \mathbf{Q}, \mathbf{Z})(\mathcal{T} \cup \{\varphi_i \Leftrightarrow P_i\}_{i\in I}) \cap \boldsymbol{\mathcal{L}}$, the propositional circumscription being made in the language $\boldsymbol{\mathcal{L}}'$, which is $\boldsymbol{\mathcal{L}}$ augmented by $\mathbf{P}$.

When $\mathbf{Q}$ is empty, we may simplify the notation as we know that $\mathbf{Z} = \mathcal{V}(\boldsymbol{\mathcal{L}})$, thus we use the notation $CIRCF(\Phi)$ for $CIRCF(\Phi; \emptyset, \mathbf{Z})$ (notice that with this short notation, we must know the vocabulary, as it does not appear here). We will see below that any formula circumscription may be written in this way, even if this is not always desirable. □

It is well known and easy to show that we could replace $\Leftrightarrow$ by $\Rightarrow$ in this definition:
$CIRC(\mathbf{P}, \mathbf{Q}, \mathbf{Z})(\mathcal{T} \cup \{\varphi_i \Leftrightarrow P_i\}_{i\in I}) \cap \boldsymbol{\mathcal{L}} = CIRC(\mathbf{P}, \mathbf{Q}, \mathbf{Z})(\mathcal{T} \cup \{\varphi_i \Rightarrow P_i\}_{i\in I}) \cap \boldsymbol{\mathcal{L}}$.

Notice also that the notion of formula circumscription subsumes the notion of circumscription as defined in proposition 4.1:

Proposition 5.2 [folklore] $\mathbf{P}, \mathbf{Q}, \mathbf{Z}$ being as in definition 4.1, we have:

$$CIRC(\mathbf{P}, \mathbf{Q}, \mathbf{Z}) = CIRCF(\mathbf{P}; \mathbf{Q}, \mathbf{Z} \cup \mathbf{P}).$$

(In $CIRCF$, the first occurrence of $\mathbf{P}$ is assimilated to the set of the *formulas* P for all the propositional symbols $P \in \mathbf{P}$.) □

[9] For the interested reader, the result given in [LS97] is true if we replace Q (our $\mathbf{Q}$) by Z (our $\mathbf{Z}$) and Q' by Z': the trick given in [dKK89] for classical circumscription (see proposition 5.9 below) works also for finite cardinality-based circumscriptions, but with varying propositions taking the place of the fixed ones, and the only way of corollary 9 of [LS97] which remains true is a consequence of this result. See [Moi98] for details.

Proof: We have $\mathcal{V}(\boldsymbol{\mathcal{L}}) = \mathbf{P} \cup \mathbf{Q} \cup \mathbf{Z}$. Let $\mathbf{P} = \{P_i\}_{i \in I}$ and let us define the set of the new propositional symbols $\mathbf{P}' = \{P'_i\}_{i \in I}$, $\boldsymbol{\mathcal{L}}'$ being the language defined by $\mathcal{V}(\boldsymbol{\mathcal{L}}') = \mathcal{V}(\boldsymbol{\mathcal{L}}) \cup \mathbf{P}'$. We have by definition $CIRCF(\mathbf{P}; \mathbf{Q}, \mathbf{Z} \cup \mathbf{P})(\mathcal{T}) = CIRC(\{P'_i\}_{i \in I}, \mathbf{Q}, \mathbf{Z} \cup \mathbf{P})(\mathcal{T} \cup \{P_i \Leftrightarrow P'_i\}_{i \in I}) \cap \boldsymbol{\mathcal{L}}$. From definition 4.1, we have obviously, $CIRC(\mathbf{P}', \mathbf{Q}, \mathbf{Z} \cup \mathbf{P})(\mathcal{T} \cup \{P_i \Leftrightarrow P'_i\}_{i \in I}) = CIRC(\mathbf{P}, \mathbf{Q}, \mathbf{Z} \cup \mathbf{P}')(\mathcal{T} \cup \{P_i \Leftrightarrow P'_i\}_{i \in I})$ and, for any $\mathcal{T} \subseteq \boldsymbol{\mathcal{L}}$, $CIRC(\mathbf{P}, \mathbf{Q}, \mathbf{Z} \cup \mathbf{P}')(\mathcal{T} \cup \{P_i \Leftrightarrow P'_i\}_{i \in I}) \cap \boldsymbol{\mathcal{L}} = CIRC(\mathbf{P}, \mathbf{Q}, \mathbf{Z})(\mathcal{T})$. $\Box$

Alternatively, we could have defined directly $CIRCF$ exactly as we have defined $CIRC$:

Definitions 5.3 Let Φ be some set of formulas, and $\mathbf{Q}, \mathbf{Z}$ be some partition of $\mathcal{V}(\boldsymbol{\mathcal{L}})$. For any μ, we define the set of formulas $\Phi_\mu = \{\varphi \in \Phi \ / \ \mu \models \varphi\} = Th(\mu) \cap \Phi$.

We define the two preference relations $\preceq_{(\Phi;\, \mathbf{Q},\, \mathbf{Z})}$ and $\prec_{(\Phi;\, \mathbf{Q},\, \mathbf{Z})}$ as follows:

$\mu \preceq_{(\Phi;\, \mathbf{Q},\, \mathbf{Z})} \nu$ iff $\mu \cap \mathbf{Q} = \nu \cap \mathbf{Q}$ and $\Phi_\mu \subseteq \Phi_\nu$, and

$\mu \prec_{(\Phi;\, \mathbf{Q},\, \mathbf{Z})}$ iff $\mu \cap \mathbf{Q} = \nu \cap \mathbf{Q}$ and $\Phi_\mu \subset \Phi_\nu$.

We define also the following relation: $\mu \simeq_{(\Phi;\, \mathbf{Q},\, \mathbf{Z})} \nu$ iff $\mu \preceq_{(\Phi;\, \mathbf{Q},\, \mathbf{Z})} \nu$ and $\nu \preceq_{(\Phi;\, \mathbf{Q},\, \mathbf{Z})} \mu$.

If $\mathbf{Q}$ is empty, we note $\preceq_\Phi$, $\prec_\Phi$ and $\simeq_\Phi$ instead of $\preceq_{(\Phi;\, \emptyset, \mathbf{Z})}$, $\prec_{(\Phi;\, \emptyset, \mathbf{Z})}$ and $\simeq_{(\Phi;\, \emptyset, \mathbf{Z})}$ respectively.

We define the set $\neg\Phi = \{\neg\varphi \ / \ \varphi \in \Phi\}$. $\Box$

From these definitions we immediately get:

Lemma 5.4 1a. $\mu \preceq_{(\Phi;\, \mathbf{Q},\, \mathbf{Z})} \nu$ iff $\mu \cap \mathbf{Q} = \nu \cap \mathbf{Q}$ and for any $\varphi \in \Phi$, if $\mu \models \varphi$ then $\nu \models \varphi$,

1b. $\mu \prec_{(\Phi;\, \mathbf{Q},\, \mathbf{Z})} \nu$ iff $\mu \cap \mathbf{Q} = \nu \cap \mathbf{Q}$, and
for any $\varphi \in \Phi$, if $\mu \models \varphi$ then $\nu \models \varphi$, and
there exists $\varphi \in \Phi$ such that $\mu \not\models \varphi$ and $\nu \models \varphi$.

1.c $\mu \prec_{(\Phi;\, \mathbf{Q},\, \mathbf{Z})} \nu$ iff $\mu \preceq_{(\Phi;\, \mathbf{Q},\, \mathbf{Z})} \nu$ and $\nu \not\preceq_{(\Phi;\, \mathbf{Q},\, \mathbf{Z})} \mu$.
$\mu \preceq_{(\Phi;\, \mathbf{Q},\, \mathbf{Z})} \nu$ iff $\mu \prec_{(\Phi;\, \mathbf{Q},\, \mathbf{Z})} \nu$ or (exclusive or) $\nu \simeq_{(\Phi;\, \mathbf{Q},\, \mathbf{Z})} \mu$.

2. For any μ, ν, we have always $\mu \preceq_{\{\varphi\}} \nu$ or $\nu \preceq_{\{\varphi\}} \mu$. Thus, we have always exactly one of the three possibilities: $\mu \prec_{\{\varphi\}} \nu$, $\nu \prec_{\{\varphi\}} \mu$, or $\mu \simeq_{\{\varphi\}} \nu$ (comparability). Notice that the strict order relation $\prec_{\{\varphi\}}$ is very particular because transitivity is trivial: we cannot have $\mu_1 \prec_{\{\varphi\}} \mu_2$ and $\mu_2 \prec_{\{\varphi\}} \mu_3$.

3. $\mu \preceq_{(\Phi;\mathbf{Q},\, \mathbf{Z})} \nu$ iff $\nu \preceq_{(\neg\Phi;\mathbf{Q},\, \mathbf{Z})} \mu$. Thus $\mu \prec_{(\Phi;\mathbf{Q},\, \mathbf{Z})} \nu$ iff $\nu \prec_{(\neg\Phi;\mathbf{Q},\, \mathbf{Z})} \mu$.

4a. $\mu \preceq_{(\Phi;\, \mathbf{Q},\, \mathbf{Z})} \nu$ iff for any $\varphi \in \Phi$, $\mu \preceq_{(\{\varphi\};\, \mathbf{Q},\, \mathbf{Z})} \nu$.

4b. $\mu \prec_{(\Phi;\, \mathbf{Q},\, \mathbf{Z})} \nu$ iff
(for any $\varphi \in \Phi$, $\mu \preceq_{(\{\varphi\};\, \mathbf{Q},\, \mathbf{Z})} \nu$, and there exists $\varphi \in \Phi$ such that $\mu \prec_{(\{\varphi\};\, \mathbf{Q},\, \mathbf{Z})} \nu$).

5. $\prec_{(\Phi;\,\mathbf{Q},\,\mathbf{Z})}$ and $\preceq_{(\Phi;\,\mathbf{Q},\,\mathbf{Z})}$ are transitive, $\prec_{(\Phi;\,\mathbf{Q},\,\mathbf{Z})}$ is irreflexive (thus it is a strict order) while $\preceq_{(\Phi;\,\mathbf{Q},\,\mathbf{Z})}$ is reflexive (it is not an order relation, because antisymmetry is missing).

   $\simeq_{(\Phi;\,\mathbf{Q},\,\mathbf{Z})}$ is an equivalence relation.

6. If $\mathbf{Z} = \emptyset$ (i.e. $\mathbf{Q} = \mathcal{V}(\boldsymbol{\mathcal{L}})$), we get $\mu \preceq_{(\Phi;\mathbf{Q},\emptyset)} \nu$ iff $\mu = \nu$.

   If $\Phi = \emptyset$, we get, for any μ and ν, $\mu \preceq_{(\emptyset;\mathbf{Q},\,\mathbf{Z})} \nu$.

   If $\Phi = \boldsymbol{\mathcal{L}}$, we get, for any μ and ν, $\mu \preceq_{(\boldsymbol{\mathcal{L}};\mathbf{Q},\,\mathbf{Z})} \nu$ iff $\mu = \nu$.

   In these three cases we get, for any μ and ν, $\mu \not\prec_{(\Phi;\mathbf{Q},\,\mathbf{Z})} \nu$, i.e. $\prec_{(\Phi;\mathbf{Q},\,\mathbf{Z})}$ has an empty graph.

□

An important consequence of these results (e.g. point 4b) is that in order to know $\prec_{(\Phi;\,\mathbf{Q},\,\mathbf{Z})}$ (which is the "useful relation", see proposition 5.5 below) we need to know more than each individual $\prec_{(\{\varphi\};\,\mathbf{Q},\,\mathbf{Z})}$, we must know all the individual $\preceq_{(\{\varphi\};\,\mathbf{Q},\,\mathbf{Z})}$, which is a much more precise information (thus harder to get).

Now comes the alternative definition of formula circumscription, directly as a preferential entailment:

Proposition 5.5 $\Phi = \{\phi_i\}_{i\in I}$ is a set of formulas in $\boldsymbol{\mathcal{L}}$. $CIRCF(\Phi;\mathbf{Q},\mathbf{Z}) = f_{\prec_{(\Phi;\,\mathbf{Q},\,\mathbf{Z})}}$. □

Proof: Let $\boldsymbol{\mathcal{L}}$, $\boldsymbol{\mathcal{L}}'$, $\mathbf{P} = \{P_i\}_{i\in I}$, $\mathbf{Z}$ and $\mathbf{Q}$ be as in definition 5.1. We know from definition 5.1 that we have, for any $\mathcal{T} \subseteq \boldsymbol{\mathcal{L}}$, $CIRCF(\Phi;\mathbf{Q},\mathbf{Z})(\mathcal{T}) = CIRC(\{P_i\}_{i\in I},\mathbf{Q},\mathbf{Z})(\mathcal{T} \cup \{\phi_i \Leftrightarrow P_i\}_{i\in I}) \cap \boldsymbol{\mathcal{L}}$.

μ being an interpretation for $\boldsymbol{\mathcal{L}}$, we define the "extension of μ" as the interpretation μ' for $\boldsymbol{\mathcal{L}}'$ defined by $\mu' = \mu \cup \{P_i \;/\; i \in I, \mu \models \phi_i\}$. μ' is a model of $\{P_i \Leftrightarrow \phi_i\}_{i\in I}$ and we even get, for any $\mathcal{T} \subseteq \boldsymbol{\mathcal{L}}$, that μ is a model of $\mathcal{T}$ iff its extension μ' is a model of $(\mathcal{T} \cup \{\phi_i \Leftrightarrow P_i\}_{i\in I})$. We define $\prec'$, the preference relation associated to $CIRC(\mathbf{P},\mathbf{Q},\mathbf{Z})$, defined in $\boldsymbol{\mathcal{L}}'$ (definitions 4.1). μ_1 being an interpretation for $\boldsymbol{\mathcal{L}}'$, we define the "restriction of μ_1" to $\boldsymbol{\mathcal{L}}$ by $\mu_1^r = \mu_1 \cap \mathcal{V}(\boldsymbol{\mathcal{L}})$. We have, for any interpretation μ for $\boldsymbol{\mathcal{L}}$: $\mu = (\mu')^r$.

From $CIRCF(\Phi;\mathbf{Q},\mathbf{Z})(\mathcal{T}) = CIRC(\{P_i\}_{i\in I},\mathbf{Q},\mathbf{Z})(\mathcal{T} \cup \{\phi_i \Leftrightarrow P_i\}_{i\in I}) \cap \boldsymbol{\mathcal{L}}$, we get that $CIRCF$ is a preferential entailment $f_{\prec''}$ defined in $\boldsymbol{\mathcal{L}}$ by $\mu \prec'' \nu$ iff $\mu' \prec' \nu'$.

As $\mu' \models P_i$ iff $P_i \in \mu'$ we get, for any interpretation μ for $\boldsymbol{\mathcal{L}}$, $\mu \models \phi_i$ iff $P_i \in \mu'$. Thus, from definitions 4.1 and 5.3, for any interpretations μ and ν for $\boldsymbol{\mathcal{L}}$, we get $\mu \prec_{(\Phi;\,\mathbf{Q},\,\mathbf{Z})} \nu$ iff $\mu' \prec' \nu'$, i.e. $\prec'' = \prec_{(\Phi;\,\mathbf{Q},\,\mathbf{Z})}$. □

A formula circumscription with an empty (graph for its) relation $\mu \prec_{(\Phi;\mathbf{Q},\,\mathbf{Z})}$ will be called a *trivial formula circumscription* (see three significative examples in lemma 5.4-6).

Notations 5.6 Notice that from propositions 5.2 and 5.5, we get that the preference relation $\prec$ associated to $CIRC(\mathbf{P},\mathbf{Q},\mathbf{Z})$ (definition 4.1) is equal to $\prec_{(\mathbf{P};\,\mathbf{Q},\,\mathbf{Z}\,\cup\,\mathbf{P})}$. We will also denote this relation by $\prec_{(\mathbf{P},\,\mathbf{Q},\,\mathbf{Z})}$.

Similarly we define the relation $\preceq_{(\mathbf{P},\ \mathbf{Q},\ \mathbf{Z})}$ as the relation $\preceq_{(\mathbf{P};\ \mathbf{Q},\ \mathbf{Z}\ \cup\ \mathbf{P})}$ of definition 5.3. This is the traditional "large" relation associated to $CIRC(\mathbf{P}, \mathbf{Q}, \mathbf{Z})$, i.e. the relation $\prec$ of definition 4.1 (more precisely as given in remark 4.2), where $\subset$ in $\mathbf{P} \cap \mu \subset \mathbf{P} \cap \nu$ is replaced by the large inclusion $\subseteq$. $\square$

Corollary 5.7 1. Any formula circumscription satisfies all the properties of preferential entailments: (CR), (CT), (Idem), (RM1), (CC1), (CNM1), (DCC), (P′), ...

2. $\prec_{(\Phi;\ \mathbf{Q},\ \mathbf{Z})}$ satisfies (sf), thus any formula circumscription satisfies also (PC) and (CUMU).

3. A formula circumscription is a preferential entailment which may be defined by one preference relation only.

4. The opposite $\succ_{(\Phi;\ \mathbf{Q},\ \mathbf{Z})}$ of $\prec_{(\Phi;\ \mathbf{Q},\ \mathbf{Z})}$ satisfies (sf). $\square$

<u>Proof:</u> 1. From propositions 3.8 (1 and 2) and 5.5.
2. From proposition 4.5 and the proof of proposition 5.5.
3. From proposition 3.12-1, indeed $\prec_{(\Phi;\ \mathbf{Q},\ \mathbf{Z})}$ is irreflexive.
4. From lemma 5.4-3), we know that the opposite $\succ_{(\Phi;\ \mathbf{Q},\ \mathbf{Z})}$ of $\prec_{(\Phi;\ \mathbf{Q},\ \mathbf{Z})}$ is the (unique) preference relation associated to $CIRCF(\neg\Phi; \mathbf{Q}, \mathbf{Z})$: $\succ_{(\Phi;\ \mathbf{Q},\ \mathbf{Z})} = \prec_{(\neg\Phi;\ \mathbf{Q},\ \mathbf{Z})}$, thus points 2 and 4 are equivalent. $\square$

A few bibliographical comments: Point 1 is folklore (except for (CC1) and (DCC) which are ours). Point 2 is closely related to Observation 3.4.11 in [Mak94]: If we take into account the equalities $f_P(\Phi) = CIRCF(\neg\Phi)$ (proposition 5.14, a result from [Poo94]) and $CIRCF(\Phi; \mathbf{P}, \mathbf{Q}) = CIRCF(\Phi \cup \mathbf{Q} \cup \neg\mathbf{Q})$ (proposition 5.9-2, an elaboration from [dKK89]), the two results are indeed equivalent. See also our bibliographical comments following proposition 4.5.

We remind here the intuitive justification of formula circumscription with respect to ordinary circumscription (see also [McC86, PM86]). Let us suppose that we want to translate the rule (with possible exceptions) *birds fly* (BF).

The traditional way is as follows[10]: The language $\mathcal{L}$ contains the propositional symbols B_i and F_i ($i \in I$), representing respectively the English sentences *individual i is a bird* and *individual i is able to fly.* We use an extended language $\mathcal{L}'$, which adds to $\mathcal{L}$ another set of propositional symbols $\{E_i\}_{i \in I}$, representing the sentence *individual i is an exception to the rule (BF).* We introduce $\mathcal{T}$ which is the set of formulas $\{(B_i \wedge \neg E_i) \Rightarrow F_i\}_{i \in I}$. Till now, we do not have translated our rule, as nothing says that the E_i's are really exceptional. This

[10]Remind that we use only propositional versions of circumscriptions while generally predicate versions are used. However, for all the points concerned in our discussion, this makes no difference at all (in [MR99], the same example is translated in terms of predicate versions of circumscriptions).

is where circumscription comes: we circumscribe the E_i's, letting the F_i's to vary. That is we use $f = CIRC(\{E_i\}_{i\in I}, \{B_i\}_{i\in I}, \{F_i\}_{i\in I})$. In this way, we have formalized the rule (BF). Indeed, we get $F_1 \in f(\mathcal{T} \sqcup B_1)$ (if all we know about *individual* 1 is that *it is a bird*, then 1 *flies*) and $F_1 \notin f(\mathcal{T} \sqcup (B_1 \wedge \neg F_1))$ (if we know that *individual* 1 *is an unflying bird*, then we do not conclude that *it flies*, i.e. we do not get a contradiction). Notice that this example shows why a formalism dealing with rules with exceptions should allow to reason non monotonically: f falsifies (MON) because $F_1 \in f(\mathcal{T} \sqcup B_1)$, $\mathcal{T} \sqcup B_1 \subseteq \mathcal{T} \sqcup (B_1 \wedge \neg F_1)$ and $F_1 \notin f(\mathcal{T} \sqcup (B_1 \wedge \neg F_1))$.

This is the way circumscription is generally used in order to formalize one rule with exceptions. We have introduced a set of auxiliary propositions (the E_i's) which complicate the language, and are not absolutely necessary when we read rule (BF) as such. There are also other problems with this method, when several rules are involved (see section 7 below).

Thus, another method is sometimes used (not as often as it should be, in our opinion, even if it is used already in [McC86], the second founding text about [predicate] circumscription).

Indeed, $(B_i \wedge \neg E_i) \Rightarrow F_i$ is equivalent to $(B_i \wedge \neg F_i) \Rightarrow E_i$ and, when we circumscribe E_i in a set $\mathcal{T}$ containing $\{(B_i \wedge \neg F_i) \Rightarrow E_i\}_{i\in I}$ and no other positive occurrence of E_i, we get $(B_i \wedge \neg F_i) \Leftrightarrow E_i$ in $f(\mathcal{T})$ (cf the comment following definition 5.1). Thus, minimizing the E_i's in $\mathcal{T}$ amounts to "minimize the formulas" $B_i \wedge \neg F_i$'s in the set of formulas $\mathcal{T} - \{(B_i \wedge \neg F_i) \Rightarrow E_i\}_{i\in I}$. And the intuitive justification of this method is not more complicated than the justification of the traditional method: we "*minimize the unflying birds*".

Precisely, we get:

$CIRC(\{E_i\}_{i\in I}, \{B_i\}_{i\in I}, \{F_i\}_{i\in I})(\mathcal{T}_1 \cup \{(B_i \wedge \neg E_i) \Rightarrow F_i\}_{i\in I}) \ \cap \ \boldsymbol{\mathcal{L}} =$
$CIRCF(\{B_i \wedge \neg F_i\}_{i\in I}; \{B_i\}_{i\in I}, \{F_i\}_{i\in I})(\mathcal{T}_1)$,

where $CIRCF$ is defined in the language $\boldsymbol{\mathcal{L}}$ with $\mathcal{V}(\boldsymbol{\mathcal{L}}) = \{B_i, F_i\}_{i\in I}$, $CIRC$ is defined in the augmented language $\boldsymbol{\mathcal{L}}'$ with $\mathcal{V}(\boldsymbol{\mathcal{L}}') = \{B_i, F_i, E_i\}_{i\in I}$ and $\mathcal{T}_1$ is any subset of $\boldsymbol{\mathcal{L}}$.

Now that we have defined formula circumscription, and also given a justification to introduce it, let us study its properties.

As formula circumscription may be expressed in terms of ordinary circumscription, we may roughly state that $CIRCF$ has almost all the properties of $CIRC$. And indeed, there is some truth in this affirmation. However, a new study is necessary to get the precise results and their precise applicability conditions, so we give here a few results about formula circumscriptions.

Firstly, in the finite case, we get also an axiom schema, very similar to $CIRCAX$ of proposition 4.10.

We use the notations 4.9, except that here there is no $\mathbf{P}$, thus we use $\mathcal{T}[\mathbf{z}, \mathbf{Q}]$ to denote $\mathcal{T}$ in which each occurrence of Z_j is replaced by φ_j, which is $\perp$ or $\top$. Additionally, If $\Phi = \{\psi_i\}_{i\in I}$, we note $\Phi[\mathbf{z}]$ for $\{\psi_i[\mathbf{z}]\}_{i\in I}$ where $\psi_i[\mathbf{z}]$ is the formula ψ_i in which each occurrence of Z_j is replaced by φ_j.

Proposition 5.8 If $\mathcal{T}$ is finite and if I is finite (a condition in fact equivalent to: Φ is finite), we have:

$CIRCF(\Phi;\ \mathbf{Q},\mathbf{Z})(\mathcal{T}) = \mathcal{T} \sqcup CIRCFAX(\Phi;\mathbf{Q},\mathbf{Z})(\mathcal{T})$ where

$CIRCFAX(\Phi;\mathbf{Q},\mathbf{Z})(\mathcal{T}) = \{(\mathcal{T}[\mathbf{z},\mathbf{Q}] \Rightarrow ((\Phi[\mathbf{z}] \Rightarrow \Phi) \Rightarrow (\Phi \Rightarrow \Phi[\mathbf{z}])))/ \text{ for any } \mathbf{z} \in \mathcal{F}(J)\}$.

The set of formulas $CIRCFAX(\Phi;\mathbf{Q},\mathbf{Z})(\mathcal{T})$ is the *formula circumscription axiom schema*, known since [McC86], with the simplification that as we are in the propositional case, we may restrict our attention to the formulas φ_j which are $\top$ or $\bot$. $\square$

As proposition 4.10, this result is well known in the circumscription literature, being a simplifying adaptation of the similar result for predicate formula circumscription. Notice that under the conditions of this proposition, $CIRCFAX(\Phi;\mathbf{Q},\mathbf{Z})(\mathcal{T})$ is a finite set: indeed, we may replace $\mathbf{z} \in \mathcal{F}(J)$ by $\mathbf{z} \in \mathcal{F}(J(\mathcal{T},\Phi))$, where $J(\mathcal{T},\Phi)$ is a finite subset of J corresponding to (indexing) the finite subset $\mathbf{Z}(\mathcal{T},\Phi)$ which is the intersection of $\mathbf{Z}$ with the finite vocabulary of $\mathcal{T} \cup \Phi$, thus $CIRCFAX(\Phi;\mathbf{Q},\mathbf{Z})(\mathcal{T})$ is equivalent to a single formula.

5.2 Eliminating fixed propositions gives yet another possible definition

Let us give here a useful result, given in [dKK89] for predicate circumscription. We give it for propositional and for propositional formula circumscription:

Proposition 5.9 $\mathbf{P},\mathbf{Q},\mathbf{Z}$ being a partition of $V(\mathcal{L})$, $\mathbf{Q}$ being equal to $\{Q_k\}_{k\in K}$, we introduce a set $\mathbf{Q}' = \{Q'_k\}_{k\in K}$ of new (not in $V(\mathcal{L})$ and all distinct) propositional symbols.

1. Then we have, for any $\mathcal{T}$ in $\mathcal{L}$:

   $CIRC(\mathbf{P},\ \mathbf{Q},\ \mathbf{Z})(\mathcal{T}) = CIRC(\mathbf{P} \cup \mathbf{Q} \cup \mathbf{Q}',\ \emptyset,\ \mathbf{Z})(\mathcal{T} \cup \{Q_k \Leftrightarrow \neg Q'_k\}_{k\in K}) \cap \mathcal{L}$

   (the second $CIRC$ being defined in the language $\mathcal{L}'$, which is $\mathcal{L}$ augmented by $\mathbf{Q}'$).
2. $CIRCF(\Phi;\ \mathbf{Q},\ \mathbf{Z}) = CIRCF(\Phi \cup \mathbf{Q} \cup \neg\mathbf{Q})$. $\square$

As this result is well known, we could omit the proof (all we have done is adapting de Kleer and Konolige's result to formula circumscription, for which things are simpler because we do not need an explicit extended vocabulary). However, let us give here a short proof, which uses an easy lemma, which it is sometimes useful to remind:

Lemma 5.10 $\preceq_{(\Phi;\ \mathbf{Q},\ \mathbf{Z})} = \preceq_{\Phi\cup\mathbf{Q}\cup\neg\mathbf{Q}}$, thus $\prec_{(\Phi;\ \mathbf{Q},\ \mathbf{Z})} = \prec_{\Phi\cup\mathbf{Q}\cup\neg\mathbf{Q}}$.

Remind (definition 5.3) that $\preceq_{\Phi\cup\mathbf{Q}\cup\neg\mathbf{Q}} = \preceq_{(\Phi\cup\mathbf{Q}\cup\neg\mathbf{Q};\ \emptyset,\ \mathbf{Z}\cup\mathbf{Q})}$ and also
$\prec_{\Phi\cup\mathbf{Q}\cup\neg\mathbf{Q}} = \prec_{(\Phi\cup\mathbf{Q}\cup\neg\mathbf{Q};\ \emptyset,\ \mathbf{Z}\cup\mathbf{Q})}$. $\square$

The lemma is an immediate consequence of the definition of $\preceq_{(\Phi;\ \mathbf{Q},\ \mathbf{Z})}$ (definition 5.3, see also lemma 5.4, points 1 and 4). Now, reminding that $CIRCF(\Phi \cup \mathbf{Q} \cup \neg\mathbf{Q};\ \emptyset,\ \mathbf{Q} \cup \mathbf{Z})$ is what we also denote by $CIRCF(\Phi \cup \mathbf{Q} \cup \neg\mathbf{Q})$, proposition 5.9-2 follows from this lemma, and proposition 5.9-1 (which is exactly de Kleer and Konolige's result, restricted to the propositional case) follows, if we use definition 5.1 and proposition 5.2. □

Proposition 5.9 shows that formally, we may get rid of the fixed propositions in any ordinary circumscription: fixing a proposition amounts to minimize the proposition together with its negation.

Thus, "the particular case" where $\mathbf{Q}$ is empty is not so particular: we may always write a formula circumscription in such a way that $\mathbf{Q}$ is empty. As sometimes we will extensively use this writing $CIRCF(\Phi)$ instead of $CIRCF(\Phi';\ \mathbf{Q}, \mathbf{Z})$, with $\Phi = \Phi' \cup \mathbf{Q} \cup \neg\mathbf{Q}$, let us give again lemma 5.4 restricted to this case.

Lemma 5.11 (**lemma 5.4 when Q is empty**)

0. $\mu \preceq_\Phi \nu$ iff $\Phi_\mu \subseteq \Phi_\nu$, thus $\mu \prec_\Phi \nu$ iff $\Phi_\mu \subset \Phi_\nu$.

1a. $\mu \preceq_\Phi \nu$ iff for any $\varphi \in \Phi$, if $\mu \models \varphi$ then $\nu \models \varphi$,

1b. $\mu \prec_\Phi \nu$ iff for any $\varphi \in \Phi$, if $\mu \models \varphi$ then $\nu \models \varphi$, and there exists $\varphi \in \Phi$ such that $\mu \not\models \varphi$ and $\nu \models \varphi$.

1.c $\mu \prec_\Phi \nu$ iff $\mu \preceq_\Phi \nu$ and $\nu \not\preceq_\Phi \mu$.

$\mu \preceq_\Phi \nu$ iff $\mu \prec_\Phi \nu$ or (exclusive or) $\nu \simeq_\Phi \mu$.

2. Nothing more ($\mathbf{Q} = \emptyset$ already in lemma 5.4).

3. $\mu \preceq_\Phi \nu$ iff $\nu \preceq_{\neg\Phi} \mu$. Thus $\mu \prec_\Phi \nu$ iff $\nu \prec_{\neg\Phi} \mu$.

4a. $\mu \preceq_\Phi \nu$ iff for any $\varphi \in \Phi$, $\mu \preceq_{\{\varphi\}} \nu$.

4b. $\mu \prec_\Phi \nu$ iff (for any $\varphi \in \Phi$, $\mu \preceq_{\{\varphi\}} \nu$, and there exists $\varphi \in \Phi$ such that $\mu \prec_{\{\varphi\}} \nu$).

5. $\prec_\Phi$ and $\preceq_\Phi$ are transitive, $\prec_\Phi$ is irreflexive (thus it is a strict order) while $\preceq_\Phi$ is reflexive (it is not an order relation, because antisymmetry is missing).

   $\simeq_\Phi$ is an equivalence relation.

6. For any μ and ν, $\mu \preceq_\emptyset \nu$ and ($\mu \preceq_{\mathcal{L}} \nu$ iff $\mu = \nu$). In these two cases $\prec_\Phi$ has an empty graph.

□

Let us give now two remarks in order to justify one aspect of the choice of notations made in the present text. Our notations may sometimes look cumbersome. We think that this is not completely avoidable. Here we will argue in favor of our notation for formula circumscriptions:

Remarks 5.12 1) We could have simplified our notations: From a formal perspective, proposition 5.9 shows that, in definition 5.1, we could use only the shorter notation $CIRCF(\Phi)$ instead of $CIRCF(\Phi; \mathbf{Q}, \mathbf{Z})$, because $CIRCF(\Phi; \mathbf{Q}, \mathbf{Z}) = CIRCF(\Phi \cup \mathbf{Q} \cup \neg \mathbf{Q})$. Thanks to proposition 5.2, we could even express any ordinary circumscription in this way: $CIRC(\mathbf{P}, \mathbf{Q}, \mathbf{Z}) = CIRCF(\mathbf{P} \cup \mathbf{Q} \cup \neg \mathbf{Q})$. But we do not think that this is always a good thing to do. One reason is that the fact that some propositions are fixed is a useful indication: In proposition 5.8, we had as applicability conditions $\mathcal{T}$ finitely axiomatizable and Φ finite, but we did not need $\mathbf{Q}$ finite. If we use the concise notation $CIRCF(\Phi)$ given above, the eventual infinity of $\mathbf{Q}$ would be transformed into an infinity of Φ, which does not seem to be a good idea: for instance, proposition 5.17 below would become inapplicable in cases when it can be applied as it stands.

Notice however that in some kinds of results, it is relatively harmless to use the concise writing $CIRCF(\Phi)$, and we will use it extensively (e.g. in almost all the section 6 and also already in subsections 5.4 and 5.5 below).

2) We could also have complicated our notations: Sometimes, mainly from a knowledge representation perspective, it would be useful to define a formula circumscription from two sets of formulas Φ and Φ'', the set Φ being minimized as in definition 5.1 and the set Φ'' being kept fixed as $\mathbf{Q}$ in definition 4.1. However, from proposition 5.9 (more precisely, from an obvious extrapolation to formulas of this result), this addition is not necessary: fixing a formula amounts to minimize this formula together with its negation: when we need to "fix" some set Φ'' of formulas, we add the set $\Phi'' \cup \neg\Phi''$ to Φ (we use this trick in subsection 7.5). Here we have decided not to introduce this complication in the notation of formula circumscriptions, even if it might be useful for many uses of these circumscriptions. $\square$

Now, we are ready to give, after definition 5.1 and proposition 5.5, a third way for defining propositional circumscription:

Definition 5.13 [Poo94, subsection 4.5] (see also [Mak94, subsections 3.3. and 3.4]) Let Φ be a set of formulas in $\mathcal{L}$. For any theory $\mathcal{T} \in \mathcal{J}$, we define the set of sets of formulas $M(\Phi, \mathcal{T})$ by: $\Psi \in M(\Phi, \mathcal{T})$ iff 1) $\Psi \subseteq \Phi$, 2) $\mathcal{T} \cup \Psi$ is consistent, and 3) Ψ is maximum (for $\subseteq$) satisfying 1) and 2).

We define the pre-circumscription $f_P(\Phi)$ by: $f_P(\Phi)(\mathcal{T}) = \bigcap_{\Psi \in M(\Phi,\mathcal{T})} (\mathcal{T} \sqcup \Psi)$.

$f_P(\Phi)$ is called the *sceptical deduction by the set Φ of Poole's defaults without constraint.* $\square$

For the reader familiar with the terminology of default logic (see [Rei80]), "Pooles defaults without constraint" are the Reiter "normal defaults" (thus the expression "without constraint") "without prerequisite" (thus the "Poole's defaults") studied in [BQQ83, Bes89], with the simplification here that only the intersection of all the extensions is considered (thus the expression "sceptical").

Proposition 5.14 [Poo94, Theorem 4.5.1] The sceptical deduction by the set Φ of Poole's defaults without constraint is equal to the formula circumscription of the set $\neg\Phi$:

$$f_P(\Phi) = CIRCF(\neg\Phi).$$

Proof: We give here, in our terms, the proof appearing in [Poo94] (proof of Theorem 4.5.1, point 3 $\Leftrightarrow$ 1, reduced to the case of propositional calculus). We consider the preference relation $\prec = \prec_{\neg\Phi}$. $\mathcal{T} \in \mathfrak{T}$.

$CIRCF(\neg\Phi)(\mathcal{T}) \subseteq f_P(\Phi)(\mathcal{T})$:

Let us suppose $\varphi \notin f_P(\Phi)(\mathcal{T})$. Thus, there exists $\Psi \in M(\mathcal{T}, \Phi)$ such that we have $\varphi \notin \mathcal{T} \sqcup \Phi = Th(\mathcal{T} \cup \Psi)$. Thus, $\mathcal{T} \cup \Psi \cup \{\neg\varphi\}$ has at least one model μ. Let us suppose $\mu \notin \mathcal{M}_\prec(\mathcal{T})$. Then there exists $\nu \in \mathcal{M}(\mathcal{T})$ such that $\nu \prec \mu$. Thus (definition 5.3 of $\prec_{\neg\Phi}$) there exists a formula $\varphi' \in \neg\Phi$ such that $\nu \models \varphi'$ and $\mu \models \neg\varphi'$. As $\mu \models \Psi$, $\Psi \subseteq \Phi$ and $\nu \prec_{\neg\Phi} \mu$, we get $\nu \models \Psi$. Thus, $\nu \models \mathcal{T} \cup \Psi \cup \{\varphi'\}$, thus $(\mathcal{T} \cup \Psi) \not\models \neg\varphi'$, thus $\neg\varphi' \notin \Psi$, i.e. $\varphi' \notin \neg\Psi$. Remind $\varphi' \in \neg\Phi$, thus $\Psi \subset \Psi \cup \{\neg\varphi'\} \subseteq \Phi$. Also, as $\mu \models \mathcal{T} \cup \Psi \cup \{\neg\varphi'\}$, we know that $\mathcal{T} \cup \Psi \cup \{\neg\varphi'\} = \mathcal{T} \cup (\Psi \cup \{\neg\varphi'\})$ is consistent. This contradicts the maximality condition for Ψ in the definition of $M(\mathcal{T}, \Phi)$, thus it cannot exist such a model ν and we get $\mu \in \mathcal{M}_\prec(\mathcal{T}) \subseteq \mathcal{M}(f_\prec(\mathcal{T}))$. Thus, there exists a model μ of $f_\prec(\mathcal{T})$ such that $\mu \models \neg\varphi$. Thus $\varphi \notin f_\prec(\mathcal{T}) = CIRCF(\neg\Phi)$.

$f_P(\Phi)(\mathcal{T}) \subseteq CIRCF(\neg\Phi)(\mathcal{T})$:

Let us suppose $\varphi \notin CIRCF(\neg\Phi)(\mathcal{T})$, thus there exists a model $\mu \in \mathcal{M}_\prec(\mathcal{T})$ such that $\mu \models \neg\varphi$. We consider the set Φ_μ (as defined in definition 5.3). We have 1) $\Phi_\mu \subseteq \Phi$, and 2) $\mathcal{T} \cup \Phi_\mu$ is consistent, because $\mu \models \mathcal{T} \cup \Phi_\mu$. Let us suppose that there exists $\psi \in \Phi - \Phi_\mu$ such that $\mathcal{T} \cup \Phi_\mu \cup \{\psi\}$ is consistent, then, $\mathcal{T} \cup \Phi_\mu \cup \{\psi\}$ has a model ν. We have then $\nu \models \Phi_\mu$, $\nu \models \psi$ and $\psi \notin \Phi_\mu$ thus $\Phi_\mu \subset \Phi_\nu$, i.e. $\mu \prec_\Phi \nu$ from lemma 5.11-0, i.e. $\nu \prec_{\neg\Phi} \mu$ from lemma 5.11-3. As $\nu \models \mathcal{T}$, this contradicts $\mu \in \mathcal{M}_\prec(\mathcal{T})$, thus nos such ψ can exist. This establishes that the set Φ_μ is maximal among the sets satisfying 1) and 2), thus $\Phi_\mu \in M(\mathcal{T}, \Phi)$. Thus, as $\mu \models \mathcal{T} \cup \Phi_\mu \cup \{\neg\varphi\}$, we get $\varphi \notin Th(\mathcal{T} \cup \Phi_\mu) = \mathcal{T} \sqcup \Phi_\mu$, thus $\varphi \notin \bigcap_{\Psi \in M(\Phi, \mathcal{T})} (\mathcal{T} \sqcup \Psi) = f_P(\Phi)(\mathcal{T})$.

Notice that we have in fact proved a result slightly stronger than what we have announced. Indeed, from this proof, it is easy to see that we have $\mathcal{M}_\prec(\mathcal{T}) = \bigcup_{\Psi \in M(\Phi, \mathcal{T})} \mathcal{M}(\mathcal{T} \cup \Psi)$, while the announced result is equivalent to $TC(\mathcal{M}_\prec(\mathcal{T})) = TC(\bigcup_{\Psi \in M(\Phi, \mathcal{T})} \mathcal{M}(\mathcal{T} \cup \Psi))$. This shows that the equivalence between formula circumscription and f_P is even slightly deeper than the pure equality $CIRCF(\neg\Phi) = f_P(\Phi)$. $\square$

Thus, as we have seen, $\prec_\Phi$ of definition 5.3 is studied in subsection 4.5 of [Poo94], and also in Observation 3.4.11 of [Mak94]. [Poo94] makes a detailed and explicit comparison with circumscription (ordinary predicate circumscription, "without fixed predicates"). However, and rather strangely, none of these two authors make the explicit comparison with formula circumscription, which is the comparison to make. May be the reason is that these two authors were not aware, at that time, of the results from [dKK89], from which we have obtained our proposition 5.9: at least this would explain the (useless) exclusion of the fixed predicates evoked above.

Notice that [Fre98] studies propositional $f_P(\Phi)$, and also $\prec_{\neg\Phi}$[11] without any reference, either explicit or implicit, to circumscription, which is, to say the least, rather strange. Indeed, the original notion of default reasoning, as defined in [Rei80], must be considerably simplified in order to get the "Poole's sceptical inference without constraints": we must restrict our attention to normal prerequisite free defaults and, more importantly, to the intersection of the classical defaults "extensions". Thus, this is no longer what is generally called "default inference". It has been observed till their introduction [BQQ83, Bes89] that the inference by normal prerequisite free defaults consists in considering maximally consistent subsets. [Poo94] and, to some extend, [Mak94], have made this point of view, and the relation with circumscription, clear and explicit in the "sceptical case". But this remains (at least in the propositional case) a method of reasoning much closely related to the circumscription formalism than to the default tradition. And the results about circumscription should not be ignored, or treated as if they were ignored. The importance of the result of Poole (and Makinson) is that it describes the restrictions that must be made on "default reasoning" in order to obtain this exact correspondance.

Once the correspondance of proposition 5.14 is obtained, this provides a new point of view from which we may consider formula circumscription, and this allows to use the literature about these particular "default inference" systems when studying formula circumscription, and conversely[12].

Let us give one example here.

Proposition 5.15 [Mak94, Observation 3.3.6] If $\neg\Phi$ is in $\mathcal{J}$, then we get, for any $\mathcal{T} \in \mathcal{J}$

$$CIRCF(\Phi)(\mathcal{T}) = \begin{cases} \mathcal{T} & \text{if } \bot \in \mathcal{T} \sqcup \neg\Phi \\ \mathcal{T} \sqcup \neg\Phi & \text{if it is consistent.} \end{cases} \quad \square$$

The case when $\neg\Phi$ is deductively closed is a rather degenerated case, a fact which was not too unexpected. This result gives precisely what happens then. Here is Makinson's proof:

Proof: $CIRCF(\Phi) = f_P(\neg\Phi)$ from proposition 5.14. We have $\neg\Phi \in \mathcal{J}$.

[11] [Fre98] considers only finite sets of formulas $\mathcal{T}$. However, this is not a great restriction because most of the results given there, including, from its own conclusion, its "main result", concern only the case when $\mathcal{V}(\mathcal{L})$ is finite also.

[12] As remarked by Poole, this correspondance is not so clear in the predicate calculus case: we have no longer exact equality between the two formalisms as they are classically described. However, even in the predicate calculus case, we may learn a lot from the remaining correspondance. Also, under a few conditions, we have exact equivalence again (see [Poo94] for details).

If $\mathcal{T} \sqcup \neg\Phi$ is consistent, then we take $\neg\Phi$ as our set Ψ in definition 5.13 and $M(\mathcal{T}, \neg\Phi) = \{\Psi\}$, thus $f_P(\neg\Phi)(\mathcal{T}) = \mathcal{T} \sqcup \neg\Phi$.

We suppose now $\perp \in \mathcal{T} \sqcup \neg\Phi$ and also $\varphi \notin \mathcal{T}$. By compactness, there exists $\psi \in \mathcal{T}$ such that $\perp \in (\neg\Phi) \sqcup \psi$, thus $\perp \in (\neg\Phi) \sqcup (\psi \wedge \varphi)$, thus $\neg\psi \vee \neg\varphi \in \neg\Phi$. As $\varphi \notin \mathcal{T}$, we know that $\mathcal{T} \cup \{\neg\varphi\}$ is consistent, thus $\mathcal{T} \cup \{\neg\varphi \vee \neg\psi\}$ is consistent. Thus, there exists $\Psi \in M(\neg\Phi, \mathcal{T})$ such that $\neg\varphi \vee \neg\psi \in \Psi$, thus $\neg\varphi \in \Psi \sqcup \psi \subseteq \Psi \sqcup \mathcal{T}$, thus $\varphi \notin \Psi \sqcup \mathcal{T}$, thus, as $\Psi \in M(\neg\Phi, \mathcal{T})$, $\varphi \notin f_P(\neg\Phi)(\mathcal{T})$ from definition 5.13. This proves: $f_P(\neg\Phi)(\mathcal{T}) \subseteq \mathcal{T}$, i.e. $CIRCF(\Phi)(\mathcal{T}) \subseteq \mathcal{T}$. As we know $\mathcal{T} \subseteq CIRCF(\Phi)(\mathcal{T})$ anyway, we get $CIRCF(\Phi)(\mathcal{T}) = \mathcal{T}$.
□

Notice that, as in proposition 5.14, we could even show easily a slightly more precised result: $\mathcal{M}_{\prec_\Phi}(\mathcal{T}) = \begin{cases} \mathcal{M}(\mathcal{T}) & \text{if } \mathcal{M}(\mathcal{T}) \cap \mathcal{M}(\neg\Phi) = \emptyset \\ \mathcal{M}(\mathcal{T}) \cap \mathcal{M}(\neg\Phi) & \text{otherwise.} \end{cases}$

Let us examine now a similar particular case:

Proposition 5.16 1. If $\Phi \in \mathcal{J}$, then we get, for any $\mathcal{T} \in \mathcal{J}$

$$CIRCF(\Phi)(\mathcal{T}) = \begin{cases} \mathcal{T} & \text{if } \mathcal{T} \models \Phi \\ Th(\mathcal{M}(\mathcal{T}) - \mathcal{M}(\Phi)) & \text{if } \mathcal{T} \not\models \Phi. \end{cases}$$

2. $CIRCF(\{\varphi\})(\mathcal{T}) = CIRCF(Th(\varphi))(\mathcal{T}) =$
$$CIRCF(\neg Th(\neg\varphi))(\mathcal{T}) = \begin{cases} \mathcal{T} & \text{if } \mathcal{T} \models \varphi \\ \mathcal{T} \sqcup \neg\varphi & \text{if } \mathcal{T} \not\models \varphi. \end{cases}$$

□

Proof: 1. $\Phi \in \mathcal{J}$ thus $\Phi_\mu = \Phi \cap Th(\mu) \in \mathcal{J}$ and $\mathcal{M}(\Phi_\mu) = \mathcal{M}(\Phi) \cup \{\mu\}$ (finite union of closed sets). Thus, $\Phi_\mu \subseteq \Phi_\nu$ iff $\mathcal{M}(\Phi_\nu) \subseteq \mathcal{M}(\Phi_\mu)$ iff $\nu \in \mathcal{M}(\Phi) \cup \{\mu\}$. Thus we get $\mu \prec_\Phi \nu$ iff $\Phi_\mu \subset \Phi_\nu$ iff $\nu \in \mathcal{M}(\Phi)$ and $\mu \notin \mathcal{M}(\Phi)$. Thus, $\mu \in \mathcal{M}_{\prec_\Phi}(\mathcal{T})$ iff
$\mu \in \mathcal{M}(\mathcal{T})$ and (for any $\nu \in \mathcal{M}(\mathcal{T})$ we have $\mu \notin \mathcal{M}(\Phi)$ or $\nu \in \mathcal{M}(\Phi)$) iff
$\mu \in \mathcal{M}(\mathcal{T})$ and ($\mu \notin \mathcal{M}(\Phi)$ or for any $\nu \in \mathcal{M}(\mathcal{T})$ we have $\nu \in \mathcal{M}(\Phi)$) iff
$\mu \in \mathcal{M}(\mathcal{T})$ and ($\mu \not\models \Phi$ or $\mathcal{M}(\mathcal{T}) \subseteq \mathcal{M}(\Phi)$).

Thus $\mathcal{M}_{\prec_\Phi}(\mathcal{T}) = \begin{cases} \mathcal{M}(\mathcal{T}) & \text{if } \mathcal{M}(\mathcal{T}) \subseteq \mathcal{M}(\Phi) \\ \mathcal{M}(\mathcal{T}) - \mathcal{M}(\Phi) & \text{if } \mathcal{M}(\mathcal{T}) \not\subseteq \mathcal{M}(\Phi), \text{ i.e.} \end{cases}$

$$CIRCF(\Phi)(\mathcal{T}) = f_{\prec_\Phi}(\mathcal{T}) = \begin{cases} \mathcal{T} & \text{if } \mathcal{T} \models \Phi \\ Th(\mathcal{M}(\mathcal{T}) - \mathcal{M}(\Phi)) & \text{if } \mathcal{T} \not\models \Phi. \end{cases}$$

2. φ in some formula in $\mathcal{L}$. Let us define $f(\mathcal{T}) = \begin{cases} \mathcal{T} & \text{if } \mathcal{T} \models \varphi \\ \mathcal{T} \sqcup \neg\varphi & \text{if } \mathcal{T} \not\models \varphi. \end{cases}$

If $\Phi = Th(\varphi)$ we get $\mathcal{M}(\Phi) = \mathcal{M}(\varphi)$ and $\mathcal{M}(\mathcal{T}) - \mathcal{M}(\Phi) = \mathcal{M}(\mathcal{T}) \cap \mathcal{M}(\neg\varphi) = \mathcal{M}(\mathcal{T} \sqcup \neg\varphi)$, thus $Th(\mathcal{M}(\mathcal{T}) - \mathcal{M}(\Phi)) = Th(\mathcal{T} \sqcup \neg\varphi) = \mathcal{T} \sqcup \neg\varphi$. Using point 1, we get $CIRCF(Th(\varphi))(\mathcal{T}) = f(\mathcal{T})$.

Also we get $\mu \prec_{\{\varphi\}} \nu$ iff $\{\varphi\}_\mu \subset \{\varphi\}_\nu$ iff $\mu \not\models \varphi$ and $\nu \models \varphi$ iff $\mu \not\models Th(\varphi)$ and $\nu \models Th(\varphi)$ iff, as seen in the proof of point 1, $\mu \prec_{Th(\varphi)} \nu$. Thus $\prec_{\{\varphi\}} = \prec_{Th(\varphi)}$, i.e. $CIRCF(\{\varphi\}) = CIRCF(Th(\varphi))$.

Finally, we know that $\mathcal{T} \sqcup Th(\neg\varphi) = \mathcal{T} \sqcup \neg\varphi$ and also that $\mathcal{T} \sqcup \neg\varphi$ is consistent iff $\mathcal{T} \not\models \varphi$. Thus, from proposition 5.15, we get $CIRCF(\neg Th(\neg\varphi))(\mathcal{T}) = f(\mathcal{T})$. Alternatively, we could easily prove directly $\prec_{\{\varphi\}} = \prec_{\neg Th(\neg\varphi)}$, which would give another proof for Makinson's result in this particular case. □

5.3 Refining our study of the properties of formula circumscription

Proposition 5.17 Any formula circumscription $CIRCF(\Phi;\ \mathbf{Q}, \mathbf{Z})$ in which Φ is finite satisfies (DC0). □

Proof: Use corollary 4.12 and definition 5.1: this gives (RM0) (subsumed by (RM1), a property of any preferential entailment) and (DC0). Alternatively, we could give a direct proof in the lines of the proof of corollary 4.12, thanks to proposition 5.8. □

Notice that this result does not extend to infinite Φ, even when $\mathbf{Q}$ is empty, as the following example shows.

Example 5.18 $\mathcal{V}(\mathcal{L}) = \{P_n\}_{n \in \mathbb{N}}$.

We define the interpretations $\mu_n = \{P_0, P_1, P_{2n}\}$, $\nu_n = \{P_0, P_1, P_{2n+1}\}$, $\mu_\omega = \{P_0, P_1\}$.

We get $\lim_{n\to\infty} \mu_n = \lim_{n\to\infty} \nu_n = \mu_\omega$.

We consider the formulas $\psi_n = P_0 \wedge P_1 \wedge P_{2n+1} \wedge \bigwedge_{k<n}(\neg P_{2k+1} \wedge \neg P_{2k})$.

Thus we get $\nu_n \in \mathcal{M}(\psi_n)$, $\mathcal{M}(\psi_n) \subseteq \mathcal{M}(P_0 \wedge P_1)$, $\mu_\omega \notin \mathcal{M}(\psi_n)$ and, if $n \neq m$, $\mathcal{M}(\psi_n) \cap \mathcal{M}(\psi_m) = \emptyset$.

We define $\prec$ by: $\mu \prec \nu$ iff $\mu = \{P_0\}$ and $\nu \in \mathcal{M}(P_0 \wedge P_1) - \bigcup_{n \in \mathbb{N}} \mathcal{M}(\psi_n)$
or $\mu = \{P_1\}$ and ($\nu = \mu_\omega$ or there exists $n \in \mathbb{N}$ such that $\nu \in \mathcal{M}(\psi_n)$).

Then $\prec$ is transitive and irreflexive.

We get, for any $n \in \mathbb{N}$, $\mu_n \in \mathcal{M}_\prec(P_1)$ because $\{P_1\} \not\prec \mu_n$ and $\nu_n \in \mathcal{M}_\prec(P_0)$ because, as $\nu_n \in \mathcal{M}(\psi_n)$, $\{P_0\} \not\prec \mu_n$.

Thus $\mu_\omega \in TC(\mathcal{M}_\prec(P_0)) \cap TC(\mathcal{M}_\prec(P_1)) = \mathcal{M}(f_\prec(P_0)) \cap \mathcal{M}(f_\prec(P_1)) = \mathcal{M}(f_\prec(P_0) \sqcup f_\prec(P_1))$.

We get also $\mathcal{M}_\prec(P_0 \vee P_1) \cap \mathcal{M}(P_0 \wedge P_1) = \emptyset$ because if $\mu \in \mathcal{M}(P_0 \wedge P_1)$ and $\{P_1\} \not\prec \mu$, then $\mu \notin \bigcup_{n \in \mathbb{N}} \mathcal{M}(\psi_n)$, thus $\{P_0\} \prec \mu$. Thus, as $\mu_\omega \in \mathcal{M}(P_0 \wedge P_1)$ and as $\mathcal{M}(P_0 \wedge P_1)$ is an open set, we have $\mathcal{M}(f_\prec(P_0) \sqcup f_\prec(P_1)) \not\subseteq TC(\mathcal{M}_\prec(P_0 \vee P_1))$, i.e. $\mathcal{M}(f_\prec(P_0) \sqcup f_\prec(P_1)) \not\subseteq \mathcal{M}(f_\prec(P_0 \vee P_1))$.

This shows that $f_\prec$ falsifies (DC0).

It remains to prove that $f_\prec$ is a formula circumscription.

We define the set of formulas $\Phi = \{\varphi \in \mathcal{L}\ /\ $ for any μ, ν such that $\mu \models \varphi$ and $\mu \prec \nu$ then $\nu \models \varphi\}$.

We will prove now that we have indeed $CIRCF(\Phi;\ \emptyset, \mathcal{V}(\mathcal{L})) = CIRCF(\Phi) = f_\prec$, i.e $\prec = \prec_\Phi$.

If $\mu \prec \nu$ then $\Phi_\mu \subseteq \Phi_\nu$ from the definition of Φ (see definition 5.3 for the definition of Φ_μ).

If $\mu = \{P_0\}$, we have $\nu \in \mathcal{M}(P_0 \wedge P_1)$, $P_0 \wedge P_1 \in \Phi$ and $\mu \not\models P_0 \wedge P_1$, thus $\Phi_\mu \subset \Phi_\nu$, i.e. $\mu \prec_\Phi \nu$.

Similarly, if $\mu = \{P_1\}$, we have also $\mu \prec_\Phi \nu$.

Let us suppose now $\mu \prec_\Phi \nu$, thus $\mu \neq \nu$.

If $\mu \in \mathcal{M}(P_0 \wedge P_1)$ then there exists φ such that $\mathcal{M}(\varphi) \subseteq \mathcal{M}(P_0 \wedge P_1)$, $\mu \models \varphi$ and $\nu \models \neg\varphi$, which contradicts $\varphi \in \Phi$, a consequence of $\mathcal{M}(\varphi) \subseteq \mathcal{M}(P_0 \wedge P_1)$. Thus we get $\mu \notin \mathcal{M}(P_0 \wedge P_1)$.

If $\mu \in \mathcal{M}(P_0 \wedge \neg P_1)$ and $\mu \neq \{P_0\}$, then there exists φ such that $\mathcal{M}(\varphi) \subseteq \mathcal{M}(P_0 \wedge \neg P_1)$, $\{P_0\} \notin \mathcal{M}(\varphi)$, $\mu \models \varphi$ and $\nu \models \neg\varphi$, which contradicts $\varphi \in \Phi$, a consequence of $\mathcal{M}(\varphi) \subseteq \mathcal{M}(P_0 \wedge \neg P_1)$ and $\{P_0\} \notin \mathcal{M}(\varphi)$.

Thus we get that if $\mu \in \mathcal{M}(P_0 \wedge \neg P_1)$, then necessarily $\mu = \{P_0\}$.

If $\mu = \{P_0\}$, then let us suppose $\nu \notin \mathcal{M}(P_0 \wedge P_1)$. Then there exists ψ such that $\mathcal{M}(\psi) \subseteq \mathcal{M}(P_0 \wedge \neg P_1)$, $\{P_0\} \in \mathcal{M}(\psi)$ and $\nu \models \neg\psi$. If we define $\varphi = (P_0 \wedge P_1) \vee \psi$, we get $\varphi \in \Phi$. Indeed, if $\mu' \prec \nu'$ and $\mu' \models \varphi$, then $\nu' \in \mathcal{M}(P_0 \wedge P_1)$ from the definition of $\prec$, thus $\nu' \models \varphi$. We have also $\mu \models \varphi$ and $\nu \not\models \varphi$, a contradiction with $\mu \prec_\Phi \nu$. Thus, if $\mu = \{P_0\}$, then $\nu \in \mathcal{M}(P_0 \wedge P_1)$.

It is also easy to see that we have $P_0 \wedge \neg\psi_n \in \Phi$. Also, $\{P_0\} \models P_0 \wedge \neg\psi_n$, thus, still with $\mu = \{P_0\}$, we get $\nu \models P_0 \wedge \neg\psi_n$, thus $\nu \notin \mathcal{M}(\psi_n)$.

We have established that, if $\mu \in \mathcal{M}(P_0)$ (remind that we know $\mu \notin \mathcal{M}(P_0 \wedge P_1)$), then $\mu = \{P_0\}$ and $\mu \prec \nu$ (see definition of $\prec$).

Let us suppose now $\mu \in \mathcal{M}(\neg P_0 \wedge P_1)$, then the same argument as above shows that we must have $\mu = \{P_1\}$ and $\nu \in \mathcal{M}(P_0 \wedge P_1)$. Now, as we have $\mu_\omega = \lim_{n\to\infty} \mu_n$, it can be shown that $TC(\cup \bigcup_{n\in\mathbb{N}} \mathcal{M}(\psi_n))$ is equal to the set $\{\mu_\omega\} \cup \bigcup_{n\in\mathbb{N}} \mathcal{M}(\psi_n)$, which is thus a closed set. Thus, if $\nu \notin \{\mu_\omega\} \cup \bigcup_{n\in\mathbb{N}} \mathcal{M}(\psi_n)$, then there exists a formula ψ such that $\mathcal{M}(\psi) \subseteq \mathcal{M}(P_0 \wedge P_1)$, $\mathcal{M}(\psi) \cap (\bigcup_{n\in\mathbb{N}} \mathcal{M}(\psi_n)) = \emptyset$ and $\nu \models \psi$. If we define $\varphi = P_1 \wedge \neg\psi$, it is easy to show that we get $\varphi \in \Phi$. Then, as we have $\{P_1\} \models \varphi$, we have also $\nu \models \varphi$ thus $\nu \not\models \psi$, a contradiction with $\nu \models \psi$. Thus, $\nu \in \{\mu_\omega\} \cup \bigcup_{n\in\mathbb{N}} \mathcal{M}(\psi_n)$ and $\{P_1\} \prec \nu$ from the definition of $\prec$.

We have established that, if $\mu \in \mathcal{M}(P_1)$ (remind that we know $\mu \notin \mathcal{M}(P_0 \wedge P_1)$), then we have $\mu = \{P_1\}$ and $\mu \prec \nu$.

Let us suppose $\mu \in \mathcal{M}(\neg P_0 \wedge \neg P_1)$. Then there exists a formula φ such that $\mathcal{M}(\varphi) \subseteq \mathcal{M}(\neg P_0 \wedge \neg P_1)$, $\mu \models \varphi$ and $\nu \not\models \varphi$. Then, as $\mathcal{M}(\varphi) \subseteq \mathcal{M}(\neg P_0 \wedge \neg P_1)$, we get $\varphi \in \Phi$, a contradiction. This proves $\mu \notin \mathcal{M}(\neg P_0 \wedge \neg P_1)$.

Thus, we have proved that in any case, if $\mu \prec_\Phi \nu$, then $\mu \prec \nu$, which, together with the previous result, gives $\prec = \prec_\Phi$. □

Proposition 5.19 (partially in [Mak94, Observation 3.3.4]) If Φ is finite, then $CIRCF(\Phi)$ satisfies (RM) and (DC). □

Notice that [Mak94] evokes neither formula circumscription nor (DC). Also Theorem 7.17 in [FL93] gives the result for f_P and (RM) (and (CUMU) by the way) and the authors seem to claim that they have found the result before Makinson (they cite Makinson for other results and not for this one, while preliminary versions of [Mak94] appeared around 1990), however, [FL93] makes no connection with circumscription, even implicitly, while Makinson gives the preference relation associated, which is a first step towards the connection with formula circumscription.

As in our context it is much more natural to use proposition 4.13, we give our own proof here.

Proof: Let $\mathcal{T}$ be a subset of $\boldsymbol{\mathcal{L}}$ and $\mathbf{P}$ a set of propositional symbols P_φ not in $\boldsymbol{\mathcal{L}}$, in one-to one correspondance with Φ. Then, from definition 4.1, we have, with $\mathcal{T}_0 = \{\varphi \Leftrightarrow P_\varphi\}_{\varphi\in\Phi}$, $CIRCF(\Phi)(\mathcal{T}) = CIRC(\mathbf{P}, \emptyset, \mathcal{V}(\boldsymbol{\mathcal{L}}))(\mathcal{T} \cup \mathcal{T}_0) \cap \boldsymbol{\mathcal{L}}$ from definition 5.1. As Φ is finite, so is $\mathbf{P}$ and from proposition 4.13-1 we know that $CIRC(\mathbf{P}, \emptyset, \mathcal{V}(\boldsymbol{\mathcal{L}}))$, defined in the language $\boldsymbol{\mathcal{L}}$ extended by $\mathbf{P}$, satisfies (RM). If $\mathcal{T} \subseteq \mathcal{T}'' \subseteq \boldsymbol{\mathcal{L}}$, then $\mathcal{T} \cup \mathcal{T}_0 \subseteq \mathcal{T}'' \cup \mathcal{T}_0$ thus from (RM) $CIRC(\mathbf{P}, \emptyset, \mathcal{V}(\boldsymbol{\mathcal{L}}))(\mathcal{T}'' \cup \mathcal{T}_0) \subseteq CIRC(\mathbf{P}, \emptyset, \mathcal{V}(\boldsymbol{\mathcal{L}}))(\mathcal{T} \cup \mathcal{T}_0) \sqcup (\mathcal{T}'' \cup \mathcal{T}_0) = CIRC(\mathbf{P}, \emptyset, \mathcal{V}(\boldsymbol{\mathcal{L}}))(\mathcal{T} \cup \mathcal{T}_0) \sqcup \mathcal{T}''$, thus $CIRCF(\Phi)(\mathcal{T}'') \subseteq CIRCF(\Phi)(\mathcal{T}'') \sqcup \mathcal{T}''$: $CIRCF$ satisfies (RM). From proposition 3.8-2 we get (DC). □

Notice the difference with proposition 5.17: Here, for a formula circumscription $CIRCF(\Phi'; \mathbf{Q}, \mathbf{Z}) = CIRCF(\Phi' \cup \mathbf{Q} \cup \neg\mathbf{Q})$ we require that $\Phi' \cup \mathbf{Q} \cup \neg\mathbf{Q}$ is finite, i.e. that $\Phi' \cup \mathbf{Q}$ is finite. This is a stronger assumption than Φ' finite, the condition of applicability of proposition 5.17. When this stronger assumption is satisfied, proposition 5.19 subsumes proposition 5.17. We already know from examples 4.14 and 4.15 that the condition Φ is finite is necessary here to get (RM) (remind that $CIRC(\mathbf{P}, \mathbf{Q}, \mathbf{Z}) = CIRCF(\mathbf{P} \cup \mathbf{Q} \cup \neg\mathbf{Q})$). Moreover, example 5.18 shows that $CIRCF(\Phi)$ may falsify (DC0) when Φ is infinite.

Proposition 5.20 1. If Φ is finite, then $\prec_{(\Phi;\, \mathbf{Q},\, \mathbf{Z})}$ satisfies (cl0).

2. If Φ is finite then $\prec_\Phi$ satisfies (cl). □

Proof: 1. See the proof of proposition 5.5 which relates $\prec_{(\Phi;\, \mathbf{Q},\, \mathbf{Z})}$ to the preference relation $\prec'$ associated to $CIRC(\mathbf{P}, \mathbf{Q}, \mathbf{Z})$ where, as Φ is finite, the added formulas $\phi_i \Leftrightarrow P_i$ are in finite number. From proposition 4.22, we know that $\prec'$ satisfies (cl0), and it is easy to show that then $\prec_{(\Phi;\, \mathbf{Q},\, \mathbf{Z})}$ satisfies (cl0).

2. From corollary 5.7-2 and proposition 5.19, together with proposition 3.14-1. □

Here is another result, which is good to remind, even if it is easy to prove. Proposition 4.3 cannot be extended fully to formula circumscription. All we have left now is the obvious:

Proposition 5.21 The more varying (or equivalently the less fixed) propositions a formula circonscription contains, the stronger it is:

$CIRCF(\Phi; \mathbf{Q}, \mathbf{Z}_1 \cup \mathbf{Z}_2)(\mathcal{T}) \models CIRCF(\Phi; \mathbf{Q} \cup \mathbf{Z}_2, \mathbf{Z}_1)(\mathcal{T})$. □

However, in the general case, we have neither $CIRCF(\Phi_1 \cup \Phi_2; \mathbf{Q}, \mathbf{Z})(\mathcal{T}) \models CIRCF(\Phi_1; \mathbf{Q}, \mathbf{Z})(\mathcal{T})$ nor $CIRCF(\Phi_1; \mathbf{Q}, \mathbf{Z})(\mathcal{T}) \models CIRCF(\Phi_1 \cup \Phi_2; \mathbf{Q}, \mathbf{Z})(\mathcal{T})$ (see example below). Indeed, from lemma 5.4-1a we know that the graph of $\preceq_{(\Phi_1 \cup \Phi_2;\ \mathbf{Q},\ \mathbf{Z})}$ is included in the graph of $\preceq_{(\Phi_1;\ \mathbf{Q},\ \mathbf{Z})}$, but we do not have such a result for the relation $\prec_{(\Phi_1;\ \mathbf{Q},\ \mathbf{Z})}$, as the following example shows.

Example 5.22 $\Phi_1 = \{A\}, \Phi_2 = \{\neg A \vee B\}, \mathbf{Z} = \mathcal{V}(\boldsymbol{\mathcal{L}}) = \{A, B\}$, thus $\Phi_1 \cup \Phi_2 = \{A, \neg A \vee B\}$.

We have: $\neg A \in CIRCF(\Phi_1; \emptyset, \mathbf{Z})(\top) = Th(\neg A)$, $\neg A \notin CIRCF(\Phi_1 \cup \Phi_2; \emptyset, \mathbf{Z})(\top) = Th(\neg A \vee \neg B)$,
and $\neg B \notin CIRCF(\Phi_1; \emptyset, \mathbf{Z})(A) = Th(A)$, $\neg B \in CIRCF(\Phi_1 \cup \Phi_2; \emptyset, \mathbf{Z})(A) = Th(A \wedge \neg B)$.

Notice however that we have $\neg B \in CIRCF(\Phi_2; \emptyset, \mathbf{Z})(\top) = Th(A \wedge \neg B)$. This, together with the preceding results, is related to the following proposition. $\Box$

Proposition 5.23 1. For any finite sets Ψ_1, Ψ_2 and $\mathcal{T}$ we have

$$CIRCF(\Psi_1;\ \mathbf{Q}, \mathbf{Z})(\mathcal{T}) \cup CIRCF(\Psi_2;\ \mathbf{Q}, \mathbf{Z})(\mathcal{T}) \models CIRCF(\Psi_1 \cup \Psi_2;\ \mathbf{Q}, \mathbf{Z})(\mathcal{T}).$$

2. For any finite sets of formulas Φ_1 and Φ_2, and any $\mathcal{T}$, we have

$$CIRCF(\Phi_1)(\mathcal{T}) \cup CIRCF(\Phi_2)(\mathcal{T}) \models CIRCF(\Phi_1 \cup \Phi_2)(\mathcal{T}).\ \Box$$

This may be enunciated as: in the conditions of applicability, "a union of circumscriptions is at least as strong as the corresponding parallel circumscription" (see also corollary 4.23-1).

Proof: We define $\Phi_i = \Psi_i \cup \mathbf{Q} \cup \neg\mathbf{Q}$ $(i \in \{1, 2\})$, then, as $CIRCF(\Psi_i;\ \mathbf{Q}, \mathbf{Z}) = CIRCF(\Phi_i)$ we may use the same proof for the two cases. Thanks to lemma 5.11-4, we get: $\mu \prec_{\Phi_1 \cup \Phi_2} \nu$ iff $\mu \preceq_{\Phi_1} \nu$ and $\mu \preceq_{\Phi_2} \nu$ and ($\nu \not\preceq_{\Phi_1} \mu$ or $\nu \not\preceq_{\Phi_2} \mu$). Thus we have: If $\mu \prec_{\Phi_1 \cup \Phi_2} \nu$ then $\mu \prec_{\Phi_1} \nu$ or $\mu \prec_{\Phi_2} \nu$. Thus we get $\mathcal{M}_{\prec_{\Phi_1}}(\mathcal{T}) \cap \mathcal{M}_{\prec_{\Phi_2}}(\mathcal{T}) \subseteq \mathcal{M}_{\prec_{\Phi_1 \cup \Phi_2}}(\mathcal{T})$. In the conditions of point 1 or 2 we know that these three sets are closed from the corresponding point of proposition 5.20. Thus we get $TC(\mathcal{M}_{\prec_{\Phi_1}}(\mathcal{T})) \cap TC(\mathcal{M}_{\prec_{\Phi_2}}(\mathcal{T})) \subseteq TC(\mathcal{M}_{\prec_{\Phi_1 \cup \Phi_2}}(\mathcal{T}))$, i.e. $\mathcal{M}(f_{\prec_{\Phi_1}}(\mathcal{T}) \cup f_{\prec_{\Phi_2}}(\mathcal{T})) \subseteq \mathcal{M}(f_{\prec_{\Phi_1 \cup \Phi_2}}(\mathcal{T}))$, i.e. $f_{\prec_{\Phi_1 \cup \Phi_2}}(\mathcal{T}) \subseteq f_{\prec_{\Phi_1}}(\mathcal{T}) \sqcup f_{\prec_{\Phi_2}}(\mathcal{T})$. $\Box$

Using $CIRC(\mathbf{P}, \mathbf{Q}, \mathbf{Z}) = CIRCF(\mathbf{P} \cup \mathbf{Q} \cup \neg\mathbf{Q})$, we get as a consequence the result given in corollary 4.23-1. Alternatively, we could have used corollary 4.23-1 in order to prove proposition 5.23.

5.4 A characterization of formula circumscription in the finite case

We begin now the study of the following "converse" of results such as corollary 5.7: when we have a given $f_\prec$, what are the conditions which assure us that $f_\prec$ is a formula circumscription. As we know that $\prec_{(\Phi;\ \mathbf{Q},\ \mathbf{Z})}$ and $\succ_{(\Phi;\ \mathbf{Q},\ \mathbf{Z})}$ satisfy (sf), we must have as a condition that $\prec$ and its opposite $\succ$ satisfy (sf). We will prove that this simple condition is indeed sufficient in the finite case. In the finite case, a relation $\prec$ satisfies (sf) iff it is transitive and irreflexive, iff its opposite $\succ$ satisfies (sf). None of these equivalences remain in the infinite case (as the infinite case is rather technical, we will postpone it to section 6).

Proposition 5.24 $V(\mathcal{L})$ is finite here.

1. (Independently also in [Cos98], and in [Fre98].) A preferential entailment $f_{\prec}$ is a formula circumscription iff $\prec$ satisfies (sf), ie iff $\prec$ is irreflexive and transitive.

2. A pre-circumscription f is a formula circumscription iff it satisfies (CR0), (DC0), (CUMU0) and (PC). $\square$

Proof: 1. $\prec$ satisfies (sf), i.e. from proposition 3.12-3, $\prec$ is is irreflexive and transitive. Also from proposition 3.12-1 we get that there is only one possible $\prec$. Let μ be an interpretation for $\mathcal{L}$ and θ_μ be the conjunction of all the literals of $\mathcal{L}$ satisfied by μ, thus $\mathcal{M}(\theta_\mu) = \{\mu\}$ and $Th(\theta_\mu) = Th(\mu)$. Let us define also $\psi_\mu = \theta_\mu \vee \bigvee_{\mu \prec \nu} \theta_\nu$ and $\Psi = \{\psi_\mu\}_{\mu \in \mathcal{M}}$. We consider the following formula circumscription: $f = CIRCF(\Psi)$, that is the circumscription of all the formulas ψ_μ, with all the propositional symbols varying. We prove $f = f_{\prec}$, remind that $\prec_\Psi$ denotes the relation naturally associated to this formula circumscription (see definition 5.3 and proposition 5.5).

If $\mu \prec \nu$ and $\mu \models \varphi_{\mu'}$, then $\mu' \prec \mu$ or $\mu' = \mu$, thus $\mu' \prec \nu$ ($\prec$ is transitive), and also $\nu \models \varphi_{\mu'}$. Finally $\nu \models \psi_\nu$ while $\mu \not\models \psi_\nu$ ($\prec$ is irreflexive) thus $\mu \prec_\Psi \nu$.

If $\mu \not\prec \nu$, then $\mu \models \varphi_\mu$ and, as $\nu \models \varphi_\mu$ iff $\mu \prec \nu$ or $\mu = \nu$, we get ($\nu \not\models \varphi_\mu$ or $\nu = \mu$), in any case: $\mu \not\prec_\Psi \nu$.

2. It suffices to remind that, as we are in the finite case, we have 1) and 2) below:

1) A pre-circumscription is a preferential entailment iff it satisfies (CR0) and (DC0), or equivalently (RM0) and (DC0): see proposition 3.8-3.

2) A preferential entailment satisfies (CUMU0) and (PC) iff it may be defined by a preference relation, which is thus unique, satisfying (sf): see proposition 3.14-1. $\square$

A few bibliographical comments are in order here. To our knowledge at the time of working, this result was new. Notice however that we knew Theorem 2, one of the fundamental results in the very interesting pioneer [Sat90][13]. This Theorem 2 constitutes an important part of proposition 5.24. Indeed, in our terms, it may be stated as: If $\mathcal{V}(\mathcal{L})$ is finite, a pre-circumscription satisfies (CR0), (DC0), (CUMU0) and (PC) iff it is a preferential entailment associated to a relation satisfying (sf) (i.e. transitive and irreflexive). This is the best [Sat90] could obtain, as it did not deal with the formula version of circumscriptions, but only with the circumscription of definition 4.1. Thus Satoh was very close to the full result, which is, as the easy proof given above shows, a rather natural extension of his result.

At the time of printing, we have discovered that the result given in Theorems 13 and 14 in [Fre98] is roughly equivalent to proposition 5.24-1. Moreover, the proof of Observation 11 in the referenced text, used as a lemma to get this result, is exactly the proof that we have given above (see also below our comment following our lemma 5.32)! This indicates probably that the proof is relatively natural. As already written, [Fre98] does not refer at all

[13] Theorem 4.13 in [Fre93] is a clone of Theorem 2 of [Sat90], not evoked there.

to circumscription, even if the object of all this paper is obviously formula circumscription[14]. Thus, at least for what concerns propositional circumscription, we could think that this result still had something new in it, after all...

However, we have discovered, roughly at the same time, that [Cos98] had also this result: it is its Theorem 7[15], and this time, the subject is clearly sated as "propositional formula circumscription". Costello's proof is close, but slightly different from our (and Freund's) proof. Again, no result related to proposition 5.24-2 is made by Costello, who acts also as if he ignored [Sat90].

Thus, this result was to be discovered now! Who is next?

[Cos98, Theorem 15] is a very interesting extension of this result. Indeed, Theorem 15, together with a well known result from [KLM90], may be expressed as follows in our terms:

"If $\mathcal{V}(\boldsymbol{\mathcal{L}})$ is finite, then any pre-circumscription f satisfying (SRM0) may be expressed thanks to a formula circumscription in an extended (finite) language."

Precisely, it is stated that there exist sets Γ and $\{\beta\}$ of formulas in this extended language such that for any set of formulas $\mathcal{T}$ in $\boldsymbol{\mathcal{L}}$, $f(\mathcal{T}) = CIRCF(\Gamma)(\mathcal{T} \sqcup \beta) \cap \boldsymbol{\mathcal{L}}$ (Def-f), where $CIRCF$ is defined in the extended language.

We refer the interested reader to [Cos98] for the proof of this result, which extends significatively the "expressive power of formula circumscription", as written by Costello. Let us just write a few indications about Costello's result. Costello refers to the preferential models of [KLM90], which are what we have called *multi-preferential entailments* in [MR98]. A multi-preferential entailment is defined as a preferential entailment, except that the *multi-preference relation* $\prec_m$ is defined in a set W of "copies of models" (see remark 3.7): this provides a simulation of the predicate calculus preferential entailment as defined in remark 3.9-1, applied to the propositional case. From [KLM90, Theorem 5.18], we know that in the finite case a pre-circumscription satisfies (CR0) and (CM0) iff it is a multi-preferential entailment $f_{\prec_m}$ where $\prec_m$ is irreflexive and transitive on W. From proposition 3.5, we know that (RM) implies (CT), which in the finite case is equivalent to (RM0) implies (CT0), and from proposition 3.8-3 we know that (RM0) is equivalent to (CR0), thus (RM0) +(CM0) is equivalent to (RM0) + (CUMU0), which is itself equivalent to (SRM0) from proposition 3.19.

Now, it is easy to show that any pre-circumscription f defined in $\boldsymbol{\mathcal{L}}$ by (Def-f) satisfies (CR0) (i.e. (RM0)) and (CM0) (i.e., in the presence of (RM0), (CUMU0)). Thus, using Costello's result reminded above, we get the following equivalence:

"If $\mathcal{V}(\boldsymbol{\mathcal{L}})$ is finite, then a pre-circumscription f in $\boldsymbol{\mathcal{L}}$ satisfies (SRM0) iff

[14][Sat90] is still not evoked, which does not help the reader to make useful connections with the previous works on the subject.

[15]At the time of printing (October 1998), we have only a quasi definitive version, from the web site of the Artificial Intelligence Journal, `http://www.elsevier.nl/locate/artint`. Minor modifications, including the numerotation, may occur in the published paper, which is abstracted in the number of August (1998).

there exists a finite language $\mathcal{L}'$ containing $\mathcal{L}$ and two sets Γ and $\{\beta\}$ of formulas in $\mathcal{L}'$ such that, for any $\mathcal{T}$ in $\mathcal{L}$, we have $f(\mathcal{T}) = CIRCF(\Gamma)(\mathcal{T} \sqcup \beta) \cap \mathcal{L}$ (Def-f), where $CIRCF$ is defined in the extended language $\mathcal{L}'$."

The reason why we do not put these last two results (delimitated by "[...]") as "**propositions**" is only that, as we have discovered Costello's result, which is the main part of these two results, only at the time of printing, we did not have the time to put here in our terms Costello's proof (or another proof) for his theorem 15. Thus we must refer the reader to [Cos98] for part of these proofs and results, while for all our propositions and theorems in sections 4, 5 and 6, either we provide a proof, or we consider the result as easy or well-known. Otherwise, these two results would be attributed a number as a "proposition", because we think that they are important.

It is interesting to compare this last result to proposition 5.24-2: we have "lost" (DC0) and (PC). In order to get a better understanding of this result (thus also of Costello's result), let us examine informally why we loose these two properties, from the two sides of the equivalence.

From the side of a pre-circumscription satisfying (CR0) +(CUMU0), i.e. a multi-preferential entailment $f_{\prec_m}$ associated to an irreflexive and transitive multi-preference relation $\prec_m$, let us give two simple examples:

Example 5.25

- $\mathcal{V}(\mathcal{L}) = \{P\}$. $f_m = f_{\prec_m}$ where $\prec_m$ is defined by: the two interpretations $\emptyset$ and $\{P\}$ have respectively 0 and 1 copy, and $\prec_m$ has an empty graph. Thus, $\prec_m$ is irreflexive and transitive. f_m is defined by: $f_m(\mathcal{T}) = Th(P)$ for $\mathcal{T} = Th(P)$ and $\mathcal{T} = Th(\top)$ and $f_m(\mathcal{T}) = Th(\bot)$ for $\mathcal{T} = Th(\bot)$ and $\mathcal{T} = Th(\neg P)$. The reader not familiar with the preferential models of [KLM90] may check directly that f_m satisfies indeed (CR0) and (CUMU0) (equivalent to (CM0) in the presence of (CR0)). However (PC) is falsified as $f_m(\neg P) = Th(\bot)$.
- $\mathcal{V}(\mathcal{L}) = \{P, Q\}$. $f_m = f_{\prec_m}$ where $\prec_m$ is defined by: the three interpretations $\emptyset$, $\{P\}$ and $\{Q\}$ have one copy, while the fourth interpretation $\{P, Q\}$ has two copies, and we have $\{P\} \prec_m \{P, Q\}_1$, $\{Q\} \prec_m \{P, Q\}_2$, and nothing else. Thus, $\prec_m$ is irreflexive and transitive. f_m is defined by: $f_m(\mathcal{T}) = \mathcal{T}$ for $\mathcal{T} \in \{Th(\bot), Th(\neg P \wedge \neg Q), Th(P \wedge \neg Q), Th(\neg P \wedge Q), Th(P \wedge Q), Th(P), Th(Q), Th(\neg P), Th(\neg Q), Th(P \Leftrightarrow \neg Q), Th(P \Leftrightarrow Q), Th(\neg P \vee \neg Q), Th(\neg P \vee Q), Th(P \vee \neg Q)\}$, $f_m(Th(P \vee Q)) = Th(P \Leftrightarrow \neg Q)$ and $f_m(\top) = Th(\neg P \vee \neg Q)$. Again, it is easy to check directly that f_m satisfies (CR0) and (CUMU0). However (DC0) is falsified as $Th(P \Leftrightarrow \neg Q) = f_m(P \vee Q) \not\subseteq f_m(P) \sqcup f_m(Q) = Th(P \wedge Q)$.

□

Thus in these two cases, no formula circumscription $CIRCF(\Gamma)$ defined in $\mathcal{L}$ can be such that $f_m(\mathcal{T}) = CIRCF(\Gamma)(\mathcal{T})$ for any $\mathcal{T}$ in $\mathcal{L}$. However, and we examine now the other side of the equivalence, this does not prevent a pre-circumscription f defined as in (Def-f) to falsify (PC) and (DC0). Indeed, it may well happen that $\mathcal{T} \sqcup \beta$ is inconsistent even if $\mathcal{T}$ is consistent, thus (PC) is not necessarily true for f. Also, it may very easily happen that $(CIRCF(\Gamma)(\mathcal{T}_1 \sqcup \beta) \sqcup CIRCF(\Gamma)(\mathcal{T}_2 \sqcup \beta)) \cap \boldsymbol{\mathcal{L}}$ is not included in $(CIRCF(\Gamma)(\mathcal{T}_1 \sqcup \beta) \cap \boldsymbol{\mathcal{L}}) \sqcup (CIRCF(\Gamma)(\mathcal{T}_2 \sqcup \beta) \cap \boldsymbol{\mathcal{L}})$, which explains why f does not necessarily satisfy (DC0). For the failure of this inclusion, let us suppose $CIRCF(\Gamma)(\mathcal{T}_1 \sqcup \beta) = Th(Q \Rightarrow P)$ and $CIRCF(\Gamma)(\mathcal{T}_2 \sqcup \beta) = Th(Q)$ with $\mathcal{V}(\boldsymbol{\mathcal{L}}) = \{P\}$. Then $(Th(Q \Rightarrow P) \sqcup Th(Q)) \cap \boldsymbol{\mathcal{L}} = Th(P \wedge Q) \cap \boldsymbol{\mathcal{L}} = Th(P)$ while $Th(P) \not\subseteq ((Th(Q \Rightarrow P) \cap \boldsymbol{\mathcal{L}}) \sqcup (Th(Q) \cap \boldsymbol{\mathcal{L}}) = Th(\top) \cap \boldsymbol{\mathcal{L}}$. □

Proposition 5.24 does not extend to the infinite case, a fact also noted in [Fre98] for the first point of this proposition (see example 6.1 below).

Clearly, various sequences of formulas may be used to define a given formula circumscription (we will examine precisely this point in subsections 5.5 and 6.2).

5.5 Two kinds of equivalence between sets of circumscribed formulas

As promised above, and before examining the characterization result in infinite case, let us examine the situation about variations of the set Φ. Precisely, we will give conditions under which two sets of formulas Φ and Φ' give rise to the same formula circumscription. From a knowledge representation perspective, and also from a formal perspective, two kinds of such "equivalences" are to be considered.

Definition 5.26 Φ and Φ' are two sets of formulas. Φ and Φ' are *c-equivalent*, noted $\Phi \equiv_c \Phi'$, iff $CIRCF(\Phi) = CIRCF(\Phi')$.

Φ and Φ' are *strongly c-equivalent*, noted $\Phi \equiv_{sc} \Phi'$, iff, for any set Φ'' of formulas, we have $CIRCF(\Phi \cup \Phi'') = CIRCF(\Phi' \cup \Phi'')$.

Clearly, if $\Phi \equiv_{sc} \Phi'$, then $\Phi \equiv_c \Phi'$. □

The interest of the strong version, from a knowledge representation perspective, comes from the fact that often (see various examples in section 7), when another informal rule, or even only another individual, is added to our knowledge, this corresponds to the addition of some formula(s) to the circumscribed formulas. If we have only the standard equivalence, we may loose this equivalence in this operation, while with the strong equivalence, we know that this equivalence is respected when we add new formula(s).

Notice that for the sake of simplicity, we consider $\mathbf{Q} = \emptyset$ in these definitions (remind that $CIRCF(\Phi) = CIRCF(\Phi; \emptyset, \mathbf{Z})$), if this is not the case, lemma 5.10 shows that we may without problem use the trick given by proposition 5.9 in order to be in this case.

We need a few definitions now.

Definitions 5.27 Let $\Phi = \{\varphi_i\}_{i \in I}$ be a set of formulas:

The $\wedge$-*closure of* Φ is the set $\Phi^\wedge = \{\bigwedge_{j \in J} \varphi_j /$ for any finite $J \subseteq I\}$.

The $\vee$-*closure* $\Phi^\vee$ is defined similarly. $\top \in \Phi^\wedge, \bot \in \Phi^\vee$ $(J = \emptyset)$.

The $\wedge\vee$-*closure of* Φ is the set $\Phi^{\wedge\vee} = (\Phi^{\wedge})^{\vee} = (\Phi^{\vee})^{\wedge}$ (the last equality comes from distributivity).

$\Phi^{\wedge}$ (respectively $\Phi^{\vee}$, or $\Phi^{\wedge\vee}$) is called a set *closed for* $\wedge$ (respectively *for* $\vee$, or *for* $\wedge$ *and* $\vee$). $\Box$

Definitions 5.28 Let f be a an application from $\mathcal{J}$ to $\boldsymbol{\mathcal{L}}$ (which may be a pre-circumscription, but not necessarily, see definition 6.34), $\prec$ be a preference relation and Φ be a set of formulas.

1. A formula φ is *accessible for* f iff there exists a theory $\mathcal{T}$ such that $\varphi \notin \mathcal{T}$ and $\varphi \in f(\mathcal{T})$.

   The set of the formulas *inaccessible for* f is

   $$I_f = \boldsymbol{\mathcal{L}} - \bigcup_{\mathcal{T}\in\mathcal{J}} (f(\mathcal{T}) - \mathcal{T}) = \bigcap_{\mathcal{T}\in\mathcal{J}} (\boldsymbol{\mathcal{L}} - (f(\mathcal{T}) - \mathcal{T})).$$

   If f is the preferential entailment $f_{\prec}$, we note $I_{\prec}$ for I_f.

2. The set of the formulas *positive for* $\prec$ is the set $Pos(\prec)$ of the formulas φ such that, if $\mu \models \varphi$ and $\mu \prec \nu$, then $\nu \models \varphi$.

   If $\prec = \prec_{\Phi}$ of definition 5.3, we note $Pos_e(\Phi)$ for the set $Pos(\prec_{\Phi})$, and we call this the set of all the formulas *positive*, in the *extended* acception, with respect to Φ.

   If $\prec = \preceq_{\Phi}$ of definition 5.3, we note $Pos_m(\Phi)$ for the set $Pos(\preceq_{\Phi})$, and we call this the set of all the formulas *positive*, in the *minimal* acception, with respect to Φ. We will see that this "minimal" acception corresponds to the most classical acception. $\Box$

A formula is inaccessible for f iff it cannot be obtained as the result of applying f to some theory not containing this formula already. One role of the inaccessible formulas for circumscriptions, evoked after example 4.20 above, is developed in [MR98] and reminded in subsection 6.3 below.

We develop in this text another important role of these inaccessible formulas with respect to formula circumscription: we will prove that in the finite case, I_f is the greatest (for set inclusion) set Ψ such that $f = CIRCF(\Phi) = CIRCF(\Psi)$ (proposition 5.33-1b below), and we will also examine the situation in the infinite case (theorem 6.15 and related results below).

Notice that, as we expect for a set of "positive formulas", $Pos(\prec)$, is always closed for $\wedge$ and $\vee$ (see proposition 5.29 below).

Notice also that we will see in section 6 that we need sometimes a third notion of "formulas positive with respect to Φ", however this third notion concerns formula circumscriptions in the infinite case, but not the propositional circumscriptions of definition 4.1.

Proposition 5.29 1. For any pre-circumscription, I_f is closed for $\wedge$.
2. For any $\prec$ satisfying (sf), we have $Pos(\prec) = I_{\prec}$.
3. For any $\prec$, the set $Pos(\prec)$ is closed for $\wedge$ and $\vee$.
4. $\Phi \subseteq Pos_m(\Phi) \subseteq Pos_e(\Phi)$.
5. $\Phi^{\wedge\vee} = Pos_m(\Phi)$ $\Box$

Proof 1. Obvious.

Notice that I_f is not necessarily stable for $\vee$ (see points 2 and 3 just below, and also remark 6.35-2).

2. $\varphi \in Pos(\prec)$, $\mathcal{T} \in \mathcal{J}$. We suppose that there exists $\mu \in \mathcal{M}(\mathcal{T}) - \mathcal{M}(\varphi)$. From (sf), there exists $\nu \in \mathcal{M}_\prec(\mathcal{T})$ such that $\nu = \mu$ or $\nu \prec \mu$. As φ is positive, we get $\nu \notin \mathcal{M}(\varphi)$. We have established: if $\mathcal{M}(\mathcal{T}) \not\subseteq \mathcal{M}(\varphi)$, then $\mathcal{M}_\prec(\mathcal{T}) \not\subseteq \mathcal{M}(\varphi)$, thus $\mathcal{M}(f_\prec(\mathcal{T})) \not\subseteq \mathcal{M}(\varphi)$. This means: if $\varphi \notin \mathcal{T}$, then $\varphi \notin f_\prec(\mathcal{T})$. This establishes $\varphi \in I_{f_\prec} = I_\prec$.

If now $\varphi \notin Pos(\prec)$, there exist $\mu \in \mathcal{M} - \mathcal{M}(\varphi) = \mathcal{M}(\neg\varphi)$ and $\nu \in \mathcal{M}(\varphi)$ such that $\nu \prec \mu$. We define $\mathcal{T} = Th(\mu) \cap Th(\varphi)$, ie $\mathcal{M}(\mathcal{T}) = \mathcal{M}(\varphi) \cup \{\mu\}$. As $\mu \notin \mathcal{M}(\varphi)$, we get $\mathcal{T} \neq Th(\varphi)$: $\mathcal{T} \subset Th(\varphi)$. As $\nu \prec \mu$, we get: $\mathcal{M}_\prec(\mathcal{T}) = \mathcal{M}_\prec(\varphi) \subseteq \mathcal{M}(\varphi)$. Thus, $\mathcal{M}(f_\prec(\mathcal{T})) \subseteq \mathcal{M}(\varphi)$, i.e. $\varphi \in f_\prec(\mathcal{T})$, and also $\varphi \notin \mathcal{T}$: We have established: φ is accessible for $f_\prec$, $\varphi \notin I_{f_\prec}$.

3. Remind that $\mathcal{M}(\varphi_1 \vee \varphi_2) = \mathcal{M}(\varphi_1) \cup \mathcal{M}(\varphi_2)$ and $\mathcal{M}(\varphi_1 \wedge \varphi_2) = \mathcal{M}(\varphi_1) \cap \mathcal{M}(\varphi_2)$. The result is then immediate.

4. Obvious.

These two inclusions may well be strict, even in the finite cases. Let us take a simple example: $\mathcal{V}(\boldsymbol{\mathcal{L}})$ is any non empty set and $\Phi = \emptyset$. Then $\preceq_\Phi$ is always satisfied and $\prec_\Phi$ has an empty graph, thus we have $Pos_m(\Phi) = Pos(\preceq_\Phi) = \{\top, \bot\}$ and $Pos_e(\Phi) = Pos(\prec_\Phi) = \boldsymbol{\mathcal{L}}$. For a more significative example, see subsection 6.2.

5. We have $\Phi \subseteq Pos_m(\Phi)$ from 4 and from 3 we get that $Pos_m(\Phi)$ is closed for $\wedge\vee$, thus $\Phi^{\wedge\vee} \subseteq Pos_m(\Phi)$.

The difficult point is then $Pos_m(\Phi) \subseteq \Phi^{\wedge\vee}$, that we prove now:

Let us suppose $\varphi \in Pos_m(\Phi) = Pos(\preceq_\Phi)$. If $\varphi = \top$ or $\varphi = \bot$, we know that φ is in $\Phi^{\wedge\vee}$. Thus, we may suppose that there exist μ and ν such that $\mu \models \varphi$ and $\nu \models \neg\varphi$. Then, $\Phi_\mu \not\subseteq \Phi_\nu$ from the definitions of $\preceq_\Phi$ and of $Pos(\preceq_\Phi)$ (definitions 5.3 and 5.28-2 respectively). This means that to any such couple (μ, ν) we may associate one formula $\varphi_{(\mu,\nu)} \in \Phi_\mu - \Phi_\nu$. Then, for any ν such that $\nu \models \neg\varphi$, $\{\mathcal{M}(\varphi_{(\mu,\nu)})\}_{\mu\models\varphi}$ is an open cover of $\mathcal{M}(\varphi)$, which is closed thus compact. Thus, there exists a finite subcover. Thus, to any such ν, we may associate a formula φ_ν, corresponding to one of these subcovers: we take for φ_ν the disjunction of the all formulas $\varphi_{(\mu,\nu)}$ involved in this finite cover. As each $\varphi_{(\mu,\nu)}$ is in Φ, we get $\varphi_\nu \in \Phi^\vee$. Also we have $\mathcal{M}(\varphi) \subseteq \mathcal{M}(\varphi_\nu)$ and $\nu \notin \mathcal{M}(\varphi_\nu)$, i.e. $\varphi \models \varphi_\nu$ and $\nu \models \neg\varphi_\nu$.

Now, $\{\mathcal{M}(\neg\varphi_\nu)\}_{\nu\models\neg\varphi}$ is an open cover of $\mathcal{M}(\neg\varphi)$ from which we may extract a finite subcover to which corresponds a formula ψ', a disjunction of formulas $\neg\varphi_\nu$. As each φ_ν is in $\Phi^\vee$, we have $\psi' \in (\neg(\Phi^\vee))^\vee = \neg((\Phi^\vee)^\wedge) = \neg(\Phi^{\wedge\vee})$. We have also $\mathcal{M}(\neg\varphi) \subseteq \mathcal{M}(\psi')$, i.e. $\neg\varphi \models \psi'$. But ψ is a disjunction of formulas $\neg\varphi_\nu$ which all satisfy $\varphi \models \varphi_\nu$, i.e. $\neg\varphi_\nu \models \neg\varphi$, thus we get $\psi' \models \neg\varphi$. Thus we get $\varphi = \neg\psi'$. This means $\varphi \in \Phi^{\wedge\vee}$.

We have established: $Pos_m(\Phi) \subseteq \Phi^{\wedge\vee}$, thus $Pos_m(\Phi) = \Phi^{\wedge\vee}$.

Clearly, we could rewrite this proof in the finite case ($\mathcal{V}(\boldsymbol{\mathcal{L}})$ finite): we could get some simplifications and some more precisions. We leave this to the interested reader. Anyway, our proof works a fortiori in the finite case. □

Let us give already a few significative examples of these kinds of "positive formulas" in the case of propositional and formula circumscriptions. This will explain the name "positive formulas" that we have chosen: these sets generalize the traditional notion of formula "positive in $\mathbf{P}$". Notice that some results are well known or immediate, but that some other results are more tricky, and the proof is then postponed later in the text.

Remarks 5.30 1. Let Φ be an arbitrary set of formulas. It seems rather natural to call all the formulas in the set $\Phi^{\wedge\vee}$ *formulas positive in* Φ, which justifies our notation $Pos_m(\Phi)$. And we think that there are also good reasons to call the formulas in the (generally) greater set $Pos_e(\Phi)$ *positive in* Φ, in an *extended acception*, thus our name for this set.

In the case of propositional circumscription, we may be more precise.

2. Indeed, let $\mathbf{P}, \mathbf{Z}$ be some partition of $\mathcal{V}(\boldsymbol{\mathcal{L}})$.

The set $(\mathbf{P} \cup \mathbf{Z} \cup \neg\mathbf{Z})^{\wedge\vee}$ is the set of the formulas *positive in* $\mathbf{P}$, in the *traditional meaning*.

Thus, let $(\mathbf{P}, \mathbf{Q}, \mathbf{Z})$ be as in definition 4.1. Remind that $f_{\prec} = CIRC(\mathbf{P}, \mathbf{Q}, \mathbf{Z}) = CIRCF(\Phi)$ where Φ is the set of *formulas* $\Phi = \mathbf{P} \cup \mathbf{Q} \cup \neg\mathbf{Q}$, i.e. $\prec = \prec_{(\mathbf{P},\ \mathbf{Q},\ \mathbf{Z})} = \prec_{\Phi}$. We consider also the relation $\preceq = \preceq_{\Phi} = \preceq_{(\mathbf{P},\ \mathbf{Q},\ \mathbf{Z})}$ (cf definition 5.3 and notations 5.6),

2a. The set $(\mathbf{P} \cup \mathbf{Q} \cup \neg\mathbf{Q} \cup \mathbf{Z} \cup \neg\mathbf{Z})^{\wedge\vee}$ is the set of the formulas positive in $\mathbf{P}$, in the traditional meaning.

The set $\Phi^{\wedge\vee} = Pos(\preceq) = Pos_m(\Phi)$ is is the set of the formulas positive in $\mathbf{P}$, in the traditional meaning, and made in the vocabulary of $\mathbf{P} \cup \mathbf{Q}$, i.e. without element of $\mathbf{Z}$.

For the (sometimes) greater set $Pos(\prec) = Pos_e(\Phi)$, see below points 2b and 2c of these remarks. Again, we hope to have given enough arguments to convince the reader that there are good reasons to call this set the set of all the *formulas positive in* $(\mathbf{P}, \mathbf{Q}, \mathbf{Z})$, in an extended acception.

Let us give already the results concerning the most significative particular cases now:

2b. If $\mathbf{Z} = \emptyset$ or if $\mathbf{P}$ is infinite, then $Pos(\prec) = Pos_m(\mathbf{P} \cup \mathbf{Q} \cup \neg\mathbf{Q}) = Pos_e(\mathbf{P} \cup \mathbf{Q} \cup \neg\mathbf{Q})$. This is the set of all the formulas *positive in* $\mathbf{P}$ in the *traditional meaning* (the formulas equivalent to a formula without any negative occurrence of an element of $\mathbf{P}$), and which do not contain any element of $\mathbf{Z}$. The proof is given below in proposition 6.32-1c.

2c. If $\mathbf{P}$ is finite and $\mathbf{Z} \neq \emptyset$, the set $Pos(\prec)$ is more complicated. In order to deal with this case, [MR98] provides a syntactical definition of the notion called there "formula positive in $\mathbf{P}$ with $\mathbf{Z}$ varying", corresponding to the "extended" acception called here $Pos(\prec)$ or $Pos_e(\mathbf{P} \cup \mathbf{Q} \cup \neg\mathbf{Q})$. See below proposition 6.32-2 where these results are reminded.

The set which could be called the set of the "formulas positive in $\mathbf{P}$ with $\mathbf{Z}$ varying", in the "classical" acception is the set noted here $Pos(\preceq) = Pos_m(\mathbf{P} \cup \mathbf{Q} \cup \neg\mathbf{Q})$. It is much simpler, being the set of all the formulas positive in $\mathbf{P}$ in the traditional meaning and without any occurrence of an element in $\mathbf{Z}$ (formulas in $\mathcal{V}(\mathbf{P} \cup \mathbf{Q})$). $\Box$

For the following results, it is convenient to establish two easy lemmas

Lemma 5.31 For any sets of formulas Φ and Ψ, if $\Phi \subseteq \Psi \subseteq \Phi^{\wedge\vee}$, we have $\preceq_\Phi = \preceq_\Psi$, thus a fortiori $\prec_\Phi = \prec_\Psi$, i.e. $CIRCF(\Phi) = CIRCF(\Psi)$. $\Box$

<u>Proof:</u> We prove $\preceq_\Phi = \preceq_{\Phi^{\wedge\vee}}$:

From lemma 5.4-1a, we get that if $\mu \preceq_\Phi \nu$ then $\mu \preceq_{\Phi^{\wedge\vee}} \nu$ (use $\mu \models \varphi_1 \vee \varphi_2$ iff $\mu \models \varphi_1$ or $\mu \models \varphi_2$ and $\mu \models \varphi_1 \wedge \varphi_2$ iff $\mu \models \varphi_1$ and $\mu \models \varphi_2$).

Now, from lemma 5.4-4a, we get that if $\Phi \subseteq \Psi$, then if $\mu \preceq_\Psi \nu$ then $\mu \preceq_\Phi \nu$, thus in particular if $\mu \preceq_{\Phi^{\wedge\vee}} \nu$ then $\mu \preceq_\Phi \nu$.

This gives $\preceq_\Phi = \preceq_{\Phi^{\wedge\vee}}$, as announced.

Thus $\preceq_\Phi = \preceq_\Psi$ iff $\preceq_{\Phi^{\wedge\vee}} = \preceq_{\Psi^{\wedge\vee}}$. Now, if $\Phi \subseteq \Psi \subseteq \Phi^{\wedge\vee}$ we get $\Phi^{\wedge\vee} = \Psi^{\wedge\vee}$. $\Box$

Lemma 5.32 (Independently also in [Fre98] for the first equality.) If $\mathcal{V}(\boldsymbol{\mathcal{L}})$ is finite, then we have $CIRCF(\Phi) = CIRCF(Pos_e(\Phi)) = CIRCF(I_{\prec_\Phi})$. $\Box$

This result is an immediate consequence of Property 5.6 in [MR98]. Also, we have discovered at the time of printing that the first equality had appeared as Observation 15 in [Fre98], with exactly the same (natural) proof that the one we had found, and that we give below.

<u>Proof:</u> Let us use the notations of the proof of proposition 5.24-1: $\mathcal{M}(\psi_\mu) = \{\nu \in \mathcal{M} \ / \ \nu = \mu \text{ or } \mu \prec \nu\}$, and $\Psi = \{\psi_\mu\}_{\mu \in \mathcal{M}}$. We have shown in this proof that if $f = f_\prec$ is a formula circumscription, i.e. if there exists some set of formulas Φ such that $f = CIRCF(\Phi)$, i.e. if $\prec = \prec_\Phi$, then $f = CIRCF(\Psi)$, i.e. $\prec_\Phi = \prec_\Psi$.

From their definition, it is clear that any ψ_μ is in $Pos(\prec) = Pos_e(\Phi)$. Thus $\Psi \subseteq Pos_e(\Phi)$. Also, $Pos_e(\Phi)$ is stable for $\vee$, thus $\Psi^\vee \subseteq Pos_e(\Phi)$. We prove the converse now. Let us suppose $\varphi \in Pos_e(\Phi)$. If $\mu \models \varphi$ and $\mu \prec \nu$ then $\nu \models \varphi$, thus $\mathcal{M}(\psi_\mu) \subseteq \mathcal{M}(\varphi)$. Thus $\bigcup_{\mu \models \varphi} \mathcal{M}(\psi_\mu) \subseteq \mathcal{M}(\varphi)$. Now, for any μ we have $\mu \in \mathcal{M}(\psi_\mu)$, thus $\mathcal{M}(\varphi) = \bigcup_{\mu \models \varphi} \{\mu\} \subseteq \bigcup_{\mu \models \varphi} \mathcal{M}(\psi_\mu)$. Thus we have $\mathcal{M}(\varphi) = \bigcup_{\mu \models \varphi} \mathcal{M}(\psi_\mu)$, i.e. $\varphi = \bigvee_{\mu \models \varphi} \psi_\mu$. Thus $\varphi \in \Psi^\vee$. This establishes $Pos_e(\Phi) \subseteq \Psi^\vee$, thus $Pos_e(\Phi) = \Psi^\vee$.

Thus we have $\Psi \subseteq Pos_e(\Phi) \subseteq \Psi^{\wedge\vee}$. From lemma 5.31 we get $CIRCF(\Psi) = CIRCF(Pos_e(\Phi))$, i.e. $CIRCF(\Phi) = CIRCF(Pos_e(\Phi))$, i.e. $\prec_\Phi = \prec_{Pos_e(\Phi)}$.

From corollary 5.7-2 and proposition 5.29-2 we know that $Pos_e(\Phi) = Pos(\prec_\Phi) = I_{\prec_\Phi}$. $\Box$

This result does not extend to the infinite case (example 6.18 below), however it extends to any propositional circumscription $CIRC(\mathbf{P}, \mathbf{Q}, \mathbf{Z})$ of definition 4.1 (propositions 6.16 and 6.32-4a).

Proposition 5.33 Φ and Ψ are sets of formulas.

1a. $\Phi \equiv_c \Psi$ iff $\prec_\Phi = \prec_\Psi$ and, if $\Phi \equiv_c \Psi$, then $Pos_e(\Phi) = Pos_e(\Psi)$.

1b. If $\mathcal{V}(\boldsymbol{\mathcal{L}})$ is finite we have the two equivalences: $\Phi \equiv_c \Psi$ iff $\prec_\Phi = \prec_\Psi$ iff $Pos_e(\Phi) = Pos_e(\Psi)$.

Moreover we have $\prec_\Phi = \prec_{Pos_e(\Phi)} = \prec_{Pos_m(\Phi)}$ and $Pos_e(\Phi)$ is the greatest (for $\subseteq$) set Ψ satisfying $\Psi \equiv_c \Phi$.

2a. $\Phi \equiv_{sc} \Psi$ iff $\preceq_\Phi = \preceq_\Psi$ iff $Pos_m(\Phi) = Pos_m(\Psi)$. Also $\preceq_\Phi = \preceq_{\Phi^{\wedge\vee}}$, thus $\prec_\Phi = \prec_{\Phi^{\wedge\vee}}$.

2b. $Pos_m(\Phi) = \Phi^{\wedge\vee}$ is the greatest (for $\subseteq$) set Ψ satisfying $\Psi \equiv_{sc} \Phi$ (cf lemma 5.31).

3. $\Phi \cup \{\varphi\} \equiv_c \Phi$ iff $\Phi \cup \{\varphi\} \equiv_{sc} \Phi$ iff $\varphi \in \Phi^{\wedge\vee}$. $\square$

Proof:

1a. The first "iff" is an immediate consequence of proposition 3.12 (points 1 and 2) and corollary 5.7-2.

If $\prec_\Phi = \prec_\Psi$, we get obviously $Pos(\prec_\Phi) = Pos(\prec_\Psi)$, i.e. $Pos_e(\Phi) = Pos_e(\Psi)$.

Notice that example 6.18 below shows that the converse is false: in the infinite case, we may have $Pos_e(\Phi) = Pos_e(\Psi)$ and $CIRCF(\Phi) \neq CIRCF(\Psi)$, i.e $\prec_\Phi \neq \prec_\Psi$.

1b. The first "iff" comes from 1a. For the second "iff", clearly if $\prec_\Phi = \prec_\Psi$, then $Pos(\prec_\Phi) = Pos(\prec_\Psi)$, i.e. $Pos_e(\Phi) = Pos_e(\Psi)$.

We suppose now $Pos_e(\Phi) = Pos_e(\Psi)$, then from lemma 5.32 we know that $\prec_\Phi = \prec_\Psi$.

The maximality of $Pos_e(\Phi)$ comes from the fact that we have anyway $\Phi \subseteq Pos_e(\Phi)$, thus, if $\Psi \equiv_c \Phi$, as we have $Pos_e(\Psi) = Pos_e(\Phi)$ from 1a, we get $\Psi \subseteq Pos_e(\Psi) = Pos_e(\Phi)$.

Notice that the result $\prec_\Phi = \prec_{Pos_e(\Phi)}$, and the maximality of $Pos_e(\Phi)$, is also an immediate consequence of property 5.6 in [MR98].

We will see below that most of the results of this point 1b do not extend to the infinite case. However, some of these results remain true in some important cases. Let us just say here that we still get $\prec_\Phi = \prec_{Pos_m(\Phi)} = \prec_{Pos_e(\Phi)}$ for propositional circumscriptions of definition 4.1 with $\Phi = \mathbf{P} \cup \mathbf{Q} \cup \neg\mathbf{Q}$ (see proposition 6.32-4a below), but not necessarily for infinite formula circumscriptions (see example 6.5 below).

2a.

First "iff", "if" part: From lemma 5.4-4a), if $\preceq_\Phi = \preceq_\Psi$, then for any set Ψ', we have $\preceq_{\Phi\cup\Psi'} = \preceq_{\Psi\cup\Psi'}$ (thus $\prec_{\Phi\cup\Psi'} = \prec_{\Psi\cup\Psi'}$, i.e. $f_{\prec_{\Phi\cup\Psi'}} = f_{\prec_{\Psi\cup\Psi'}}$).

First "iff", "only if" part: We suppose $\preceq_\Phi \neq \preceq_\Psi$. Clearly, if $\prec_\Phi \neq \prec_\Psi$, then $\Phi \not\equiv_c \Psi$ thus $\Phi \not\equiv_{sc} \Psi$. Let us suppose then that we have $\prec_\Phi = \prec_\Psi$ and $\preceq_\Phi \neq \preceq_\Psi$. Thus, there exists μ and

ν such that we have e.g. $\mu \preceq_\Phi \nu$, $\nu \preceq_\Phi \mu$, and $\mu \not\preceq_\Psi \nu$, $\nu \not\preceq_\Psi \mu$. Now, as $\mu \neq \nu$, there exists a formula φ such that $\mu \models \varphi$ and $\nu \not\models \varphi$. We consider the sets $\Phi \cup \{\varphi\}$ and $\Psi \cup \{\varphi\}$. We get, from lemma 5.4-4b, $\nu \prec_{\Phi\cup\{\varphi\}} \mu$. Also, $\nu \not\prec_{\Psi\cup\{\varphi\}} \mu$ (indeed $\nu \not\preceq_\Psi \mu$ thus $\nu \not\preceq_{\Psi\cup\{\varphi\}} \mu$). This establishes $\Psi \cup \{\varphi\} \not\equiv_c \Phi \cup \{\varphi\}$, thus $\Psi \not\equiv_{sc} \Phi$.

Second "iff", "only if" part: Obviously, if $\preceq_\Phi = \preceq_\Psi$, then $Pos(\preceq_\Phi) = Pos(\preceq_\Psi)$, i.e. $Pos_m(\Phi) = Pos_m(\Psi)$.

Second "iff", "if" part: it is a consequence of the second sentence of this point 2a, examined just below. Indeed, let us suppose that we know that $\preceq_\Phi = \preceq_{\Phi^{\wedge\vee}}$. Thus if $Pos(\preceq_\Phi) = Pos(\preceq_\Psi)$, i.e. if $\Phi^{\wedge\vee} = \Psi^{\wedge\vee}$, we get $\preceq_\Phi = \preceq_{\Phi^{\wedge\vee}} = \preceq_{\Psi^{\wedge\vee}} = \preceq_\Psi$.

Second sentence: From lemma 5.31 and its proof, we already know that $\preceq_\Phi = \preceq_{\Phi^{\wedge\vee}}$, thus $\prec_\Phi = \prec_{\Phi^{\wedge\vee}}$.

2b. From proposition 5.29-5 we know that $Pos_m(\Phi) = \Phi^{\wedge\vee}$.

Now, if $\Psi \equiv_{sc} \Phi$ we know from 2a above that we have $\Psi^{\wedge\vee} = \Phi^{\wedge\vee}$. As $\Psi \subseteq \Psi^{\wedge\vee}$, we get $\Psi \subseteq \Phi^{\wedge\vee}$, which establishes the maximality (for $\subseteq$) of the set $\Phi^{\wedge\vee}$.

3. $\Phi \cup \{\varphi\} \equiv_c \Phi$ iff $\Phi \cup \{\varphi\} \equiv_{sc} \Phi$ iff $\varphi \in \Phi^{\wedge\vee}$.

First "iff": Let us suppose now that we have $\Phi \cup \{\varphi\} \equiv_c \Phi$, i.e. $\prec_{\Phi\cup\{\varphi\}} = \prec_\Phi$, i.e., from lemma 5.4-4a, that $\mu \preceq_{\{\varphi\}} \nu$ whenever $\mu \prec_\Phi \nu$, and also that we have $\preceq_{\Phi\cup\{\varphi\}} \neq \preceq_\Phi$. Then, from lemma 5.4-4b, there must exist some μ, ν such that $\mu \preceq_\Phi \nu$, $\nu \preceq_\Phi \mu$, $\mu \not\preceq_{\{\varphi\}} \nu$, $\nu \not\preceq_{\{\varphi\}} \mu$, which is impossible from lemma 5.4-2. This means that if $\Phi \cup \{\varphi\} \equiv_c \Phi$, then we must have also $\preceq_{\Phi\cup\{\varphi\}} = \preceq_\Phi$, thus $\Phi \cup \{\varphi\} \equiv_{sc} \Phi$.

Second "iff": As we have $\varphi \in \Phi^{\wedge\vee}$ iff $(\Phi \cup \{\varphi\})^{\wedge\vee} = \Phi^{\wedge\vee}$, point 2a above gives the result. $\square$

We address now various problems which are technically more complicated than the preceding ones.

6 More technical results about circumscriptions

6.1 Characterization of formula circumscription (infinite case)

Remind that the finite case has already been solved above (see proposition 5.24). The infinite case is much more complicated and needs yet a few more technical definitions and results, which may be skipped by a reader not too mathematically oriented...

Firstly, here is an example showing that a naive extension to the infinite case of the first point of proposition 5.24 does not hold. Notice that the second point does not extend so easily either: we need more that full (CR), even together with full (RM), (DC), (CUMU) and (PC), to be sure that we have a formula circumscription (see [MR99] for a counter-example).

Example 6.1 $\mathcal{V}(\boldsymbol{\mathcal{L}}) = \{P_i\}_{i\in \mathbb{N}}$. $\nu_i = \{P_0, P_1, \cdots, P_i\}$ $(i \in \mathbb{N})$, $\nu = \mathcal{V}(\boldsymbol{\mathcal{L}})$, $\mu = \{P_1\}$. We define the preference relation $\prec$ by $\mu \prec \nu_n$ for any $n \in \mathbb{N}$, and otherwise $\mu' \not\prec \nu'$.

Clearly, $\prec$ satisfies (sf) and its opposite $\succ$ also. However, it cannot exist any set Φ of formulas such that $CIRCF(\Phi) = f_\prec$. We will prove rigorously this point below, which comes from the fact that we have $\lim_{i\to\infty} \nu_i = \nu$, but that we require $\mu \not\prec \nu$ and $\nu \not\prec \nu_i$ (see example 6.22). □

At the time of printing, we have discovered example 8 in [Fre98] and example 1 in [Cos98], which are significantly more intricate than example 6.1, and that we will examine below (see example 6.21). These two examples are in fact the same example. [Fre98] does not evoke circumscription, but it proves in an ad-hoc way that it is not a f_P (which is equivalent to prove that it is not a formula circumscription from proposition 5.14). [Cos98] deals with formula circumscription, but it is not concerned in proving whether it is a formula circumscription or not: all it does is proving that some (arbitrarily chosen?) reflexive relation associated to this preferential entailment cannot be a $\preceq_\Phi$. Thus, in some way, our example still remains the first one about formula circumscription...

We need a few more technical definitions now.

Definitions 6.2 $\prec$ is some preference relation and Φ is some set of formulas.

1. For any μ, we define the sets of interpretations $M_\prec(\mu) = \{\nu \ / \ \mu \prec \nu\}$ (set of *successors* of μ), $m_\prec(\mu) = \{\nu \ / \ \nu \prec \mu\}$ (set of *predecessors* of μ), $C_\prec(\mu) = M_\mu \cup m_\mu$ (set of elements *comparable* with μ).
2. We note M_Φ, m_Φ and C_Φ respectively for $M_{\prec_\Phi}$, $m_{\prec_\Phi}$ and $C_{\prec_\Phi}$ ($\prec_\Phi$ as in definition 5.3).
3. We define the equivalence relation $\mu \equiv_\prec \nu$ iff $M_\prec(\mu) = M_\prec(\nu)$ and $m_\prec(\mu) = m_\prec(\nu)$. Again, we note $\equiv_\Phi$ for $\equiv_{\prec_\Phi}$.

   Remind (definition 5.3) that we note $\mu \simeq_\Phi \nu$ iff $\Phi_\mu = \Phi_\nu$.
4. If $\equiv'$ is some equivalence relation among $\mathcal{M}$, we note $\equiv'[\mu]$ for the equivalence class of μ. □

Remarks 6.3 1. Obviously we have that $\mu \simeq_\Phi \nu$ implies $\mu \equiv_\Phi \nu$, while the converse is generally false.

2. $\simeq_\Phi[\mu]$ is closed in $\mathcal{M}$, and more precisely $\simeq_\Phi[\mu] = \mathcal{M}(\Phi_\mu \cup \{\neg\varphi \ / \ \varphi \in \Phi - Th(\mu)\})$.

   The proof is easy: remind from definition 5.3, that $\Phi_\mu = \Phi_\nu$ iff $\nu \models \Phi_\mu$ and $\nu \not\models \varphi$ for any $\varphi \in \Phi - \Phi_\mu = \Phi - Th(\mu)$.

   Notice that $\{\nu \ / \ \mu \preceq_\Phi \nu\}$ is closed also, being equal to $\mathcal{M}(\Phi_\mu)$.

   However, the set $\equiv_\Phi[\mu]$ is not necessarily closed, i.e. is not necessarily the set of all the models of some theory (see examples 6.5 and 6.24 below). □

Lemma 6.4 Φ is a set of formulas. If $\Phi_\mu \subseteq \Phi_\nu$, then $\mu \prec_\Phi \nu$ or $\mu \equiv_\Phi \nu$. □

Proof: By the definitions of $\prec_\Phi$, $\preceq_\Phi$ and $\simeq_\Phi$ (definition 5.3) we have $\Phi_\mu \subseteq \Phi_\nu$ iff $\mu \preceq_\Phi \nu$ iff ($\mu \prec_\Phi \nu$ or $\mu \simeq_\Phi \nu$). Then, use remark 6.3-1. □

Here is an example showing that, contrarily to the finite case, we cannot always take the set $Pos(\prec) = I_\prec$ as our set Φ, even if $f_\prec$ is a formula circumscription (cf points 1 of proposition 5.33). This shows that we must find another set in the infinite case.

Example 6.5 $\mathcal{V}(\boldsymbol{\mathcal{L}}) = \{P_i\}_{i\in \mathbb{N}}$. $\nu_i = \{P_0, P_1, \cdots, P_i\}$ $(i \in \mathbb{N})$, $\nu = \mathcal{V}(\boldsymbol{\mathcal{L}})$, $\mu = \{P_1\}$. We define the preference relation $\prec$ by $\nu \prec \nu_n$ and $\mu \prec \nu_n$ for any $n \in \mathbb{N}$, and otherwise $\mu' \not\prec \nu'$. Notice that, as in example 6.1, we have $\lim_{i\to\infty} \nu_i = \nu$, but we require $\mu \not\prec \nu$.

$\prec$ is obviously (sf) and its opposite $\succ$ also. $f_\prec$ is a formula circumscription, indeed $f_\prec = CIRCF(\Phi)$ with $\Phi = \{\varphi \in Pos(\prec) \;/\; \mu \models \varphi \text{ iff } \nu \models \varphi\}$.

We have here $M_\prec(\mu) = M_\prec(\nu) = \{\nu_i\}_{i\in \mathbb{N}}$ and $m_\prec(\mu) = m_\prec(\nu) = \emptyset$, thus $\mu \equiv_\prec \nu$.

If $\varphi \in (Pos(\prec))_\mu$, then $M_\prec(\mu) = M_\prec(\nu) \subseteq \mathcal{M}(\varphi)$ thus $\nu \in \mathcal{M}(\varphi)$. Thus $(Pos(\prec))_\mu \subseteq (Pos(\prec))_\nu$. As we have $P_0 \in Pos(\prec)$, $\nu \models P_0$ and $\mu \not\models P_0$, we get $(Pos(\prec))_\nu \subset (Pos(\prec))_\mu$ and $\mu \prec_{Pos(\prec)} \nu$. This shows $\prec \neq \prec_{Pos(\prec)}$, i.e. $CIRCF(Pos(\prec)) \neq f_\prec = CIRCF(\Phi)$.

Thus, in this example, $Pos_e(\Phi) = Pos(\prec)$ cannot be taken as set of formulas to be circumscribed: $CIRCF(\Phi) \neq CIRCF(Pos_e(\Phi))$, which shows that we need a more refined definition in the infinite case: see definition 6.10 below.

Notice also that the set $\equiv_\prec [\nu_1] = \{\nu_i\}_{i\in \mathbb{N}}$ does not contain ν, thus it is not a closed set (cf the last remark 6.3-2). □

Lemma 6.6 Φ and Ψ are sets of formulas. If $\Phi \subseteq \Psi \subseteq Pos_e(\Phi)$, and if $\Psi_\mu \subseteq \Psi_\nu$, then $\mu \prec_\Phi \nu$ or $\mu \equiv_\Phi \nu$. □

Proof: We suppose $\Phi \subseteq \Psi \subseteq Pos_e(\Phi)$. If $\Psi_\mu \subseteq \Psi_\nu$, then a fortiori $\Phi_\mu \subseteq \Phi_\nu$. If $\Phi_\mu \subset \Phi_\nu$, we get $\mu \prec_\Phi \nu$ and if $\Phi_\mu = \Phi_\nu$, we get $\mu \simeq_\Phi \nu$ thus $\mu \equiv_\Phi \nu$. □

We must introduce now two (rather technical but essential) definitions, $\overline{\prec}$ and $Pos_r(\prec)$:

Definition 6.7 $\prec$ being a preference relation, we define the preference relation $\overline{\prec}$ by:

$$\mu \,\overline{\prec}\, \nu \quad \text{iff} \quad \left\{ \begin{array}{l} \text{for any formulas } \varphi, \psi \text{ such that } \mu \models \varphi \text{ and } \nu \models \psi, \\ \text{there exist } \mu' \in \mathcal{M}(\varphi) \text{ and } \nu' \in \mathcal{M}(\psi) \text{ such that } \mu' \prec \nu'. \end{array} \right. \square$$

Remarks 6.8

1. If $\mu \prec \nu$, then $\mu \,\overline{\prec}\, \nu$ (choose $\mu' = \mu$ and $\nu' = \nu$ in the definition of $\overline{\prec}$).
2. If $\mathcal{V}(\boldsymbol{\mathcal{L}})$ is enumerable, we have (proof straightforward):

   $\mu \,\overline{\prec}\, \nu$ iff there exist two sequences $(\mu_i)_{i\in \mathbb{N}}$ and $(\nu_i)_{i\in \mathbb{N}}$ such that $\lim_{i\to\infty} \mu_i = \mu$, $\lim_{i\to\infty} \nu_i = \nu$ and, for any $i \in \mathbb{N}$, $\mu_i \prec \nu_i$.
3. If $\mathcal{V}(\boldsymbol{\mathcal{L}})$ is finite, we have $\overline{\prec} \;= \prec$ (immediate consequence of 2).

   See more about the finite case in the comments following theorem 6.15 below. □

Proposition 6.9 $\prec$ is a preference relation, Φ is a set of formulas.

1. If $\mu \overline{\prec} \nu$ then $(Pos(\prec))_\mu \subseteq (Pos(\prec))_\nu$.
2. If $\mu \overline{\prec_\Phi} \nu$ then $\mu \prec_\Phi \nu$ or $\mu \equiv_\Phi (\nu)$. $\square$

Proof: If $\mu \overline{\prec} \nu$, then, for any $\varphi \in (Pos(\prec))_\mu \subseteq Th(\mu)$, we know that for any $\psi \in Th(\nu)$ there exist $\mu' \in \mathcal{M}(\varphi)$ and $\nu' \in \mathcal{M}(\psi)$ such that $\mu' \prec \nu'$. As $\varphi \in Pos(\prec)$, we get $\nu' \in \mathcal{M}(\varphi)$, thus $\nu \in TC(\mathcal{M}(\varphi))$ – indeed, as the set of all the $\mathcal{M}(\psi')$ for all the $\psi' \in \boldsymbol{\mathcal{L}}$ is an open base of neighbourhood, we have shown that any open set containing ν has a non empty intersection with $\mathcal{M}(\varphi)$ – and we know that we have $TC(\mathcal{M}(\varphi)) = \mathcal{M}(\varphi)$. Thus $\varphi \in (Pos(\prec))_\nu$.

The result about the case of a formula circumscription, i.e. $\prec = \prec_\Phi$, thus $Pos(\prec) = Pos_e(\Phi)$, follows immediately, using lemma 6.6. $\square$

Definition 6.10 $\prec$ being a preference relation, we define the set of the formulas *positive for* $\prec$, in the *restricted* acception, by:

$Pos_r(\prec) = \{\varphi \in Pos(\prec) \ / \ \text{ for any } \mu, \nu, \text{ if } \mu \overline{\prec} \nu, \ \mu \not\prec \nu \text{ and } \nu \models \varphi, \text{ then } \mu \models \varphi\}$.

If Φ is a set of formulas, then we note $Pos_r(\Phi)$ for the set $Pos_r(\prec_\Phi)$. $\square$

Proposition 6.11

1. $Pos_r(\prec) \subseteq Pos(\prec)$. Moreover, if $\mu \overline{\prec} \nu$ then $(Pos_r(\prec))_\mu \subseteq (Pos_r(\prec))_\nu$.
2. If Φ and Ψ are sets of formulas such that $\Phi \subseteq \Psi \subseteq Pos_e(\Phi)$, then $\Phi \equiv_c \Psi$ iff, for any μ, ν, if $\mu \equiv_\Phi \nu$ then $\Psi_\mu \not\subset \Psi_\nu$ (strict inclusion, notice that by symmetry we could clearly add "and $\Psi_\nu \not\subset \Psi_\mu$"). $\square$

Proof: 1. $Pos_r(\prec) \subseteq Pos(\prec)$ is clear from definition 6.10. Now, from the definition of Φ_μ (definition 5.3) and from proposition 6.9-1, we get then $(Pos_r(\prec))_\mu \subseteq (Pos_r(\prec))_\nu$.

2. Let us suppose $\Phi \subseteq \Psi \subseteq Pos_e(\Phi)$ and also $\mu \prec_\Phi \nu$, then $\Phi_\mu \subset \Phi_\nu$ thus $\Psi_\nu - \Psi_\mu \neq \emptyset$. Moreover we have $\Psi \subseteq Pos_e(\Phi)$ thus $\Psi_\mu \subseteq \Psi_\nu$, which establishes $\Psi_\mu \subset \Psi_\nu$, i.e. $\mu \prec_\Psi \nu$.

If $\mu \prec_\Psi \nu$, i.e. $\Psi_\mu \subset \Psi_\nu$, then from lemma 6.6 $\mu \prec_\Phi \nu$ or $\mu \equiv_\Phi \nu$. Also, we have, from the preceding result, $\prec_\Psi = \prec_\Phi$ iff $\mu \prec_\Phi \nu$ whenever $\Psi_\mu \subset \Psi_\nu$, which establishes the equivalence. $\square$

Here is an interesting result, which was not completely obvious from the definitions.

Proposition 6.12

1. For any set of formulas Φ, we have $\Phi \subseteq Pos_r(\Phi)$.
   $Pos_r(\Phi)$ is stable for $\wedge$ and $\vee$. Thus: $Pos_m(\Phi) \subseteq Pos_r(\Phi) \subseteq Pos_e(\Phi)$.
2. If $\mathcal{V}(\boldsymbol{\mathcal{L}})$ is finite, then $Pos_r(\prec) = Pos(\prec)$, thus $Pos_r(\Phi) = Pos_e(\Phi)$. $\square$

Proof: 1. Let us suppose $\varphi \in \Phi, \mu \not\prec_\Phi \nu, \nu \in \mathcal{M}(\varphi)$ and $\mu \overline{\prec} \nu$. Then, $(Pos_e(\Phi))_\mu \subseteq (Pos_e(\Phi))_\mu$ thus, as $\Phi \subseteq Pos_e(\Phi)$, $\Phi_\mu \subseteq \Phi_\nu$. If $\mu \notin \mathcal{M}(\varphi)$ then $\Phi_\mu \subset \Phi_\nu$, i.e. $\mu \prec_\Phi \nu$, a contradiction. Thus, $\varphi \in Pos_r(\Phi)$.

The stability for $\wedge$ and $\vee$ of the set $Pos_r(\Phi)$, and more generally for any set $Pos_r(\prec)$, is an easy consequence of the definition of $Pos_r(\prec)$. The two inclusions follow: remind that $Pos_m(\Phi) = \Phi^{\wedge\vee}$ and $Pos_e(\Phi) = Pos(\prec_\Phi)$.

2. If $\mathcal{V}(\boldsymbol{\mathcal{L}})$ is finite, then the definitions 6.7 and 6.10 may be simplified: $\mu \mathrel{\overline{\prec}} \nu$ iff $\mu \prec \nu$, thus $Pos_r(\prec) = Pos(\prec)$. Taking $\prec = \prec_\Phi$ gives $Pos_r(\Phi) = Pos(\prec_\Phi) = Pos_e(\Phi)$. $\square$

We are close to our goal now, we need just to introduce one property of the preference relation.

Definition 6.13 Let $\prec$ be a preference relation. We say that $\prec$ satisfies the property of *formula circumscription* (**(fc)** for short), if the following condition holds:
(fc): If $(Pos_r(\prec))_\mu \subseteq (Pos_r(\prec))_\nu$, then $\mu \prec \nu$ or $\mu \equiv_\prec \nu$. $\square$

Lemma 6.14 If an irreflexive and antisymmetrical preference relation satisfies (fc), then it is transitive. $\square$

Proof: Let us suppose that $\prec$ is irreflexive, antisymmetrical, and satisfies (fc). Let us suppose also $\mu_1 \prec \mu_2$ and $\mu_2 \prec \mu_3$. Then $\mu_1 \mathrel{\overline{\prec}} \mu_2$ and $\mu_2 \mathrel{\overline{\prec}} \mu_3$ from remark 6.8-1, thus $(Pos_r(\prec))_{\mu_1} \subseteq (Pos_r(\prec))_{\mu_2}$ and $(Pos_r(\prec))_{\mu_2} \subseteq (Pos_r(\prec))_{\mu_3}$ from proposition 6.11-1. Thus $(Pos_r(\prec))_{\mu_1} \subseteq (Pos_r(\prec))_{\mu_3}$ and, from (fc), we get $\mu_1 \prec \mu_3$ or $\mu_1 \equiv_\prec \mu_3$. Now, if $\mu_1 \equiv_\prec \mu_3$, we get $\mu_2 \prec \mu_1$ (from $\mu_2 \prec \mu_3$), which contradicts either antisymmetry (if $\mu_1 \neq \mu_2$) or irreflexivity (if $\mu_1 = \mu_2$). Thus, we must have $\mu_1 \prec \mu_3$. $\square$

Theorem 6.15 A preferential entailment $f_\prec$ is a formula circumscription iff the preference relation $\prec$ is irreflexive, antisymmetrical and satisfies (fc).

A pre-circumscription f is a formula circumscription iff $f = f_{\prec_f}$ and $\prec_f$ is antisymmetrical and satisfies (fc). $\square$

Remind from lemma 6.14 that transitivity is a consequence of these conditions. Moreover, we get from this theorem, together with corollary 5.7 (2 and 3), that if $\prec$ is irreflexive, antisymmetrical and satisfies (fc), then $\prec$ and its opposite satisfy (sf), and also that the opposite of $\prec$ satisfies (fc)) (remind that from irreflexivity we are in a case of uniqueness of the preference relation $\prec$ associated to $f_\prec$).

Before giving the proof of this theorem, let us examine the (easy) finite case.

We suppose $\mathcal{V}(\boldsymbol{\mathcal{L}})$ finite here, thus $\overline{\prec} = \prec$ and $Pos_r(\prec) = Pos(\prec)$ (see remark 6.8-3 and proposition 6.12-2).

(fc) in this case is thus: if $(Pos(\prec))_\mu \subseteq (Pos(\prec))_\nu$, then $\mu \prec \nu$ or $\mu \equiv_\prec \nu$. From lemma 6.14, we know that $\prec$ is transitive. Thus $\prec$ satisfies (sf) and its opposite $\succ$ also from proposition 3.12-3.

Conversely, let us suppose that $\prec$ is (sf), i.e. that it is irreflexive and transitive. Then $\prec$ is clearly antisymmetrical. From lemma 5.32, we know that $\prec = \prec_{Pos(\prec)}$ thus from lemma

6.4 we get that if $(Pos(\prec))_\mu \subseteq (Pos(\prec))_\nu$, then $\mu \prec \nu$ or $\mu \equiv_\prec \nu$, which is (fc) in the finite case.

Thus, if $\mathcal{V}(\boldsymbol{\mathcal{L}})$ is finite, $\prec$ satisfies irreflexivity, antisymmetry and (fc) iff $\prec$ is transitive and irreflexive, i.e. iff $\prec$ satisfies (sf).

Proof of theorem 6.15:

First sentence:

If $f_\prec = CIRCF(\Phi)$, then $\prec = \prec_\Phi$ and $\prec$ is irreflexive and transitive, thus antisymmetrical. If we have $(Pos_r(\prec))_\mu \subseteq (Pos_r(\prec))_\nu$, as we know $\Phi \subseteq Pos_r(\Phi)$ from proposition 6.12, we get $\Phi_\mu \subseteq \Phi_\nu$, i.e. $\mu \preceq_\Phi \nu$. If we have also $\nu \preceq_\Phi \mu$, then we have $\mu \simeq_\Phi \nu$ thus $\mu \equiv_\prec \nu$ from the first remark 6.3. This establishes (fc).

Conversely, let us suppose that $\prec$ is irreflexive and antisymmetrical and satisfies (fc). We want to prove that $f_\prec$ is a formula circumscription, i.e. that there exists some set Φ of formulas such that $\mu \prec \nu$ iff $\Phi_\mu \subset \Phi_\nu$. We will prove that we may take the set $Pos_r(\prec)$ as our set Φ.

If $\mu \prec \nu$, then $(Pos(\prec))_\mu \subseteq (Pos(\prec))_\nu$ thus (see the proof of proposition 6.11-1) $(Pos_r(\prec))_\mu \subseteq (Pos_r(\prec))_\nu$. If $(Pos_r(\prec))_\mu = (Pos_r(\prec))_\nu$, then from (fc) we get $\nu \prec \mu$ or $\nu \equiv_\prec \mu$. If $\nu \equiv_\prec \mu$, as we have $\mu \prec \nu$ anyway, be get $\mu \prec \mu$, a contradiction with irreflexivity. Otherwise we have $\nu \prec \mu$, which, as we have $\mu \prec \nu$ anyway, contradicts antisymmetry. Thus we must have $(Pos_r(\prec))_\mu \subset (Pos_r(\prec))_\nu$.

Conversely, let us suppose that $(Pos_r(\prec))_\mu \subset (Pos_r(\prec))_\nu$ and also $\mu \not\prec \nu$. Then, there exists $\varphi \in (Pos_r(\prec))_\nu - (Pos_r(\prec))_\mu$, also, from (fc), we get $\mu \equiv_\prec \nu$.

We prove now that M_μ must be a closed set: Otherwise, there exists $\nu' \in TC(M_\mu) - M_\mu$.

From $\nu' \in TC(M_\mu)$ we get $\mu \mathrel{\overline{\prec}} \nu'$, and also, using $\mu \equiv_\prec \nu$, $\nu \mathrel{\overline{\prec}} \nu'$. Thus, from proposition 6.11-1, $(Pos_r(\prec))_\nu \subseteq (Pos_r(\prec))_{\nu'}$. As $\varphi \in (Pos_r(\prec))_\nu \subseteq (Pos_r(\prec))_{\nu'}$, we get $\nu' \in \mathcal{M}(\varphi)$ and, from definition 6.10, we get $\mu \in \mathcal{M}(\varphi)$, a contradiction with $\varphi \notin (Pos_r(\prec))_\mu$ and $\varphi \in (Pos_r(\prec))_\nu \subseteq Pos_r(\prec)$.

Now, let us consider some $\nu' \in M_\nu$. We get $\nu \prec \nu'$ thus there exists $\varphi'_{\nu'} \in (Pos_r(\prec))_{\nu'} - (Pos_r(\prec))_\nu$: indeed, we have shown above that if $\mu \prec \nu$, then $(Pos_r(\prec))_\mu \subset (Pos_r(\prec))_\nu$. Let us consider the set $\{\mathcal{M}(\varphi'_{\nu'}) \;/\; \nu' \in M_\nu\}$. This set is an open cover of the compact M_ν (remind that $\mu \equiv_\prec \nu$ thus $M_\mu = M_\nu$ and we have shown that M_μ is closed, thus compact in this topology) thus there exists a finite subcover. As $Pos_r(\prec)$ is stable for $\vee$, there exists a formula $\varphi_\nu \in Pos_r(\prec)$ such that $M_\mu \subseteq \mathcal{M}(\varphi_\nu)$ and $\nu \notin \mathcal{M}(\varphi_\nu)$: indeed, there exists a finite set of formulas $\{\psi_i\}_{i \in I}$ such that $M_\nu \subseteq \bigcup_{i \in I} \mathcal{M}(\psi_i)$ and, for any $i \in I$, $\psi_i \in Pos_r(\prec)$ and $\nu \notin \mathcal{M}(\psi_i)$; we may choose $\varphi_\nu = \bigvee_{i \in I} \psi_i$. Let φ' be the formula $\varphi \wedge \neg\varphi_\nu$ and B be the subset of $\mathcal{M}$: $B = (\bigcup_{\nu' \in \mathcal{M}(\varphi')} m_{\nu'}) - \mathcal{M}(\varphi)$.

We prove now that this set B is closed: Otherwise, there exists $\mu' \in TC(B) - B$ and we get $\mu' \notin \mathcal{M}(\varphi)$: indeed, $B \subseteq \mathcal{M}(\neg\varphi)$ thus $TC(B) \subseteq TC(\mathcal{M}(\neg\varphi)) = \mathcal{M}(\neg\varphi)$. Let ν' be in $\mathcal{M}(\varphi')$. If $\mu' \precsim \nu'$, then $(Pos_r(\prec))_{\mu'} \subseteq (Pos_r(\prec))_{\nu'}$ (proposition 6.11-1), thus by (fc), $\mu' \prec \nu'$ or $\mu' \equiv_\prec \nu'$, thus, as $\mu' \notin B$, $\mu' \equiv_\prec \nu'$. As $\nu' \in \mathcal{M}(\varphi)$ and $\varphi \in Pos_r(\prec)$, we get $\mu' \in \mathcal{M}(\varphi)$, a contradiction. Thus $\mu' \not\precsim \nu'$, which means that there exist $\psi'_{\mu'}, \varphi'_{\mu'}$ such that $\mu' \in \mathcal{M}(\psi'_{\mu'}), \nu' \in \mathcal{M}(\psi'_{\nu'})$ and, if $\mu'' \in \mathcal{M}(\psi'_{\mu'}), \nu'' \in \mathcal{M}(\psi'_{\nu'})$, then $\mu'' \not\precsim \nu''$. $\mathcal{M}(\varphi')$ is compact, thus there exists a formula ψ' such that $\mu' \in \mathcal{M}(\psi')$ and, if $\mu'' \in \mathcal{M}(\psi'), \nu'' \in \mathcal{M}(\varphi')$, then $\mu'' \not\precsim \nu''$. As our ν'', we may take ν' here. We have shown that, for any $\nu' \in \mathcal{M}(\varphi')$, and any $\mu'' \in \mathcal{M}(\psi')$, we have $\mu'' \not\precsim \nu'$, thus, from the definition of B, we have proved $\mathcal{M}(\psi') \cap B = \emptyset$ a contradiction with $\mu' \in TC(B)$. B is then closed, thus compact.

If $\mu' \in B$, then $(Pos_r(\prec))_\mu \not\subseteq (Pos_r(\prec))_{\mu'}$. Indeed, otherwise we would get $\mu \prec \mu'$ or $\mu \equiv_\prec \mu'$ from (fc), for any $\nu' \in \mathcal{M}(\varphi')$ such that $\mu' \prec \nu'$ we would get $\mu \prec \nu'$ (remind that from lemma 6.14, $\prec$ is transitive), thus $\nu' \in M_\mu$, which contradicts $\mathcal{M}(\varphi') \cap M_\mu = \emptyset$, an immediate consequence of the definition of φ'. Thus, there exists a formula $\psi_{\mu'} \in (Pos_r(\prec))_\mu - (Pos_r(\prec))_{\mu'}$. From the compactness of B, there exists $\psi \in (Pos_r(\prec))_\mu$ such that $B \subseteq \mathcal{M}(\neg\psi)$ (same argument as above, for the existence of φ_ν).

We define $\psi' = \psi \wedge \neg\varphi'$ and we show now that $\psi' \in Pos_r(\prec)$. If $\mu' \in \mathcal{M}(\psi')$ and $\mu' \prec \mu''$, then $\mu'' \in \mathcal{M}(\psi)$ because $\psi \in Pos_r(\prec) \subseteq Pos(\prec)$. If $\mu'' \in \mathcal{M}(\varphi')$, as $\mu' \in m_{\mu''}$ (because $\mu' \prec \mu''$) and $\mu' \notin B$ (because $\mu' \in \mathcal{M}(\psi)$ and $B \subseteq \mathcal{M}(\neg\psi)$), we get $\mu' \in \mathcal{M}(\varphi)$ from the definition of B. Thus $\mu' \in \mathcal{M}(\varphi_\nu)$ because $\mu' \in \mathcal{M}(\neg\varphi')$. Thus, $\varphi_\nu \in Pos(\prec)$, $\mu'' \in \mathcal{M}(\varphi_\nu)$, which contradicts $\mu'' \in \mathcal{M}(\varphi')$. Thus, we get $\mu'' \in \mathcal{M}(\psi')$ which proves $\psi' \in Pos(\prec)$.

Let us suppose now that $\mu' \precsim \nu'$, $\mu' \not\prec \nu'$, and $\nu' \in \mathcal{M}(\psi')$. Then $\nu' \in \mathcal{M}(\psi)$ and, as $\psi \in Pos_r(\prec)$, $\mu' \in \mathcal{M}(\psi)$. If $\mu' \notin \mathcal{M}(\psi')$, then $\mu' \in \mathcal{M}(\varphi')$ thus $\mu' \in \mathcal{M}(\varphi)$ and, as $\varphi \in Pos_r(\prec) \subseteq Pos(\prec)$, $\nu' \in \mathcal{M}(\varphi)$. As $\nu' \notin \mathcal{M}(\varphi')$, we get $\nu' \in \mathcal{M}(\varphi_\nu)$ and, as $\varphi_\nu \in Pos_r(\prec)$, $\mu' \in \mathcal{M}(\varphi_\nu)$, which contradicts $\mu' \in \mathcal{M}(\varphi')$.

Thus we get $\psi' \in (Pos_r(\prec))_\mu - (Pos_r(\prec))_\nu$, which contradicts our initial hypothesis.

We have established $\mu \prec \nu$ iff $(Pos_r(\prec))_\mu \subset (Pos_r(\prec))_\nu$, i.e. (see definition 5.3 and proposition 5.5) $f_\prec = CIRCF(Pos_r(\prec))$.

Second sentence:

Use the first sentence, together with proposition 3.16, reminding that any formula circumscription satisfies (PC) and, being a preferential entailment, satisfies also (DCC), (P′).

The interest of this last formulation is that as anything here is defined in terms of f, if we have a pre-circumscription described by the value of $f(\mathcal{T})$ for any $\mathcal{T} \in \mathcal{J}$, we may find whether it is a formula circumscription or not. Notice that we get also the following result:

A pre-circumscription f is a formula circumscription iff it satisfies (DCC), (P$'$), (PC), and the preference relation $\prec_f$ is antisymmetrical and satisfies (fc).

In order to use this result, or the second sentence of the proposition, it may also be useful to remind propositions 3.8 (last sentence of point 2, and point 3), 3.14, and 3.16. $\square$

Notice that theorem 6.15 solves a problem stated as open in [Mak94]. In the remarks following his Observation 3.4.11, Makinson evokes this unsolved problem and states that, from proposition 5.19 (his Observation 3.3.4) we know that sometimes an infinite set Φ may be necessary. It is hard to see how the author came to this conclusion as no counter-example to (RM) for a formula circumscription with an infinite Φ is evoked in [Mak94] (see example 6.26 below). Anyway, this seems to indicate that Makinson was conscious that the finite case was easy, but that the infinite case could be more difficult. In fact, as we have seen above, the main difficulty does not come from the possible (and not too unexpected) infinity of Φ, but rather from the fact that there exist formula circumscriptions $f = f_\prec$ for which the set $Pos(\prec) = I_f$ of their inaccessible formulas cannot be taken as the set of the formulas to be circumscribed (see example 6.5 completed in example 6.23, and remark 6.17, see also example 6.28 below for a comment about a "difficult case"). Ordinary classical propositional circumscription $CIRC(\mathbf{P}, \mathbf{Q}, \mathbf{Z})$ falsifies (RM) as soon as $\mathbf{P} \cup \mathbf{Q}$ is infinite (and $\mathbf{P}$ is not empty), while in this case we may well choose (in place of the even more obvious set $\mathbf{P} \cup \mathbf{Q} \cup \neg\mathbf{Q}$) the set of the inaccessible formulas as our set of formulas to circumscribe, and we know also a syntactical description of this set (see subsection 6.2 below): this fact shows that the falsification of (RM) is not a good indicator of the complexity of the problem.

Proposition 6.16 Let $\prec$ be a preference relation.
$f_\prec$ is a formula circumscription iff $f_\prec = CIRCF(Pos_r(\prec))$.
Moreover, in this case, $Pos_r(\prec)$ is the greatest set (for $\subseteq$) Φ such that $CIRCF(\Phi) = f_\prec$.
If Φ is some set of formulas, for any set of formulas Ψ, we have
$CIRCF(\Phi) = CIRCF(\Psi)$ iff $Pos_r(\Phi) = Pos_r(\Psi)$. $\square$

Proof: We have seen in the proof of theorem 6.15 that if $f_\prec = CIRCF(\Phi)$, then we have $CIRCF(\Phi) = CIRCF(Pos_r(\prec))$. Moreover, we know from proposition 6.12-1 that $Pos_r(\Phi) = Pos_r(\prec_\Phi)$ is the greatest possible set Ψ (for $\subseteq$) such that $CIRCF(\Phi) = CIRCF(\Psi)$ (i.e. $\Phi \equiv_c \Psi$).
The last result follows easily. $\square$

Remark 6.17 We do not have the same equivalence with $Pos(\prec)$ instead of $Pos_r(\prec)$:
Obviously, if $f_\prec = f_{\prec'}$ with $\prec$ and $\prec'$ irreflexive (which is always the case for formula circumscriptions) we have $Pos(\prec) = Pos(\prec')$.
However, we may have two different formula circumscriptions $f_\prec = CIRCF(\Phi)$ and $f_{\prec'} = CIRCF(\Phi')$ such that $Pos(\prec) = Pos(\prec')$ (i.e. $Pos_e(\Phi) = Pos_e(\Phi')$). $\square$

To establish this result, which completes (negatively) lemma 5.32 and proposition 5.33-1a above, we need only to give the following example:

Example 6.18 We consider the following interpretations: $\mu, \nu, \mu_i (i \in \mathbb{N})$ with $\lim_{i\to\infty} \mu_i = \mu$ and no other limits with respect to these interpretations.

We consider the following two preference relations:

$\prec$: $\mu' \prec \nu'$ iff ($\mu' = \mu$ or $\mu' = \nu$) and there exists $i \in \mathbb{N}$ such that $\nu' = \mu_i$.

$\prec'$: $\mu' \prec' \nu'$ iff $\mu' \prec \nu'$ or ($\mu' = \nu$ and $\nu' = \mu$).

We get $Pos(\prec) = Pos(\prec')$ because if $\varphi \in Pos(\prec)$ and $\nu \models \varphi$ then $\mu_n \models \varphi$ for any $n \in \mathbb{N}$ thus $\mu \models \varphi$.

Now, $\prec$ and $\prec'$ are irreflexive and different, thus $f_\prec \neq f_{\prec'}$.

It remains to show that $f_\prec$ and $f_{\prec'}$ are two formula circumscriptions.

Indeed, it is easy to show that $Pos_r(\prec') = Pos(\prec')$ is a "good" set of formulas for $f_{\prec'}$: $CIRCF(Pos(\prec')) = f_{\prec'}$.

Now, it is also easy to show (we leave the details to the reader) that $Pos_r(\prec)$ is a "good" set for $f_\prec$, i.e. $f_\prec = CIRCF(Pos_r(\prec))$.

$Pos_r(\prec) = \{\varphi \in Pos(\prec) \ / \ \mu \models \varphi \text{ iff } \nu \models \varphi\} \neq Pos(\prec)$.

Thus, this is a case when $CIRCF(\Phi) \neq CIRCF(Pos_e(\Phi))$ (see lemma 5.32). Indeed, from proposition 6.16 we know that $Pos_r(\prec)$ is the greatest set Ψ such that $f_\prec = CIRCF(\Psi)$ and we have just shown that $Pos_r(\prec) \neq Pos(\prec)$, which, as we have $Pos_r(\prec) \subseteq Pos(\prec)$ anyway, is equivalent to $Pos_r(\prec) \subset Pos(\prec)$, i.e., with $\Phi = Pos_r(\prec)$, $Pos_r(\Phi) \subset Pos_e(\Phi)$. From propositions 6.12-1 and 6.16, we get then $CIRCF(Pos_e(\Phi)) \neq CIRCF(\Phi)$. □

At the time of printing we have discovered Theorem 8 [Cos98] (see note 15 page 47), which is presented as another "characterization of what minimizing infinite sets of formulas can capture", which means that this is presented as a characterization result of propositional formula circumscription. We argue now against this presentation of the result (not against the result itself, which is correct and may have some interest). In our terms, this result is very similar to (and simpler than) proposition 5.29-5:

Proposition 6.19 As in [MR98], let us call a set E of interpretations *finishing for a preference relation* $\prec$ iff $\nu \in E$ whenever $\mu \in E$ and $\mu \prec \nu$.

1. For any set of formulas Φ, a set E is finishing for $\preceq_\Phi$ iff it is an intersection of unions of sets $\mathcal{M}(\varphi)$ where each φ is in Φ.

2. [Cos98, Theorem 8] A preference relation $\preceq$ which is reflexive and transitive is equal to $\preceq_\Phi$ for some set of formulas Φ iff any set finishing for $\preceq$ can be obtained as an intersection of unions of sets $\mathcal{M}(\varphi)$, where each φ is in $Pos(\preceq)$. □

[Cos98] gives a proof, to which we refer the interested reader. We provide here our own proof (which is not so different by the way), which takes its inspiration in our proof of proposition 5.29-5, but is much simpler as we allow infinite unions and intersections here, so we do not need to use compactness. We prefer to give firstly another related result in point 1, which explains what happens, making clear the connection between Costello's result (our point 2) and proposition 5.29-5 (a direct proof of point 2 alone would clearly be shorter).

Proof:

1. It is obvious that any union and any intersection of finishing sets is a finishing set. From definition 5.28 we see that a formula φ is in $Pos(\preceq)$ iff $\mathcal{M}(\varphi)$ is finishing for $\preceq$. Thus, as any $\varphi \in \Phi$ is in $Pos(\preceq_\Phi)$ from proposition 5.29-4, we get that any intersection of unions of sets $\mathcal{M}(\varphi)$ where all the φ's are in Φ is a set finishing for $\preceq_\Phi$.

We prove now the other way:

Let us suppose E is finishing for $\preceq_\Phi$. If $E = \emptyset$ or $E = \mathcal{M}$, we know that E is the union, or the intersection, of 0 elements. Thus, we may suppose that there exist μ and ν such that $\mu \in E$ and $\nu \notin E$. Then, $\Phi_\mu \not\subseteq \Phi_\nu$ from the definitions of $\preceq_\Phi$ and of finishing sets. This means that to any such couple (μ, ν) we may associate one formula $\varphi_{(\mu,\nu)} \in \Phi_\mu - \Phi_\nu$. Then, for any ν such that $\nu \notin E$, we have: $E \subseteq \bigcup_{\mu \in E} \mathcal{M}(\varphi_{(\mu,\nu)})$ (1) , where any $\varphi_{(\mu,\nu)}$ is in $\Phi_\mu \subseteq \Phi$ (2) , and, as $\varphi_{(\mu,\nu)} \notin \Phi_\nu$, we get $\nu \notin \mathcal{M}(\varphi_{(\mu,\nu)})$, thus $\nu \notin \bigcup_{\mu \in E} \mathcal{M}(\varphi_{(\mu,\nu)})$ (3). From (1) we get $E \subseteq \bigcap_{\nu \notin E} \bigcup_{\mu \in E} \mathcal{M}(\varphi_{(\mu,\nu)})$. From (3) we get, for any $\nu' \notin E$, $\nu' \notin \bigcap_{\nu \notin E} \bigcup_{\mu \in E} \mathcal{M}(\varphi_{(\mu,\nu)})$.

Thus we have $E = \bigcap_{\nu \notin E} \bigcup_{\mu \in E} \mathcal{M}(\varphi_{(\mu,\nu)})$, where, from (2), any $\varphi_{(\mu,\nu)}$ is in Φ.

2. Clearly, for any set of formulas Φ, the set of all the intersections of unions of $\mathcal{M}(\varphi)$, where φ is in Φ is the set of all the intersections of unions of $\mathcal{M}(\varphi)$, where φ is in $Pos(\preceq_\Phi) = \Phi^{\wedge\vee}$.

If $\preceq = \preceq_\Phi$, as we know $\preceq_\Phi = \preceq_{Pos(\preceq_\Phi)}$ from proposition 5.33-2a, we get from point 1 that the sets finishing for $\preceq$ are all the intersections of unions of $\mathcal{M}(\varphi)$, where φ is in $Pos(\preceq_\Phi)$.

Conversely, if $\preceq$ is a preference relation such that there exists no set Φ of formulas such that the sets finishing for $\preceq$ are exactly all the intersections of unions of sets $\mathcal{M}(\varphi)$ where each φ is in Φ, then a fortiori, for $\Phi = Pos(\preceq)$, we have that the sets finishing for $\preceq$ do not coincide with all the intersections of unions of sets $\mathcal{M}(\varphi)$ where each φ is in $Pos(\preceq)$. Thus, from point 1, we get that $\preceq$ cannot be equal to any $\preceq_\Psi$. $\square$

We have shown how this result follows from the definition of $\preceq_\Phi$. We will show now that this result does not help to determine whether a given pre-circumscription f, or even a preferential entailment $f_\prec$, is or is not a formula circumscription. Indeed, when we are given a pre-circumscription f, or a preferential entailment associated to an irreflexive preference relation $\prec$, we are not given a reflexive preference relation $\preceq$ such that $f = f_{\prec'}$ where $\prec'$ is defined by $\mu \prec' \nu$ iff $\mu \preceq \nu$ and $\nu \not\preceq \mu$. Using proposition 3.16, together with the various characterization results given in this text, in propositions 3.8 (last sentence of point 2, and point 3) and 3.14, and also with the results known for formula circumscriptions, if we are given a pre-circumscription f which is candidate to be a formula circumscription, it is easy to define its associated irreflexive preference relation $\prec_f$. Then, using theorem 6.15, or proposition 6.16, we may (even if it is not always an easy task) determine whether it is a formula circumscription or not. While from proposition 6.19, we do not see how to do this: to any irreflexive preference relation $\prec$ may be associated a fairly great number (generally non enumerable as soon as $\mathcal{V}(\boldsymbol{\mathcal{L}})$ is infinite, and otherwise proposition 5.24 is enough) of reflexive preference relations $\preceq$, which should all be tested by this not too friendly property of the finishing sets. It is not better (and possibly worse) than trying all the sets of formulas Φ and checking directly whether $f = CIRCF(\Phi)$ – i.e checking whether $\mu \prec_f \nu$ iff $\Phi_\mu \subset \Phi_\nu$ – or not. We provide below various examples in which theorem 6.15 or proposition 6.16 allow to determine whether a given $f_\prec$ is a formula circumscription or not, while we do not see any way of using proposition 6.19 for this purpose: see examples 6.20, 6.21, 6.24, 6.25 and 6.26. [Cos98] does not provide any example of the use of theorem 8 in order to determine whether a given pre-circumscription or preferential entailment is a formula circumscription or not.

Notice that we could also use any alternative definition given in this text such as in proposition 5.14. But we would not call such results "characterization results". And, for example, we do not see how a discussion such as the one we had just above about a comment made by Makinson could be conducted from proposition 6.19.

Let us begin by a simple example in order to justify above comments.

Example 6.20 $\mathcal{V}(\mathcal{L})$ is infinite, f is the identity, thus $f = f_\prec$ where $\prec$ has an empty graph.

If we want to use proposition 6.19, we must consider all the reflexive and transitive preference relations $\preceq$ such that $\mu \prec \nu$ iff $\mu \preceq \nu$ and $\nu \not\preceq \mu$. As $\mathcal{M}$ is not enumerable, the set of all these relations $\preceq$ is not enumerable: it includes (among many other possibilities) all the relations $\preceq$ defined by: $\mu_0 \preceq \nu_0$ for two different given interpretations μ_0 and ν_0, $\mu \preceq \mu$ for any μ, and nothing else. And, for each of these relations $\preceq$, we should test the rather hard to test condition given in proposition 6.19.

This is an example where it is easier to check directly all the sets of formulas Φ and to wait one such that $CIRCF(\Phi) = f$. If we are lucky, we begin by a good one (indeed, $\emptyset$ and $\mathcal{L}$ are among the good ones!), but even otherwise, this is not more complicated than using proposition 6.19.

With theorem 6.15 or here indifferently with proposition 6.16, all we have to do is to show that $\overline{\prec} = \prec$ (obvious), thus $Pos_r(\prec) = Pos(\prec)$, and $Pos(\prec) = \mathcal{L}$ (obvious also). Then, if we use theorem 6.15, we only need to check that $\mathcal{L}_\mu \subseteq \mathcal{L}_\nu$ implies $\mu \prec \nu$ or $\mu \equiv_\prec \nu$, which is immediate: $\mathcal{L}_\mu = Th(\mu)$, thus $\mathcal{L}_\mu \subseteq \mathcal{L}_\nu$ implies $\mu = \nu$, thus $\mu \equiv_\prec \nu$. If we want to use proposition 6.16 instead of the theorem, it is not more complicated: we check whether $f = CIRCF(\mathcal{L})$ or not, obviously the answer is yes, as $CIRCF(\mathcal{L})$ is the identity in $\mathcal{J}$. $\square$

Let us give now an example evoked above, coming from the recent literature:

Example 6.21 (cf example 8 in [Fre98] and example 1 in [Cos98]). $\mathcal{V}(\mathcal{L}) = \{P_i\}_{i \in \mathbb{N}}$. S is the subset of $\mathcal{M}$ containing all the interpretations $\nu_i = \{P_i\}$, for any $i \in \mathbb{N}$. [Fre98] (once translated in terms of formula circumscription) defines $\prec$ by $\mu \prec \nu$ iff $\mu \notin S$ and $\nu \in S$. [Cos98] does not evoke $\prec$ and defines only the reflexive relation $\preceq$ by $\mu \preceq \nu$ iff $\mu \prec \nu$ or $\mu \equiv_\prec \nu$, i.e. iff (($\mu \notin S$ and $\nu \in S$) or ($\mu \notin S$ and $\nu \notin S$) or ($\mu \in S$ and $\nu \in S$). Then clearly we have $\mu \prec \nu$ iff $\mu \preceq \nu$ and $\nu \not\preceq \mu$.

As Freund's proof is rather intricate and is completely ad-hoc, let us give our own proof. Any interpretation $\nu_i = \{P_i\}$ is the limit of a sequence of interpretations not in S (e.g. $\{P_i\} = \lim_{n\to\infty}\{P_i, P_n\}$). Also the interpretation $\mu_0 = \emptyset$ is not in S, but it is the limit of interpretations of S: $\mu_0 = \lim_{i\to\infty} \nu_i$). Thus, from remark 6.8-2 we get $\nu_i \mathrel{\overline{\prec}} \mu_0$. From proposition 6.11-1 we get $(Pos_r(\prec))_{\nu_i} \subseteq (Pos_r(\prec))_{\mu_0}$. However, we have neither $\nu_i \prec \mu_0$ (by the definition of $\prec$) nor $\mu_i \equiv_\prec \mu_0$ ($m_{\nu_i} = \mathcal{M} - S$ while $m_{\mu_0} = \emptyset$). $\prec$ falsifies (fc), thus $f_\prec$ is not a formula circumscription from theorem 6.15.

[Cos98] is not concerned by the fact of knowing whether $f_\prec$ is a formula circumscription or not, however it shows, using its theorem 8 (proposition 6.19-2) that $\preceq$ as it has defined it does not correspond to any $\preceq_\Phi$. As clearly we could have chosen many other reflexive and transitive relations $\preceq'$ such that $\mu \prec \nu$ iff $\mu \preceq' \nu$ and $\nu \not\preceq' \mu$, this is far from proving that $f_\prec$ is not a formula circumscription.

For the interested reader let us see more precisely what happens in this example. $\prec$ is ranked with 2 ranks: we may define the rank (see proposition 3.21) by $r(\mu) = 0$ iff $\mu \notin S$ and $r(\mu) = 1$ iff $\mu \in S$. Thus $f = f_\prec$ satisfies (RatM1) and (DR), also $\prec$ satisfies (sf) (finite number or ranks) thus f satisfies (CUMU) and (PC). We have $TC(S) = S \cup \{\mu_0\}$ (indeed, μ_0 is the only element not in S which may be obtained as limit of elements in S) and $TC(\mathcal{M} - S) = \mathcal{M}$ (see above). Thus from remark 6.8-2, we get $\mu \mathrel{\overline{\prec}} \mu'$ iff $\mu' \in S \cup \{\mu_0\}$. Thus, ($\mu \mathrel{\overline{\prec}} \mu'$ and $\mu \not\prec \nu$) iff (($\mu \in S$ and $\mu' \in S$) or ($\mu' = \mu_0$)). $Pos(\prec) = Th(S) \cup \{\bot\}$ (indeed, no formula except $\top$ is satisfied by all the interpretations not in S) and $Pos_r(\prec) = \{\top, \bot\}$. As $CIRCF(\{\top, \bot\}) = identity$ we get that f cannot be a formula circumscription from proposition 6.16. □

Let us re-examine three already given examples, in the light of these new definitions and results:

Example 6.22 (example 6.1 continued):
In example 6.1, we get clearly $\mu \mathrel{\overline{\prec}} \nu$. Also, if $\varphi \in Pos_r(\prec))_\mu$, then $\varphi \in Pos(\prec)$ and $\mu \in \mathcal{M}(\varphi)$, thus $M_\mu \subseteq \mathcal{M}(\varphi)$, thus, as ν is limit of the ν_n's, $\nu \in \mathcal{M}(\varphi)$, thus $\varphi \in Pos_r(\prec))_\nu$: this establishes $(Pos_r(\prec))_\mu \subseteq (Pos_r(\prec))_\nu$.

However, we have neither $\mu \prec \nu$ nor $\mu \equiv_\prec \nu$ (indeed $M_\nu = \emptyset \neq M_\mu$). Thus $\prec$ falsifies (fc). □

Example 6.23 (example 6.5 continued):
In example 6.5, we have $\mu \mathrel{\overline{\prec}} \nu$ and also $Pos_r(\prec) = \Phi = \{\varphi \in Pos(\prec) \ / \ \mu \models \varphi \text{ iff } \nu \models \varphi\}$.
□

Example 6.24 $\mathcal{V}(\boldsymbol{\mathcal{L}})$ is infinite, μ_1, μ_2 and μ_3 are three distinct interpretations for $\boldsymbol{\mathcal{L}}$, $\prec$ is defined by: $\mu \prec \nu$ iff $\mu = \mu_1$ and $\nu = \mu_2$.

It is immediate to see that $\overline{\prec} = \prec$, $Pos(\prec) = Pos_r(\prec)$ and that $\prec$ is irreflexive, antisymmetrical and it is easy to see that $\prec$ satisfies (fc). Thus $f = f_\prec$ is a formula circumscription from theorem 6.15, and, from proposition 6.16 we may take $Pos(\prec)$ as our set Φ of circumscribed formulas: $f = CIRCF(Pos(\prec))$.

Now, $\equiv_{\prec}[\mu_3] = \mathcal{M} - \{\mu_1, \mu_2\}$ is not a closed set.

This example shows that, as announced in remark 6.3-2, the set $\equiv_{\Phi}[\mu]$ is not necessarily closed.

Notice that example 2.2 1 in [Sch97] (appearing already as examples 1.3-1 and 1.9-1 in [Sch92]) is similar. Schlechta gives it as an example of a preferential entailment falsifying (RM) and where $\prec$ falsifies (cl). Indeed, it is clear that $\mathcal{M}_{\prec}(\top) = \mathcal{M} - \{\mu_2\}$, thus $\mathcal{M}(f(\top)) = TC(\mathcal{M}_{\prec}(\top)) = \mathcal{M}$: $\prec$ falsifies (cl). From proposition 3.14-1 we get that $f_{\prec}$ must falsify (RM). This is easy to check directly (see also [Sch92, Sch97]): $\mathcal{T} = Th(\top), \mathcal{T}'' = Th(\{\mu_1, \mu_2\})$. Then $f(\mathcal{T}) = \mathcal{T}$ and $f(\mathcal{T}'') = Th(\mu_1)$. Thus, $\mathcal{T} \subseteq \mathcal{T}''$ while $f(\mathcal{T}'') = Th(\mu_1) \not\subseteq f(\mathcal{T}) \sqcup \mathcal{T}'' = \mathcal{T}'' = Th(\{\mu_1, \mu_2\}) = Th(\mu_1) \cap Th(\mu_2)$: f falsifies (RM). Apparently Schlechta was unaware of the fact that his example is an example of formula circumscription, thus that it is yet another example of a formula circumscription falsifying (RM). Moreover, it is an "easy" formula circumscription, as $\prec = \overline{\prec}$ thus $Pos_r(\prec) = Pos(\prec)$.

As a last comment about this example, we show now that even in such an "easy case" of formula circumscription, we cannot always choose for our relation $\preceq_{\Phi}$ the reflexive relation $\preceq$ defined by

$\mu \preceq \nu$ iff ($\mu \prec \nu$ or $\mu \equiv_{\prec} \nu$) (Def-$\preceq$).

This illustrates one reason of the difficulty of using proposition 6.19-2 as a characterization result of formula circumscriptions. Indeed, this choice of $\preceq$ seems the most natural candidate for a reflexive relation. By the way, this is the proposal made in [Cos98, example 1], which is a (rather particular) case where it works. Thus, the present example shows that it is not clear which reflexive relation must be considered in order to check the condition of proposition 6.19-2.

So, let us define $\preceq$ as in (Def-$\preceq$), in the case of the present example. We have already seen that $\equiv_{\prec}[\mu_3]$ is not closed, and precisely we get $\equiv_{\prec}[\mu_3] = \mathcal{M} - \{\mu_1, \mu_2\}$, thus $TC(\equiv_{\prec}[\mu_3]) = \mathcal{M}$ and the same is true for any μ different from μ_1 and μ_2 instead of μ_3. Thus, $\mu \preceq \nu$ iff (($\mu = \mu_i$ and $\nu = \mu_j$ for any $(i,j) \in \{(1,1),(1,2),(2,2)\}$) or $\{\mu, \nu\} \cap \{\mu_1, \mu_2\} = \emptyset$). Thus, $Pos(\preceq) = \{\top, \bot\}$ and, as $CIRCF(\{\top, \bot\}) = identity$, we get $f_{\prec} \neq CIRCF(\{\top, \bot\})$. From lemma 5.31 together with proposition 5.29-5 (or alternatively from proposition 6.19-2), this shows that $\preceq$ cannot be equal to any $\preceq_{\Phi}$. □

Example 6.25 (example 5.18 continued):

In example 5.18, we get $Pos_r(\prec) = Pos(\prec)$: indeed, we have shown that $f_{\prec} = CIRCF(\Phi)$ where Φ is the set called now $Pos(\prec)$. If $\mu \prec \nu$ then $\mu = \{P_0\}$ or $\mu = \{P_1\}$. The two sets $M_{\{P_0\}}$ and $M_{\{P_1\}}$ are closed sets, thus we get $\overline{\prec} = \prec$.

Notice that the fact that $\overline{\prec} = \prec$ and $Pos_r(\prec) = Pos(\prec)$ shows that this example is not "so complicated", which indicates that it is "relatively easy" for a formula circumscription to falsify (DC0).

Another comment about this example: we have already shown in example 5.18 that $f_{\prec}$ is indeed a formula circumscription, by a "lucky guess" of the set Φ. Now that we have theorem 6.15 and proposition 6.16, we may get this set Φ without any guess. Indeed, as

$Pos_r(\prec) = Pos(\prec)$, we know that $f_\prec$ is a formula circumscription iff the set $Pos(\prec)$ (which is also known as the set $I_\prec$) may be chosen as our set Φ. This shows the power of our characterization results. $\Box$

Example 6.26 [Mak94] gives, after Observation 3.4.8, an example of a preferential entailment $f_\prec$ which falsifies (RM) (thus $\prec$ falsifies (cl)). The example is more complicated than Schlechta's example, (see example 6.24 above), and we prove now that Makinson's example is not a formula circumscription. This constitutes another illustration of the power of theorem 6.15.

Makinson's example is as follows: $\mathcal{V}(\boldsymbol{\mathcal{L}}) = \{P_i\}_{i\in \mathbb{N}}$, $\nu = \mathcal{V}(\boldsymbol{\mathcal{L}})$. $\prec$ is defined by $\mu \prec \nu$ for any finite subset μ of $\mathcal{V}(\boldsymbol{\mathcal{L}})$ and nothing else. Obviously, $\prec$ satisfies (sf) and its opposite relation also, and $\prec$ is transitive and irreflexive.

Makinson shows that (RM) is falsified by choosing $\mathcal{T} = Th(\top)$ and $\mathcal{T}'' = Th(\{P_i\}_{i\geq 1})$ (the P_i's being considered as formulas here). Thus $\nu \models P_0$ and for any finite μ, $\mu \not\models \mathcal{T}''$, thus $\mathcal{M}_\prec(\mathcal{T}'') = \{\nu\} = \mathcal{M}(f_\prec(\mathcal{T}''))$ and $P_0 \in f_\prec(\mathcal{T}'')$. Also, $\mathcal{M}_\prec(\mathcal{T}) = \mathcal{M} - \{\nu\}$ thus $\mathcal{M}(f_\prec(\mathcal{T})) = TC(\mathcal{M}_\prec(\mathcal{T})) = \mathcal{M}$, i.e. $f_\prec(\mathcal{T}) = \mathcal{T}$. We have $\mathcal{T} \subseteq \mathcal{T}''$ while $f_\prec(\mathcal{T}) \sqcup \mathcal{T}'' = \mathcal{T} \sqcup T'' = \mathcal{T}''$ and $P_0 \notin \mathcal{T}''$ thus $f_\prec(\mathcal{T}'') \not\subseteq f_\prec(\mathcal{T}) \sqcup \mathcal{T}''$.

We show now that $f_\prec$ is not a formula circumscription. From our results, it is almost immediate: let us choose $\mu' = \nu - \{P_0\}$. As $\mu' = \lim_{n\to\infty}\{P_1, P_2, \cdots P_n\}$, where each interpretation $\{P_1, P_2, \cdots P_n\}$ is a finite subset of $\mathcal{V}(\boldsymbol{\mathcal{L}})$, we get $\mu' \overline{\preceq} \nu$ from remark 6.8-2. From proposition 6.11-1 we get $(Pos_r(\prec))_{\mu'} \subseteq (Pos_r(\prec))_\nu$. However, we have neither $\mu' \prec \nu$ (by the definition of $\prec$) nor $\mu' \equiv_\prec \nu$ ($m_{\mu'} = \emptyset$ while m_ν is the set of all the finite subsets of $\mathcal{V}(\boldsymbol{\mathcal{L}})$). $\prec$ falsifies (fc), thus $f_\prec$ is not a formula circumscription from theorem 6.15.

As in the preceding examples, we do not see how proposition 6.19 could be used in order to show that $f_\prec$ is (in examples 6.20, 6.24 and 6.25) or is not (as in the present example) a formula circumscription.

For the interested reader, let us give now a few more results, in order to see precisely what is the situation here. $\overline{\preceq}$ is defined by $\mu' \prec \nu$ for any interpretation μ' for $\boldsymbol{\mathcal{L}}$ and nothing else. Indeed, $\mathcal{V}(\boldsymbol{\mathcal{L}})$ is enumerable, thus we may use remark 6.8-2, and any interpretation μ' is limit of some finite subsets of $\mathcal{V}(\boldsymbol{\mathcal{L}})$. $Pos(\prec)$ is the set of all the formulas which have no finite subset of $\mathcal{V}(\boldsymbol{\mathcal{L}})$ for model, together with the formulas in $Th(\nu)$. As any consistent formula has some finite subset of $\mathcal{V}(\boldsymbol{\mathcal{L}})$ for model, we get $Pos(\prec) = Th(\nu) \cup \{\bot\}$.

We have ($\mu' \overline{\preceq} \mu''$ and $\mu' \not\prec \mu''$) iff (μ' is an infinite subset of $\mathcal{V}(\boldsymbol{\mathcal{L}})$ and $\mu'' = \nu$). Thus, from definition 6.10, $Pos_r(\prec) = \{\varphi \in Th(\nu) \ / \ \mu \models \varphi \text{ for any infinite } \mu\} \cup \{\bot\}$. Thus $Pos_r(\prec) = \{\top, \bot\}$ and $(Pos_r(\prec))_\mu = \{\top\}$ for any interpretation μ and $(Pos_r(\prec))_{\mu'} \subseteq (Pos_r(\prec))_{\mu''}$ is satisfied for any interpretations μ' and μ''. And, clearly, we do not have always $\mu' \prec \mu''$ or $\mu' \equiv_\prec \mu''$ (choose e.g. $\mu' = \nu, \mu'' \neq \nu$). Not only $\prec$ does not satisfy (fc), but it is "very far from satisfying (fc)". Let us consider now proposition 6.16: once we know $Pos_r(\prec) = \{\top, \bot\}$, it suffices to remark that $f_\prec \neq CIRCF(\{\top, \bot\}) = identity$. In some way, we have shown that "if $f_\prec$ was a formula circumscription, then it should be the trivial formula circumscription", thus it would satisfy (RM). This shows why this example is far from being an example of formula circumscription falsifying (RM) (see our comment following the proof of theorem 6.15). $\square$

Now, in order to justify in part the complexity of the definitions given before theorem 6.15, let us make the following remark.

Remarks 6.27 1. As proved by example 6.28 below, we cannot replace property (fc) by any of the easier following properties in theorem 6.15 ($\prec$ is supposed transitive and irreflexive):

(fc$_0$): If $(Pos(\prec))_\mu \subseteq (Pos(\prec))_\nu$, then $\mu \prec \nu$ or $\mu \equiv_\prec \nu$.

(fc$_{00}$): If $\mu \mathrel{\overline{\preceq}} \nu$, then $\mu \prec \nu$ or $\mu \equiv_\prec \nu$.

If $\prec$ satisfies (fc$_0$), from proposition 6.9-1, we know that $\prec$ satisfies (fc$_{00}$).

2. However we know, from propositions 6.9-1 and 6.11 together with theorem 6.15 that if an irreflexive and transitive $\prec$ falsifies (fc$_0$) or (fc$_{00}$), then $f_\prec$ cannot be a formula circumscription. $\square$

We have already used the trick given in point 2, with (fc$_{00}$), in order to show that example 6.26 is not a formula circumscription. As (fc$_{00}$) is much simpler than (fc$_0$), which is itself much simpler than (fc), this shows that it is "relatively easy" to prove that such examples are not a formula circumscriptions.

We give now an example which is not so easy to that respect.

Example 6.28 $V(\boldsymbol{\mathcal{L}}) = \{P_i\}_{i \in \mathbb{N}}$.

We define the following interpretations: $\mu_n = \{P_{2i+1}\}_{i \leq n}$, $\mu'_n = \{P_{2i}\}_{i \leq n}$,
$\mu'^k_n = \mu'_n \cup \{P_{2(n+k)+1}\}$,
$\mu = \{P_{2i+1}\}_{i \in \mathbb{N}}$, $\mu' = \{P_{2i}\}_{i \in \mathbb{N}}$, and $\nu = \emptyset$.

Let us call S the set of all these interpretations.

We define $\prec$ by: 1) For any n, k in $\mathbb{N}$, $\mu_n \prec \mu'^k_n$ and $\mu'_n \prec \mu'^k_n$,
2) $\mu \prec \nu$, $\nu \prec \mu'$, $\mu \prec \mu'$, and
3) nothing else.

We have: $\lim_{k\to\infty} \mu'^k_n = \mu'$, $\lim_{n\to\infty} \mu_n = \mu$, and $\lim_{n\to\infty} \mu'_n = \mu'$. With regard to the preceding interpretations, there exist no other limits, thus S is a closed set.

Notice also that $\prec$, and its opposite relation $\succ$, satisfy (sf).

For any $n \in \mathbb{N}$, we have also $\mu_n \equiv_\prec \mu'_n$.

If $(Pos(\prec))_{\nu'} \subseteq (Pos) \prec))_{\nu''}$ and $\nu' \not\prec \nu''$, then ν' and ν'' must belong to the closed set S, more precisely we have either $\nu' = \nu'' = \mu'_n$, or $\nu' = \nu'' = \mu'$ or $\nu' = \mu_n$ and $\nu'' = \mu'_n$. Then, it is easy to check that $\prec$ satisfies (fc$_0$), thus also (fc$_{00}$).

If $\varphi \in (Pos_r(\prec))_\nu$, then $\mu' \in \mathcal{M}(\varphi)$ thus there exists $N \in \mathbb{N}$ such that for any $n \in \mathbb{N}$, $n \geq N$ implies $\mu'_n \in \mathcal{M}(\varphi)$. As for any $n \geq N$ we have 1) $\mu_n \mathrel{\overline{\preceq}} \mu'_n$ from remark 6.8-2, 2) $\mu'_n \in \mathcal{M}(\varphi)$, 3) $\mu'_n \not\prec \mu_n$, and as we have also $\varphi \in Pos_r(\prec)$, we get $\mu_n \in \mathcal{M}(\varphi)$.

Then, we get $\mu \in \mathcal{M}(\varphi)$. We have thus $\mu \prec \nu$ and $(Pos_r(\prec))_\mu = (Pos_r(\prec))_\nu$, which shows that $f_\prec \neq CIRCF(Pos_r(\prec))$ thus, from proposition 6.16, $f_\prec$ cannot be a formula circumscription. □

We introduce now yet another set of formulas and its associated property for $\prec$. Fortunately, we will see that in the "interesting cases", neither this set nor this property are new, they just provide another way to see what happens. We could have taken directly these alternative set and property in our proof of the characterization theorem, but, even if these definitions with a " $'$ " may look simpler than the definitions without the " $'$ ", the proof of theorem 6.15 would be exactly as complicated. And we think that the definitions without the " $'$ " are more "natural" after all.

Definitions 6.29 $\prec$ being a preference relation, we define the set

$$Pos'_r(\prec) = \{\varphi \in Pos(\prec) \ / \ \text{for any interpretations } \mu \text{ and } \nu, \text{ if } \mu \not\prec \nu,\ (Pos(\prec))_\mu \subseteq (Pos(\prec))_\nu \text{ and } \nu \models \varphi, \text{ then } \mu \models \varphi\}.$$

We call (fc$'$) the following property of a preference relation $\prec$:
(fc$'$): for any μ, ν, if $(Pos'_r(\prec))_\mu \subseteq (Pos'_r(\prec))_\nu$, then $\mu \prec \nu$ or $\mu \equiv_\prec \nu$. □

Proposition 6.30 1. If $f_\prec$ is a formula circumscription, then $Pos_r(\prec) = Pos'_r(\prec)$.

2. In any case, $Pos'_r(\prec) \subseteq Pos_r(\prec)$ and (fc$'$) implies (fc). Also, $Pos'_r(\prec) = (Pos'_r(\prec))^{\wedge\vee}$.

3. If $\prec$ is irreflexive and transitive (thus in particular if $f_\prec$ is a formula circumscription), then (fc$'$) is equivalent to (fc). □

Proof: 1. From proposition 6.16, we know that $f_\prec = CIRCF(Pos_r(\prec))$. If $\varphi \in Pos_r(\prec)$, $\nu \models \varphi$, $\mu \not\prec \nu$ and $(Pos(\prec))_\mu \subseteq (Pos(\prec))_\nu$, then, as $Pos_r(\prec) \subseteq Pos(\prec)$, we get $(Pos_r(\prec))_\mu \subseteq (Pos_r(\prec))_\nu$ and, as $\mu \not\prec \nu$, we get $(Pos_r(\prec))_\mu \not\subset (Pos_r(\prec))_\nu$, thus $(Pos_r(\prec))_\mu = (Pos_r(\prec))_\nu$. Thus, $\varphi \in (Pos_r(\prec))_\mu$ and $\mu \models \varphi$. Thus $Pos_r(\prec) \subseteq Pos'_r(\prec)$.

The inclusion $Pos'_r(\prec) \subseteq Pos_r(\prec)$ is an immediate consequence of the fact that, if $\mu \,\overline{\prec}\, \nu$, then $(Pos(\prec))_\mu \subseteq (Pos(\prec))_\nu$ (proposition 6.9-1).

2. If $\varphi \in Pos'_r(\prec)$, $\mu \not\prec \nu$, $\mu \,\overline{\prec}\, \nu$ and $\nu \models \varphi$, then $(Pos(\prec))_\mu \subseteq (Pos(\prec))_\nu$ from proposition 6.9-1 and $\mu \models \varphi$. Thus $\varphi \in Pos_r(\prec)$.

Now, we suppose (fc$'$) and $(Pos_r(\prec))_\mu \subseteq (Pos_r(\prec))_\nu$, then, as $Pos'_r(\prec) \subseteq Pos_r(\prec)$, we get $(Pos'_r(\prec))_\mu \subseteq (Pos'_r(\prec))_\nu$ thus, from (fc$'$) we get $\mu \prec \nu$ or $\mu \equiv_\prec \nu$: we have (fc).

The stability for $\wedge$ and $\vee$ of $Pos'_r(\prec)$ is immediate from the definition.

3. Irreflexivity and transitivity give antisymmetry, thus if we have (fc), $f_\prec$ is a formula circumscription (theorem 6.15) and $Pos_r(\prec) = Pos'_r(\prec)$ (point 1 above) thus (fc) and (fc$'$) are identical. □

6.2 Description of the sets of "positive formulas" associated to $CIRC(\mathbf{P}, \mathbf{Q}, \mathbf{Z})$

After all these more or less technical results, let us provide to the patient reader a few simple and illustrative examples. We describe syntactically all the "sets of positive formulas" defined in this text which correspond to a propositional circumscription. Notice that some of the results proved here have already been stated above (e.g. in remarks 5.30-2 and in the proof of proposition 5.33-1b).

Notations 6.31 If $\mathbf{Y}'$ is a subset of $\mathcal{V}(\mathcal{L})$, let $\mathbf{Y}$ be some finite and consistent sets of literals from $\mathbf{Y}'$ (each element of $\mathbf{Y}$ is either an atom in $\mathbf{Y}'$ or the negation of an atom in $\mathbf{Y}'$, and if a literal Y is in $\mathbf{Y}$ then its complementary $\neg Y$ is not in $\mathbf{Y}$).

We define the two formulas $\bigvee(\mathbf{Y}) = \bigvee_{\varphi \in \mathbf{Y}} \varphi$ and $\bigwedge(\mathbf{Y}) = \bigwedge_{\varphi \in \mathbf{Y}} \varphi$. If $\mathbf{Y} = \emptyset$, as usual $\bigvee(\mathbf{Y}) = \bot$ and $\bigwedge(\mathbf{Y}) = \top$.

If, for any $Y \in \mathbf{Y}'$, $\mathbf{Y}$ contains either Y or $\neg Y$, the set of literals $\mathbf{Y}$ is *complete* in $\mathbf{Y}'$. $\Box$

Proposition 6.32 The case of propositional circumscription $CIRC(\mathbf{P}, \mathbf{Q}, \mathbf{Z})$ (see also remarks 5.30):

0. $\prec$ is the preference relation associated to $CIRC(\mathbf{P}, \mathbf{Q}, \mathbf{Z})$ (see definition 4.1), that is also the preference relation associated to $CIRCF(\mathbf{P}; \mathbf{Q}, \mathbf{P} \cup \mathbf{Z})$ (see propositions 5.2 and 5.5) or equivalently to $CIRCF(\Phi)$ where Φ is the set of formulas $\Phi = \mathbf{P} \cup \mathbf{Q} \cup \neg\mathbf{Q}$ (see proposition 5.9-2).

   Thus $\prec = \prec_{(\mathbf{P},\ \mathbf{Q},\ \mathbf{Z})} = \prec_\Phi$.

   We consider also $\preceq = \preceq_{(\mathbf{P},\ \mathbf{Q},\ \mathbf{Z})} = \preceq_\Phi$ (see notations 5.6).

   We have $Pos(\preceq) \subseteq Pos(\prec) = I_\prec$ (proposition 5.29, points 4 and 2 respectively).

   Also, from the equality $CIRC(\mathbf{P}, \mathbf{Q}, \mathbf{Z}) = CIRCF(\Phi)$ we have

   $Pos(\prec) = Pos_e(\Phi)$ and $Pos(\preceq) = Pos_m(\Phi) = \Phi^{\wedge\vee}$ (see remarks 5.30-2 and proposition 5.29-5).

1a. If $\varphi \in Pos(\prec)$, then φ is positive in $\mathbf{P}$ (traditional meaning, i.e. $\varphi \in (\mathbf{P} \cup \mathbf{Q} \cup \neg\mathbf{Q} \cup \mathbf{Z} \cup \neg\mathbf{Z})^{\wedge\vee}$).

1b. $\Phi^{\wedge\vee} \subseteq Pos(\prec)$.

   Moreover $\varphi \in Pos(\preceq)$ iff $\mathcal{V}(\varphi) \subseteq \mathbf{P} \cup \mathbf{Q}$ and φ is positive in $\mathbf{P}$ (traditional meaning), i.e.,

   $Pos(\preceq) = \Phi^{\wedge\vee}$.

1c. If $\mathbf{P}$ is infinite, or if $\mathbf{Z} = \emptyset$, then $Pos(\prec) = Pos(\preceq)$ is the set of all the formulas made in the vocabulary $\mathbf{P} \cup \mathbf{Q}$ (meaning without any element of $\mathbf{Z}$) which are positive in $\mathbf{P}$ (traditional meaning), i.e. $Pos(\prec) = Pos(\preceq) = \Phi^{\wedge\vee}$.

2. If $\mathbf{P}$ is finite, then $Pos(\prec)$ is the set of all the formulas in $\mathcal{V}(\mathcal{L})$ which are disjunctions of formulas of the kind $\bigwedge(\mathbf{P}_a) \wedge \bigwedge(\mathbf{Q}_l) \wedge (\bigwedge(\mathbf{Z}_l) \vee \bigvee(\mathbf{P} - \mathbf{P}_a))$, for any subset $\mathbf{P}_a$ of $\mathbf{P}$ and any finite sets $\mathbf{Q}_l$ and $\mathbf{Z}_l$ made of literals of $\mathbf{Q}$ and of $\mathbf{Z}$ respectively.

   Alternatively, we may also describe $Pos(\prec)$ as the set of all the formulas in $\mathcal{V}(\mathcal{L})$ which are conjunctions of formulas of the kind $\bigvee(\mathbf{P}_a) \vee \bigvee(\mathbf{Q}_l) \vee (\bigvee(\mathbf{Z}_l) \wedge \bigwedge(\mathbf{P} - \mathbf{P}_a))$.

   If $\mathcal{V}(\mathcal{L})$ is finite, we may consider only the sets of literals $\mathbf{Q}_l$ complete in $\mathbf{Q}$ and $\mathbf{Z}_l$ complete in $\mathbf{Z}$.

3. C_μ is a closed set iff $\mathbf{P}$ is finite.

4a. $Pos_r(\prec) = Pos(\prec)$, i.e. $Pos_r(\Phi) = Pos_e(\Phi)$.

   Thus $\prec = \prec_\Phi = \prec_{Pos_m(\Phi)} = \prec_{Pos_e(\Phi)}$, i.e. $CIRCF(\Phi) = CIRCF(Pos_m(\Phi)) = CIRCF(Pos_e(\Phi))$.

4b. If $\mathbf{P}$ is infinite, or if $\mathbf{Z} = \emptyset$, then $Pos_r(\prec) = Pos(\prec) = Pos(\preceq) = \Phi^{\wedge\vee}$.

□

Proof:

1a. Let us suppose $\varphi \in Pos(\prec) = I_\prec$ and $\varphi = \varphi_1 \vee \cdots \vee \varphi_n$ a *reduced disjunctive normal form* of φ. This means that each φ_i is a conjunction of literals, the φ_i's are all distinct and, if φ' is a conjunction of literals such that $\varphi_i \models \varphi'$ and $\varphi_i \neq \varphi'$, then $\varphi' \not\models \varphi$.

We suppose also that $\neg P$ appears in φ_i, for some $P \in \mathbf{P}$. We call φ'_i the conjunction (eventually $\top$) of all the other literals of φ_i. Let μ be a model of φ'_i. If $\mu \not\models P$, then $\mu \models \varphi_i$ thus $\mu \models \varphi$. Otherwise, with $\nu = \mu - \{P\}$, we have $\nu \prec \mu$ and $\nu \models \varphi_i$ thus $\nu \models \varphi$ and, as $\varphi \in Pos(\prec)$, again $\mu \models \varphi$. Thus, we have $\mathcal{M}(\varphi'_i) \subseteq \mathcal{M}(\varphi)$, a contradiction with the assumption that we started from a reduced form of φ.

1b. This comes from $Pos(\preceq) = Pos_m(\Phi) = \Phi^{\wedge\vee}$ and $Pos_m(\Phi) \subseteq Pos_e(\Phi) = Pos(\prec)$ (proposition 5.29-4 and -5).

Let us give also a direct proof of the first part of the result, as an illustration of what happens in the case of propositional circumscription:

We suppose $\varphi \in (\mathbf{P} \cup \mathbf{Q} \cup \neg\mathbf{Q})^{\wedge\vee}$, i.e. $\mathcal{V}(\varphi) \subseteq \mathbf{P} \cup \mathbf{Q}$ and φ is positive in $\mathbf{P}$ in the traditional meaning.

Let $\varphi_1 \vee \cdots \vee \varphi_n$ be a reduced disjunctive normal form of φ.

If $\varphi = \varphi_1$, φ is a conjunction of atoms in $\mathbf{P}$ and (possibly) other literals from $\mathbf{Q}$. Let μ be a model of φ and $\mu \prec \mu'$. Then, we define $\mu'' = (\mu' \cap (\mathbf{P} \cup \mathbf{Q})) \cup (\mu \cap \mathbf{Z})$. We have $\mu \subseteq \mu''$ thus, from the nature of φ, we have also $\mu'' \models \varphi$ and, as φ does not contain any $Z \in \mathbf{Z}$, and as $\mu' \cap (\mathcal{V}(\mathcal{L}) - \mathbf{Z}) = \mu'' \cap (\mathcal{V}(\mathcal{L}) - \mathbf{Z})$, $\mu' \models \varphi$.

If φ is a disjunction of φ_i's, each of these φ_i is in $Pos(\prec)$ from the preceding case, and, as
$Pos(\prec) = I_\prec$ is closed for $\vee$, $\varphi \in Pos(\prec)$.

1c. If $\mathbf{Z} = \emptyset$, then $\mathcal{V}(\varphi) \subseteq \mathbf{P} \cup \mathbf{Q}$ and we have already the conclusion from 1a and 1b.

We suppose now that $\mathbf{P}$ is infinite and $\mathbf{Z} \neq \emptyset$. From 1a and 1b we know already that if $\mathcal{V}(\varphi) \subseteq \mathbf{P} \cup \mathbf{Q}$, we have $\varphi \in Pos(\prec)$ iff φ is positive in $\mathbf{P}$. We suppose now $\mathcal{V}(\varphi) \not\subseteq \mathbf{P} \cup \mathbf{Q}$, and $\varphi_1 \vee \cdots \vee \varphi_n$ is a reduced normal disjunctive form of $\varphi \in Pos(\prec)$. Let Z be some element in $\mathbf{Z}$ appearing in φ_i. Let us call φ'_i the conjunction (eventually $\top$) of all the literals of φ_i which do not contain an element of $\mathbf{Z}$. Let P be some element of $\mathbf{P}$ not appearing in φ. Let μ be a model of φ'_i, then there exists a model μ' of φ_i such that $\mu' \cap (\mathbf{P} \cup \mathbf{Q}) = \mu \cap (\mathbf{P} \cup \mathbf{Q})$. Let us define also $\nu = \mu \cup \{P\}$. Then, $\mu' \prec \nu$ thus $\nu \models \varphi$. P does not appear in φ and $(\mu \cup \nu) - (\mu \cap \nu) = \{P\}$, thus $\mu \models \varphi$. Thus, $\mathcal{M}(\varphi'_i) \subseteq \mathcal{M}(\varphi)$, which contradicts the fact that we started from a reduced form of φ.

2. $\mathbf{P}$ is finite, thus, for any $\mathbf{P}_a \subseteq \mathbf{P}$, $\mathbf{P}_a$ and $\mathbf{P} - \mathbf{P}_a$ are finite.

a) If $\varphi = \bigwedge(\mathbf{P}_a) \wedge \bigwedge(\mathbf{Q}_l) \wedge (\bigwedge(\mathbf{Z}_l) \vee \bigvee(\mathbf{P} - \mathbf{P}_a))$, for some subset $\mathbf{P}_a$ of $\mathbf{P}$ and some finite consistent sets $\mathbf{Q}_l$ and $\mathbf{Z}_l$ made of literals of $\mathbf{Q}$ and of $\mathbf{Z}$ respectively, then $\varphi \in Pos(\prec)$:

Let $\mu \models \varphi$ and $\mu \prec \nu$. Then, as $\mu \cap \mathbf{P} \subseteq \nu \cap \mathbf{P}$, $\nu \models \bigwedge(\mathbf{P}_a)$. Also, as $\mu \cap \mathbf{Q} = \nu \cap \mathbf{Q}$, $\nu \models \bigwedge(\mathbf{Q}_l)$. Finally, as $\mu \cap \mathbf{P} \subset \nu \cap \mathbf{P}$, $\nu \models \bigvee(\mathbf{P} - \mathbf{P}_a)$ and we get $\nu \models \varphi$. This establishes $\varphi \in Pos(\prec)$.

Thus, as $Pos(\prec)$ is stable for $\vee$, any disjunction of such formulas is in $Pos(\prec)$.

b) Let us suppose $\varphi \in Pos(\prec)$ and $\varphi_1 \vee \cdots \vee \varphi_n$ is a reduced normal disjunctive form of φ.

We may write φ_i under the form $\bigwedge(\mathbf{P}_a) \wedge \bigwedge(\mathbf{Q}_l) \wedge \bigwedge(\mathbf{Z}_l)$ for some subset $\mathbf{P}_a$ of $\mathbf{P}$ and some finite consistent sets $\mathbf{Q}_l$ and $\mathbf{Z}_l$ of literals respectively from $\mathbf{Q}$ and from $\mathbf{Z}$. We define $\varphi'_i = \bigwedge(\mathbf{P}_a) \wedge \bigwedge(\mathbf{Q}_l) \wedge \bigvee(\mathbf{P} - \mathbf{P}_a)$ if $\mathbf{P}_a \neq \mathbf{P}$ (if $\mathbf{P}_a = \mathbf{P}$, φ_i is already as we want it). We suppose also $\varphi_i \neq \top$ (otherwise φ is already as we want it, being an empty disjunction). Let $\nu \in \mathcal{M}(\varphi'_i)$. Let $\mu \in \mathcal{M}(\varphi_i)$, we define $\mu' = \mathbf{P}_a \cup (\nu \cap \mathbf{Q}) \cup (\mu \cap \mathbf{Z})$. Then $\mu' \cap \mathbf{P} = \mathbf{P}_a \subset \nu \cap \mathbf{P}$ and $\mu' \cap \mathbf{Q} = \nu \cap \mathbf{Q}$, thus $\mu' \prec \nu$. Also, $\mu' \models \varphi_i$ from the definition of μ', thus $\mu' \models \varphi$ and, as $\varphi \in Pos(\prec)$, $\nu \models \varphi$. Thus, $\mathcal{M}(\varphi_i) \subseteq \mathcal{M}(\varphi)$. We define $\varphi' = \varphi \vee (\bigvee_{i=1}^{n} \varphi'_i)$, we get then $\mathcal{M}(\varphi) = \mathcal{M}(\varphi')$, i.e. $\varphi = \varphi'$, thus $\varphi = \bigvee_{i=1}^{n}(\varphi_i \vee \varphi'_i)$. Now, $\varphi_i \vee \varphi'_i = \bigwedge(\mathbf{P}_a) \wedge \bigwedge(\mathbf{Q}_l) \wedge (\bigwedge(\mathbf{Z}_l) \vee \bigvee(\mathbf{P} - \mathbf{P}_a))$: φ can be written as a disjunction of formulas of the form indicated in the present proposition.

Now, if $\mathcal{V}(\boldsymbol{\mathcal{L}}) = \mathbf{P} \cup \mathbf{Q} \cup \mathbf{Z}$ is finite, it is obvious that we may restrict our attention to the formulas $\bigwedge(\mathbf{Q}_l)$ and $\bigwedge(\mathbf{Z}_l)$ corresponding to sets of literals $\mathbf{Q}_l$ and $\mathbf{Z}_l$ which are complete respectively in $\mathbf{Q}$ and in $\mathbf{Z}$. Indeed, any formula $\bigwedge(\mathbf{Q}'_l)$, for any set of literals $\mathbf{Q}'_l$ from $\mathbf{Q}$, is the disjunction of some formulas $\bigwedge(\mathbf{Q}_l)$, where all the $\mathbf{Q}_l$'s of the disjunction are sets of literals complete in $\mathbf{Q}$. And the same applies for $\mathbf{Z}$.

But we may prove more in this case (see also [MR98]): the set of all the formulas $\bigwedge(\mathbf{P}_a) \wedge \bigwedge(\mathbf{Q}_l) \wedge (\bigwedge(\mathbf{Z}_l) \vee \bigvee(\mathbf{P} - \mathbf{P}_a))$, for all the subsets $\mathbf{P}_a$ of $\mathbf{P}$ and all the sets of literals $\mathbf{Q}_l$ complete for $\mathbf{Q}$ and $\mathbf{Z}_l$ complete for $\mathbf{Z}$ defines an $\vee$-*basis* of the set $Pos(\prec)$. This means that this set is the (unique) smallest (for $\subseteq$) set $\mathcal{X}$ such that $Pos(\prec) = \mathcal{X}^{\vee}$.

We will prove this fact (announced and proved in an partially informal way in [MR98]). For any set of formulas $\mathcal{X}$, we call (as in [MR98]) $\mathcal{X}_{\vee}$ the set $\{\varphi \;/\; \varphi \in \mathcal{X}, \varphi \notin (X - \{\varphi\})^{\vee}\}$.

We have that $\mathcal{X}_\vee$ is the smallest set $\mathcal{Y}$ such that $\mathcal{X}^\vee = \mathcal{Y}^\vee$, precisely: $\mathcal{X}^\vee = (\mathcal{X}_\vee)^\vee$ and if $\mathcal{X}^\vee = \mathcal{Y}^\vee$ then $\mathcal{X}_\vee \subseteq \mathcal{Y}$. We call $X_\vee$ the $\vee$-basis of the set of formulas $\mathcal{X}$. Thus, we will prove that $(Pos(\prec))_\vee$ is the set described above.

Let us call $\mathcal{X}$ this set, i.e. the set of all the formulas $\bigwedge(\mathbf{P}_a)\wedge\bigwedge(\mathbf{Q}_l)\wedge(\bigwedge(\mathbf{Z}_l)\vee\bigvee(\mathbf{P}-\mathbf{P}_a))$, for all the subsets $\mathbf{P}_a$ of $\mathbf{P}$ and all the sets of literals $\mathbf{Q}_l$ complete for $\mathbf{Q}$ and $\mathbf{Z}_l$ complete for $\mathbf{Z}$.

As already noticed, it is clear that we have $\mathcal{X}^\vee = Pos(\prec)$ $(= I_{f_\prec})$.

Let us suppose $\varphi \in \mathcal{X}$, $\varphi = \varphi_1 \vee \cdots \vee \varphi_n$ where each φ_i is distinct from φ and is in $\mathcal{X}$, and also $\varphi = \bigwedge(\mathbf{P}_a) \wedge \bigwedge(\mathbf{Q}_l) \wedge (\bigwedge(\mathbf{Z}_l) \vee \bigvee(\mathbf{P} - \mathbf{P}_a))$, for some subset $\mathbf{P}_a$ of $\mathbf{P}$ and two sets of literals $\mathbf{Q}_l$ complete for $\mathbf{Q}$ and $\mathbf{Z}_l$ complete for $\mathbf{Z}$. Let μ be such that $\mu \cap \mathbf{P} = \mathbf{P}_a$, $\mu \models \bigwedge(\mathbf{Q}_l)$ and $\mu \models \bigwedge(\mathbf{Z}_l)$. Then, $\mu \models \varphi$ thus there exists $i \in \{1, \cdots, n\}$ such that $\mu \models \varphi_i = \bigwedge(\mathbf{P}'_a) \wedge \bigwedge(\mathbf{Q}'_l) \wedge (\bigwedge(\mathbf{Z}'_l) \vee \bigvee(\mathbf{P} - \mathbf{P}'_a))$ where $\mathbf{P}'_a \subseteq \mathbf{P}$ and $\mathbf{Q}_l$ and $\mathbf{Z}_l$ are two sets of literals complete for $\mathbf{Q}$ and $\mathbf{Z}_l$ respectively.

As $\mu \cap \mathbf{P} = \mathbf{P}_a$ and $\mu \models \bigwedge(\mathbf{P}'_a)$, we get $\mathbf{P}'_a \subseteq \mathbf{P}_a$.

As $\mu \models \bigwedge(\mathbf{Q}_l)$ and $\mu \models \bigwedge(\mathbf{Q}'_l)$, we get $\mathbf{Q}_l = \mathbf{Q}'_l$.

If $\mathbf{P}_a = \mathbf{P}'_a$, $\mu \not\models \bigvee(\mathbf{P} - \mathbf{P}'_a)$ thus $\mu \models \bigwedge(\mathbf{Z}'_l)$ and $\mathbf{Z}_l = \mathbf{Z}'_l$. In this case we get $\varphi = \varphi_i$, a contradiction with our hypothesis.

Thus, we must have $\mathbf{P}'_a \subset \mathbf{P}_a$. Let us define $\mu' = \mu - (\mathbf{P}_a - \mathbf{P}'_a)$, we have $\mu' \models \varphi_i$, thus $\mu' \models \varphi$ and $\mu' \models \bigwedge(\mathbf{P}_a)$, a contradiction.

This shows that our hypothesis does not hold, i.e. no formula in $\mathcal{X}$ may be a disjunction of at least two different formulas in $\mathcal{X}$: $\mathcal{X}$ is the $\vee$-basis of $Pos(\prec)$.

The results for conjunctions instead of disjunction are proved in the same way (the duality is perfect here). In particular, if $\mathcal{V}(\boldsymbol{\mathcal{L}})$ is finite, we establish in the same way that the set $\mathcal{Y}$ of all the formulas $\bigvee(\mathbf{P}_a) \vee \bigvee(\mathbf{Q}_l) \vee (\bigvee(\mathbf{Z}_l) \wedge \bigwedge(\mathbf{P} - \mathbf{P}_a))$, for any set $\mathbf{P}_a \subseteq \mathbf{P}$ and any sets of literals $\mathbf{Q}_l$ and $\mathbf{Z}_l$, complete in $\mathbf{Q}$ and in $\mathbf{Z}$ respectively, is the $\wedge$-*basis* of $Pos(\prec)$: $(Pos(\prec))_\wedge = \mathcal{Y}$ (replace $\vee$ by $\wedge$ in above definition of the $\vee$-basis). This means that $\mathcal{Y}^\wedge = Pos(\prec)$ and a set $\mathcal{X}$ is such that $\mathcal{X}^\wedge = Pos(\prec)$ iff $(Pos(\prec))_\wedge \subseteq \mathcal{X} \subseteq Pos(\prec)$. Any such set $\mathcal{X}$ can be taken as the set X such that $CIRC(\mathbf{P}, \mathbf{Q}, \mathbf{Z}) = f_X$ (see definition 6.34 and proposition 6.39 below, and [MR98]).

3. Let us suppose $\mu \cap \mathbf{P}$ finite. Then $m_\mu = \mathcal{M}(\{\bigvee_{P \in \mu \cap \mathbf{P}} \neg P\} \cup \{\neg P \ / \ P \in \mathbf{P} - \mu\} \cup (\mu \cap \mathbf{Q}) \cup \{\neg Q \ / \ Q \in \mathbf{Q} - \mu\}$. Thus, m_μ is a closed set, being the set of all the models of some theory.

Similarly, if $\mathbf{P} - \mu = \mathbf{P} - (\mu \cap \mathbf{P})$ is finite, then M_μ is a closed set. Thus, if $\mathbf{P}$ is finite, $C_\mu = m_\mu \cup M_\mu$ is a closed set.

Let us suppose $\mu \cap \mathbf{P}$ infinite, for example $\{P_n\}_{n \in \mathbb{N}} \subseteq \mu \cap \mathbf{P}$. We define the interpretations $\mu_n = \mu - \{P_n\}$. Then we have $\lim_{n \to \infty} \mu_n = \mu$ and $\mu_n \in m_\mu$. Thus $\mu \in TC(m_\mu)$.

Similarly, if $\mathbf{P} - \mu$ is infinite, then $\mu \in TC(M_\mu)$. Thus, as clearly $\mu \notin C_\mu$, if $\mathbf{P}$ is infinite, we get $\mu \in TC(C_\mu) - C_\mu$: C_μ is not a closed set.

4a. We know from proposition 6.11-1 above that it suffices to prove $Pos(\prec) \subseteq Pos_r(\prec)$.

Let φ be in $Pos(\prec)$, and two interpretations be such that $\mu \overline{\prec} \nu$, $\mu \not\prec \nu$, and $\nu \models \varphi$. As $(\mathbf{P} \cup \mathbf{Q} \cup \neg\mathbf{Q}) \subseteq Pos(\prec)$, from proposition 6.9-1 we get $(\mathbf{P} \cup \mathbf{Q} \cup \neg\mathbf{Q})_\mu \subseteq (\mathbf{P} \cup \mathbf{Q} \cup \neg\mathbf{Q})_\nu$, i.e. $\mu \cap \mathbf{P} \subseteq \nu \cap \mathbf{P}$ and $\mu \cap \mathbf{Q} = \nu \cap \mathbf{Q}$. As $\mu \not\prec \nu$. we must have in fact $\mu \cap \mathbf{P} = \nu \cap \mathbf{P}$. We split now the proof in two cases.

Case 1: Let us suppose $\mathbf{P}$ finite.

We define $\psi = \bigwedge(\mu \cap \mathbf{P}) \wedge \bigwedge(\neg(\mathbf{P} - \mu))$. Then $\mu \in \mathcal{M}(\psi), \nu \in \mathcal{M}(\psi)$ and, as $\mu \overline{\prec} \nu$, there exist μ' and ν' in $\mathcal{M}(\psi)$ such that $\mu' \prec \nu'$. Thus, $\mu' \cap \mathbf{P} \subset \nu' \cap \mathbf{P}$, a contradiction with $\mu' \models \psi$ and $\nu' \models \psi$ which forces $\mu' \cap \mathbf{P} = \nu' \cap \mathbf{P}$. Thus we cannot have our hypothesis if $\mathbf{P}$ is finite.

Case 2: $\mathbf{P}$ is infinite here.

From 1c. above, we already know that we have: $Pos(\prec) = Pos(\preceq) = (\mathbf{P} \cup \mathbf{Q} \cup \neg\mathbf{Q})^{\wedge\vee}$. Thus, it is obvious that we have $\mu \models \varphi$[16].

Cases 1 and 2: Thus, in any case, $\varphi \in Pos_r(\prec)$ from definition 6.10. This proves: $Pos(\prec) \subseteq Pos_r(\prec)$, thus $Pos(\prec) = Pos_r(\prec)$, i.e. $Pos_r(\Phi) = Pos_e(\Phi)$.

We deduce now easily $\prec = \prec_\Phi = \prec_{Pos_m(\Phi)} = \prec_{Pos_e(\Phi)}$. Indeed, we know already $\prec = \prec_\Phi = \prec_{Pos_m(\Phi)}$ from propositions 5.29-5 and 5.33-2a and $\prec = \prec_{Pos_r(\prec)}$ from proposition 6.16, and we have just proved $Pos_e(\Phi) = Pos(\prec) = Pos_r(\prec)$.

4b. If P is infinite or $\mathbf{Z} = \emptyset$, we know from 1c above that $Pos(\prec) = Pos(\preceq) = (\mathbf{P} \cup \mathbf{Q} \cup \neg\mathbf{Q})^{\wedge\vee}$. From 4a we know $Pos_r(\prec) = Pos(\prec)$. □

Let us give here a small additional property, in order to get a better understanding of the structure of one of the sets considered:

Remark 6.33 Let S be the set of the formulas positive in $\mathbf{P}$ (in the traditional meaning) and made in the vocabulary $\mathbf{P} \cup \mathbf{Q}$, i.e. $S = (\mathbf{P} \cup \mathbf{Q} \cup \neg\mathbf{Q})^{\wedge\vee}$.

If $\mathbf{P}$ is infinite, then, if we have two distinct formulas φ and ψ in S such that $\varphi \models \psi$, we may always find a third formula φ' in S, distinct from φ and from ψ, such that $\varphi \models \varphi'$ and $\varphi' \models \psi$. Indeed, as $\mathbf{P}$ is infinite, there exists some symbol $P \in \mathbf{P}$ which appears neither in φ nor in ψ and we may take the formula $\varphi \vee (\psi \wedge P)$ as our formula φ'.

This result is clearly no longer true if $\mathbf{P}$ is finite: Taking e.g $\mathbf{P} = \{P_1, P_2\}, \mathbf{Q} = \{Q\}$, and any set for $\mathbf{Z}$, we may consider for instance $\varphi = P_1 \wedge \neg Q$ and $\psi = (P_1 \vee P_2) \wedge \neg Q$. We have $\varphi \in S$, $\psi \in S$, $\varphi \neq \psi$ and $\varphi \models \psi$, and there exists no formula $\varphi' \in S$, distinct from φ and ψ, such that $\varphi \models \varphi'$ and $\varphi' \models \psi$. □

6.3 Propositional circumscription from what they cannot do

We give now in this "technical section" the complete proof of a result already announced in [MR98] (see preliminary discussion above, after example 4.20). We examine precisely

[16] We can show in the same way that more generally, for any formula circumscriptions $CIRCF(\Phi)$ (i.e. for any set of formulas Φ), if $Pos(\prec_\Phi) = Pos(\preceq_\Phi)$, then $Pos_r(\prec_\Phi) = Pos(\prec_\Phi) = Pos(\preceq_\Phi)$.

when (and how) propositional circumscriptions may be expressed easily in terms of their inaccessible formulas.

We need a definition and a few results.

Definition 6.34 [SF96, MR98] An *X-mapping* f is some mapping from $\mathcal{J}$ to $\mathcal{L}$ such that there exists some subset X of $\mathcal{L}$ such that $\varphi \in f(\mathcal{T})$ iff $(\mathcal{T} \sqcup \varphi) \cap X \subseteq Th(\mathcal{T})$.

The X-mapping f associated to a set of formulas X will be noted f_X.

If f_X is the X-mapping defined by the set of formulas X, we note I_X for I_{f_X} (see definition 5.28-1 for the definition of I_f).

As with pre-circumscriptions, if $\mathcal{T} \notin \mathcal{J}$, we will define $f_X(\mathcal{T})$ as $f_X(Th(\mathcal{T}))$, thus X-mappings may be also considered as mappings from $\mathcal{L}$ to $\mathcal{L}$. □

Remarks 6.35 1. Beware that X-mappings are generally not pre-circumscriptions. It is immediate to show that they respect extensivity: for any $\mathcal{T}$, we have $\mathcal{T} \subseteq f_X(\mathcal{T})$ thus, as $f_X(\mathcal{T}) = f_X(Th(\mathcal{T}))$, $Th(\mathcal{T}) \subseteq f_X(\mathcal{T})$. However, $f_X(\mathcal{T})$ is generally not deductively closed. Indeed, it is easy to show that $f_X(\mathcal{T})$ is a union of (possibly more than one) elements of $\mathcal{J}$, thus f_X may be considered as a mapping f from $\mathcal{J}$ to the set of the unions of elements of $\mathcal{J}$ (see [MR98] where these mappings are called *S-mappings* when, as here, they satisfy $\mathcal{T} \subseteq f(\mathcal{T})$).

2. Let us remind here a result from [MR98], which completes a comment made above in the proof of proposition 5.29-1: If $\mathcal{V}(\mathcal{L})$ is finite, and if the pre-circumscription f is an X-mapping, then if I_f is also stable for $\vee$, f is a preferential entailment. □

Proposition 6.36 Any X-mapping f_X satisfies the following properties, where $X, \mathcal{T}, \mathcal{T}'', \mathcal{T}'''$ are subsets of $\mathcal{L}$ (not necessarily in $\mathcal{J}$) while $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_i$ are in $\mathcal{J}$.

1. *strong cumulative transitivity*: If $\mathcal{T}_1 \subseteq f_X(\mathcal{T})$ and $\mathcal{T}_2 \subseteq f_X(\mathcal{T} \cup \mathcal{T}_1)$ then $\mathcal{T}_1 \sqcup \mathcal{T}_2 \subseteq f_X(\mathcal{T})$ **(sCT)**.

2. [SF96] *cumulative transitivity* for general mappings from $\mathcal{L}$ to $\mathcal{L}$:

   If $\mathcal{T} \subseteq \mathcal{T}'' \subseteq f_X(\mathcal{T})$ then $f_X(\mathcal{T}'') \subseteq f_X(\mathcal{T})$ $(\mathbf{CT}_g)$

   Thus, when an X-mapping is a pre-circumscription, it satisfies (CT) as we have defined it in definition 3.2.

3. $\bigcap_{i \in I} f_X(\mathcal{T}_i) \subseteq f_X(\bigcap_{i \in I} \mathcal{T}_i)$.

   Thus, when an X-mapping is a pre-circumscription, it satisfies (CR∞) as we have defined it in definition 3.2.

4. *restricted cumulative monotony*: If $\mathcal{T} \subseteq \mathcal{T}'' \subseteq \mathcal{T}_1 \subseteq f_X(\mathcal{T})$, then $\mathcal{T}_1 \subseteq f_X(\mathcal{T}'')$ **(rCM)** .

   This is a kind of "(CM) restricted to the theories".

5. An X-mapping is a pre-circumscription iff it satisfies (CM), iff it satisfies (CM1).

   [SF96] Thus, any X-mapping which is a pre-circumscription satisfies (CUMU). □

A small bibliographical comment: Point 2 and the second sentence of point 5 are from [SF96]. Points 1 and 4 appear in [MR98], and our "restricted cumulative monotony" is different from the restricted cumulativity as introduced in [SF96][17].

Proof:

0. Let us give a small obvious "lemma", useful for our proofs:
From definition 6.34, we get, for any $\mathcal{T}$, $f_X(\mathcal{T}) \cap X \subseteq Th(\mathcal{T}) \cap X$.

1. We suppose $\mathcal{T}_1 \subseteq f_X(\mathcal{T})$ and $\mathcal{T}_2 \subseteq f_X(\mathcal{T} \cup \mathcal{T}_1)$, where $\mathcal{T}_1$ and $\mathcal{T}_2$ are in $\mathfrak{T}$. Let us suppose $\varphi \in \mathcal{T}_1 \sqcup \mathcal{T}_2$. Then there exist $\varphi_1 \in \mathcal{T}_1$ and $\varphi_2 \in \mathcal{T}_2$ such that $\varphi = \varphi_1 \wedge \varphi_2$, thus $\mathcal{T} \sqcup \varphi \subseteq \mathcal{T} \sqcup \mathcal{T}_1 \sqcup \varphi_2$, from $\mathcal{T}_2 \subseteq f_X(\mathcal{T} \cup \mathcal{T}_1)$, we know that $(\mathcal{T} \sqcup \mathcal{T}_1 \sqcup \varphi_2) \cap X \subseteq \mathcal{T} \sqcup \mathcal{T}_1$, thus $(\mathcal{T} \sqcup \varphi) \cap X \subseteq (\mathcal{T} \sqcup \mathcal{T}_1) \cap X$. Now, for any $\psi \in \mathcal{T} \sqcup \mathcal{T}_1$ there exists $\varphi_1' \in \mathcal{T}_1$ such that $\psi \in \mathcal{T} \sqcup \varphi_1'$, from $\mathcal{T}_1 \subseteq f_X(\mathcal{T})$ we know that $(\mathcal{T} \sqcup \varphi_1') \cap X \subseteq Th(\mathcal{T})$, thus $(\mathcal{T} \sqcup \mathcal{T}_1) \cap X \subseteq Th(\mathcal{T})$, thus $(\mathcal{T} \sqcup \varphi) \cap X \subseteq Th(\mathcal{T})$. i.e. $\varphi \in f_X(\mathcal{T})$. Thus $\mathcal{T}_1 \sqcup \mathcal{T}_2 \subseteq f_X(\mathcal{T})$: f_X satisfies (sCT).

We show now that a consequence of (sCT) when f_X is a pre-circumscription is (CT). This proof is useless as anyway f_X satisfies (CT_g) from point 2 proved below, and in this case (CT_g) is (CT), but we give it in order to justify the name "strong cumulative transitivity". We think in fact that for mappings from $\mathcal{T}$ (or $\boldsymbol{\mathcal{L}}$) to $\boldsymbol{\mathcal{L}}$ such as the X-mappings, the "interesting property" corresponding to the idea of (CT) is (sCT).

We suppose now that f_X is a pre-circumscription and that $\mathcal{T} \subseteq \mathcal{T}'' \subseteq f_X(\mathcal{T})$. We have $\mathcal{T}_1 = Th(\mathcal{T}'') \in \mathfrak{T}$ and, as $f_X(\mathcal{T}) \in \mathfrak{T}$, we get $\mathcal{T}_1 \subseteq f_X(\mathcal{T})$. $Th(\mathcal{T}'') = \mathcal{T} \sqcup \mathcal{T}_1$. Let us define also $\mathcal{T}_2 = f_X(\mathcal{T}'') = f_X(\mathcal{T} \sqcup \mathcal{T}'')$, we have $\mathcal{T}_2 \in \mathfrak{T}$ because f_X is a pre-circumscription. Thus by (sCT), we get $\mathcal{T}_1 \sqcup \mathcal{T}_2 \subseteq f_X(\mathcal{T})$, i.e. $\mathcal{T}'' \sqcup f_X(\mathcal{T}'') \subseteq f_X(\mathcal{T})$. As $\mathcal{T}'' \subseteq f_X(\mathcal{T}'')$ from remark 6.35-1, we get $\mathcal{T}'' \sqcup f_X(\mathcal{T}'') = f_X(\mathcal{T}'')$, thus $f_X(\mathcal{T}'') \subseteq f_X(\mathcal{T})$: f_X satisfies (CT).

2. We suppose $\mathcal{T} \subseteq \mathcal{T}'' \subseteq f_X(\mathcal{T})$ and $\varphi \in f_X(\mathcal{T}'')$. Then $(\mathcal{T}'' \sqcup \varphi) \cap X \subseteq Th(\mathcal{T}'')$ and, from $\mathcal{T} \subseteq \mathcal{T}''$, $(\mathcal{T} \sqcup \varphi) \cap X \subseteq Th(\mathcal{T}'') \cap X$. From $\mathcal{T}'' \subseteq f_X(\mathcal{T})$ we get $Th(\mathcal{T}'') \cap X \subseteq f_X(\mathcal{T}) \cap X$ and from point 0 above $f_X(\mathcal{T}) \cap X \subseteq Th(\mathcal{T}) \cap X$. Thus $(\mathcal{T} \sqcup \varphi) \cap X \subseteq Th(\mathcal{T})$: $\varphi \in f_X(\mathcal{T})$. Thus $f_X(\mathcal{T}'') \subseteq f_X(\mathcal{T})$: f_X satisfies (CT_g). If f_X is a pre-circumscription, we get (CT) as we have defined it in definition 3.2.

3. Let us suppose $\varphi \in \bigcap_{i \in I} f_X(\mathcal{T}_i)$, then for any $i \in I$, $(\mathcal{T}_i \sqcup \varphi) \cap X \subseteq \mathcal{T}_i$, thus $\bigcap_{i \in I}((\mathcal{T}_i \sqcup \varphi) \cap X) \subseteq \bigcap_{i \in I} \mathcal{T}_i$, thus $(\bigcap_{i \in I}(\mathcal{T}_i \sqcup \varphi)) \cap X \subseteq \bigcap_{i \in I} \mathcal{T}_i$, thus $((\bigcap_{i \in I} \mathcal{T}_i) \sqcup \varphi) \cap X \subseteq \bigcap_{i \in I} \mathcal{T}_i$, i.e. $\varphi \in f_X(\bigcap_{i \in I} \mathcal{T}_i)$.

[17] One way of this badly named restricted cumulativity is a variation around the property called LLE in [KLM90]: $f_X(\mathcal{T}) = f_X(Th(\mathcal{T}))$. We do not have elucidated what is the other way, which has nothing to do with (CT_g) or (rCM).

When f_X is a pre-circumscription, we have called this property (CR∞).

4. We suppose $\mathcal{T} \subseteq \mathcal{T}'' \subseteq \mathcal{T}_1 \subseteq f_X(\mathcal{T})$, then for any $\varphi \in \mathcal{T}_1$ we have $(\mathcal{T} \sqcup \varphi) \cap X \subseteq Th(\mathcal{T})$. Let us suppose $\varphi \in \mathcal{T}_1$ and $x \in (\mathcal{T}'' \sqcup \varphi) \cap X$, then, as $\mathcal{T}'' \subseteq \mathcal{T}_1$, $x \in \mathcal{T}_1 \sqcup \varphi = \mathcal{T}_1$ and, as $\mathcal{T}_1 \subseteq f_X(\mathcal{T})$, $x \in f_X(\mathcal{T}) \cap X$, thus $x \in Th(\mathcal{T}) \cap X$ from above result. As $\mathcal{T} \subseteq \mathcal{T}''$, we get $x \in Th(\mathcal{T}'')$. This establishes $(\mathcal{T}'' \sqcup \varphi) \cap X \subseteq Th(\mathcal{T}'')$, i.e. $\varphi \in f_X(\mathcal{T}'')$: f_X satisfies (rCM).

5. We suppose that f_X is a pre-circumscription here and also $\mathcal{T} \subseteq \mathcal{T}'' \subseteq f_X(\mathcal{T})$. As f_X is a pre-circumscription, $\mathcal{T}_1 = f_X(\mathcal{T})$ is in $\mathscr{T}$ and we have $\mathcal{T} \subseteq \mathcal{T}'' \subseteq \mathcal{T}_1 \subseteq f_X(\mathcal{T})$, we apply (rCM) and we get $f_X(\mathcal{T}'') \subseteq f_X(\mathcal{T})$: f_X satisfies (CM). Thus f_X satisfies (CM1).

Conversely, let us suppose that f_X is an X-mapping satisfying (CM1) (more precisely, the writing we have given for (CM1) in definitions 3.3, applied here to the mapping f_X which is not supposed to be necessarily a pre-circumscription, we should write "(CM1_g)" but we do not think this is necessary here). Then, from point 2, we know that f_X satisfies (CT$_g$). Notice that it is not immediate to get that f_X satisfies thus (CUMU1) (same remark as for (CM1)). We need a short proof because f_X is not supposed to be a pre-circumscription: if $\varphi \in f_X(\mathcal{T})$, then $\mathcal{T} \sqcup \varphi \in f_X(\mathcal{T})$, thus we get (CT$1_g$): if $\varphi \in f_X(\mathcal{T})$, then $f_X(\mathcal{T} \sqcup \varphi) \subseteq f_X(\mathcal{T})$. Together with (CM1), this gives (CUMU1): if $\varphi \in f_X(\mathcal{T})$ then $f_X(\mathcal{T}) = f_X(\mathcal{T} \sqcup \varphi)$.

Let us suppose $\varphi \in f_X(\mathcal{T})$ and $\psi \in f_X(\mathcal{T})$. Then, from (CUMU1), $f_X(\mathcal{T}) = f_X(\mathcal{T} \sqcup \varphi)$ thus $\psi \in f_X(\mathcal{T} \sqcup \varphi)$. Thus, using (CUMU1) again, $f_X((\mathcal{T} \sqcup \varphi) \sqcup \psi) = f_X(\mathcal{T} \sqcup \varphi) = f_X(\mathcal{T})$. As $(\mathcal{T} \sqcup \varphi) \sqcup \psi = \mathcal{T} \sqcup \varphi \wedge \psi$, and as from remark 6.35-1 we have $\varphi \wedge \psi \in f_X(\mathcal{T} \sqcup \varphi \wedge \psi)$, we get $\varphi \wedge \psi \in f_X(\mathcal{T})$. As it is obvious from definition 6.34 that if $\varphi \in f_X(\mathcal{T})$ then $Th(\varphi) \subseteq f_X(\mathcal{T})$, this establishes $f_X(\mathcal{T}) \in \mathscr{T}$: f_X is a pre-circumscription. $\square$

A consequence is that a pre-circumscription which may be expressed as an X-mapping must satisfy (CUMU) and (CR∞).

Here is an easy lemma:

Lemma 6.37 For any pre-circumscription f, and any $\mathcal{T} \in \mathscr{T}$, we have $f(\mathcal{T}) \subseteq f_{I_f}(\mathcal{T})$. $\square$

Proof: f_{I_f} is the X-mapping (definition 6.34) defined by the set I_f (definition 5.28-1).

This result is an immediate consequence of the definitions of X-mappings f_X and of the set I_f of the inaccessible formulas for a pre-circumscription f. $\square$

Proposition 6.38 For any X-mapping, we have $f = f_{I_f}$. Moreover, I_f is the greatest (for $\subseteq$) set Y such that $f = f_Y$. $\square$

X-mappings may then be considered as mappings based upon their inaccessible formulas, and let us say then that the "X" in the name X-mapping comes from "InaCCeSSible"...

Proof: Notice that this result is a consequence of theorem 3.2 of [MR98]. We give an easier direct proof here, as we do not need all the general results about X-mappings given in [MR98]. Let us suppose $f = f_X$ for some set X. Let us suppose $\varphi \notin I_f$: there exists $\mathcal{T} \in \mathcal{J}$ such that $\varphi \in f(\mathcal{T}) - \mathcal{T}$. As $f(\mathcal{T}) = f_X(\mathcal{T})$, we get from definition 6.34 that $\mathcal{T} \sqcup \varphi - \mathcal{T} \subseteq \boldsymbol{\mathcal{L}} - X$, thus we have proved: $\boldsymbol{\mathcal{L}} - I_f \subseteq \boldsymbol{\mathcal{L}} - X$, i.e. $X \subseteq I_f$.

Let us suppose now $\varphi \in f(\mathcal{T})$, with $\mathcal{T} \in \mathcal{J}$. Then, $\mathcal{T} \sqcup \varphi - \mathcal{T} \subseteq f(\mathcal{T}) - \mathcal{T} \subseteq \boldsymbol{\mathcal{L}} - I_f$, thus $(\mathcal{T} \sqcup \varphi) \cap I_f \subseteq \mathcal{T} = Th(\mathcal{T})$, i.e. $\varphi \in f_{I_f}(\mathcal{T})$ from definition 6.34.

Let us suppose now $\varphi \in f_{I_f}(\mathcal{T})$, with $\mathcal{T} \in \mathcal{J}$. Then, from definition 6.34, we get $(\mathcal{T} \sqcup \varphi) \cap I_f \subseteq \mathcal{T}$, i.e. $\mathcal{T} \sqcup \varphi - \mathcal{T} \subseteq \boldsymbol{\mathcal{L}} - I_f$ and from above result $\boldsymbol{\mathcal{L}} - I_f \subseteq \boldsymbol{\mathcal{L}} - X$. Thus $\varphi \in f_X(\mathcal{T}) = f(\mathcal{T})$.

These last two results show: $f_X = f_{I_f}$.□

Notice that if we particularize this result for pre-circumscriptions, we get that a pre-circumscription f is an X-mapping iff $f = f_{I_f}$.

Thus, expressing a pre-circumscription f (or even a more general mapping from $\boldsymbol{\mathcal{L}}$ to $\boldsymbol{\mathcal{L}}$) in terms of X-mapping is a (rather natural even if it may seem somehow paradoxical) way to express f thanks to its inaccessible formulas. This explains the title of this subsection.

Before giving our result about propositional circumscription, let us remind a last general result about X-mappings, which will precise the nature of the set of the inaccessible formulas:

Proposition 6.39 [MR98, theorem 3.2-3] For any set of formulas X, we have $I_X = X^\wedge$. □

<u>Proof:</u> Remind the definition of I_X in definition 6.34. We know from proposition 6.38 that we have $X \subseteq I_X$.

We have already remarked that any set I_f (thus I_X) is stable for $\wedge$.

Thus, we have $X^\wedge \subseteq I_X$.

We want to prove the converse.

Let us suppose that $\varphi \in I_X$ for some set X of formulas. Let us consider the theory $\mathcal{T} = Th(Th(\varphi) \cap X)$. Then we have $(\mathcal{T} \sqcup \varphi) \cap X = Th(\varphi) \cap X \subseteq \mathcal{T}$ thus $\varphi \in f_X(\mathcal{T})$ from definition 6.34 thus $\varphi \in \mathcal{T}$ because $\varphi \in I_X = I_{f_X}$. Thus we have $Th(\varphi) = \mathcal{T}$.

By compactness, we know that there exists a finite subset $\{\psi_i\}_{i \in I}$ of $\mathcal{T} = Th(\varphi) \cap X$ such that $\bigwedge_{i \in I} \psi_i \models \varphi$. We have $\varphi \models \psi_i$ because $\psi_i \in Th(\varphi) \cap X$ and $Th(Th(\varphi) \cap X) = Th(\varphi)$, thus we get $\varphi = \bigwedge_{i \in I} \psi_i$. Now, each ψ_i being in X, we get $\varphi \in X^\wedge$. □

Here is our detailed result, which states precisely which propositional circumscriptions (of definition 4.1) are X-mappings, and which describes then the set of the inaccessible formulas.

Theorem 6.40 A propositional circumscription is equal to an X-mapping iff $\mathbf{Z}$ is empty or $\mathbf{P}$ is finite.

When a propositional circumscription $f = CIRC(\mathbf{P}, \mathbf{Q}, \mathbf{Z})$ is an X-mapping, for any set of formulas X, we have $f = f_X$ iff $X^\wedge = Pos(\prec_{(\mathbf{P},\, \mathbf{Q},\, \mathbf{Z})})$. □

Notice that this result concerns propositional circumscriptions of definition 4.1 and that in this case we have $Pos(\prec_{(\mathbf{P},\, \mathbf{Q},\, \mathbf{Z})}) = Pos_e(\mathbf{P} \cup \mathbf{Q} \cup \neg\mathbf{Q})$. Also, as any X-mapping satisfies

(CR∞) (proposition 6.36-3), a consequence of this result is "the positive part" of proposition 4.19. Thus, together with example 4.20, this provides a proof of proposition 4.19 without any restriction on $\mathcal{V}(\boldsymbol{\mathcal{L}})$.

Proof:

$f = CIRC(\mathbf{P}, \mathbf{Q}, \mathbf{Z})$ and $\prec = \prec_{(\mathbf{P},\ \mathbf{Q},\ \mathbf{Z})}$. We know that $I_f = I_\prec = Pos(\prec)$ from proposition 5.29-2.

1. We suppose $\mathbf{Z} = \emptyset$ here.

Let us suppose there exists $\varphi \in f_{I_f}(\mathcal{T}) - f(\mathcal{T})$. As $\varphi \notin f(\mathcal{T})$, there exists $\mu \in \mathcal{M}_\prec(\mathcal{T})$ such that $\mu \models \neg\varphi$. Let us define the sets of formulas $\mathcal{A}_0 = (\mu \cap \mathbf{Q}) \cup \{\neg Q \ / \ Q \in \mathbf{Q} - \mu\}$ and $\mathcal{A}_1 = \mathcal{T} \cup \{\neg P \ / \ P \in \mathbf{P} - \mu\} \cup \mathcal{A}_0$.

Let $\mu' \in \mathcal{M}(\mathcal{A}_1)$, then, as $\mu' \in \mathcal{M}(\mathcal{A}_0)$, we get $\mu' \cap \mathbf{Q} = \mu \cap \mathbf{Q}$ and also $\mu' \cap \mathbf{P} \subseteq \mu \cap \mathbf{P}$. Moreover, $\mu' \in \mathcal{M}(\mathcal{T})$, thus, either $\mu' = \mu$ or $\mu' \prec \mu$ (remind that $\mathbf{Z}$ is empty). As $\mu \in \mathcal{M}_\prec(\mathcal{T})$, we get $\mu' = \mu$. As clearly $\mu \in \mathcal{M}(\mathcal{A}_1)$, we have proved $\mathcal{M}(\mathcal{A}_1) = \{\mu\}$.

As $\mu \models \neg\varphi$, we have $\mathcal{A}_1 \models \neg\varphi$ thus there exists a finite subset $\mathcal{A}'$ of $\mathcal{A}_1$ such that $\mathcal{A}' \models \neg\varphi$, thus there exists a formula $\gamma = \neg P_1 \wedge \cdots \wedge \neg P_n \wedge \psi$ (with $P_i \in \mathbf{P}$) where ψ is a conjunction of literals in $\mathbf{Q}$ such that $\mathcal{T} \cup \{\gamma\} \models \neg\varphi$, i.e. $\mathcal{T} \cup \{\varphi\} \models \neg\gamma$. We have $\neg\gamma = P_1 \vee \cdots \vee P_n \vee \neg\psi$.

Thus, $\neg\gamma \in (\mathbf{P} \cup \mathbf{Q} \cup \neg\mathbf{Q})^{\wedge\vee}$ and, from propositions 5.29 and 6.32-1b, $(\mathbf{P} \cup \mathbf{Q} \cup \neg\mathbf{Q})^{\wedge\vee} \subseteq Pos(\prec) = I_{f_\prec}$. Thus $\neg\gamma \in Pos(\prec) = I_{f_\prec}$.

As we have also $\neg\gamma \in \mathcal{T} \sqcup \varphi$, we get $\neg\gamma \in \mathcal{T}$ because $\varphi \in f_{I_f}(\mathcal{T})$ from the definition of f_X (definition 6.34).

We get a contradiction: $\mu \in \mathcal{M}(\mathcal{T})$ and $\mu \models \gamma$. Thus, there exists no such formula φ, i.e. we have, for any $\mathcal{T} \in \mathcal{J}$, $f_{I_f}(\mathcal{T}) \subseteq f(\mathcal{T})$.

From lemma 6.37 we get the equality, thus $f_{I_f} = f$.

2. We suppose now $\mathbf{P}$ finite (no conditions for $\mathbf{Q}$ and $\mathbf{Z}$). Again, let us suppose there exists $\varphi \in f_{I_f}(\mathcal{T}) - f(\mathcal{T})$. Again, as $\varphi \notin f(\mathcal{T})$, there exists $\mu \in \mathcal{M}_\prec(\mathcal{T})$ such that $\mu \models \neg\varphi$. Let us define the formula ψ as the conjunction of all the literals in $\mathbf{P}$ satisfied by μ: $\psi = (\bigwedge_{P \in \mu \cap \mathbf{P}} P \wedge \bigwedge_{P \in \mathbf{P} - \mu} \neg P)$. We define the sets of formulas $\mathcal{A}_0 = (\mu \cap \mathbf{Q}) \cup \{\neg Q \ / \ Q \in \mathbf{Q} - \mu\}$, $\mathcal{A}_1 = \mathcal{T} \cup \{\neg P \ / \ P \in \mathbf{P} - \mu\} \cup \mathcal{A}_0$ (as above), and also $\mathcal{A}_2 = (\mu \cap \mathbf{Z}) \cup \{\neg Z \ / \ Z \in \mathbf{Z} - \mu\}$.

If $\mu' \in \mathcal{M}(\mathcal{A}_1)$, then $\mu' \cap \mathbf{P} \subseteq \mu \cap \mathbf{P}$, $\mu' \cap \mathbf{Q} = \mu \cap \mathbf{Q}$ and $\mu' \in \mathcal{M}(\mathcal{T})$. If $\mu' \cap \mathbf{P} \subset \mu \cap \mathbf{P}$, we get $\mu' \prec \mu$, a contradiction with $\mu \in \mathcal{M}_\prec(\mathcal{T})$. Thus $\mu' \cap \mathbf{P} = \mu \cap \mathbf{P}$ and $\mu' \models \psi$. We have proved: $\mathcal{A}_1 \models \psi$. Thus there exists a finite subset $\mathcal{A}'$ of $\mathcal{A}_1$ such that $\mathcal{A}' \models \psi$, thus there exists a conjunction γ_1 of literals in $\mathbf{P} \cup \mathbf{Q}$ such that $\mu \models \gamma_1$ and $\mathcal{T} \cup \{\gamma_1\} \models \neg\psi$.

Let us consider the set of formulas $A_3 = \mathcal{T} \cup \mathcal{A}_0 \cup \mathcal{A}_2 \cup \{\psi, \varphi\}$. μ is the only model of $\mathcal{T} \cup \mathcal{A}_0 \cup \mathcal{A}_2 \cup \{\psi\}$ and $\mu \models \neg\varphi$, thus A_3 is inconsistent and there exists a finite subset of A_3 which is inconsistent, which means that there exists a conjunction γ_2 of literals in $\mathbf{Q} \cup \mathbf{Z}$ such that $\mathcal{T} \cup \{\varphi, \psi, \gamma_2\}$ is inconsistent and $\mu \models \gamma_2$.

We suppose now $\mu \cap \mathbf{P} \neq \emptyset$.

Let ψ' be a formula in the vocabulary $\mathbf{P}$ such that, for any interpretation μ', we have $\mu' \models \psi'$ iff $\mu' \cap \mathbf{P} \subset \mu \cap \mathbf{P}$ (as $\mathbf{P}$ is finite, we know that such a formula ψ' exists). We define also $\varphi' = (\neg\psi' \vee \neg\gamma_1) \wedge (\neg\psi \vee \neg\gamma_1 \vee \neg\gamma_2)$, i.e. $\neg\varphi' = (\psi' \wedge \gamma_1) \vee (\psi \wedge \gamma_1 \wedge \gamma_2)$.

$\mu \models (\psi \wedge \gamma_1 \wedge \gamma_2)$, thus $\mu \models \neg\varphi'$, thus $\varphi' \notin \mathcal{T}$.

Let ν be such that $\nu \in \mathcal{M}(\mathcal{T} \sqcup \varphi)$ and $\nu \models \neg\varphi'$. If $\nu \models \neg\psi$, then $\nu \models \mathcal{T} \cup \{\neg\psi, \gamma_1\}$, which contradicts $\mathcal{T} \cup \{\gamma_1\} \models \neg\psi$. If $\nu \models \psi$, as $\nu \cap \mathbf{P} = \mu \cap \mathbf{P}$, we get $\nu \models \neg\psi'$, thus $\nu \models \mathcal{T} \cup \{\varphi, \psi, \gamma_2\}$, a contradiction with the inconsistency of this set.

Thus we have $\varphi' \in \mathcal{T} \sqcup \varphi$.

Let us suppose that there exist interpretations ν, ν' such that $\nu \models \varphi', \nu \prec \nu'$ and $\nu' \models \neg\varphi'$. Then $\nu' \models \gamma_1$ and, as $\nu \cap \mathbf{Q} = \nu' \cap \mathbf{Q}$ and also as the literals of γ_1 in $\mathbf{P}$ are negative, we get $\nu \models \gamma_1$. As we have also $\nu \models \varphi'$ and $\models \varphi' \Rightarrow \neg\gamma_1$, we get a contradiction.

We have proved that if $\nu \models \varphi'$ and $\nu \prec \nu'$ then $\nu' \models \varphi'$, i.e. $\varphi' \in Pos(\prec) = I_f$.

Thus, as $\varphi' \in \mathcal{T} \sqcup \varphi$, and $\varphi \in f_{I_f}(\mathcal{T})$, we get $\varphi' \in \mathcal{T}$ (as in point 1), a contradiction with $\varphi' \notin \mathcal{T}$ proved above.

We conclude as in point 1.

3. If $\mathbf{P}$ is infinite and $\mathbf{Z} \neq \emptyset$ then example 4.20 shows that f falsifies (CR∞), thus it cannot be an X-mapping.

4. Thus, example 4.20 shows that the translation of circumscriptions with varying propositions in terms of X-mappings, described already in [MR98] for the finite case, cannot be extended to the infinite case, except in the "easy case" when $\mathbf{P}$ is finite. We have just described precisely the situation in these "easy cases":

a) When no varying proposition is present, the circumscriptions $CIRC(\mathbf{P}, \mathbf{Q}, \emptyset)$ are indeed X-mappings, and we may choose the set of the formulas positive in $\mathbf{P}$ as our set X, which is the set $Pos(\prec) = Pos(\preceq) = (\mathbf{P} \cup \mathbf{Q} \cup \neg\mathbf{Q})^{\wedge\vee}$ (see proposition 6.32-1c). This is not too surprising because it is well known that this is the set of the formulas inaccessible by such a circumscription.

b) When $\mathbf{P}$ is finite, the circumscriptions $CIRC(\mathbf{P}, \mathbf{Q}, \mathbf{Z})$ are also X-mappings, and we may choose the set of the formulas positive in $\mathbf{P} \cup \mathbf{Q} \cup \neg\mathbf{Q}$ in the extended acception: $Pos(\prec) = Pos_e(\mathbf{P} \cup \mathbf{Q} \cup \neg\mathbf{Q})$, see proposition 6.32-2 for a syntactical description of this set.

c) In these two cases a) and b), which are the only ones in which $CIRC(\mathbf{P}, \mathbf{Q}, \mathbf{Z})$ may be expressed as an X-mapping, we may choose as our set X any set such that $X^{\wedge} = Pos(\prec) = I_{\prec}$, and these sets are the only possible sets (see propositions 6.38 and 6.39).

When $\mathcal{V}(\mathcal{L})$ is finite (then $\mathbf{P}$ is finite and $CIRC(\mathbf{P}, \mathbf{Q}, \mathbf{Z})$ is an X-mapping) we know a syntactical description of the smallest possible set X such that $f_X = CIRC(\mathbf{P}, \mathbf{Q}, \mathbf{Z})$: see the $\wedge$-basis of $Pos(\prec)$ in this case in the proof of proposition 6.32-2.

Results b) and c) were not known before [MR98]. □

We have not settled the case of predicate circumscription yet, but this result, and example 4.20, give very strong indications that the same phenomena should appear: varying predicates should very likely be forbidden (except in easy cases), if we want to express a predicate circumscription in terms of X-mappings. This would explain why Suchenek, the first one [Suc93] who expressed the mixed predicate circumscriptions in terms of what is called here X-mapping, never considered varying predicates. Suchenek's results are restricted to finitely axiomatizable universal theories $\mathcal{T}$, and only universal sentences (sentences of the kind $\forall x_1 \cdots \forall x_n\ \psi$ where ψ is without quantifier) of $f(\mathcal{T})$ are considered. The restriction to universal theories was to be expected, because we need cumulativity (see proposition 6.36-5), and it is known that if we do not make any restrictions on the theories considered, the mixed predicate circumscriptions falsify cumulativity, while they satisfy cumulativity when restricted to universal theories, finitely axiomatizable or not (see note 6 page 18). As to the restriction of the kind of formulas considered, it seems that this restriction also is indeed necessary. However, it does not seem that the restriction to finitely axiomatizable theories is mandatory.

6.4 The respective roles of the various kinds of "positive formulas"

Let us recapitulate the results of this text which concern our various kinds of "positive formulas" (the reader interested in the role of "positive formulas" in preferential entailments and circumscriptions should consult also the recent and interesting [Eng98][18], in which positive formulas are called "upward persistent").

Recapitulation of the roles of the various sets of positive formulas:
In the following, we consider a formula circumscription $f = f_{\prec} = CIRCF(\Phi)$, where $\prec = \prec_{\Phi}$.
We consider also the "large" relation $\preceq = \preceq_{\Phi}$ (see definitions 5.3).

When we talk of a "propositional circumscription", we consider $f = f_{\prec} = CIRC(\mathbf{P}, \mathbf{Q}, \mathbf{Z})$.
Remind that any propositional circumscription is a particular formula circumscription:
$f = CIRCF(\Phi)$ where $\Phi = \mathbf{P} \cup \mathbf{Q} \cup \neg\mathbf{Q}$, $\prec = \prec_{(\mathbf{P},\,\mathbf{Q},\,\mathbf{Z})}$ and $\preceq = \preceq_{(\mathbf{P},\,\mathbf{Q},\,\mathbf{Z})}$ (see notations 5.6).

1. For any formula circumscription $f = f_{\prec}$, the set $I_f = I_{\prec} = Pos(\prec)$ is the set of the formulas inaccessible for f: for any $\varphi \in Pos(\prec)$, we get $\varphi \in f(\mathcal{T})$ only if φ is already a consequence of $\mathcal{T}$ (proposition 5.29-2).

2. For propositional circumscriptions $CIRC(\mathbf{P}, \mathbf{Q}, \mathbf{Z})$, this set $I_f = Pos(\prec)$ can be used in order to describe f in terms of X-mappings (i.e. to describe in a natural way f thanks to its inaccessible formulas) only when $\mathbf{P}$ is finite, or $\mathbf{Z}$ is empty. If $\mathbf{P}$ is infinite and $\mathbf{Z}$ not empty, the set I_f cannot be used in this way (theorem 6.40).

[18] Notice however that [Eng98] considers only circumscriptions without any varying predicate. Moreover most of its results about circumscriptions concern what is called "the finite case", which in fact corresponds to finite propositional circumscriptions, still without varying propositions. The interest of this text comes from its very general approach of positive formulas and related notions.

When the circumscription is an X-mapping, i.e. when $\mathbf{P}$ is finite, or $\mathbf{Z}$ is empty, then for any set of formulas X we have $f = f_X$ iff $X^\wedge = Pos(\prec)$ (propositions 6.38 and 6.39, already reminded in theorem 6.40).

3. Again in the case of propositional circumscription, we have given a syntactical description of this set $I_f = Pos(\prec)$ in any case (see proposition 6.32-1c and -2).

   When $\mathcal{V}(\boldsymbol{\mathcal{L}})$ is finite, there exists a smallest (for $\subseteq$) set X satisfying $f_X = CIRC(\mathbf{P}, \mathbf{Q}, \mathbf{Z})$, for which we have given a syntactical description (see proof of proposition 6.32-2, finite case). For any set of formulas Y, we have $f_Y = CIRC(\mathbf{P}, \mathbf{Q}, \mathbf{Z})$ iff $X \subseteq Y \subseteq Pos(\prec)$.

4. For any formula circumscription $f = f_\prec$, the set $Pos(\preceq) = Pos_m(\Phi)$, for which we know an easy syntactical definition $Pos_m(\Phi) = \Phi^{\wedge\vee}$, is the greatest set (for $\subseteq$) which is strongly elementarily equivalent to Φ. This means that $\Psi = Pos_m(\Phi)$ is the greatest set Ψ such that, for any set of formulas Ψ', we have $CIRCF(\Phi \cup \Psi') = CIRCF(\Psi \cup \Psi')$ (propositions 5.29 and 5.33-2a).

   Moreover, we have $CIRCF(\Phi \cup \Psi') = CIRCF(\Psi \cup \Psi')$ for any set Ψ' iff $\Phi^{\wedge\vee} = \Psi^{\wedge\vee}$ (proposition 5.33-2a again).

5. For any formula circumscription $f = f_\prec$, the set $\Psi = Pos_r(\prec) = Pos_r(\Phi)$ is the greatest (for $\subseteq$) set Ψ such that $CIRCF(\Phi) = CIRCF(\Psi)$.

   Moreover, we have $CIRCF(\Phi) = CIRCF(\Psi)$ iff $Pos_r(\Phi) = Pos_r(\Psi)$ (proposition 6.16).

6. For any propositional circumscription $CIRC(\mathbf{P}, \mathbf{Q}, \mathbf{Z})$, the sets $Pos(\prec)$ and $Pos_r(\prec)$ are equal, thus we know a syntactical description of $Pos_r(\prec)$ in this case (proposition 6.32, point 4a for the result, points 1c and 2 for the syntactical description).

7. For propositional circumscriptions $CIRC(\mathbf{P}, \mathbf{Q}, \mathbf{Z})$, in the cases when $\mathbf{P}$ is finite, the set
   $Pos(\prec) = Pos_e(\Phi)$ plays two roles:

   It satisfies 1) $f = CIRCF(\Phi) = f_{Pos_e(\Phi)}$ and 2) $f = CIRCF(\Phi) = CIRCF(Pos_e(\Phi))$.

   However, we may make much greater modifications to this set if we consider role 2 than if we consider role 1.

   Indeed, we have, for any set X, $f_X = f = CIRCF(\Phi)$ iff $X^\wedge = Pos_e(\Phi)$.

   However, for any set Ψ satisfying $\Psi^{\wedge\vee} = Pos_e(\Phi)$, we have $f = CIRCF(\Phi) = CIRCF(\Psi)$ (see proposition 5.33-2a, reminding that, if $\Phi \equiv_{sc} \Psi$, then $\Phi \equiv_c \Psi$).

   And, if $\mathbf{Z}$ is not empty, this still does not exhaust all the modifications we can do. Indeed, in this case we have $f = CIRCF(\Phi) = CIRCF(\Psi)$ iff $Pos_e(\Phi) = Pos_e(\Psi)$ and we may well have this equality, even if $\Psi^{\wedge\vee} \neq \Phi^{\wedge\vee}$: only, in this case, we do not necessarily have $CIRCF(\Phi \cup \Psi') = CIRCF(\Psi \cup \Psi')$.

8. For propositional circumscriptions $CIRC(\mathbf{P}, \mathbf{Q}, \mathbf{Z})$, in the cases when $\mathbf{P}$ is infinite or $\mathbf{Z}$ is empty, we have only one set of "positive formulas" $Pos_r(\prec) = Pos(\prec) = Pos(\preceq)$, i.e. $Pos_r(\Phi) = Pos_e(\Phi) = Pos_m(\Phi) = \Phi^{\wedge\vee}$ (proposition 6.32-4b).

   Thus, when $\mathbf{Z} = \emptyset$, we have only one set (let us design it by its syntactical description $\Phi^{\wedge\vee}$) to consider, and this set may be used for the two roles reminded in point 7. We get then $f_X = f$ iff $X^{\wedge} = \Phi^{\wedge\vee}$ and $CIRCF(\Phi) = CIRCF(\Psi)$ iff $\Psi^{\wedge\vee} = \Phi^{\wedge\vee}$.

□

$\Phi^{\wedge\vee}$ is the easier set of "formulas positive with respect to Φ" to think about. Next comes the set $Pos(\prec) = Pos_e(\Phi)$, but the set $Pos_r(\Phi)$ is not so easy to grasp. However, we hope to have convinced the reader that it plays an important role also, and that there are good reasons to call it a set of "positive formulas" for some aspects of this rather intuitive and vague notion. The notion of "sets strongly equivalent" seems very important to us (we hope that anybody recognizes that the notion of "sets basically equivalent" has an importance). Indeed, if we want to define some combinations of formulas from two sets (see section below 7) corresponding to the combination of the rules associated to these respective sets, we think that it is important to know what exactly is the exact nature of the "sets" considered, i.e. when two sets are considered as "equivalent". If we require only basic equivalence, it may happen that adding a few new formulas to these two sets breaks the equivalence, which may be unexpected. Adding formulas comes easily if we want to add new rules, or even, as we are in the propositional case, new individuals (see next section).

7 The meaning of some of the logical properties of circumscriptions

In the following, when we write "we know that *Tweety is ...*", we mean "all we know a priori about Tweety is that *Tweety is ...*".

7.1 Formula versions versus full versions

We give now a few words in order to compare the various versions, for any property. In the full versions, such as (RM), any amount (finite or not) of knowledge may be considered. In a formula version such as (RM1), we may start from some basic unrestricted knowledge $\mathcal{T}$, to which we are allowed to add only a finite amount of knowledge. In the formula-only version such as (RM0), we must use only finite amounts of knowledge.

7.2 A short reminder about cumulativity

Remind that (CT) is a property of any circumscription, even in the predicate calculus case, while, in the predicate calculus case, (CM) is true only in some cases: universal theories, mixed circumscriptions only, and no varying function. However, it makes sense to consider such circumscriptions, due to the importance of (CUMU) and of the circumscriptions concerned. Moreover we have seen that when, as in the present text, we restrict our attention to the propositional case, any circumscription satisfies (CUMU).

Cumulativity allows to make "full use of lemmas": Adding to $\mathcal{T}_1$ a set of formulas $\mathcal{T}_2$ which can already been deduced from $\mathcal{T}_1$ (meaning $\mathcal{T}_2 \subseteq f(\mathcal{T}_1)$) does not modify the result of a circumscription f. This is important from a knowledge representation perspective: if, from what we know as certain, we can conclude that *the bird Tweety flies*, then, if we add to our certain informations that *Tweety flies*, the set of conclusions is unmodified. This is also important when effective computation is considered, as in these cases we do not need to compute f again, because we know that $f(\mathcal{T}_1 \cup \mathcal{T}_2) = f(\mathcal{T}_1)$.

In this situation (CT) alone says only that we do not need to consider formulas outside of what could already been concluded before the addition of $\mathcal{T}_2$ to our certain knowledge, because $f(\mathcal{T}_1 \cup \mathcal{T}_2) \subseteq f(\mathcal{T}_1)$. This is better than nothing, but clearly not as powerful as cumulativity.

7.3 Reverse monotony, case reasoning and conjunctive coherence

(RM): the more things are known with certainty (the bigger $\mathcal{T}$ is), the less **new results** the circumscription f produces: If $\mathcal{T}_1 \subseteq \mathcal{T}_2$, we cannot deduce more from $f(\mathcal{T}_2)$ that what can be deduced from $f(\mathcal{T}_1)$ together with $\mathcal{T}_2$ itself. This justifies the name "reverse monotony". (RM), which is not implied by monotony, goes "against monotony" to some extend, while preserving the basic property $\mathcal{T} \subseteq f(\mathcal{T})$ of pre-circumscriptions.

Also, for any pre-circumscription, (RM1) is equivalent to the *deduction principle*, introduced in the context of circumscription and preferential entailment in [Sho88]: if ψ is in $f(\mathcal{T} \cup \{\varphi\})$, then $\varphi \Rightarrow \psi$ is in $f(\mathcal{T})$.

The other way (the "easy way" for classical logic) of the full "deduction theorem", namely if $\varphi \Rightarrow \psi$ is in $f(\mathcal{T})$, then ψ is in $f(\mathcal{T} \cup \{\varphi\})$, is equivalent to (MON1), thus it is falsified by all the interesting circumscriptions [Sho88].

Finally, remind (proposition 3.5) that, for pre-circumscriptions, (RM1) is equivalent to (CR1) and that (RM1) entails (CC1) while (RM) entails (CR) and (CC). For preferential entailments we have more (proposition 3.8-1): (RM1), (CR) and (CC1) are always satisfied, and (RM) is equivalent to (CC∞), and even to (CC) if $\mathcal{V}(\boldsymbol{\mathcal{L}})$ is enumerable.

(CR1): If, when we know that *Tweety is a bird*, we can conclude that *Tweety flies*, and similarly when we know that *Tweety is a bat*, then, if we know that *Tweety is a bird or a bat*, we can still conclude that *Tweety flies.*

(CC1): If we know that *Tweety is a red bird*, then we cannot conclude anything about Tweety that could not be concluded from the situations in which we know that *Tweety is red* together with the situations in which we know that *Tweety is a bird.*

7.4 Disjunctive coherence versus disjunctive rationality

We use the formula versions ("unary versions" in which one theory appears) below because this facilitates the readability of the examples, but in fact our examples use the formula-only versions (DC0) and (DR0). Remind that only (DC0) is satisfied by all the circumscriptions studied in this text ((DC1) being generally falsified), while even the simplest version (DR0) of the badly named "disjunctive rationality" is falsified by circumscriptions, except in very elementary cases.

(DC1): If we know that *Tweety is a bird or a bat*, then we cannot conclude anything about Tweety that could not be concluded from the situations in which we know that *Tweety is a bird* together with the situations in which we know that *Tweety is a bat.*

From (CR1) we know that anything that can be concluded from the situations in which we know that *Tweety is a bird* and also from the situations in which we know that *Tweety is a bat* could also be concluded in the situations in which we know that *Tweety is a bird or a bat.* This shows how (DC1) completes (CR1): circumscriptions stay in the "interval" delimited by these two properties.

(DR1): Notice that if $\mathcal{T}, \mathcal{T}_1$ and $\mathcal{T}_2$ are in $\mathscr{T}$, we know that $\mathcal{T} \subseteq \mathcal{T}_1 \cup \mathcal{T}_2$ iff $\mathcal{T} \subseteq \mathcal{T}_1$ or $\mathcal{T} \subseteq \mathcal{T}_2$. Thus, (DR) (definition 3.2) can also be expressed as: $f(\mathcal{T}_1 \cap \mathcal{T}_2) \subseteq f(\mathcal{T}_1)$ or $f(\mathcal{T}_1 \cap \mathcal{T}_2) \subseteq f(\mathcal{T}_2)$.

Similarly, for (DR1) and (DR0) (definition 3.3). Thus, (DR1) means that any set of formulas which can be concluded from a situation $\mathcal{T} \sqcup \psi_1 \vee \psi_2$, that is any set $X \subseteq f(\mathcal{T} \sqcup \psi_1 \vee \psi_2)$, must be included in totality in $f(\mathcal{T} \sqcup \psi_1)$ or in $f(\mathcal{T} \sqcup \psi_2)$.

For instance, if we consider everything that can be concluded from *Tweety is a bird or a bat*, then either all this can be concluded from *Tweety is a bird* alone, or this can be concluded from *Tweety is a bat* alone. Contrarily to what happens with disjunctive coherence, we are not allowed to "combine the two elementary situations" in order to get conclusions from a disjunction. All we can do is to take each of the two "elementary situations" individually (and this amounts to take only one of these two situations). We do not think that this behavior is really "rational". We give now an example involving circumscription.

Instead of using directly the propositional circumscription of definition 4.1, it is much better to use formula circumscription of definition 5.1: one reason, already noticed in the beginning of section 5, is that this avoids the artificial use of "exception propositional symbols" E_i. Another reason will be given at the end of subsection 7.5.

We consider only one rule with exceptions *birds fly* (BF) here. Suppose that our language $\mathcal{L}$ contains only B_i, F_i and R_i $(i \in I)$ as propositional symbols (read "*i is a bird*", "*i is able to fly*" and "*i is red*" respectively). Then we have the choice, with circumscription, to fix the R_i's or to let them vary. With $f = CIRCF(\{B_i \wedge \neg F_i\}_{i\in I}; \{R_i, B_i\}_{i\in I}, \{F_i\}_{i\in I},)$, the R_i's are fixed, while with $f' = CIRCF(\{B_i \wedge \neg F_i\}_{i\in I}; \{B_i\}_{i\in I}, \{F_i, R_i\}_{i\in I})$, the R_i's are allowed to vary (B_i's fixed in these two cases). f translates the rule (BF) with an emphasis put on the fact that we do not want that the color has any impact on the ability of a bird to fly, and f' translates this same rule, without this emphasis. With f', if we have some informations about the color and the ability of a bird to fly, we are ready to force Tweety to change its color in order to allow it to fly, as any normal bird should do. The adequation of this translation may depend on the context. It is generally a good policy to allow any proposition to vary, except a few particular ones (one reason why B_i's are fixed here is that we do not want the contraposition of (BF), as it is generally not desirable to get the systematic contrapositions of the rules with exceptions).

Here are a few examples: Our basic set contains only B_1: $\mathcal{T} = \{B_1\}$ (*Tweety*, denoted by the index 1, *is a bird*). We consider two particular situations $\mathcal{T}_1$ and $\mathcal{T}_2$, in which we have got some certain informations about *Tweety*, its color and its ability to fly. In $\mathcal{T}_1$, *Tweety is red iff it flies* while in $\mathcal{T}_2$, *Tweety is red or (exclusive or, denoted by $\oplus$) it flies.* $\mathcal{T}_1 = \mathcal{T} \sqcup R_1 \Leftrightarrow F_1$, $\mathcal{T}_2 = \mathcal{T} \sqcup R_1 \oplus F_1$. We get clearly $f(\mathcal{T}_j) = \mathcal{T}_j$ for $j \in \{1, 2\}$: as R_1 is fixed, F_1 also. Thus, $F_1 \notin f(\mathcal{T}_1)$ and $F_1 \notin f(\mathcal{T}_2)$. Indeed, in each case, from what we know about *Tweety*, concluding that *Tweety flies* would force its color to be *red* (in $\mathcal{T}_1$) or to be *not red* (in $\mathcal{T}_2$). This behavior is an expected consequence of our desire that the color of a bird does not change just to make it fly. f is a good way of translating rule (BF).

We get also $F_1 \in f(\mathcal{T})$: *a generic bird flies.* Again, this is as it should be.

As $R_1 \oplus F_1$ is equivalent to $\neg(R_1 \Leftrightarrow F_1)$, we have $\mathcal{T}_1 \cap \mathcal{T}_2 = Th(\mathcal{T})$. Thus, we get $f(\mathcal{T}_1 \cap \mathcal{T}_2) \not\subseteq f(\mathcal{T}_1) \cup f(\mathcal{T}_2)$: f falsifies (DR) (this corresponds to example 4.30).

This illustrates the big difference between (DR) and (DC). As any circumscription, f satisfies (DC0): notice that $f(\mathcal{T}_1) \sqcup f(\mathcal{T}_2) = \mathcal{T}_1 \sqcup \mathcal{T}_2 = Th(\perp)$.

Thus, we see why (DC0) does not prevent the (expected) behavior that we have described, while no formalism satisfying disjunctive rationality can have this behavior. We consider this as an example showing that disjunctive rationality is not a desirable property, while disjunctive coherence does not have such a bug. [KLM90] gives a very similar example, attributed to Lehmann and Ginsberg, *in defense* of a property (that they call "negation rationality") close to (DR0). They do not develop enough their example to be convincing, and we think on the contrary that such an example is a strong argument *against* disjunctive rationality or related properties. Indeed, we have shown that we cannot translate the rule (BF), with an emphasis put on the fact that we do not want the color change just to make a bird fly, by using a pre-circumscription f satisfying (DR0). It is because circumscriptions (except very elementary ones) falsify (DR0) that circumscription allows to translate such a rule.

The following passage is written only for the sake of comparison with f and to provide another way in which traditional circumscriptions falsify disjunctive rationality, corresponding to example 4.31. Let us examine f' now. We get, for the cases considered above: $F_1 \in f'(\mathcal{T}_1)$ and $F_1 \in f'(\mathcal{T}_2)$ and indeed we can no longer exhibit a falsification of disjunctive rationality in these cases. Notice that we get also $R_1 \in f'(\mathcal{T}_1)$: *Tweety* must be *red* in this case. similarly $\neg R_1 \in f'(\mathcal{T}_2)$. This is an illustration of the fact that, with f', we may sometimes get new informations about the *color* of a bird, if this allows it to *fly*.

However, f' also falsifies (DR) as can be shown by the following example, involving three birds. We must consider at least three birds because the inclusion relation among the subsets of a set with at most two elements is ranked[19] (cf example 4.31).

$\mathcal{T}$ is $\{B_1, B_2, B_3, \neg F_1 \vee \neg F_2 \vee \neg F_3, F_1 \vee F_2 \vee F_3\}$ (there are *exactly three birds*, and, in order to make the example not too trivial, we know that there exists at least *one flying bird* and *one unflying bird*). $\mathcal{T}_1 = \mathcal{T} \sqcup F_2 \Leftrightarrow F_3$, $\mathcal{T}_2 = \mathcal{T} \sqcup F_1 \Leftrightarrow F_2$.

We get $f'(\mathcal{T}_j) = \mathcal{T}_j$ $(j \in \{1, 2\})$: For $\mathcal{T}_1$ there are two models, one in which only 1 *flies*, and one in which only 2 and 3 *fly*, and these two models are incomparable for the set inclusion of the extensions of $\{B_1 \wedge \neg F_1, B_2 \wedge \neg F_2, B_3 \wedge \neg F_3\}$ in these two models. The same happens for $\mathcal{T}_2$, with a model in which only 3 *flies* and a model in which only 1 and 2 *fly*. Thus, $F_2 \notin f'(\mathcal{T}_1)$ and $F_2 \notin f'(\mathcal{T}_2)$.

Now, we have $\mathcal{T}_1 \cap \mathcal{T}_2 = \mathcal{T} \sqcup ((F_2 \Leftrightarrow F_3) \vee (F_1 \Leftrightarrow F_2)) = \mathcal{T} \sqcup (F_1 \oplus F_3)$. Thus, $F_2 \in f'(\mathcal{T}_1 \cap \mathcal{T}_2)$: from what we know in $\mathcal{T}_1 \cap \mathcal{T}_2$, nothing prevents 2 from *flying*. This shows that f' also falsifies disjunctive rationality. Again, $\mathcal{T}_1 \sqcup \mathcal{T}_2 = Th(\perp)$, which explains why disjunctive coherence is not falsified in this case. This second example gives an even stronger argumentation against disjunctive rationality: the three conclusions obtained here are what should be expected from the rule (BF)[20], and we could not have this behavior if f' should satisfy disjunctive rationality.

7.5 Coherent non monotony versus rational monotony

In (CNM1), if φ does not contradict $f(\mathcal{T})$, we are certain that no conclusion ψ in $f(\mathcal{T})$ can contradict $f(\mathcal{T} \sqcup \varphi)$. This explains the name: it is not a kind of restricted monotony because we do not get $f(\mathcal{T}) \subseteq f(\mathcal{T} \sqcup \varphi)$ as in (RatM1), but at least we know that the eventual non monotony produced when adding φ remains coherent with our previous conclusions.

Any circumscription falsifies rational monotony (except a few very elementary circumscriptions). Let us add a few words here about (RatM) versus (CNM). We develop this point because it is a hot topic at the time of the printing of this text. Indeed, some researchers consider that the falsification of rational monotony is a drawback for a formalism, which means that they reject circumscriptions as an appropriate way of formalizing common sense reasoning. This is not our point of view, and we explain why. Remind that, in the presence of (PC), (RatM) implies (DR), thus, our argumentation against (DR) in the preceding subsection was already an (indirect) argumentation against (RatM) also. However, we think

[19] As B_i's are fixed in f' also, we could again consider one individual only, but this individual may no longer be supposed to be necessarily a *bird*.

[20] At least if the rule is not understood with numerical considerations. If we want that there are the least possible number of flying birds, the answers are not the expected ones in the cases of $\mathcal{T}_1$ and $\mathcal{T}_2$. Circumscription is not adapted to translate directly rules with numerical considerations, but only rules with "sets considerations": we prefer the models in which we cannot eliminate any exception to a rule without adding another exception elsewhere. For the use of circumscription for translating rules when we want to count the exceptions, see [Moi98].

that a common sense example illustrating directly why (RatM) is not desirable either is important here, for our discussion about the common sense aspects of the logical properties of circumscriptions given in sections 4 and 5.

We have the two rules with exceptions (BF) and *penguins do not fly* (PNF), also *all penguins are birds* (PB). Taking propositions P_i, B_i, F_i ($i \in I$) to represent respectively *penguins, birds* and *flying animals*, we want our circumscription f to have the following behavior. $\mathcal{T}$ is $\{P_i \Rightarrow B_i\}_{i \in I}$. $F_1 \in f(\mathcal{T} \sqcup B_1)$ and $\neg F_1 \in f(\mathcal{T} \sqcup P_1)$ (a *"generic bird" flies* while a *"generic penguin" does not fly*). $(\mathcal{T} \sqcup P_1) \equiv (\mathcal{T} \sqcup (P_1 \wedge B_1))$, thus, as any circumscription f satisfies (CNM1) we get: $\neg P_1 \in f(\mathcal{T} \sqcup B_1)$ (if we know that *Tweety is a bird*, then we are forced to conclude that *it is not a penguin*).

We get also $F_1 \notin f(\mathcal{T} \sqcup P_1)$, thus (RatM1) gives $\neg P_1 \in f(\mathcal{T} \sqcup B_1)$: in the presence of rational monotony also, we are forced to conclude that *birds are not penguins* (with exceptions).

Now, we want to translate the rule without exceptions *all steamer-ducks are birds*, with (BF) and also we do not want to conclude anything about the ability of *steamer-ducks* to *fly* (these Patagonian birds are roughly equally divided into flying subspecies and non flying ones). Then, taking propositions S_i to represent *steamer-ducks*, we want our f to have the following behavior, where $\mathcal{T} = \{S_i \Rightarrow B_i\}_{i \in I}$: $F_1 \in f(\mathcal{T} \sqcup B_1)$ (as in the penguin example) and $F_1 \notin f(\mathcal{T} \sqcup S_1)$ (if we know that *Tweety is a steamer-duck*, then we do not want to conclude that *it flies*). This "cancellation of inheritance" may be easily simulated by some adequate circumscription. Again $(\mathcal{T} \sqcup S_1) \equiv (\mathcal{T} \sqcup (S_1 \wedge B_1))$, thus, if f is a pre-circumscription satisfying (RatM1) we get: $\neg S_1 \in f(\mathcal{T} \sqcup B_1)$. We get the new rule: "*birds are not steamer-ducks*" (with possible exceptions).

On the contrary, the fact that f satisfies (CNM1) does not necessarily enforce this rule. This explains why, in fact, (RatM) is really stronger than (CNM).

This example illustrates clearly the difference between the two properties:

Coherent non monotony gives us as an unescapable result that

birds are not penguins (with possible exceptions).

Rational monotony forces the same rule, plus another one:

birds are not steamer-ducks (with possible exceptions).

The behavior of (CNM) looks better than the behavior of (RatM), and thus "rational monotony" has got a misleading name. Indeed, as any species of bird is likely to retract at least one of the properties shared by the "generic birds", if f satisfies rational monotony, each time we have enough informations to conclude that *Tweety is a bird*, but not enough to conclude that it belongs to some already known species, we are forced to conclude that

Tweety does not belong to any already known species of birds. We call this the "rara avis paradox", and consider this as a serious drawback. This is loosely related to, but significantly different from, the "lottery paradox" (see e. g. [Poo89]).

Even if rational monotony is worse, coherent non monotony falls prey to a milder version of the paradox: we are forced to conclude that *Tweety* does not belong to any species which contradicts (even with exceptions) some property of generic birds. However, this milder version looks much more reasonable: in the absence of any other information, it may be accepted as normal to consider that a given "generic bird" does not belong to a species which contradicts (even with exceptions) some property of the generic birds.

Here is an example of circumscription having the "good behavior" advocated above. $\mathcal{T} = \{S_i \Rightarrow B_i\}_{i \in I}$, $f = CIRCF(\{B_i \wedge \neg F_i,\ S_i \wedge F_i, \neg S_i \vee \neg F_i\}_{i \in I}; (B_i)_{i \in I}, (F_i, S_i)_{i \in I})$. The intuition is that *non flying birds* $B_i \wedge \neg F_i$'s should be minimized, while *flying steamer-ducks* should be "fixed", which amounts to minimize $S_i \wedge F_i$ and its negation together, in the line of proposition 5.9.

Here are a few significative results:

$F_1 \in f(\mathcal{T} \sqcup B_1), \neg F_1 \notin f(\mathcal{T} \sqcup B_1)$ (*a generic bird flies*), $\neg F_1 \notin f(\mathcal{T} \sqcup S_1)$, $F_1 \notin f(\mathcal{T} \sqcup S_1)$ (we do not know whether *a generic steamer-ducks belongs to a flying or to an unflying subspecies*). We get also $S_1 \notin f(\mathcal{T} \sqcup B_1)$ and $\neg S_1 \notin f(\mathcal{T} \sqcup B_1)$: we do not know whether *a generic bird is a steamer-duck or not.* We have used the possibility of falsifying rational monotony.

Notice that we get also: $S_1 \notin f(\mathcal{T} \sqcup B_1 \wedge \neg F_1)$, and $\neg S_1 \notin f(\mathcal{T} \sqcup B_1 \wedge \neg F_1)$: a "*generic unflying bird*" is constrained neither to be *steamer duck* nor *not to be a steamer-duck*, which again, is an expected answer.

The traditional method does not take advantage of the fact that circumscription is not constrained to conform to the unwanted rule (RatM0). Let us remind the traditional method here: it makes use of exceptional propositions E_1^1 and E_1^2 and of the *prioritized circumscription* $f' = CIRC((E_i^2), (B_i), (E_i^1, F_i, S_i)) \sqcup CIRC((E_i^1), (E_i^2, B_i), (F_i, S_i))$ [McC86]. Two (kinds of) formulas are added to any given $\mathcal{T}$: $\mathcal{T}_E = \mathcal{T} \cup \{(B_i \wedge \neg E_i^1) \Rightarrow F_i,\ (S_i \wedge \neg E_i^2) \Rightarrow E_i^1\}_{i \in I}$ (in our example $I = \{1\}$) and we use $f'(\mathcal{T}_E)$ instead of $f(\mathcal{T})$. The "exceptional" E_1^1 gives one formula $(\neg E_1^1)$ true in $f'(\mathcal{T}_E \sqcup B_1)$ and false in $f'(\mathcal{T}_E \sqcup S_1)$, thus (CNM1) (satisfied by f' which is a preferential entailment) suffices to enforce the unwanted $\neg S_1 \in f'(\mathcal{T}_E \sqcup B_1)$.

7.6 A new proposal to deal with the penguins example

Let us end this section with another example of the use of circumscriptions, in order to deal with the penguins. This discussion is important in this text, for two reasons. Firstly, it shows that considering properties such as the ones studied in this text does not suffice to guarantee that an appropriate use of circumscription is made: we must also study carefully the situation considered if we want to be sure that our formalization in terms of circumscription is really adequate. Here, the problem cannot be expressed in terms of general properties of the kind of the properties studied above: the problem is more fundamental, and comes directly from what we want to express. Secondly, it shows that in many cases one single "ordinary" circumscription suffices in order to translate a set of rules. This goes against the tradition which would use a prioritized circumscription in this case also, that is a union of ordinary circumscriptions. We think that this is often the case, and that this reenforces the importance of our study of the properties of circumscriptions: from all the properties and counter-properties that we have seen, nothing says that circumscription is not, a priori, a good method for translating sets of rules with exceptions. And indeed, we show now that ordinary formula circumscriptions suffice (and even are better than some more complicated traditional proposals) to translate most of the sets of rules.

The rules considered here are the rules already given above: *birds fly* (with exceptions) (BF), *penguins do not fly* (with exceptions) (PNF) and *all penguins are birds* (PB), with the propositions B_i, P_i and F_i given above.

The traditional way to translate these rules is to use again the prioritized circumscription $f' = CIRC((E_i^2)_{i\in I}, (B_i)_{i\in I}, (E_i^1, F_i, P_i)_{i\in I}) \sqcup CIRC((E_i^1)_{i\in I}, (E_i^2, B_i)_{i\in I}, (F_i, P_i)_{i\in I})$, precisely to use $f'(\mathcal{T}_E)$ instead of $f(\mathcal{T})$, where $\mathcal{T}_E$ is defined now as $\mathcal{T}_E = \mathcal{T} \sqcup (\{(B_i \wedge \neg E_i^1) \Rightarrow F_i,\ (P_i \wedge \neg E_i^2) \Rightarrow \neg F_i\}_{i\in I}))$ ($\mathcal{T}$ still contains $\{P_i \Rightarrow B_i\}_{i\in I}$). Notice that from its definition, f' is the preferential entailment associated to the preference relation $\prec'$ defined as follows: $\mu \prec' \nu$ iff $(\mu \cap \{E_i^2\}_{i\in I} \subset \nu \cap \{E_i^2\}_{i\in I}$ and $\mu \cap \{B_i\}_{i\in I} = \nu \cap \{B_i\}_{i\in I})$ or $(\mu \cap \{E_i^1\}_{i\in I} \subset \nu \cap \{E_i^1\}_{i\in I}$ and $\mu \cap \{E_i^2, B_i\}_{i\in I} = \nu \cap \{E_i^2, B_i\}_{i\in I})$.

We consider, contrarily to what is generally written, that this translation is not appropriate in such a case, and we explain why:

Our set $\mathcal{T}$ concerns two individuals, and is restricted to the minimum possible in this case: $\mathcal{T} = \{P_1 \Rightarrow B_1, P_2 \Rightarrow B_2\}$. Let us consider the following two models of $\mathcal{T}_E$ with $i = \{1, 2\}$.

$\mu = \{B_1, B_2, P_1, E_1^1, E_2^1\}$, $\nu = \{B_1, B_2, P_1, F_1, F_2, E_1^2\}$. Then, the preference relation $\prec'$ associated to f' gives $\mu \prec' \nu$. Indeed, as $\mu \cap \{E_1^2, E_2^2\} \subset \nu \cap \{E_1^2, E_2^2\}$, E_i^1's are not even considered. We prefer that individual 1 (the penguin), does not fly, *against* the fact that individual 2 (the bird) flies.

We do not think that this is appropriate. Indeed, the higher priority given to the minimization of E_i^2 with respect to the minimization of E_i^1 is uncontroversially justified only when the same individual i is considered. With these two models, this is not so: we prefer that the *penguin "individual 1" does not fly*, even if this prevents *one other "generic" bird (individual 2)* from *flying*. And the same would happen if instead of just one individual 2, we would have a flock of birds $\{B_2, \cdots B_{1001}\}$ having the same properties: in order to make *one single penguin* to conform to its rule (PNF), we should prevent *one thousand generic birds* to conform to their own rule (BF). We consider this as a bug. The priority should work only individual by individual: *individual 1 should not fly* because *it is a penguin*, and the fact that *it is also a bird*, as any *penguin*, should not prevent it from *not flying*. Indeed, otherwise the rule (PNF) could never been applied, and it is reasonable to assume that if this rule about *penguins* has been given, then it should be applied and take precedence over any contradictory rule concerning the less specific category of generic *birds*. This notion of "preference by specificity" is a natural way of reasoning (see e.g. [Moi90]). But we see no real reason to prefer that *individual 1 does not fly*, against the fact that *the individuals* $2, \cdots, 1001$ *fly*, we would like better to have no preference[21] in this case. This is why we make here another proposal, which still gives priority to the most specific rule in the cases when it is uncontroversially justified, without presenting the unwanted behavior described above.

We use the following formula circumscription:
$f = CIRCF(\{P_i \wedge F_i, B_i \wedge (P_i \vee \neg F_i)\}_{i \in I}; (B_i)_{i \in I}, (F_i, P_i)_{i \in I})$.

The general rule that we propose for combining a formula circumscription associated to the tuple of formulas $(\varphi_i^1)_{i \in I}$ and the tuple of formulas $(\varphi_i^2)_{i \in I}$, with higher priority given to φ_i^2, is to use the formula circumscription associated to the tuple of triples of formulas $(\varphi_i^1 \wedge \varphi_i^2, \varphi_i^2, \varphi_i^1 \vee \varphi_i^2)_{i \in I}$. This is possible even when φ_i^1 and φ_i^2 are not contradictory, and in this case the behavior is much better than the behavior of the classical prioritized circumscription. And this is interesting, because it is not always easy to detect the cases when two rules are contradictory: it is much better to have a solution which works in any case.

In our example, let us replace rule (BF) by
birds have a beak (with exceptions) (BBeak).
Then, φ_i^1 would be $B_i \wedge \neg Beak_i$, which it is not contradictory with $\varphi_i^2 = P_i \wedge F_i$ (we leave rule (PNF) as it is). We consider that $CIRCF(\{\varphi_i^1 \wedge \varphi_i^2, \varphi_i^2, \varphi_i^1 \vee \varphi_i^2\}_{i \in I}; (B_i)_{i \in I}, (P_i, Beak_i, F_i)_{i \in I})$ is a good way of translating rules (BBeak) and (PNF) together, for sets containing $(P_i \Rightarrow B_i)_{i \in I}$, and that this gives acceptable answers for sets $\mathcal{T}$ in which $Beak_i$ does not contradict $\neg F_i$ and also for sets in which $Beak_i$ and $\neg F_i$ are contradictory. As this is not the main purpose of the present report, we do not elaborate more on the subject in order to explain why we consider this proposal as appropriate, and we leave the interested reader to make his own opinion on the subject, concentrating our attention to the case where there is a contradiction.

When there is a contradiction, as for rules (BF) and (PNF), our triple vanishes into a pair: $(\varphi_i^2, \varphi_i^1 \vee \varphi_i^2)_{i \in I}$, and this pair is what we have proposed above, because $(P_i \wedge F_i) \vee (B_i \wedge \neg F_i)$ is equivalent to $B_i \wedge (P_i \vee \neg F_i)$, provided we suppose also $P_i \Rightarrow B_i$. We give here one justification of our claim that this method is really preferable to the classical prioritized circumscription: The main difference, with respect to the associated preference relations, is as follows: as seen above, for classical prioritized circumscription f', with the two models $\mu = \{B_1, B_2, P_1, E_1^1, E_2^1\}$ and $\nu = \{B_1, B_2, P_1, F_1, F_2, E_1^2\}$ given above we prefer μ to ν. This is no longer the case with our proposal f: if we consider the interpretations in the "useful language" (without the E_i^j's), we get $\mu_0 = \{B_1, B_2, P_1\}$ and $\nu_0 = \{B_1, B_2, P_1, F_1, F_2\}$ and

[21] Remind (cf note 20 page 88) that we do not count the exceptions, thus no preference should be expected here.

the preference relation $\prec$ associated to f does not choose between these two models of $\mathcal{T}''$, thus $f(\mathcal{T}'') = \mathcal{T}''$. The preference is given to rule (PNF) about penguins against rule (BF) about birds only when the same individual is concerned. The drawback detected above is suppressed.

To illustrate why we think that the behavior of our proposal conforms to what could be expected from the two rules (BF) and (PNF) in the presence of rule (PB), we give a few significative basic examples, where $\mathcal{T} = \{P_1 \Rightarrow B_1\}$:

$F_1 \in f(\mathcal{T} \sqcup B_1), \neg F_1 \notin f(\mathcal{T} \sqcup B_1)$ (*a generic bird flies*), $\neg F_1 \in f(\mathcal{T} \sqcup P_1)$, $F_1 \notin f(\mathcal{T} \sqcup P_1)$ (*a generic penguin does not fly*). We get also $\neg P_1 \in f(\mathcal{T} \sqcup B_1)$ (*a generic bird is not a penguin*) which, as explained above, is unavoidable because any circumscription satisfies (CNM1).

Notice that we get also: $\neg F_1 \in f(\mathcal{T} \sqcup B_1 \wedge (P_1 \vee \neg F_1))$, which is as it should be: if we know that *Tweety is a bird which is either a penguin or unable to fly*, then *Tweety is unable to fly*.

All these results about "*individual 1*" are also true with the classical prioritized circumscription $f'(\mathcal{T}_E)$, the two solutions being really different only when several individuals are considered. Notice that what we have said for the properties of circumscriptions applies to prioritized circumscriptions (with some variations in the conditions of applicability). And here, the most significative difference between the two proposals $f(\mathcal{T})$ and $f'(\mathcal{T}_E)$ does not seem to come from a difference in behavior with respect to the properties studied in this text. This is an example showing that we must regard also beyond such properties in order to decide which translation is better. Indeed, the property asking to have no preference between the models μ_0 and ν_0 given above does not seem to be expressible in general terms such as those of (RM) or (CNM), even if it describes some precise behavior that we want that our formalization respects.

8 Conclusion

We have given three kinds of results.

1) We have given the list of the main properties of the reasoning through circumscription known to us at this time. This subject is surprisingly little known, and in the predicate calculus case the results are not still fully precised (however see [MR99] for various results). Thus, we have studied thoroughly the (infinite) propositional case as a first step towards the full solution for the predicate calculus case. This case also was very badly known, as shown by some recently published literature dealing with propositional circumscription, even in the finite case. We have made a systematic comparison with what is known for the predicate calculus case. Roughly, the properties are the same. The big difference comes from the applicability conditions of some precise property or version of a property. Our study of the propositional case is very detailed, it includes the formula (propositional) circumscription, and we have taken care to give exactly the version of each property which holds, and in which

cases. The conclusion of this first study is that nothing seems to prevent circumscription from translating sets of rules with exceptions.

2) In our study of formula propositional circumscription, we have provided the first known characterization theorem. We have also studied carefully when two sets of formulas give rise to the same circumscription. We have shown that two equivalences are to be examined. In the basic equivalence, the sets give rise to the same circumscription, but adding some given formulas to these two sets may break the equivalence. This is why a stronger equivalence, preserved when adding a set of formulas to our two sets, must also been considered. And we have shown that this strong equivalence corresponds exactly to have the same closure for $\wedge$ and $\vee$. We have also defined precisely the kind of "closure" which corresponds to the basic equivalence. This gives rise to two other sets of "positive formulas", which were never studied seriously before to our knowledge (except partially in our [MR98]). We have shown that for ordinary circumscription and also in other cases, these two sets coincide, and this set is easy to describe. This is the set of the formulas which can never be obtained as a result of the application of the formula circumscription (the formulas inaccessible for this circumscription). For ordinary circumscription we have given a syntactical description of this set. This set may sometimes be used also in order to give a rather paradoxical description of the circumscription from their inaccessible formulas. For general formula circumscription, these last two sets are distinct, thus we get a third, more complex, set. We have shown that in any case, two sets Φ and Ψ of formulas give rise to the same circumscription if and only if this "third set of formulas positive with respect to a given set of formulas" is the same one if we start from Φ or from Ψ.

3) In order to illustrate the utility of our study of circumscription, we have given an intuitive interpretation for some of the main logical properties studied here. These results help understanding the behavior of the inference by circumscription, which is an important matter. This may help to decide whether circumscription is adapted to a given situation, and also which precise circumscription must be used. We have provided an example with a circumscription which shows that the traditional methods do not make full use of the properties and counter-properties of the circumscriptive inference. We have also given an example showing that we must still regard sometimes beyond such properties in order to be sure that the formalization chosen for a given set of rules corresponds to what is expected. The examples we have given also show that a unique formula circumscription may do a better job than some more complicated unions of classical circumscriptions which are traditionally advocated in the literature. Thus, our study of the logical properties of classical formula circumscription takes a greater importance. We have shown, in small examples, how we can combine naturally the sets associated to the translation of two rules in order to get the set associated to a combination of these two rules. Thus, our study of the "equivalences" between sets of formulas to be circumscribed takes a greater importance.

As a conclusion of our study, we hope that we have given enough indications and enough examples involving circumscriptions in order to show that circumscription is indeed a good way to formalize rules with exceptions.

Still a lot of work must be done. Even in the propositional case, it is possible that some property or counter-property, important from a knowledge representation point of view, is missing. And the predicate calculus case needs much more investigations, which could take some inspiration in the present work.

Finally, the problem of which circumscription (or possibly union or any other combination of circumscriptions) is the most appropriate to translate a given set of rules with exceptions should be re-examined in the light of these properties. Indeed, these properties were almost completely ignored when the first proposals, which are still considered as the classical ones, have been made. We have exhibited two simple cases in which the traditional methods are inadequate. We have given a few "real examples", but clearly this is far from sufficient in order to solve this complex subject, and our goal was mainly to convince the reader that this is an important open problem. We do not pretend to have solved this problem. But we pretend to have given some tracks in order to solve it.

We think that the small examples should be re-examined carefully before trying to solve (by chance?) greater examples. We would advocate research in this direction: You have several (let us say two to begin with) informal "rules" with exceptions. You know how to translate each of these rules in terms of a formula circumscription, that is you know how to associate a set of formulas to be circumscribed to each of these rules (notice that even this step is not simple). You consider now these rules together. The problem is to design precise combinations of these sets of formulas which give automatically the new set of formulas which corresponds to the translation of all the rules together. We have given very partial examples, in the case when one rule should take precedence. But more examples should be considered, and more complex combinations should be designed. We think that this is a largely unexplored way, which should be at last studied carefully, if circumscription is to be considered as a serious candidate for translating such sets of rules.

So, even if our study brings many precise results about circumscription, we think that we have asked more questions than solved problems. However, we think that our work is important in order to begin to attack the main problem circumscription is faced with: how to translate a set of informal rules in terms of circumscription without being forced to do "manually" much part of the work in order to find the good circumscription. And, to this respect, all the three kinds of results we have given are important. The logical properties may help in finding the adequate set to be used in order to translate one rule, or also to describe the adequate combination of sets to translate a combination of rules. The study about the equivalences between sets allows to know precisely which sort of sets are considered here. Indeed, two sets strongly equivalent may be assimilated, but not necessarily two sets which are only "basically equivalent". The small examples studied show that we should not necessarily stick to the traditional methods, as they have some serious bugs. Also, these examples show that simple solutions do exist.

As a last word, we would like to emphasize the fact that all this study may be pursued in the propositional case. Indeed, we have shown that these problems exist even in this "simple" case, and that finding solutions even in this case is more complex that it could seem. In our opinion, the predicate calculus case should be examined later, and should not bring really new tools, with respect to the problem evoked here: how to translate a set of rules, in order to obtain expected (or if not "expected" when the situation is too complex for ordinary human reasoning, at least seriously justifiable) answers.

If this problem has a solution, then circumscription will be considered as a valuable tool. And then, the problem of its automatization will become important. We think also that our work about the "equivalences" between sets of formulas and our syntactical study of the "positive" formulas may be used in order to help this automatization.

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