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A decomposition method to solve variational inequalities. Study of the symmetric case

Roberto L.V. González* Edmundo Rofman†
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Abstract

We study in this work the solution of coupled systems of symmetric variational inequalities. We present a decomposition method which allows us to solve the original problem dealing with simpler problems comprising variational inequalities on smaller convex sets.

Key-words: variational inequalities, decomposition methods, optimization, numerical solution, iteration algorithm.

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Sur les solutions des inéquations variationnelles par une méthode de décomposition. L'étude du cas symétrique.

Résumé

On considère ici la solution de systèmes d'inéquations variationnelles couplées. On présente une méthode de décomposition qui permet de résoudre le problème originel à travers la solution de problèmes plus simples posés sur des convexes plus petits.

Mots-clé : inéquations variationnelles, méthodes de décomposition, optimisation, solution numérique, algorithmes itératifs.

Contents

1 Introduction

In this work we present a decomposition method to solve symmetric variational inequalities. Preliminary versions of the method and some applications can be seen in [6], [7] and [12]-[15]. Our procedure, which stems from the general principles analyzed in [10], was developed to solve the set of junction problems which were presented in [9]. There, junction problems were modelized using variational inequalities. Let us remark that junction problems have been intensely studied in recent years (see [2] and [8]) and that variational inequalities is in itself a subject of permanent interest (see as recent samples of this interest the publications [4] and [5]).

When the form $a(\cdot,\cdot)$ is bilinear and symmetric, these problems have an especial characteristic: They are equivalent to an optimization problem on a convex set. So, basically the problem is: Find $\bar{u} \in K$ such that

$$J(\bar{u}) = \min_{v \in K} J(v). \tag{1}$$

In the cases studied in this paper, the convex set K can be parametrized by an auxiliary variable in the following form

$$K = \bigcup_{v_I \in K_I} \hat{K}(v_I), \tag{2}$$

where v_I is the auxiliary variable and K_I is also a convex set.

The decomposition of K given by (2) implies that the original problem can be decomposed in a set of partial optimization problems defined on the sets $\hat{K}(v_I)$. In a second step of the method it is found the privileged $\bar{v}_I$ such that

$$\min_{v \in K} J(v) = \min_{v \in \hat{K}(\bar{v}_I)} J(v).$$

The paper is organized of the following form: In section 2 we present the original variational inequality and its reformulation in terms of optimization. In section 3 we present the methodology of decomposition and the solution by hierarchical optimization; also an iterative algorithm is described and its convergence is proved. In the Appendix some analytical properties of the auxiliary function and related items are proved.

2 Variational problem

Let V be a Hilbert space, we consider on $V \times V$ a bilinear, coercive, continuous and symmetric form a, i.e. there exist $\alpha > 0$, $\beta > 0$ such that

$$\left| \begin{array}{ll} a(v,v) \geq \alpha \|v\|_V^2 & \forall v \in V, \\ |a(v,u)| \leq \beta \|v\|_V \|u\|_V & \forall v,u \in V, \\ a(u,v) = a(v,u) & \forall v,u \in V. \end{array} \right. \tag{3}$$

We also consider a continuous linear form L defined by $L(v) = (f,v)$, where $f \in V$ and $(\cdot,\cdot)$ denotes the inner product on V. Let K be a non empty, bounded, closed and convex set of V.

The original problem consists of the resolution of the following variational inequality:

$$\text{Find } \bar{u} \in K \text{ such that } a(\bar{u}, v - \bar{u}) \geq (f, v - \bar{u}) \qquad \forall v \in K\,. \tag{4}$$

Let A be the linear operator associated to the bilinear form a, i.e.

$$a(v,u) = (Av, u) \ \forall u \in V$$

and A^* the adjoint of A. Since A is monotone, hemicontinuous and coercive on K, and K is closed, the variational inequality has a unique solution which we denote by $\bar{u}$ (see [3] and [11]).
We consider the following functional defined on V

$$J(v) = \tfrac{1}{2} a(v,v) - L(v) \qquad \forall v \in V\,. \tag{5}$$

As the form a is symmetric, inequality (4) is the necessary and sufficient condition that must be satisfied at the point that realizes the minimum of the functional J on the set K, therefore the equivalent problem can be stated as:

$$\boxed{\mathbf{P}} \qquad \text{Find } \bar{u} \in K \text{ such that } J(\bar{u}) = \min_{v \in K} J(v)\,.$$

3 Solution by a decomposition method

In order to determine the element $\bar{u}$ by a decomposition method, we will analyze the case where the convex set K is the union of a family of convex sets, and on this family we will propose a hierarchical decomposition of the problem.

3.1 Preliminary definitions

Definition 1 *Let $K \subset V$ be a non empty convex set, φ a real function, then*

1. *φ is a convex function on K if $\forall \lambda \in (0,1)\,,\ \forall x_1, x_2 \in K$,*
$$\varphi(\lambda x_1 + (1-\lambda) x_2) \leq \lambda \varphi(x_1) + (1-\lambda)\varphi(x_2)\,.$$
2. *φ is a strictly convex function on K if $\forall \lambda \in (0,1)\,,\ \forall x_1, x_2 \in K,\ x_1 \neq x_2\,,$*
$$\varphi(\lambda x_1 + (1-\lambda) x_2) < \lambda \varphi(x_1) + (1-\lambda)\varphi(x_2)\,.$$
3. *φ is a strongly convex function on K if there exists $\delta > 0$ (named coercivity coefficient) such that $\forall \lambda \in (0,1)\,,\ \forall x_1, x_2 \in K$,*
$$\varphi(\lambda x_1 + (1-\lambda) x_2) \leq -\delta\lambda(1-\lambda)\|x_1 - x_2\|^2 + \lambda \varphi(x_1) + (1-\lambda)\varphi(x_2)\,.$$

Definition 2 *Let $E : \mathrm{Dom}(E) \to V$ be an operator, if $W \subset \mathrm{Dom}(E) \subset V$ then*

1. *E is monotone on W if*
$$(E(u) - E(v), u - v) \geq 0 \ \ \forall u, v \in W\,.$$
2. *E is strongly monotone on W if there exist $\eta > 0$ such that*
$$(E(u) - E(v), u - v) \geq \eta \|u - v\|^2 \ \ \forall u, v \in W\,.$$

3.2 Decomposition of the convex K

Let X_I be a Hilbert space (we will call it the *intermediate space*). We will suppose that there exists a bounded, closed and convex set $K_I \subset X_I$, and an application which to each $v_I \in K_I$ assigns the subset $\hat{K}(v_I)$, where $\hat{K}(v_I)$ is a bounded, closed and convex subset of the space V, in such a way that the convex set K verifies the following decomposition

$$K = \bigcup_{v_I \in K_I} \hat{K}(v_I). \tag{6}$$

Definition 3 *Let $Y, Z \subset K$, we define*

$$d_1(Y,Z) = \max\left(\sup_{y\in Y}\inf_{z\in Z}\|y-z\| \;,\; \sup_{z\in Z}\inf_{y\in Y}\|y-z\|\right),$$

$$d_2(Y,Z) = \inf_{y\in Y}\inf_{z\in Z}\|y-z\| \,.$$

Remark 1 *It can be seen that d_1 is a distance between subsets of K named the Hausdorff distance.*

Properties of the family of convex sets $\hat{K}(v_I)$

We will suppose that the family of convex sets $\{\hat{K}(v_I) : v_I \in K_I\}$ verifies the following hypotheses:

1. $\forall u_I, v_I \in K_I$ and $\forall \lambda \in (0,1)$
$$\lambda \hat{K}(u_I) + (1-\lambda)\hat{K}(v_I) \subset \hat{K}(\lambda u_I + (1-\lambda) v_I). \tag{7}$$

2. There exist positive constants c and γ such that $\forall u_I, v_I \in K_I$,
$$d_1\left(\hat{K}(u_I), \hat{K}(v_I)\right) \le c\|u_I - v_I\|_{X_I}, \tag{8}$$
$$d_2\left(\hat{K}(u_I), \hat{K}(v_I)\right) \ge \gamma\|u_I - v_I\|_{X_I}. \tag{9}$$

3. For each $v_I \in K_I$ and for each $v \in \hat{K}(v_I)$ there exists a linear and continuous operator
$$T_v : X_I \to V$$
such that
$$\text{if } v_I \in K_I, \delta v_I \in X_I \text{ and } v_I + \delta v_I \in K_I \Rightarrow v + T_v(\delta v_I) \in \hat{K}(v_I + \delta v_I). \tag{10}$$
Besides, the family of operators $(T_v)_{v\in\hat{K}(v_I)}$ is uniformly continuous in norm with respect to the parameter v, i.e.
$$\|T_v - T_{\tilde{v}}\| \le C\|v - \tilde{v}\|_V \qquad \forall v, \tilde{v} \in K. \tag{11}$$

3.3 Solution by hierarchical optimization

Taking into account (6), we can decompose hierarchically the problem $\mathbf{P}$ of finding $\bar{u} \in K$ such that $\bar{u}$ minimizes the functional J in K. Under the hypothesis (6) we have the following problem of *concatenated minima*

$$\min_{v \in K} J(v) = \min_{v_I \in K_I} \min_{v \in \hat{K}(v_I)} J(v) . \tag{12}$$

We introduce the function

$$J_I(v_I) = \min_{v \in \hat{K}(v_I)} J(v) . \tag{13}$$

Remark 2 *We will show in the Appendix that the function J_I is convex on the set K_I and that this function J_I has unique minimum at a point $\bar{u}_I \in K_I$.*

Definition 4 *We denote by $\bar{u}(v_I)$ the unique point that realizes the minimum of J in $\hat{K}(v_I)$.*

Remark 3 *We will see in the Appendix that $\bar{u}(v_I)$ is a Lipschitz continuous function with respect to the parameter v_I.*

Now (12) becomes:

$$\min_{v_I \in K_I} \min_{v \in \hat{K}(v_I)} J(v) = \min_{v_I \in K_I} J_I(v_I) . \tag{14}$$

Definition 5 *We introduce now the following problem $\mathbf{P}_I$:*

$$\boxed{\mathbf{P}_I} \qquad \text{Find } \bar{u}_I \in K_I \text{ such that } J_I(\bar{u}_I) = \min_{v_I \in K_I} J_I(v_I) .$$

Remark 4 *From (6) – (14) it follows that the problem $\mathbf{P}_I$ is a hierarchical optimization problem (see [10]) because it is composed by two concatenated optimizations. Problem $\mathbf{P}_I$ is generally easier to solve than problem $\mathbf{P}$, (because usually, K_I is simpler than K).*

Remark 5 *We will see later that $\bar{u}(\bar{u}_I)$ is the unique solution of the variational inequality (4). Then, we conclude that $\mathbf{P}_I$ is equivalent to $\mathbf{P}$ in the sense that*

- $\min_{v_I \in K_I} J_I(v_I) = \min_{v \in K} J(v)$,
- *the point $\bar{u}_I$ which realizes the minimum of J_I gives the solution of the original problem $\mathbf{P}$ through the relation $\bar{u} = \bar{u}(\bar{u}_I)$.*

We formalize these properties in the following:

Theorem 1

1. *Let $\hat{u}_I$ be such that $\bar{u} \in \hat{K}(\hat{u}_I)$ (where $\bar{u}$ is the solution of $\mathbf{P}$) then $\hat{u}_I$ is solution of problem $\mathbf{P}_I$.*
2. *If $\bar{u}_I$ is solution of $\mathbf{P}_I$ then $\bar{u}(\bar{u}_I)$ is the solution of problem $\mathbf{P}$.*

Proof.

1. By definition of $\bar{u}$
$$J(\bar{u}) \leq J(u) \quad \forall u \in K\,,$$
then, for the restriction to the space $\hat{K}(u_I)$ we also have
$$J(\bar{u}) \leq J(u) \quad \forall u \in \hat{K}(u_I)\,.$$
So,
$$J(\bar{u}) \leq J_I(u_I) \qquad \forall u_I \in K_I\,. \tag{15}$$
In particular, for $\hat{u}_I$ we have
$$J(\bar{u}) \leq J(u) \quad \forall u \in \hat{K}(\hat{u}_I)\,. \tag{16}$$
As $\bar{u} \in \hat{K}(\hat{u}_I)$, the relation (16) implies that
$$J(\bar{u}) = J_I(\hat{u}_I)\,. \tag{17}$$
From (15) and (17) we get that $\hat{u}_I$ is optimal for $\mathbf{P}_I$, i.e.
$$J_I(\hat{u}_I) \leq J_I(u_I) \;\forall u_I \in K_I\,.$$

2. By definition of $\bar{u}_I$ we have
$$J_I(\bar{u}_I) \leq J_I(u_I) \quad \forall u_I \in K_I\,.$$
By definition of $\bar{u}(u_I)$ and taking into account that $J_I(u_I) \leq J(u) \quad \forall u \in \hat{K}(u_I)$, we have
$$J(\bar{u}(\bar{u}_I)) = J_I(\bar{u}_I) \leq J_I(u_I) \leq J(u) \quad \forall u \in \hat{K}(u_I)\,. \tag{18}$$
Considering (6) and (18) we get
$$J(\bar{u}(\bar{u}_I)) \leq J(u) \quad \forall u \in K\,,$$
which proves the optimality of $\bar{u}(\bar{u}_I)$.

□

3.4 Coupled variational inequalities system

Our aim in this section is to transform the hierarchical optimization problem obtained in the previous section into a coupled system of variational inequalities. We will deal with the following problems:

- the problem $\min\limits_{v \in \hat{K}(v_I)} J(v)$ which allows us to compute the function J_I
- the problem $\mathbf{P}_I : \min\limits_{v_I \in K_I} J_I(v_I)$.

Both problems will be transformed into an equivalent variational inequality.

3.4.1 The computation of J_I as a variational problem

Taking into account the differentiability of the function J (see Appendix) and the necessary and sufficient condition of optimality, the minimum problem

$$J_I(v_I) = \min_{v \in \hat{K}(v_I)} J(v) \tag{19}$$

is equivalent to find $\bar{u}(v_I) \in \hat{K}(v_I)$ such that

$$\left(\frac{\partial J}{\partial v}(\bar{u}(v_I)), v - \bar{u}(v_I)\right) \geq 0 \quad \forall v \in \hat{K}(v_I).$$

Taking into account that $\dfrac{\partial J}{\partial v}(\bar{u}(v_I)) = A\bar{u}(v_I) - f$, we have

$$(A\bar{u}(v_I) - f, v - \bar{u}(v_I)) \geq 0 \qquad \forall v \in \hat{K}(v_I).$$

So, the minimum problem (19) is equivalent to the resolution of the following variational inequality:

$$a(\bar{u}(v_I), v - \bar{u}(v_I)) \geq (f, v - \bar{u}(v_I)) \qquad \forall v \in \hat{K}(v_I), \bar{u}(v_I) \in \hat{K}(v_I). \tag{20}$$

Remark 6 *The variational inequality* (20) *is similar to* (4) *except that the original set K is replaced by $\hat{K}(v_I)$. As $\hat{K}(v_I)$ is closed and convex, we have that there exists a unique solution $\bar{u}(v_I)$ of*

$$\boxed{\mathbf{VI}} \quad (A\bar{u}(v_I) - f, v - \bar{u}(v_I)) \geq 0 \quad \forall v \in \hat{K}(v_I).$$

3.4.2 The variational inequality associated to the problem $\mathbf{P}_I$.

We will prove in the Appendix that the function J_I is differentiable. This property allows us to associate a variational inequality to the point that minimizes J_I in K_I. This variational inequality results from considering the necessary condition of minimum and the property of differentiability of J_I, i.e. $\bar{u}_I \in K_I$ minimizes J_I iff it is a solution of

$$\boxed{\mathbf{VI}_I} \quad \left(T^*_{\bar{u}(\bar{u}_I)}(A\bar{u}(\bar{u}_I) - f), v_I - \bar{u}_I\right) \geq 0 \quad \forall v_I \in K_I.$$

Definition 6 *We define the operator $B : K_I \to X_I$ in the following form*

$$B(u_I) = T^*_{\bar{u}(u_I)}(A\bar{u}(u_I) - f). \tag{21}$$

Remark 7 *We will see in the Appendix that B is strongly monotone and hemicontinuous, then the $\mathbf{VI}_I$ has unique solution $\bar{u}_I \in K_I$.*

The following theorem establishes the relations that exist between the variational inequalities $\mathbf{VI}_I$ and **VI**.

Theorem 2

1. *Let $\bar{u}$ be a solution of* (4) *and let $\hat{u}_I \in K_I$ such that $\bar{u} \in \hat{K}(\hat{u}_I)$ then $\hat{u}_I$ is a solution of $\mathbf{VI}_I$.*
2. *If $\bar{u}_I$ is a solution of $\mathbf{VI}_I$ then $\bar{u}(\bar{u}_I)$ is a solution of* (4).

Proof.

1. Let $\bar{u}$ be a solution of (4), i.e.
$$(A\bar{u} - f, v - \bar{u}) \geq 0 \quad \forall v \in K. \tag{22}$$
Since $K = \bigcup_{u_I \in K_I} \hat{K}(u_I)$, then there exists $\hat{u}_I \in K_I$ such that $\bar{u} \in \hat{K}(\hat{u}_I)$, and then we have
$$(A\bar{u} - f, v - \bar{u}) \geq 0 \quad \forall v \in \hat{K}(\hat{u}_I).$$
In other words,
$$\bar{u} = \bar{u}(\hat{u}_I).$$
Let us prove that $\hat{u}_I$ is solution of $\mathbf{VI}_I$. We set
$$\begin{aligned}\left(T^*_{\bar{u}(\hat{u}_I)}(A\bar{u}(\hat{u}_I) - f), v_I - \hat{u}_I\right) &= \left(A\bar{u}(\hat{u}_I) - f, T_{\bar{u}(\hat{u}_I)}(v_I - \hat{u}_I)\right) \\ &= \left(A\bar{u}(\hat{u}_I) - f, \bar{u}(\hat{u}_I) + T_{\bar{u}(\hat{u}_I)}(v_I - \hat{u}_I) - \bar{u}(\hat{u}_I)\right),\end{aligned}$$
but $v = \bar{u}(\hat{u}_I) + T_{\bar{u}(\hat{u}_I)}(v_I - \hat{u}_I) \in \hat{K}(v_I)$, so
$$\left(T^*_{\bar{u}(\hat{u}_I)}(A\bar{u}(\hat{u}_I) - f), v_I - \hat{u}_I\right) = \left(A\bar{u}(\hat{u}_I) - f, v - \bar{u}(\hat{u}_I)\right).$$
Finally, taking into account (22), we have that $\hat{u}_I$ verifies
$$\left(T^*_{\bar{u}(\hat{u}_I)}(A\bar{u}(\hat{u}_I) - f), v_I - \hat{u}_I\right) \geq 0 \quad \forall v_I \in K_I.$$

2. Let $\bar{u}_I$ be a solution of $\mathbf{VI}_I$ then
$$J_I(u_I) - J_I(\bar{u}_I) \geq 0 \quad \forall u_I \in K_I,$$
consequently
$$J(\bar{u}(\bar{u}_I)) = J_I(\bar{u}_I) \leq J_I(u_I) \quad \forall u_I \in K_I$$
and since
$$J_I(u_I) \leq J(u) \quad \forall u \in \hat{K}(u_I),$$
we get
$$J(\bar{u}(\bar{u}_I)) \leq J(u) \quad \forall u \in \hat{K}(u_I).$$
As u_I is arbitrary, from (6) we obtain
$$J(\bar{u}(\bar{u}_I)) \leq J(u) \quad \forall u \in K$$
and so, using the necessary conditions of optimality, we get
$$(A\bar{u}(\bar{u}_I) - f, u - \bar{u}(\bar{u}_I)) \geq 0 \quad \forall u \in K.$$

□

Remark 8 *The previous theorem reduces the original problem* $\mathbf{P}$ *to find the solution of the coupled variational inequalities system:*
$$\left|\begin{array}{lll}\left(T^*_{\bar{u}(\bar{u}_I)}(A\bar{u}(\bar{u}_I) - f), u_I - \bar{u}_I\right) \geq 0 & \forall u_I \in K_I, & \boxed{\mathbf{VI}_I} \\ (A\bar{u}(u_I) - f, u - \bar{u}(u_I)) \geq 0 & \forall u \in \hat{K}(u_I). & \boxed{\mathbf{VI}}\end{array}\right.$$

3.5 Iterative solution of the decomposition procedure

The system $\mathbf{VI}_I - \mathbf{VI}$ can be solved using the following iterative method.

3.5.1 Description of the iterative algorithm

The backbone of the procedure is the following one: given a tentative point $u_I^\nu \in K_I$ and using the information given by the element $\bar{u}\,(u_I^\nu)$ (which is computed in terms of u_I^ν through the solution of **VI**), we modify this point u_I^ν in order to satisfy the condition $\mathbf{VI}_I$.

Let us denote by $\Pr(u, \Omega)$ the projection of a point u on a closed convex set Ω, i.e.

$$\|\Pr(u,\Omega) - u\| \leq \|w - u\|\,, \quad \forall w \in \Omega\,,\ \Pr(u,\Omega) \in \Omega\,.$$

Specifically, the algorithm has the following structure:

Algorithm

$\boxed{1}$ Give $u_I^o \in K_I$, $\rho > 0$, $\nu = 0$

$\boxed{2}$ Solve **VI**, obtaining $u^\nu = \bar{u}(u_I^\nu)$

$\boxed{3}$ Compute $T_{\bar{u}(u_I^\nu)}$

$\boxed{4}$ Compute $B^\nu = T^*_{\bar{u}(u_I^\nu)}\,(A\bar{u}(u_I^\nu) - f)$

$\boxed{5}$ Compute $u_I^{\nu+1} = \Pr(u_I^\nu - \rho B^\nu, K_I)$
set $\nu = \nu + 1$, and go to $\boxed{2}$.

Analysis of the algorithm

This algorithm generates a sequence (u_I^ν, u^ν) which converges to the solution $(\bar{u}_I, \bar{u})$ of $(\mathbf{VI}_I - \mathbf{VI})$ for all $\rho < \bar{\rho}$, being $\bar{\rho}$ a suitable positive number.
At step 2, given $u_I^\nu \in K_I$ we solve the **VI**

$$(A\bar{u}(u_I^\nu) - f, v - \bar{u}(u_I^\nu)) \geq 0 \qquad \forall v \in \hat{K}(u_I^\nu)\,,$$

obtaining in that way the unique solution $u^\nu = \bar{u}(u_I^\nu)$. At step 3, for these $u_I^\nu \in K_I$ and the associated $u^\nu = \bar{u}(u_I^\nu) \in \hat{K}(u_I^\nu)$, we compute $T_{\bar{u}(u_I^\nu)}$. At step 4 we compute the vector $B^\nu = B(u_I^\nu)$, where B is the strongly monotone operator defined by (21). To describe step 5 we introduce the following applications Q and M.

Definition 7

- *We define for $\rho > 0$ the application $Q : K_I \to X_I$ in the following form*

$$Q\,(u_I) = u_I - \rho B\,(u_I)\,.$$

- *We also define the application $M : X_I \to K_I$ in the following form*

$$M(u_I) = \Pr(Q\,(u_I)\,, K_I)\,.$$

At step 5 we compute the element $u_I^{\nu+1}$ applying the operator M, i.e. we define

$$u_I^{\nu+1} = M u_I^{\nu}.$$

3.5.2 Convergence of the algorithm

Let us define $\Xi := \sup\limits_{\substack{u_I \in K_I \\ \hat{u}_I \in K_I}} \frac{\|B(u_I) - B(\hat{u}_I)\|}{\|u_I - \hat{u}_I\|}$. Since B is Lipschitz continuous and strongly monotone (see Appendix) we have that Ξ is finite and $\forall \tilde{v}_I \in K_I$, $\forall v_I \in K_I$ it is verified that

$$(B(\tilde{v}_I) - B(v_I), \tilde{v}_I - v_I) \geq 2\beta_{J_I} \|v_I - \tilde{v}_I\|^2. \tag{23}$$

Consequently, we obtain

$$\frac{2\beta_{J_I}}{\Xi} < 1.$$

Using these parameters we can prove the following:

Proposition 1 *If* $0 < \rho < \dfrac{4\beta_{J_I}}{\Xi^2}$, *then* Q *and* M *are contractive operators and* M *has a unique fixed point* $\bar{u}_M \in K_I$.

Proof. Let $u_I \in K_I$, $\hat{u}_I \in K_I$, we have

$$Q(u_I) = u_I - \rho B(u_I),$$

$$Q(\hat{u}_I) = \hat{u}_I - \rho B(\hat{u}_I).$$

We will estimate the difference

$$\begin{aligned} \|Q(u_I) - Q(\hat{u}_I)\|^2 &= \|u_I - \rho B(u_I) - (\hat{u}_I - \rho B(\hat{u}_I))\|^2 \\ &= \|u_I - \hat{u}_I\|^2 + \rho^2 \|B(u_I) - B(\hat{u}_I)\|^2 - 2\rho(u_I - \hat{u}_I, B(u_I) - B(\hat{u}_I)). \end{aligned}$$

Since B is strongly monotone, from (23) we obtain

$$\|Q(u_I) - Q(\hat{u}_I)\|^2 \leq \left(1 - 4\rho\beta_{J_I} + \rho^2\Xi^2\right) \|u_I - \hat{u}_I\|^2.$$

Therefore, if $0 < \rho < \dfrac{4\beta_{J_I}}{\Xi^2}$, there exists $\sigma < 1$ such that

$$\|Q(u_I) - Q(\hat{u}_I)\| \leq \sigma \|u_I - \hat{u}_I\|$$

and then Q is contractive. Moreover

$$\|\Pr(Q(u_I), K_I) - \Pr(Q(\hat{u}_I), K_I)\| \leq \|Q(u_I) - Q(\hat{u}_I)\|,$$

consequently

$$\|M(u_I) - M(\hat{u}_I)\| \leq \sigma \|u_I - \hat{u}_I\|$$

and then M has a unique fixed point $\bar{u}_M$.

□

Lemma 1 *The fixed point* $\bar{u}_M$ *of* M *is the solution of the* $\mathbf{VI}_I$, *i.e.*

$$\left(T^*_{\bar{u}(\bar{u}_M)}(A\bar{u}(\bar{u}_M) - f), v_I - \bar{u}_M\right) \geq 0 \qquad \forall v_I \in K_I$$

and $\bar{u}(\bar{u}_M)$, *defined as the solution of* **VI**, *is the solution of the original variational inequality* (4).

Proof. Let $\bar{u}_M$ be the fixed point of M. We use the following equivalence

$$\bar{u}_M = \Pr\left(\bar{u}_M - \rho B(\bar{u}_M), K_I\right) \quad \Leftrightarrow \quad ((\bar{u}_M - \rho B(\bar{u}_M)) - \bar{u}_M, u_I - \bar{u}_M) \leq 0 \quad \forall u_I \in K_I\,.$$

Then we have $\forall u_I \in K_I$

$$(-\rho B(\bar{u}_M), u_I - \bar{u}_M) = \left(-\rho T^*_{\bar{u}(\bar{u}_M)}\left(A\bar{u}\left(\bar{u}_M\right) - f\right), u_I - \bar{u}_M\right) \leq 0\,,$$

and so, since $\rho > 0$

$$\left(T^*_{\bar{u}(\bar{u}_M)}\left(A\bar{u}\left(\bar{u}_M\right) - f\right), u_I - \bar{u}_M\right) \geq 0 \quad \forall u_I \in K_I\,.$$

Consequently, $\bar{u}_M$ is solution of $\mathbf{VI}_I$. By the uniqueness of solution we have $\bar{u}_M = \bar{u}_I$; finally by virtue of theorem 2, $\bar{u}(\bar{u}_I)$ is the solution of (4).

□

The following theorem summarizes the properties of the algorithm:

Theorem 3 *If* $0 < \rho < \dfrac{4\beta_{J_I}}{\Xi^2}$, *the algorithm generates a sequence* $\{u_I^\nu, \bar{u}(u_I^\nu)\}$ *such that* u_I^ν *converges to the unique solution* $\bar{u}_I$ *of* $\mathbf{VI}_I$ *and* $\bar{u}(u_I^\nu)$ *converges to* $\bar{u}(\bar{u}_I)$, *the unique solution of the original problem* (4).

Proof. As M is contractive, the sequence u_I^ν converges to $\bar{u}_I$, the fixed point of M. By the continuity mentioned in remark 3, the sequence $\bar{u}(u_I^\nu)$ converges to $\bar{u}(\bar{u}_I) = \bar{u}$.

□

Appendix

Properties of the function J

Convexity

We consider the function $J(v) = \frac{1}{2}a(v,v) - L(v)\ \forall v \in V$.

Proposition 2 *$J(\cdot)$ is strongly convex in V.*

Proof. The coercivity of the bilinear form a implies that J is a strongly convex function. Let us consider $v_1 \in V$ and $v_2 \in V$, $v_1 \neq v_2$ and $0 < \lambda < 1$,

$$
\begin{aligned}
J(\lambda v_1 + (1-\lambda) v_2) &= \tfrac{1}{2}a(\lambda v_1 + (1-\lambda) v_2, \lambda v_1 + (1-\lambda) v_2) - L(\lambda v_1 + (1-\lambda) v_2) \\
&= \tfrac{1}{2}\left(\lambda^2 a(v_1,v_1) + 2\lambda(1-\lambda) a(v_1,v_2) + (1-\lambda)^2 a(v_2,v_2)\right) - \lambda L(v_1) - (1-\lambda) L(v_2) \\
&= \tfrac{1}{2}\left(\lambda^2 - \lambda\right) a(v_1,v_1) + \lambda\left(\tfrac{1}{2}a(v_1,v_1) - L(v_1)\right) + \lambda(1-\lambda) a(v_1,v_2) \\
&\quad + \tfrac{1}{2}\left((1-\lambda)^2 - (1-\lambda)\right) a(v_2,v_2) + (1-\lambda)\left(\tfrac{1}{2}a(v_2,v_2) - L(v_2)\right) \\
&= -\tfrac{1}{2}\lambda(1-\lambda) a(v_2 - v_1, v_2 - v_1) + \lambda J(v_1) + (1-\lambda) J(v_2) \\
&\leq -\tfrac{\alpha}{2}\lambda(1-\lambda)\|v_2 - v_1\|^2 + \lambda J(v_1) + (1-\lambda) J(v_2).
\end{aligned}
$$

Then, J is strongly convex and therefore is strictly convex.

□

Differentiability

Lemma 2 *The function $J : V \to \Re$ is Frèchet differentiable $\forall v \in V$, being*

$$
\frac{\partial J}{\partial v}(v) = Av - f. \tag{24}
$$

Proof.

$$
\begin{aligned}
J(v + \delta v) &= \tfrac{1}{2}a(v + \delta v, v + \delta v) - L(v + \delta v) \\
&= \tfrac{1}{2}a(v,v) - L(v) + \tfrac{1}{2}a(\delta v, v) + \tfrac{1}{2}a(v, \delta v) + \tfrac{1}{2}a(\delta v, \delta v) - L(\delta v) \\
&= J(v) + a(v, \delta v) + \tfrac{1}{2}a(\delta v, \delta v) - L(\delta v),
\end{aligned}
$$

then

$$
J(v + \delta v) - J(v) = a(v, \delta v) + \tfrac{1}{2}a(\delta v, \delta v) - L(\delta v) = (Av - f, \delta v) + \tfrac{1}{2}(A\delta v, \delta v).
$$

So, we can write

$$
J(v + \delta v) = J(v) + (Av - f, \delta v) + o(\delta v),
$$

where $(A\delta v, \delta v) = o(\delta v)$ because by definition of A it results

$$
|(A\delta v, \delta v)| \leq \|A\delta v\|\ \|\delta v\| \leq \|A\|\ \|\delta v\|^2,
$$

$$\lim_{\|\delta v\|\to 0} \frac{(A\delta v, \delta v)}{\|\delta v\|} = 0 .$$

Therefore, $J(v)$ is Frèchet differentiable in v and (24) holds.

□

Proposition 3 $\dfrac{\partial J}{\partial v}$ *is strongly monotone and hemicontinuous in V.*

Proof.

1. Since $J(\cdot)$ is strongly convex and F-differentiable (see [1]) there exists $\beta_J > 0$ such that

$$J(v) - J(\tilde{v}) \geq \left(\frac{\partial J}{\partial v}(\tilde{v}), v - \tilde{v}\right) + \beta_J \|v - \tilde{v}\|^2 ,$$

$$J(\tilde{v}) - J(v) \geq \left(\frac{\partial J}{\partial v}(v), \tilde{v} - v\right) + \beta_J \|v - \tilde{v}\|^2 .$$

   Adding both expressions, we obtain

$$\left(\frac{\partial J}{\partial v}(\tilde{v}) - \frac{\partial J}{\partial v}(v), \tilde{v} - v\right) \geq 2\beta_J \|v - \tilde{v}\|^2 , \tag{25}$$

   and then, $\dfrac{\partial J}{\partial v}$ is strongly monotone.

2. Let $v, \tilde{v} \in K,\ t \in (0,1)$ and $v_t = (1-t)v + t\tilde{v}$, then

$$\left(\frac{\partial J}{\partial v}(v_t), \tilde{v} - v\right) = (Av_t - f, \tilde{v} - v) = (Av - f, \tilde{v} - v) + t\,(A(\tilde{v} - v), \tilde{v} - v) .$$

   Hence

$$\left(\frac{\partial J}{\partial v}(v_t), \tilde{v} - v\right) - \left(\frac{\partial J}{\partial v}(v), \tilde{v} - v\right) = t\,(A(\tilde{v} - v), \tilde{v} - v) .$$

   Since $|(A(\tilde{v} - v), \tilde{v} - v)| \leq \beta \|\tilde{v} - v\|^2$ and the convex set K is bounded, if we make $t \to 0^+$, we obtain the continuity of the application $t \to \left(\dfrac{\partial J}{\partial v}(v + t(\tilde{v} - v)), \tilde{v} - v\right)$, and so we get that $\dfrac{\partial J}{\partial v}$ is hemicontinuous in V.

□

Lemma 3

1. *The functional $J : V \to \Re$ is weakly lower semi-continuous in V .*
2. *If K is a not empty, bounded, closed and convex set, then there exists a unique element $\tilde{v}$ which minimizes J in K.*

Proof.

1. Let $(v_n)_{n\in N}$ be such that $v_n \rightharpoonup v$ in V. We will show that
$$\underline{\lim}_{n\to\infty} J(v_n) \geq J(v)\,.$$
Since J is strictly convex and *F-differentiable*, then we have
$$J(v_n) - J(v) \geq \big(J'(v), v_n - v\big) \qquad \forall n \in N\,,$$
where
$$J'(v) = Av - f$$
and since $v_n \rightharpoonup v$ in V we get that
$$\lim_{n\to\infty} \big(J'(v), v_n - v\big) = 0\,.$$
and therefore $\underline{\lim}_{n\to\infty} J(v_n) \geq J(v)$.

2. Since J is strictly convex, the element which realizes the minimum is unique. We get the existence of it from (i) and using the fact that the set K is bounded.

□

We will prove now, using only the hypothesis (8) that the application $v_I \to \bar{u}(v_I)$ is Hölder continuous in K_I.

Proposition 4 *If $\bar{u}(v_I)$ is the point which realizes the minimum of J in $\hat{K}(v_I)$, then $\bar{u}(v_I)$ is a Hölder continuous function with respect to the parameter v_I, i.e.*
$$\|\bar{u}(v_I) - \bar{u}(u_I)\| \leq C\,\|v_I - u_I\|^{\frac{1}{2}}\,. \tag{26}$$

Proof. Let $u_I \in K_I$ and $v_I \in K_I$, by definition of $\bar{u}(u_I)$ and $\bar{u}(v_I)$ we have
$$(A\bar{u}(u_I) - f, v - \bar{u}(u_I)) \;\geq\; 0 \quad \forall v \in \hat{K}(u_I)\,, \tag{27}$$
$$(A\bar{u}(v_I) - f, v - \bar{u}(v_I)) \;\geq\; 0 \quad \forall v \in \hat{K}(v_I)\,. \tag{28}$$
By definition of d_1 and (8) we have that $\exists\ \hat{u} \in \hat{K}(v_I)$ such that
$$\|\hat{u} - \bar{u}(u_I)\|_V \leq d_1(\hat{K}(u_I), \hat{K}(v_I)) \leq c\,\|u_I - v_I\|_{X_I} \tag{29}$$
and $\exists\ \hat{v} \in \hat{K}(u_I)$ such that
$$\|\hat{v} - \bar{u}(v_I)\|_V \leq d_1(\hat{K}(u_I), \hat{K}(v_I)) \leq c\,\|u_I - v_I\|_{X_I}\,. \tag{30}$$
By virtue of (27) and (28) we have
$$(A\bar{u}(u_I) - f, \hat{v} - \bar{u}(u_I)) \;\geq\; 0\,, \tag{31}$$
$$(A\bar{u}(v_I) - f, \hat{u} - \bar{u}(v_I)) \;\geq\; 0\,. \tag{32}$$

From (31) and (32) we obtain

$$(A\bar{u}\,(u_I)\,,\hat{v}-\bar{u}\,(u_I))-(A\bar{u}(v_I),\bar{u}(v_I)-\hat{u})\geq(f,\hat{u}-\bar{u}\,(u_I)+\hat{v}-\bar{u}\,(v_I))$$

then

$$\begin{aligned}(A\,(\bar{u}\,(u_I)-\bar{u}\,(v_I))\,,\bar{u}\,(v_I)-\bar{u}\,(u_I))&\\ +\,(A\bar{u}\,(u_I)\,,\hat{v}-\bar{u}\,(v_I))&\\ +\,(A\bar{u}\,(v_I)\,,\hat{u}-\bar{u}\,(u_I))&\geq(f,\hat{u}-\bar{u}\,(u_I)+\hat{v}-\bar{u}\,(v_I))\,.\end{aligned}\tag{33}$$

Considering that K is bounded and the relations (3), (29) and (30), we obtain the following estimates

$$|(f,\hat{u}-\bar{u}\,(u_I))|\leq\|f\|\;\|\hat{u}-\bar{u}\,(u_I)\|\leq C\;\|u_I-v_I\|_{X_I}$$

$$|(f,\hat{v}-\bar{u}\,(v_I))|\leq\|f\|\;\|\hat{v}-\bar{u}\,(v_I)\|\leq C\;\|u_I-v_I\|_{X_I}$$

$$|(A\bar{u}\,(u_I)\,,\hat{v}-\bar{u}\,(v_I))|\leq C\;\;\|\hat{v}-\bar{u}\,(v_I)\|\leq C\;\|u_I-v_I\|_{X_I}$$

$$|(A\bar{u}\,(v_I)\,,\hat{u}-\bar{u}\,(u_I))|\leq C\;\;\|\hat{u}-\bar{u}\,(u_I)\|\leq C\;\|u_I-v_I\|_{X_I}$$

$$(A\,(\bar{u}\,(u_I)-\bar{u}\,(v_I))\,,\bar{u}\,(v_I)-\bar{u}\,(u_I))\leq-\alpha\,\|\bar{u}\,(u_I)-\bar{u}\,(v_I)\|^2\,.$$

So, from (33) we get

$$\alpha\,\|\bar{u}\,(u_I)-\bar{u}\,(v_I)\|^2\leq C\;\|u_I-v_I\|\;,$$

or, in the equivalent form

$$\|\bar{u}(v_I)-\bar{u}(u_I)\|\leq C\;\|v_I-u_I\|^{\frac{1}{2}}\;.$$

□

If we use now the hypotheses (10) and (11) we can strengthen the previous continuity result to a Lipschitz continuity result.

Proposition 5 *If $\bar{u}(u_I)$ is the point that realizes the minimum of J in $\hat{K}(u_I)$, then $\bar{u}(u_I)$ is a Lipschitz continuous function with respect to the parameter u_I, i.e. there exists a positive constant k_S which verifies, $\forall u_I\in K_I$ and $\forall\delta u_I\in X_I$ such that $u_I+\delta u_I\in K_I$*

$$\|\bar{u}\,(u_I+\delta u_I)-\bar{u}\,(u_I)\|\leq k_S\,\|\delta u_I\|\;.\tag{34}$$

Proof. To simplify the notation we will use in the following paragraphs a common letter C to denote a generic constant independent of any particular point $u\in K$, $u_I\in K_I$, etc. (i.e. those constants C depend only on the data of the problem: the form $a\,(\cdot,\cdot)$, the convex K, etc.).

Let $u_I\in K_I$, $u_I+\delta u_I\in K_I$. By definition of $\bar{u}\,(u_I)$ and $\bar{u}\,(u_I+\delta u_I)$ we have

$$(A\bar{u}\,(u_I)-f,v-\bar{u}\,(u_I))\;\;\geq\;\;0\quad\forall v\in\hat{K}\,(u_I)\tag{35}$$

$$(A\bar{u}\,(u_I+\delta u_I)-f,v-\bar{u}\,(u_I+\delta u_I))\;\;\geq\;\;0\quad\forall v\in\hat{K}\,(u_I+\delta u_I)\,.\tag{36}$$

We put, respectively in (36) and (35), the following vectors

$$v = \bar{u}(u_I) + T_{\bar{u}(u_I)}(\delta u_I) \in \hat{K}(u_I + \delta u_I)$$

and

$$v = \bar{u}(u_I + \delta u_I) + T_{\bar{u}(u_I+\delta u_I)}(-\delta u_I) \in \hat{K}(u_I) .$$

In this form we obtain

$$\left(A\bar{u}(u_I + \delta u_I) - f, \bar{u}(u_I + \delta u_I) - \bar{u}(u_I) - T_{\bar{u}(u_I)}(\delta u_I)\right) \leq 0, \tag{37}$$

$$\left(A\bar{u}(u_I) - f, \bar{u}(u_I + \delta u_I) - \bar{u}(u_I) - T_{\bar{u}(u_I+\delta u_I)}(\delta u_I)\right) \geq 0. \tag{38}$$

Substracting (37) – (38) we get

$$\begin{aligned}
&(A\bar{u}(u_I + \delta u_I) - A\bar{u}(u_I), \bar{u}(u_I + \delta u_I) - \bar{u}(u_I)) \\
&\quad \leq \left(A\bar{u}(u_I + \delta u_I), T_{\bar{u}(u_I)}(\delta u_I)\right) - \left(A\bar{u}(u_I), T_{\bar{u}(u_I+\delta u_I)}(\delta u_I)\right) \\
&\qquad + \left(f, \left(T_{\bar{u}(u_I+\delta u_I)} - T_{\bar{u}(u_I)}\right)(\delta u_I)\right) \\
&\quad = \left(A\bar{u}(u_I + \delta u_I) - A\bar{u}(u_I), T_{\bar{u}(u_I)}(\delta u_I)\right) + \left(A\bar{u}(u_I), \left(T_{\bar{u}(u_I)} - T_{\bar{u}(u_I+\delta u_I)}\right)(\delta u_I)\right) \\
&\qquad + \left(f, \left(T\bar{u}(u_I + \delta u_I) - T_{\bar{u}(u_I)}\right)(\delta u_I)\right).
\end{aligned}$$

Then, by virtue of (3), we have

$$\begin{aligned}
&\alpha \left\|\bar{u}(u_I + \delta u_I) - \bar{u}(u_I)\right\|^2 \\
&\quad \leq \left\|A\bar{u}(u_I + \delta u_I) - A\bar{u}(u_I)\right\| \left\|T_{\bar{u}(u_I)}\right\| \left\|\delta u_I\right\| + \left\|A\bar{u}(u_I)\right\| \left\|T_{\bar{u}(u_I)} - T_{\bar{u}(u_I+\delta u_I)}\right\| \left\|\delta u_I\right\| \\
&\qquad + \|f\| \left\|T_{\bar{u}(u_I+\delta u_I)} - T_{\bar{u}(u_I)}\right\| \left\|\delta u_I\right\| \\
&\quad \leq \beta \left\|\bar{u}(u_I + \delta u_I) - \bar{u}(u_I)\right\| \left\|T_{\bar{u}(u_I)}\right\| \left\|\delta u_I\right\| + \left\|A\bar{u}(u_I)\right\| C \left\|\bar{u}(u_I + \delta u_I) - \bar{u}(u_I)\right\| \left\|\delta u_I\right\| \\
&\qquad + \|f\| C \left\|\bar{u}(u_I + \delta u_I) - \bar{u}(u_I)\right\| \left\|\delta u_I\right\|,
\end{aligned}$$

therefore

$$\alpha \left\|\bar{u}(u_I + \delta u_I) - \bar{u}(u_I)\right\| \leq \left(\beta \left\|T_{\bar{u}(u_I)}\right\| + C \left\|A\bar{u}(u_I)\right\| + C\|f\|\right) \left\|\delta u_I\right\| . \tag{39}$$

So, we obtain that (39) is equivalent to

$$\left\|\bar{u}(u_I + \delta u_I) - \bar{u}(u_I)\right\| \leq k_S \left\|\delta u_I\right\|,$$

where k_S is a constant independent of u_I and δu_I.

□

Properties of the function J_I

J_I was defined in (13) as $J_I(v_I) = \min_{v \in \hat{K}(v_I)} J(v)$. So, by virtue of the properties of J and $\hat{K}(v_I)$, the function J_I is well defined.

Convexity

Proposition 6 *If the hypothesis* (7) *holds, i.e.*

$$\lambda \hat{K}(\bar{v}_I) + (1-\lambda)\hat{K}(v_I) \subset \hat{K}(\lambda \bar{v}_I + (1-\lambda) v_I) ,$$

then $J_I(\cdot)$ is a convex function in K_I .

Proof. Let $v_I \neq \bar{v}_I \in K_I$, from (7) we have

$$J_I(\lambda \bar{v}_I + (1-\lambda) v_I) = \min_{v \in \hat{K}(\lambda \bar{v}_I + (1-\lambda) v_I)} J(v) \leq \min_{v \in \lambda \hat{K}(\bar{v}_I) + (1-\lambda)\hat{K}(v_I)} J(v) . \tag{40}$$

There exist $v_1 \in \hat{K}(\bar{v}_I)$ and $v_2 \in \hat{K}(v_I)$ such that

$$J(v_1) = J_I(\bar{v}_I) \qquad \text{and} \qquad J(v_2) = J_I(v_I) ,$$

then considering the convexity of J we get

$$J(\lambda v_1 + (1-\lambda) v_2) \leq \lambda J(v_1) + (1-\lambda) J(v_2)$$

or in an equivalent way

$$J(\lambda v_1 + (1-\lambda) v_2) \leq \lambda J_I(\bar{v}_I) + (1-\lambda) J_I(v_I) .$$

Hence

$$\min_{v \in \lambda \hat{K}(\bar{v}_I) + (1-\lambda)\hat{K}(v_I)} J(v) \leq J(\lambda v_1 + (1-\lambda) v_2) \leq \lambda J_I(\bar{v}_I) + (1-\lambda) J_I(v_I) .$$

Taking again (40), we have

$$J_I(\lambda \bar{v}_I + (1-\lambda) v_I) \leq \lambda J_I(\bar{v}_I) + (1-\lambda) J_I(v_I) . \tag{41}$$

□

Remark 9 *If the following property holds,*

$$v_I \neq \bar{v}_I \Rightarrow \hat{K}(v_I) \cap \hat{K}(\bar{v}_I) = \emptyset,$$

then J_I is strictly convex. The proof of this property is entirely similar to the previous one and it is omitted for the sake of brevity.

Proposition 7 *If the hypothesis* (9) *holds, i.e.*

$$d_2(\hat{K}(v_I), \hat{K}(\bar{v}_I)) \geq \gamma \left\| v_I - \bar{v}_I \right\| ,$$

then $J_I(\cdot)$ is strongly convex in K_I.

Proof. Let $v_I \neq \bar{v}_I \in K_I$, by (7) we have

$$J_I(\lambda\bar{v}_I + (1-\lambda)v_I) = \min_{v\in\hat{K}(\lambda\bar{v}_I+(1-\lambda)v_I)} J(v) \leq \min_{v\in\lambda\hat{K}(\bar{v}_I)+(1-\lambda)\hat{K}(v_I)} J(v)\,. \tag{42}$$

There exist $v_1 \in \hat{K}(\bar{v}_I)$ and $v_2 \in \hat{K}(v_I)$ such that

$$J(v_1) = J_I(\bar{v}_I) \quad \text{and} \quad J(v_2) = J_I(v_I)\,.$$

By proposition 2 we have

$$J(\lambda v_1 + (1-\lambda)v_2) \leq -\tfrac{\alpha}{2}\lambda(1-\lambda)\|v_2 - v_1\|^2 + \lambda J(v_1) + (1-\lambda)J(v_2)$$

and from (9) we get

$$\|v_1 - v_2\| \geq d_2(\hat{K}(v_I), \hat{K}(\bar{v}_I)),$$

then

$$J(\lambda v_1 + (1-\lambda)v_2) \leq -\tfrac{\alpha}{2}\gamma\lambda(1-\lambda)\|v_I - \bar{v}_I\|^2_{X_I} + \lambda J_I(\bar{v}_I) + (1-\lambda)J_I(v_I)\,.$$

Hence

$$\min_{v\in\lambda\hat{K}(\bar{v}_I)+(1-\lambda)\hat{K}(v_I)} J(v) \leq -\tfrac{\alpha}{2}\gamma\lambda(1-\lambda)\|v_I - \bar{v}_I\|^2_{X_I} + \lambda J_I(\bar{v}_I) + (1-\lambda)J_I(v_I)\,.$$

Considering (42) we get

$$J_I(\lambda\bar{v}_I + (1-\lambda)v_I) \leq -\tfrac{\alpha}{2}\gamma\lambda(1-\lambda)\|v_I - \bar{v}_I\|^2_{X_I} + \lambda J_I(\bar{v}_I) + (1-\lambda)J_I(v_I) \tag{43}$$

and therefore J_I is strongly convex in v_I.

□

Proposition 8 *There exists a unique point $\bar{u}_I \in K_I$ where the function J_I realizes the minimum on K_I.*

Proof. Let $\bar{u}$ be the minimum of J in K, since $K = \bigcup_{u_I\in K_I} \hat{K}(u_I)$, there exists $\hat{u}_I$ such that $\bar{u} \in \hat{K}(\hat{u}_I)$, then

$$J_I(\hat{u}_I) = \min_{u\in\hat{K}(\hat{u}_I)} J(u) \leq \min_{u\in K} J(u) = J(\bar{u}) \leq \min_{u\in\hat{K}(w_I)} J(u) = J_I(w_I) \qquad \forall w_I \in K_I\,,$$

which proves the optimality of $\hat{u}_I$, i.e. $\bar{u}_I = \hat{u}_I$.

We have proven the existence of $\bar{u}_I \in K_I$ that minimizes J_I. From the strictly convexity of J_I we get the uniqueness of the minimizing point.

□

Continuity

The function J_I can be written as $J_I(v_I) = J(\bar{u}(v_I))$, consequently it inherits the property of continuity in the variable v_I from the function $\bar{u}(v_I)$. We have the following:

Proposition 9 *$J_I(\cdot)$ is continuous in K_I and if hypotheses (10) and (11) holds, $J_I(\cdot)$ is Lipschitz continuous.*

Proof. Being $J_I(v_I) = \min\limits_{v \in \hat{K}(v_I)} J(v)$, there exists $\bar{u}(v_I) \in \hat{K}(v_I)$ such that

$$J_I(v_I) = J(\bar{u}(v_I)) .$$

We consider now $v_I + \delta v_I \in K_I$ and $\bar{u}(v_I + \delta v_I) \in \hat{K}(v_I + \delta v_I)$. We have

$$J_I(v_I + \delta v_I) = J(\bar{u}(v_I + \delta v_I))$$

and consequently

$$|J_I(v_I + \delta v_I) - J_I(v_I)| = |J(\bar{u}(v_I + \delta v_I)) - J(\bar{u}(v_I))| .$$

But J is Lipschitz continuous with respect to the parameter v, then there exists L_J such that

$$|J(\bar{u}(v_I + \delta v_I)) - J(\bar{u}(v_I))| \leq L_J \|\bar{u}(v_I + \delta v_I) - \bar{u}(v_I)\| .$$

Taking into account (26) we get, as $\|\delta v_I\| \to 0$.

$$|J_I(v_I + \delta v_I) - J_I(v_I)| \to 0 .$$

The proof of Lipschitz continuity is obvious and it is omitted for the sake of brevity.

□

Differentiability

Proposition 10 *The function $J_I : K_I \to \Re$ is Frèchet differentiable in K_I, and*

$$\frac{\partial J_I}{\partial v_I}(v_I) = T^*_{\bar{u}(v_I)}(A\,\bar{u}(v_I) - f) .$$

Proof. Let $v_I \in K_I$ and let δv_I be an admissible increment, i.e. $v_I + \delta v_I \in K_I$. We have

$$\left| \begin{array}{l} J_I(v_I) = \min\limits_{v \in \hat{K}(v_I)} J(v) = J(\bar{u}(v_I)) \\ \\ J_I(v_I + \delta v_I) = \min\limits_{v \in \hat{K}(v_I + \delta v_I)} J(v) = J(\bar{u}(v_I + \delta v_I)) . \end{array} \right. \tag{44}$$

For the sake of simplicity we use the following notation

$$\hat{u}_o = \bar{u}(v_I) \qquad \hat{u}_1 = \bar{u}(v_I + \delta v_I) .$$

Let $T_v : K_I \to V$ be the linear and continuous operator described in section 3.2. Since $v_I \in K_I$ and $v_I + \delta v_I \in K_I$, considering that $\hat{u}_o \in \hat{K}(v_I)$ we obtain $\hat{u}_o + T_{\hat{u}_o}\delta v_I \in \hat{K}(v_I + \delta v_I)$ and so

$$J_I(v_I + \delta v_I) \leq J(\hat{u}_o + T_{\hat{u}_o}\delta v_I) . \tag{45}$$

Since $v_I + \delta v_I \in K_I$ and $v_I = (v_I + \delta v_I) + (-\delta v_I) \in K_I$, considering that $\hat{u}_1 \in \hat{K}(v_I + \delta v_I)$, we obtain $\hat{u}_1 + T_{\hat{u}_1}(-\delta v_I) \in \hat{K}(v_I)$, then

$$J_I(v_I) \leq J(\hat{u}_1 - T_{\hat{u}_1}\delta v_I) . \tag{46}$$

The function J is Frèchet differentiable, property which allows us to express

$$J(\hat{u}_o + T_{\hat{u}_o}\delta v_I) = J(\hat{u}_o) + (A\hat{u}_o - f, T_{\hat{u}_o}\delta v_I) + o(T_{\hat{u}_o}\delta v_I), \tag{47}$$

$$J(\hat{u}_1 - T_{\hat{u}_1}\delta v_I) = J(\hat{u}_1) - (A\hat{u}_1 - f, T_{\hat{u}_1}\delta v_I) + o(T_{\hat{u}_1}\delta v_I). \tag{48}$$

From (44), (45) and (47) it follows that

$$J_I(v_I + \delta v_I) - J_I(v_I) \leq (A\hat{u}_o - f, T_{\hat{u}_o}\delta v_I) + o(T_{\hat{u}_o}\delta v_I). \tag{49}$$

From (44), (46) and (48) it follows that

$$\begin{aligned} J_I(v_I + \delta v_I) - J_I(v_I) &\geq J(\hat{u}_1) - J(\hat{u}_1 - T_{\hat{u}_1}\delta v_I) \\ &= (A\hat{u}_1 - f, T_{\hat{u}_1}\delta v_I) + o(T_{\hat{u}_1}\delta v_I). \end{aligned} \tag{50}$$

Taking into account that

$$A\hat{u}_1 - f = A\hat{u}_o - f + A(\hat{u}_1 - \hat{u}_o),$$

we obtain that

$$\begin{aligned} &(A\hat{u}_1 - f, T_{\hat{u}_1}\delta v_I) \\ &\quad = (A\hat{u}_o - f, T_{\hat{u}_o}\delta v_I) + (A\hat{u}_o - f, T_{\hat{u}_1}\delta v_I - T_{\hat{u}_o}\delta v_I) + (A(\hat{u}_1 - \hat{u}_o), T_{\hat{u}_1}\delta v_I). \end{aligned} \tag{51}$$

For $\hat{u}_o = \bar{u}(v_I) \in \hat{K}(v_I)$ and $\hat{u}_1 = \bar{u}(v_I + \delta v_I) \in \hat{K}(v_I + \delta v_I)$, we get by virtue of (34) that

$$\|\hat{u}_1 - \hat{u}_o\| \leq C\|\delta v_I\|,$$

besides

$$\begin{aligned} |(A\hat{u}_o - f, T_{\hat{u}_1}\delta v_I - T_{\hat{u}_o}\delta v_I)| &\leq \|A\hat{u}_o - f\| \, \|T_{\hat{u}_1}\delta v_I - T_{\hat{u}_o}\delta v_I\| \\ &\leq \|A\hat{u}_o - f\| \, \underbrace{\|T_{\hat{u}_1} - T_{\hat{u}_o}\|}_{\leq C\|\delta v_I\|} \, \|\delta v_I\| = o(\delta v_I) \end{aligned}$$

$$|(2A(\hat{u}_1 - \hat{u}_o), T_{\hat{u}_1}\delta v_I)| \leq 2\|A\| \, \underbrace{\|\hat{u}_1 - \hat{u}_o\|}_{\leq C\|\delta v_I\|} \, \|T_{\hat{u}_1}\| \, \|\delta v_I\| = o(\delta v_I).$$

We have seen that the last two terms of (51) are of order $o(\delta v_I)$, so we have proven that

$$(A\hat{u}_1 - f, T_{\hat{u}_1}\delta v_I) = (A\hat{u}_o - f, T_{\hat{u}_o}\delta v_I) + o(\delta v_I). \tag{52}$$

Moreover, from the continuity of T_v we get

$$o(T_{\hat{u}_o}\delta v_I) = o(\delta v_I), \qquad o(T_{\hat{u}_1}\delta v_I) = o(\delta v_I).$$

Considering (52) we can write (49) and (50) in the following way

$$(A\hat{u}_o - f, T_{\hat{u}_o}\delta v_I) + o(\delta v_I) \leq J_I(v_I + \delta v_I) - J_I(v_I) \leq (A\hat{u}_o - f, T_{\hat{u}_o}\delta v_I) + o(\delta v_I).$$

We conclude that

$$J_I(v_I + \delta v_I) - J_I(v_I) = \left(T^*_{\hat{u}_o}(A\hat{u}_o - f), \delta v_I\right) + o(\delta v_I)$$

and therefore $J_I(v_I)$ is Frèchet differentiable with respect to v_I and

$$\frac{\partial J_I}{\partial v_I}(v_I) = T^*_{\bar{u}(v_I)}(A\,\bar{u}(v_I) - f).$$

□

Lemma 4 *The operator*

$$\begin{aligned} & B : K_I \to X_I \\ & v_I \to B(v_I) = T^*_{\bar{u}(v_I)} \left(A\,\bar{u}(v_I) - f \right) \end{aligned}$$

is strongly monotone.

Proof. Considering the fact that $J_I(\cdot)$ is a strongly convex F-differentiable function (see [1]), there exists $\beta_{J_I} > 0$ such that

$$J_I(v_I) - J_I(\tilde{v}_I) \geq \left(\frac{\partial J_I}{\partial v_I}(\tilde{v}_I), v_I - \tilde{v}_I \right) + \beta_{J_I} \|v_I - \tilde{v}_I\|^2$$

$$J_I(\tilde{v}_I) - J_I(v_I) \geq \left(\frac{\partial J_I}{\partial v_I}(v_I), \tilde{v}_I - v_I \right) + \beta_{J_I} \|v_I - \tilde{v}_I\|^2 .$$

Adding both expressions we obtain

$$\left(\frac{\partial J_I}{\partial v_I}(\tilde{v}_I), \tilde{v}_I - v_I \right) - \left(\frac{\partial J_I}{\partial v_I}(v_I), \tilde{v}_I - v_I \right) \geq 2\beta_{J_I} \|v_I - \tilde{v}_I\|^2 ,$$

then

$$\left(B(\tilde{v}_I) - B(v_I), \tilde{v}_I - v_I \right) \;\; \geq \;\; 2\beta_{J_I} \|v_I - \tilde{v}_I\|^2 .$$

This proves that $B(v_I) = T^*_{\bar{u}(v_I)} \left(A\,\bar{u}(v_I) - f \right)$ is strongly monotone.

□

Proposition 11 *The operator B is Lipschitz continuous in K_I, i.e. there exists k_B such that*

$$\|B(u_I + \delta u_I) - B(u_I)\| \leq k_B \|\delta u_I\| .$$

Proof. We have

$$\begin{aligned} B(u_I + \delta u_I) - B(u_I) &= T^*_{\bar{u}(u_I + \delta u_I)} \left(A\bar{u}(u_I + \delta u_I) - f \right) - T^*_{\bar{u}(u_I)} \left(A\bar{u}(u_I) - f \right) \\ &= \left(T^*_{\bar{u}(u_I + \delta u_I)} - T^*_{\bar{u}(u_I)} \right) \left(A\bar{u}(u_I + \delta u_I) - f \right) \\ & \quad + T^*_{\bar{u}(u_I)} \left(A\bar{u}(u_I + \delta u_I) - A\bar{u}(u_I) \right) . \end{aligned}$$

then

$$
\begin{aligned}
&\|B(u_I+\delta u_I)-B(u_I)\| \\
&\quad\le \left\|T^{**}_{\bar{u}(u_I+\delta u_I)}-T^{**}_{\bar{u}(u_I)}\right\| \|A\bar{u}(u_I+\delta u_I)-f\| + \left\|T^{**}_{\bar{u}(u_I)}(A\bar{u}(u_I+\delta u_I)-A\bar{u}(u_I))\right\| \\
&\quad\le C\|\bar{u}(u_I+\delta u_I)-\bar{u}(u_I)\|(\|A\bar{u}(u_I+\delta u_I)\|+\|f\|)+\left\|T^{**}_{\bar{u}(u_I)}\right\|\beta\|\bar{u}(u_I+\delta u_I)-\bar{u}(u_I)\| \\
&\quad\le (Ck_S(C+\|f\|)+C\beta k_S)\|\delta u_I\|.
\end{aligned}
$$

Therefore

$$\|B(u_I+\delta u_I)-B(u_I)\|\le k_B\|\delta u_I\|.$$

□

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