Checking the Convexity of Polytopes and the Planarity of Subdivisions

Olivier Devillers, Giuseppe Liotta, Franco Preparata, Roberto Tamassia

INSTITUT NATIONAL DE RECHERCHE EN INFORMATIQUE ET EN AUTOMATIQUE

Checking the Convexity of Polytopes and the Planarity of Subdivisions

Olivier Devillers — Giuseppe Liotta — Franco P. Preparata — Roberto Tamassia

N° 3527

Octobre 1998

THÈME 2

Rapport de recherche

ISSN 0249-6399

Checking the Convexity of Polytopes and the Planarity of Subdivisions

Olivier Devillers , Giuseppe Liotta , Franco P. Preparata , Roberto Tamassia

Thème 2 — Génie logiciel
et calcul symbolique
Projet Prisme

Rapport de recherche n° 3527 — Octobre 1998 — 31 pages

Abstract: This paper considers the problem of verifying the correctness of geometric structures. In particular, we design simple optimal checkers for convex polytopes in two and higher dimensions, and for various types of planar subdivisions, such as triangulations, Delaunay triangulations, and convex subdivisions. Their performance is analyzed also in terms of the algorithmic degree, which characterizes the arithmetic precision required.

Key-words: Computational geometry, Delaunay triangulation, convex hull, exact arithmetic.

Adresses: O. Devillers, INRIA.
G. Liotta, Dipartimento di Informatica e Sistemistica, Universitá di Roma "La Sapienza", Via Salaria 113, Roma 00198, Italy.
F. Preparata and R. Tamassia, Center for Geometric Computing, Department of Computer Science, Brown University, 115 Waterman Street, Providence, RI 02912.
Research supported in part by the National Science Foundation under grant CCR–9732327, by the U.S. Army Research Office under grant DAAH04–96–1–0013, by the EC ESPRIT Long Term Research Project ALCOM-IT under contract 20244, by the CNR Project "Geometria Computazionale Robusta con Applicazioni alla Grafica ed al CAD," and by CGAL under contract 21957. Work by the first author performed in part while he was visiting the Center for Geometric Computing at Brown University. Work by the second author performed in part while he was with the Center for Geometric Computing at Brown University.

Unité de recherche INRIA Sophia Antipolis
2004, route des Lucioles, B.P. 93, 06902 Sophia Antipolis Cedex (France)
Téléphone : 04 92 38 77 77 - International : +33 4 92 38 77 77 —Fax : 04 92 38 77 65 - International : +33 4 92 38 77 65

Verification de la convexité de polytopes et de la planarité de sundivisions

Résumé : Dans cet article, on s'intéresse au problème de la vérification de la validité de certaines structures géométriques. En particulier, nous proposons un vérificateur, simple et optimal, pour les polytopes convexes en dimensions deux et supérieures, et pour divers types de subdivisions planaires telles que triangulations, triangulations de Delaunay ou subdivisions convexes. Les performances sont également analysées en termes de degré algorithmique qui caractérise la précision requise pour l'arithmétique utilisée.

Mots-clés : Géométrie algorithmique, triangulation de Delaunay, enveloppe convexe, arithmétique exacte.

1 Introduction

The development of checkers for geometric structures is justified by the expectation that it is easier to evaluate the quality of the output than the correctness of the algorithm producing it, and is further motivated by the increasing availability of geometric software on the Internet (see, e.g., [1]), and by the emerging client-server distributed models of geometric computing over the Web (see, e.g., [2]). Mehlhorn *et al.* [21] identify three fundamental features of a good checker: *correctness*, *simplicity*, and *efficiency*.

In this paper, we consider checkers for subdivisions in two and higher dimensions. In particular, we consider two-dimensional planar subdivisions and convex polytopes in a fixed dimension d. The subdivision to be checked can be either a primitive structure, or a derived structure computed from a primitive one (e.g., the Voronoi diagram of a set of sites in the plane). This paper considers checkers for various types of planar subdivisions and for convex polytopes in two and higher dimensions. We usually assume that the embedding (circular ordering of the edges around each vertex) of the input subdivision or polytope is given as part of the input. When this is not the case, i.e., no embedding information is provided, we call the checker a *strong checker*.

We advocate a new requirement that good geometric checkers should satisfy, and present simple and efficient strong checkers for several structures for which only checkers have been so far designed. Specifically, our contributions can be summarized as follows.

1. As an additional measure of effectiveness for a checker, we adopt the notion of *degree* [18, 19, 5], which takes into account the number of bits required by the checker to carry out error-free computations. A good checker should have degree no higher than the problem at hand allows. We give lower bounds on the degree of checkers for planar subdivisions and convex polytopes.

2. We present a checker for convex polytopes that involves novel ideas and has the potential to lead to a simpler implementation than the one given in [21]. Our checker works in any dimension and recursively reduces the verification of a d-dimensional polytope to the verification of an associated $(d-1)$-dimensional polytope. The checker is optimal with respect to both the time complexity and the degree. The design of our checker for convex polytopes reveals new combinatorial and geometric properties that may be of independent interest.

3. We present linear-time optimal-degree checkers for triangulations, convex subdivisions, and general planar subdivisions. Such checkers use as subroutines elementary graph algorithms and do not require to test the planarity of the underlying input graph.

4. Extending the above results on checkers, we present linear-time optimal-degree strong checkers for triangulations and convex subdivisions. This solves significant special cases of an open problem mentioned by Kirkpatrick [15] on the existence of an $o(n \log n)$ algorithm to verify the planarity of a geometric graph with n vertices.

5. As a further application, we give linear-time optimal-degree strong checkers for Delaunay triangulations, locally-minimum-weight triangulations and Delaunay diagrams.

Near the completion of our investigations, we became aware of two ongoing projects on the design of checkers for planar subdivisions, including triangulations and convex subdivisions. A manual [20] describing the functionality of C++ functions that implement checkers for Delaunay triangulations, Voronoi diagrams and convex planar subdivisions is available from Mehlhorn's Web page. A manuscript in progress [22] contains characterizations of triangulations and convex planar subdivisions similar to those of the present paper.

Since these two concomitant and independent research efforts address the same class of problems, minor overlaps are unavoidable. However, our approach is markedly different (although equivalent) to that of [22, 20]. In addition, we show that our checkers can be made strong while maintaining the same efficiency as the (non strong) checkers of [22, 20].

The rest of this paper is organized as follows. Preliminaries and definitions are given in Section 2. Our checker for convex polytopes is presented in Section 3. Section 4 is devoted to checkers for triangulations, convex subdivisions, and general planar subdivisions. Strong checkers are studied in Section 5. The degree of checkers is explored in Section 6. Finally extension and open problems are discussed in Section 7.

2 Definitions

2.1 Geometric graphs, ordered graphs, and planarity

A d-dimensional *geometric graph* is a graph drawn with straight-line edges in d-dimensional space, i.e., a graph whose vertices have d-dimensional coordinates. In the following, we often denote with Γ a geometric graph, and with G its underlying combinatorial structure. To simplify the notation, and when the context is unambiguous, we may denote with G both the geometric graph and its underlying combinatorial structure.

A two-dimensional geometric graph Γ is *planar* if it has no crossing edges, i.e., any two edges of Γ intersect only at a common vertex. For every planar graph G, there exists a planar geometric graph Γ with underlying graph G, i.e., every planar graph admits a planar straight-line drawing (see, e.g., [11]). However, a geometric graph with an underlying planar graph is not necessarily planar.

A planar geometric Γ graph determines a *planar subdivision* S, i.e., a partition of the plane into regions called faces. Planar subdivision S is said to be *induced* by Γ. A planar subdivision is *convex* if the boundary of each face is a convex polygon. A planar subdivision S induced by geometric graph Γ is *maximal* if there is no other planar geometric graph Γ' such that Γ is a subgraph of Γ'. In a maximal planar subdivision, the boundary of each internal face is a triangle, and the boundary of the external face is a convex polygon. A maximal planar subdivision is also called a *triangulation*.

A graph G is *ordered* (or G has an *ordering*) if for each vertex v of G, a circular ordering of the edges incident on v is given. The ordering of a graph is usually denoted with Ψ. The *natural ordering* of a two-dimensional geometric graph Γ is given by the clockwise circular sequence of the edges incident on each vertex. The *natural ordering* of a three-dimensional convex polytope is similarly defined.

An ordering Ψ of a graph induces a set of directed *circuits*, where the edge (v, w) following (u, v) in a circuit is the successor of (u, v) in the circular ordering of the edges incident on v. Every edge of the graph is traversed by exactly two circuits, once in each direction.

Let Γ be a geometric graph, let Ψ be its natural ordering, and let v be a vertex of Γ with largest y-coordinate. Let (u, v) and (v, w) be two edges of Γ incident on v such that the wedge defined by two rays emanating from v and passing through u and w contains all other edges incident on v. We call *outer circuit* of Γ the circuit induced by Ψ that contains edges (u, v) and (v, w). The remaining circuits induced by Ψ are said to be *internal circuits*. If Γ is planar, then the circuits induced by

Ψ are the boundaries of the faces of the subdivision S induced by Γ. In particular, the outer circuit is the boundary of the external face traversed clockwise, and the internal circuits are the boundaries of the internal faces traversed counterclockwise.

The ordering Ψ of an ordered graph G is *planar* if there exists a planar geometric graph Γ whose underlying graph is G and whose natural ordering is Ψ. A planar ordering of G is associated with a planar (topological) embedding of G. A graph G admits a planar ordering only if it is planar. Also, the number of distinct planar orderings (topological embeddings) of a planar graph can be super-exponential.

2.2 Polytopes

A 3-*dimensional polytope* is the (convex) 3-dimensional intersection of 3-dimensional half-spaces.

A *simplicial* 3-*dimensional polyhedron* Γ is a 3-dimensional geometric graph with a natural planar ordering that induces faces that are 3-cycles. The faces induced by the planar ordering correspond to the facets of Γ. Γ is *convex* if it is the boundary of a convex polytope. It is *locally convex* if for each edge (p_1, p_2) $(p_i = (x_i, y_i, z_i))$, with incident facets $p_1p_2p_3$ and $p_1p_2p_4$, the simplex (p_1, p_2, p_3, p_4) lies in the interior half-space of each of the two planes supporting the facets. We assume that facets of 3-dimensional polyhedra and edges of 2-dimensional polygons are open sets.

A *simplicial* d-*dimensional polyhedron* is a simplicial $(d-1)$-dimensional surface without boundary. A d-*dimensional polytope* is the (convex) d-dimensional intersection of d-dimensional half-spaces.

2.3 Degree of Geometric Algorithms and Problems

The numerical computations of a geometric algorithm are basically of two types: tests (predicates) and constructions. Tests are associated with branching decisions in the algorithm that determine the flow of control, whereas constructions are needed to produce the output data of the algorithm.

Approximations in the execution of constructions may be acceptable depending on the resolution required by the application. On the other hand, approximations in the execution of tests may produce an incorrect branching of the algorithm and give rise to structurally incorrect results. The exact-computation paradigm [24] therefore requires that tests be executed with total accuracy.

Geometric algorithms can be studied on the basis of the complexity of their test computations. Any such computation consists of evaluating the sign of an algebraic expression over the input variables, constructed using an adequate set of operators, such has $\{+, -, \times, \div, \sqrt[2]{}\}$. This can be reduced to the evaluation of the signs of multivariate polynomials derived from the expression.

We make here the reasonable assumption that input data be dimensionally consistent, i.e., that, if an entity with the physical dimension of a length is represented with b bits, then one with the dimension of an area be represented with $2b$ bits, and so on. Under the hypothesis of dimensional consistency (where point coordinates are b-bit entries), a polynomial expressing a test is homogeneous because all of its monomials must have the same physical dimension.

A primitive variable is an input variable of the algorithm and has conventional arithmetic degree 1. The arithmetic degree of a polynomial expression E is the common arithmetic degree of its monomials. The arithmetic degree of a monomial is the sum of the arithmetic degrees of its variables. An algorithm has *degree* d if its test computations involve the evaluation of multivariate polynomials of arithmetic degree at most d [18, 19] (a related concept is defined in [5]). A problem Π has *degree* d if any algorithm that solves Π has degree at least d.

3 Checkers for Convex Polytopes

In this section, we describe the design of a checker for convex polytopes in a fixed dimension d.

Because of its appeal to intuition, we consider the question for $d \leq 3$ (specifically for $d = 2$ and $d = 3$) and indicate later how to extend the result to higher dimensions. In Subsections 3.2 and 3.3, we simplify the notation and use the term polyhedron to denote a simplicial[1] 3-dimensional *polyhedron* and the term *polytope* to denote a 3-dimensional polytope.

The input to the checker is:

- a 3-dimensional geometric graph Γ;
- an ordering Ψ of Γ such that every circuit induced by Ψ has three edges.

The task of the checker is to verify that Γ is the boundary of a convex polytope with natural ordering Ψ.

3.1 Previous Results

Let Γ be a locally convex polygon. Given an edge e of Γ, the line through e divides the plane into two half-planes. The closed half-plane containing the two edges incident

[1] Any polytope can be viewed as simplicial by trivial face triangulation

on e is said to be the *negative* half-plane of e. The other half-space is said to be *positive*. The *core* of Γ, denoted by $\chi(\Gamma)$, is the interior of the convex polygon obtained as the intersection of all negative half-planes defined by the edges of Γ. Note that the above definition of core is different from that of the kernel of a simple (i.e., not intersecting) polygon.

Given a facet f of a locally convex polyhedron Γ, the plane supporting f divides the space into two half-spaces. We call *negative* the half-space that contains the facets adjacent to f. The *core* of Γ, denoted by $\chi(\Gamma)$, is the interior of the convex polytope $\chi(\Gamma)$ obtained as the intersection of all negative half-spaces defined by the facets of Γ.

Mehlhorn et al. [21] have proved results equivalent to those stated in the theorems and lemmas that follow.

Lemma 1 [21] *Let Γ be a locally convex polygon and let q be any point of $\chi(\Gamma)$. Every ray emanating from q intersects the same number of edges and/or vertices of Γ.*

Theorem 2 [21] *A locally convex polygon Γ is globally convex if and only if a ray emanating from a point of $\chi(\Gamma)$ intersects Γ at a single edge or at a single vertex.*

Lemma 3 [21] *Let Γ be a locally convex polyhedron and let q be any point of $\chi(\Gamma)$. Every ray emanating from q and intersecting Γ only at facets intersects the same number of facets of Γ.*

Theorem 4 [21] *A locally convex polyhedron Γ is the boundary of a convex polytope if and only if any ray emanating from a point of $\chi(\Gamma)$ and intersecting a facet of Γ does not have any other intersection with Γ.*

Based on Theorem 4, Mehlhorn et al. [21] check whether a locally convex polyhedron is the boundary of a convex polytope by first computing a point q of $\chi(\Gamma)$ and a ray r emanating from from q and passing through the centroid of an arbitrarily chosen facet, and then checking that no other facet is intersected by r. Clearly, this checker runs in time linear in the number of vertices, and is therefore optimal.

3.2 A New Convexity Criterion

We start by characterizing the global convexity of a locally convex polygon. Let Γ be a locally convex polygon, given as a circular sequence of vertices. A vertex v of Γ is said to be a *2-seam vertex* of Γ if:

- the intersection of the negative half-planes defined by the two edges sharing v is contained in one of the closed half-planes defined by the vertical (i.e., parallel to the y-axis) line ℓ through v, and
- the upper (according to the positive y-direction) edge incident on v is contained in one of the open half-planes defined by ℓ.

The 2-*seam* of Γ is the set of its 2-seam vertices. The following theorem is straightforward.

Theorem 5 *A locally convex polygon is globally convex if and only if its* 2*-seam consists of two vertices.*

We now discuss the more complicate case of checking the convexity of a (3-dimensional) polyhedron. Let Γ be a locally convex polyhedron.

An edge e of Γ is said to be a *seam edge* of Γ if:

- e is not vertical,
- the intersection of the negative half-spaces defined by the two facets sharing e is contained in one of the closed half-spaces defined by the vertical (i.e., parallel to the z-axis) plane Π through e, and
- the upper (according to the positive z-direction) facet incident on e is contained in one of the open half-spaces defined by Π.

The *seam* of Γ, denoted by $\sigma(\Gamma)$, is the subgraph of Γ induced by its seam edges.

Lemma 6 *For each vertex v of $\sigma(\Gamma)$ there are at least two seam edges incident on v.*

Proof We shall constructively show that, given a seam edge (u, v), we can find another seam edge (v, u'). Consider a sphere S whose center is v and whose radius is small enough so that any other vertex of Γ and edge of Γ not incident to v is outside S. The intersection between S and Γ defines a spherical polygon P on S.

The local convexity of Γ induces local convexity on P. We observe that seam edges of Γ incident to v correspond to vertices of P that have a supporting meridian (i.e., a great circle in a vertical plane). Hence, denoting by w be the vertex of P defined as the intersection point between (u, v) and S, we have that w has a supporting meridian. Thus, starting at w it suffices to follow the sequence of edges

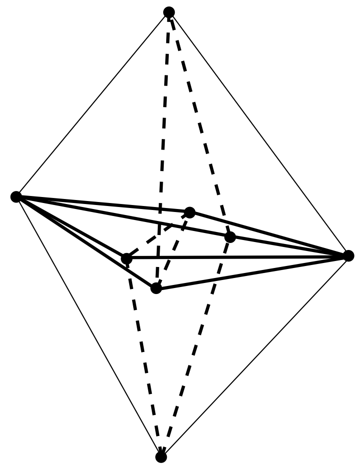

Figure 1: A locally convex polygon that has a degenerate seam. The seam edges are drawn with thick lines.

on P to find another vertex w' of P with supporting meridian, which corresponds to a new seam edge (v, u'). ■

We say that the seam $\sigma(\Gamma)$ is *degenerate* if it has a vertex with more than two seam edges incident on it. Figure 1 shows a degenerate seam.

Lemma 7 *If $\sigma(\Gamma)$ is degenerate, then Γ is not globally convex.*

Proof Let e_1, e_2, and e_3 be three seam edges incident on the same vertex v. Edges e_1, e_2, and e_3 have three vertical supporting planes Π_1, Π_2, and Π_3, respectively. Let W be the common intersection of the negative half-spaces defined by Π_1, Π_2, and Π_3. Without loss of generality, let Π_1 and Π_2 be the planes delimiting W. In order to be globally convex, Γ must be inside W. However, Π_3 is not internal to W. If it is external to W, so is e_3; if it coincides with, say, Π_1, then e_3 is still external to W, otherwise it would violate the seam edge definition. In either case, we conclude that Γ is not globally convex. ■

The following lemma reveals the combinatorial structure of the seam $\sigma(\Gamma)$. In the lemma, the term disjoint cycles has to be intended in the graph sense.

Lemma 8 *If $\sigma(\Gamma)$ is nondegenerate, then it is a collection of disjoint cycles.*

Proof Since $\sigma(\Gamma)$ is nondegenerate, every vertex of $\sigma(\Gamma)$ has at most two incident seam edges by definition. By Lemma 6, every vertex of $\sigma(\Gamma)$ has at least two incident seam edges. ∎

By *z-projection* of a geometric object o (point, edge, face) we mean its projection parallel to the z-axis and we denote it as o'.

Lemma 9 *Let $\sigma(\Gamma)$ be nondegenerate, let q be a point of $\chi(\Gamma)$, let r be any nonvertical ray emanating from q, and let r' be the z-projection of r. If the intersection of r with Γ is at k facets, then the intersection of r' with the z-projection $\sigma'(\Gamma)$ of $\sigma(\Gamma)$ is at k edges and/or vertices.*

Proof Assume that the intersection of ray r with Γ is at the k facets $f_1, \ldots, f_k$, $k \geq 2$. Consider the vertical plane α containing r, and let $\alpha^\star \subset \alpha$ be the half-plane containing r and limited by the vertical line ℓ through q.

Let p_i be the intersection of r with f_i. Points p_i $(i = 1, \ldots, k)$ belong to a closed curve in α (since Γ is a surface without boundary). Since Γ is a locally convex polyhedron, such closed curve in α is a locally convex polygon and ℓ traverses the core of such polygon. As a consequence, each point p_i belongs to a distinct polygonal chain in $\alpha^\star$ (see Figure 2 for the case $k = 2$).

Starting from p_i and proceeding on the corresponding polygonal chain by increasing z we reach a point of maximum z in $\alpha^\star$, denoted t_i (either t_i is a local maximum in α, or t_i is on ℓ). Analogously, if we proceed from p_i by decreasing z we will reach a point of minimum z in $\alpha^\star$, denoted b_i. If t_i is not a point of ℓ, then it admits a horizontal supporting line in α; otherwise, the angle δ formed by the ascending polygonal line at t_i with the vertical by q is $\delta > \pi/2$ (see Figure 2). Analogously, b_i either admits a horizontal supporting line or the angle of the corresponding angle is $\delta' < \pi/2$. We conclude that traversing downwards the subchain γ_i from t_i to b_i we reach a point s_i which is the first to admit a vertical supporting line in $\alpha^\star$, i.e., s_i belongs to the seam. Observe that, because of the local convexity of Γ, there is no other point on γ_i having a supporting vertical line.

Thus, for each γ_i in $\alpha^\star$ intersected by r there is a unique point s_i that belongs to the seam. Hence, since intersection is preserved by projection, each s_i projects to a point on the z-projection r' of r. Also, because $\sigma(\Gamma)$ is nondegenerate, it cannot happen that two distinct s_i and s_j project to the same point on r', which proves the lemma. ∎

The above results allow us to derive a new characterization of convex polytopes and reduce by one the dimensionality of the criterion given by Theorem 4.

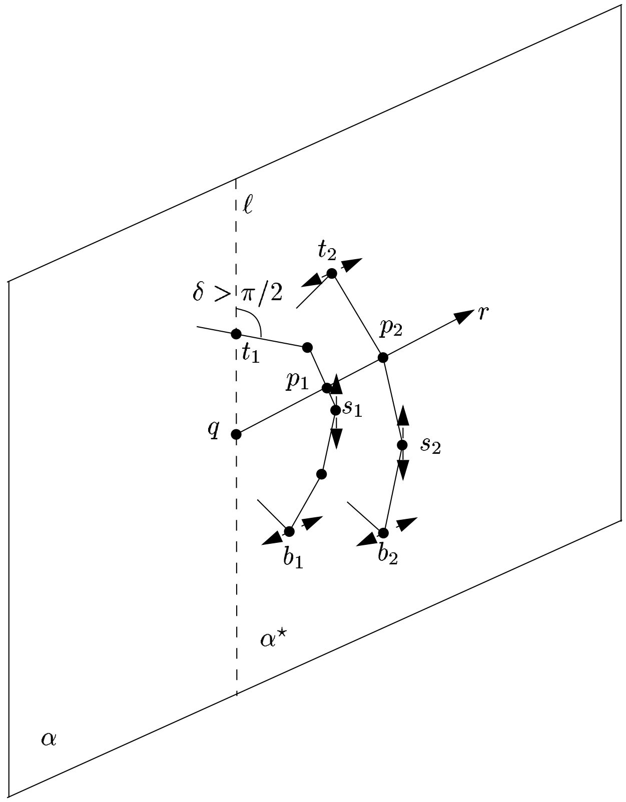

Figure 2: Example of cross section in plane $\alpha^\star$ for the proof of Lemma 9.

Theorem 10 *A 3-dimensional simplicial polyhedron Γ is the surface of a convex polytope if and only if the following two conditions hold:*

1. *Γ is locally convex,*
2. *$\sigma'(\Gamma)$ is a globally convex polygon.*

Proof We first show that if Γ is globally convex, then Conditions 1, and 2 are satisfied. Clearly, if Γ is globally convex, it is also locally convex. Also, the z-projection of a globally convex polyhedron is globally convex and coincides with the z-projection of its seam.

We now prove that if Conditions 1, and 2 are satisfied, then Γ is globally convex. By Condition 2, and because of Lemma 8, we have that $\sigma(\Gamma)$ is nondegenerate. Also, since $\sigma'(\Gamma)$ is a globally convex polygon, its core coincides with the polygon itself, i.e. $\chi(\sigma'(\Gamma)) = \sigma'(\Gamma)$.

We begin by showing that the core of Γ, $\chi(\Gamma)$, is nonempty. Take any point q' of $\chi(\sigma'(\Gamma))$ and consider a vertical plane Π through q'. The intersection between Π and Γ is a closed curve, since Γ is a surface without boundary. Let γ be such a curve. Observe that γ is a locally convex polygon since, by Condition 1, Γ is locally convex. Also, each 2-seam vertex of γ corresponds to a seam edge of Γ that is vertically projected to $\sigma'(\Gamma)$. Since Π intersects $\sigma'(\Gamma)$ at two edges and/or vertices, we have that γ can have only two 2-seam vertices. Thus, by Theorem 5, γ is a globally convex polygon. Hence, a vertical line ℓ through q' intersects γ only twice. We take any point along ℓ and in the interior of polygon γ and call it q. Observe that rotating the vertical plane Π around ℓ, one can define infinitely many globally convex polygons, given by the intersection of the rotated plane and Γ. Also notice that q lies in the interior of any of such globally convex polygons. We can conclude that q is on the negative side of all facets of Γ, and thus it belongs to $\chi(\Gamma)$.

We are now in the position of show that Γ is globally convex. Consider point q and a nonvertical ray r emanating from q such that r intersects Γ only at facets (clearly, there always exists such a ray). Let k be the number of facets of Γ intersected by r. We prove that $k = 1$ and thus, by Theorem 4, Γ is globally convex.

Let r' be the z-projection of r. Ray r' originates from point q' that is a point of $\chi(\sigma'(\Gamma))$ by construction. By Lemma 9, the intersection of r' with the z-projection $\sigma'(\Gamma)$ of $\sigma(\Gamma)$ consists of k edges and/or vertices. Since $\sigma'(\Gamma)$ is globally convex, by Lemma 1 every ray emanating from q' intersects $\sigma'(\Gamma)$ the same number of times. Also, by Theorem 2, r' intersects $\sigma'(\Gamma)$ at exactly one edge and/or vertex, that is $k = 1$. ∎

3.3 Checking Algorithm

Theorem 10 allows us to check whether a 3-dimensional simplicial locally convex polyhedron is the boundary of a convex polytope by testing the z-projection of its seam $\sigma'(\Gamma)$ for global convexity using Theorem 5.

The pseudocode of our checker for 3-dimensional convex polytopes is given in Fig. 3.

Lemma 11 *Algorithm Check-3-Polytope correctly checks whether a polyhedron is the boundary of a convex polytope.*

Proof The certificate describing the topology is used in the local convexity test. It identifies the seam edges, and the two vertices of each such edges are tested

```
Algorithm Check-3-Polytope

r ← 0
foreach vertex v ∈ Γ do
  d(v) ← 0
foreach edge e ∈ Γ do
  if e is not locally convex
    then return false
    else if e is a seam edge
      then let w[0] and w[1] be the two endpoints of e
        for i ← 0,1 do
          v ← w[i]
          u ← w[(i + 1) mod 2]
          if d(v) = 2
           then return false
           else if d(v) = 0
             then d(v) ← 1
               p(v) ← u
             else { d(v) = 1 }
               d(v) ← 2
                if v is a right-2-seam vertex (based on p(v) and u)
                  then if r = 0
                    then r ← 1
                    else return false
return true
```

Figure 3: Checker for 3-dimensional convex polyhedra.

for memberships in the 2-seam. Since, among the vertices of the 2-seam there are identical numbers of x-minimum vertices and x-maximum vertices, it suffices to verify that there is only one x-maximum vertex.

The algorithm examines each vertex that is the endpoint of a seam edge and counts the number of seam edges incident on it by means of variable $d(v)$. If $d(v) > 2$ then the seam is degenerate and, by Lemma 7, Algorithm Check-3-Polytope concludes that the input is not the boundary of a convex polytope.

Otherwise, the algorithm verifies whether the z-projection of v belongs to the 2-seam and has a local maximum abscissa, by comparing the x-coordinate of v with that of the other two endpoints u and $p(v)$ of the seam edges incident on v. Namely, if the x-value of v is larger than the x-values of both u and $p(v)$, then we say that v belongs to the *right-2-seam*. If the x-value of v is equal to the abscissa of one of its neighbors in the seam, say $p(v)$, and larger the abscissa of u, then v belongs to the right-2-seam only if the y-value of v is larger or equal to the y-value of $p(v)$. Otherwise, v does not belong to the right-2-seam.

When a vertex is determined to be in the right-2-seam if the variable r has value 0, then it is set to 1. Otherwise, if the value of r is found to be already 1 (which means that there are at least two vertices in the right-2-seam), Algorithm Check-3-Polytope concludes that the input is not the boundary of a convex polytope. ∎

Theorem 12 *Let Γ be a locally convex polyhedron with n vertices in 3-dimensional space. Checker Check-3-Polytope verifies whether Γ is the boundary of a convex polytope in $O(n)$ time.*

3.4 Extension to d-dimensional Space

The approach described in the previous sections can be generalized to higher dimensions. Let Γ be a simplicial polyhedron in d dimensions. The idea is to prove that Γ is convex if and only if its seam $\sigma_d(\Gamma)$ is nondegenerate and convex, which is done by recursive application of the seam technique. A j-facet of Γ is a simplex defined by j linearly independent points; an ordinary facet is a $(d-1)$-facet, a $(d-2)$-facet is called a *ridge*.

A ridge r of Γ is said to be a *seam ridge* of Γ if:

- r is not vertical (i.e. parallel to the dth coordinate axis),
- the intersection of the negative half-spaces defined by the two facets sharing r is contained in one of the closed half-spaces defined by the vertical plane Π through r, and
- the upper facet incident on r is contained in one of the open half-spaces defined by Π.

The *seam* of Γ, denoted by $\sigma_d(\Gamma)$, is the subgraph of Γ induced by its seam ridges. We can show that for each $(d-3)$-facet g of $\sigma_d(\Gamma)$, there are at least two seam ridges incident on g. The proof is similar to that of the three-dimensional case. Let S be the locus of points within distance δ from g, for suitably small δ, and consider a cross section within S by a three-dimensional flat perperdicular to g. The resulting three-dimensional intersection exactly reproduces the situation of Lemma 6 and we can conclude similarly.

We say that the seam $\sigma_d(\Gamma)$ is *degenerate* if it has a $(d-2)$-facet with more than two seam ridges incident on it. Lemma 7 applies also in higher dimension: If the seam is degenerate, then Γ is not globally convex. Lemma 9 also generalizes easily. Given a nonvertical (not parallel to the d-th coordinate) ray r, we consider the vertical two-dimensional plane α containing r, and we conclude that to any intersection of r and Γ there corresponds an intersection of the vertical projection of r with $\sigma_d(\Gamma)$ (see Figure 2).

We can now state the generalization of Theorem 10:

Theorem 13 *A d-dimensional simplicial polyhedron Γ is the surface of a convex polytope if and only if the following two conditions hold:*

1. *Γ is locally convex,*
2. *The projection of the seam $\sigma_d(\Gamma)$ is a globally convex $d-1$-dimensional polyhedron.*

Proof The proof is similar to that of Theorem 10. If Γ is convex the two conditions holds easily. If the two conditions holds, then a point in the core of Γ can be found from a point in the core of the projection of $\sigma_d(\Gamma)$. The conclusion derives from the generalization of Lemma 9. ∎

Applying recursively Theorem 13 and using the fact that local convexity of the seam is implied by the local convexity of the polyhedron, we have the following theorem, where $\sigma_j(\Gamma)$ denotes the seam of the projection of $\sigma_{j+1}(\Gamma)$ on the first j coordinates.

Theorem 14 *A d-dimensional simplicial polyhedron Γ is the surface of a convex polytope if and only if the following conditions hold:*

1. *Γ is locally convex,*
2. *The seams $\sigma_j(\Gamma)$ are non degenerate for $3 \leq j \leq d$.*
3. *The 2-seam $\sigma_2(\Gamma)$ is globally convex, that is, it consists of only two vertices.*

Theorem 14 is the basis of the checker for d-dimensional convex polytopes given in Fig. 4.

Theorem 15 *Let Γ be a locally convex polyhedron with n vertices in d-dimensional space. For constant d, checker Check-d-Polytope verifies whether Γ is the boundary of a convex polytope in $O(n)$ time.*

Algorithm *Check-d-Polytope*

$\sigma_2 \leftarrow \emptyset$
for $j = 0$ to $d-2$
  foreach j-facet $f \in \Gamma$ **do** $d(f) \leftarrow 0$
foreach ridge $r \in \Gamma$ **do**
  if r is not locally convex
    then return false
  else if r in d-seam
    then $\sigma' \leftarrow \{r\}$
      for $j = d$ downto 3
        $\sigma \leftarrow \sigma'$; $\sigma' \leftarrow \emptyset$
        foreach $(j-1)$-facet g of a j-facet f of σ **do**
          if $d(g) = 0$ **then** $d(g) \leftarrow f$
          else if $d(g) = full$ **then return** false $\{\sigma_j(\Gamma)$ degenerate$\}$
          else $\{d(g) \notin \{0, full\}\}$
            if g is in $j-1$-seam   {orientation test on $f \cup d(g)$}
              then $\sigma' \leftarrow g \cup \sigma'$; $d(g) \leftarrow full$
      $\sigma_2 \leftarrow \sigma_2 \cup \sigma'$ {2-seam vertices of r are in σ'}
if $|\sigma_2| \neq 2$ **return** false $\{\sigma_2(\Gamma)$ not globally convex$\}$
return true

Figure 4: Checker for d-dimensional convex polyhedra.

4 Checkers for Planar Subdivisions

In this section we present checkers that verify different classes of 2-dimensional geometric graphs. The input to the checkers is

- a 2-dimensional geometric graph Γ;
- the natural ordering Ψ of Γ.

The checkers perform the following tasks, respectively:

- verify whether Γ is a planar triangulation;
- verify whether Γ is a convex planar subdivision;

4.1 Planar Orderings

A building block of such checkers is an algorithm that tests whether an ordering Ψ of a graph G is planar. A linear-time algorithm for answering the question was given by Kirkpatrick in [15]. His algorithm considers the circuits induced by Ψ and adds to G a new vertex v_c for each induced circuit c and a new edge (v_c, w) for each vertex w of c. The resulting augmented graph G^* is planar if and only if the ordering Ψ is planar. Thus, the planarity of Ψ can be checked by running a planarity-testing algorithm (e.g., [14]) on G^*. Besides its theoretical interest, however, this algorithm may be not be the most suited for practical applications, since the implementation of a linear-time planarity testing algorithm is complex and requires sophisticated data structures.

We show that there exists a much simpler solution to the problem of testing in linear time whether an ordering Ψ of a connected graph G is planar. Our algorithm exploits basic results in planarity theory [16, 13]. Namely, we determine the circuits induced by Ψ and check whether their number is equal to $E - V + 2$, where V and E are the number of vertices and edges of G, respectively. Clearly, this takes linear time.

Lemma 16 *Let Γ be a two-dimensional connected geometric graph with a planar natural ordering Ψ. If Γ is not planar (i.e., it has crossing edges), then at least one circuit of Γ induced by Ψ is not planar (i.e., it is a self-intersecting polygon).*

Proof Let G be the graph underlying Γ. Ordering Ψ induces a planar embedding of G whose faces are the circuits of Γ induced by Ψ. The proof is by induction on the number of faces of G (i.e., circuits of Γ). Clearly, if G has only one face f (f is in the case the external face) and Γ is not planar, then the induced circuit of Γ associated with f is a self-intersecting polygon.

Suppose now that the lemma holds for k faces and that G has $k+1$ faces. Let f be an internal face associated with a circuit c of Γ that shares one or more edges with the outer circuit. If c is self-intersecting, then we are done. Otherwise, remove the edges of c from the outer circuit of Γ, and let Γ' be the resulting geometric graph. If Γ' is not planar, then Γ', and thus also Γ, has a self-intersecting circuit by the inductive hypothesis. Otherwise, the outer circuit of Γ is a self-intersecting circuit. ■

4.2 Triangulations

By Lemma 16, verifying whether a connected geometric graph Γ with a planar natural ordering is a planar can be reduced to testing separately whether the circuits induced by Ψ are simple (i.e., not self-intersecting) polygons. Lemma 16 also provides an alternative and shorter proof of a result of [10] (Lemma 4.5), as the following lemma shows.

Lemma 17 *A two-dimensional connected geometric graph Γ with natural ordering Ψ is planar and induces a triangulation if and only if:*

1. *Ψ is planar;*
2. *all the internal circuits induced by Ψ are triangles; and*
3. *the outer circuit induced by Ψ is a convex polygon.*

Proof Follows from Lemma 16 and from the fact that no triangle can be a self-intersecting polygon. ■

We recall that the input to our checker is a two-dimensional geometric graph Γ with n vertices and the natural ordering Ψ of Γ. The checker constructs the circuits induced by Ψ and then performs the following sequence of simple checks. It fails as soon as one of the checks fails, and succeeds if all the checks succeed. Checks 1 and 6 have degree 2. The other checks have degree 1.

1. Check that Ψ is the natural ordering of Γ.
2. Check that ordering Ψ is planar.
3. Check that Γ is connected.
4. Check that Γ has no more than $3n - 6$ edges.
5. Check that all internal circuits induced by Ψ are triangles.
6. Check that the outer circuit induced by Ψ is a convex polygon.

Theorem 18 *There exists an optimal checker for planar triangulations that runs in linear time.*

4.3 Convex Subdivisions

In order to study checkers for other types of planar subdivisions, it is worth noticing that the planarity of the natural ordering does not in general imply the planarity of the subdivision itself. For example, Figure 5 shows a triconnected geometric graph (obtained by removing two edges from a triangulation) that has a planar ordering but is itself not planar.

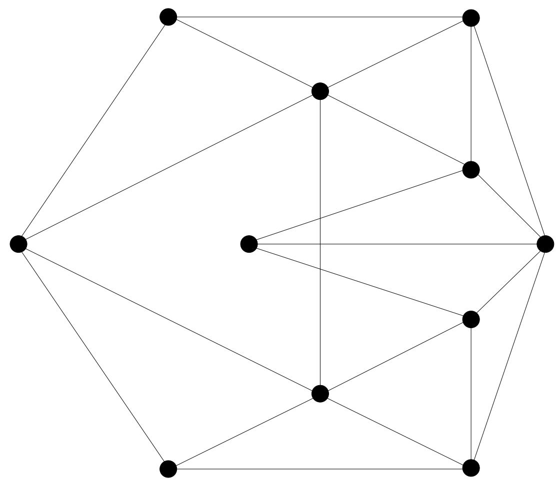

Figure 5: A triconnected geometric graph whose ordering is planar, but which is itself not planar.

Lemma 17 can be generalized to convex planar subdivisions.

Lemma 19 *A two-dimensional connected geometric graph* Γ *with natural ordering* Ψ *is planar and induces a convex planar subdivision if and only if:*

1. *ordering* Ψ *is planar; and*
2. *all the circuits induced by* Ψ *are convex polygons.*

Theorem 20 *There exists an optimal checker for convex planar subdivisions that runs in linear time.*

5 Strong Checkers for Planar Subdivisions

In this section, we present strong checkers for planar subdivisions. The input to the checkers is a 2-dimensional geometric graph Γ (the natural ordering is not provided). The checkers perform the following tasks, respectively:

- verify whether Γ is a planar triangulation;
- verify whether Γ is a convex planar subdivision.

Our strong checkers compute the natural ordering of the input geometric graph and then use one of the checkers presented in Section 4.

Theorem 21 [15] *Let Γ be a geometric graph with n vertices such that its underlying graph G is planar and has $\lambda(G)$ distinct planar orderings. There exists an algorithm that either computes the natural ordering of Γ in $O(n + \log \lambda(G))$ time or fails. If it fails, then the natural ordering of Γ is not planar.*

5.1 Triangulations

Our strong checker for planar triangulations exploits Theorem 21, Lemma 36, and the fact that for the underlying graph G of a triangulation, we have $\lambda(G) = 2$. Let Γ be a geometric graph with n vertices. By Theorem 21, we can compute the ordering of Γ in $O(n)$ time or else we can conclude that Γ is not a triangulation (we reach this conclusion either when the running time of Kirkpatrick's algorithm exceeds $O(n)$ or when Kirkpatrick's algorithm fails in $O(n)$-time). Once the ordering of Γ is computed, we apply the checker of Theorem 18.

Theorem 22 *There exists an optimal strong checker for planar triangulations that runs in linear time.*

Based on the above theorem, checkers that verify other geometric structures can be easily devised. Two applications of Theorem 22 are given below.

Corollary 23 *There exists an strong checker for Delaunay triangulations that runs in linear time.*

Proof First, we verify that Γ is planar and induces a triangulation S using the checker of Theorem 22. To verify that S is a Delaunay triangulation, it suffices to check whether for every triangle $T = \triangle(a, b, c)$ of S, the disk through a, b, c contains any of the opposite vertices in the triangles sharing one edge with T. Clearly, this can be done in linear time. Also, the above in-circle test can be executed with a degree 4 algorithm (see, e.g. [18, 19]). ∎

A *locally minimum-weight triangulation* is a triangulation such that for every edge shared by two triangles $\triangle(a,b,c)$ and $\triangle(a,b,d)$, edge bd is the shortest diagonal of the quadrilateral with vertices a,b,c,d. Locally minimum-weight triangulations have been extensively studied for their relationship to minimum-weight triangulations (see, e.g., [17]).

Corollary 24 *There exists an optimal strong checker for locally minimum-weight triangulations that runs in linear time.*

5.2 Convex Subdivisions

The optimal time complexity of the strong checkers of Theorems 22, 23, and 24 relies on the fact that $\lambda(G) = 2$ for the underlying graph of a triangulation. The next lemma shows that for convex planar subdivisions, $\lambda(G)$ is simply exponential.

Lemma 25 *Let G be the underlying graph of a convex planar subdivision. Then the number $\lambda(G)$ of topologically distinct planar orderings of G is $O(2^n)$, where n is the number of vertices of G.*

Proof The proof is based on the results of [8, 9], where the SPQR-trees of planar graphs that admit a planar straight-line drawing with convex faces are characterized. ∎

By combining Theorem 21, Lemma 36, Lemma 25, and Theorem 20 we obtain the following.

Theorem 26 *There exists an optimal strong checker for convex planar subdivisions that runs in linear time.*

A straightforward application of Theorem 26 is to verify whether a convex planar subdivision is the geometric dual of the Voronoi diagram of its vertices (notice that such dual, also called *Delaunay diagram*, is not a planar triangulation).

Corollary 27 *There exists an strong checker for Delaunay diagrams that runs in linear time.*

6 Degree of Checkers

6.1 Lower Bounds

In this section, we give lower bounds on the degree of checkers for planar subdivisions and convex polytopes.

Theorem 28 [18, 19] *The degree of the problem of evaluating a predicate expressed by a polynomial P is the maximum arithmetic degree of the irreducible factors of P that change sign over their domain.*

Theorem 29 *A (strong) checker for* 3*-dimensional convex polytopes has degree at least* 3.

Proof Let Γ be an input of the checker, such that Γ is isomorphic to K_4. Let $(p_i = (x_i, y_i, z_i))$ $(i = 1, \ldots, 4)$ the four vertices of Γ. Γ is globally convex if and only if its four vertices are not co-planar. This is equivalent to specifying that the determinant Δ_{1234} defined below be non-zero.

$$\Delta_{1234} = \begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}$$

Since points p_1, p_2, p_3 and p_4 are independent, Δ_{1234} is an irreducible polynomial over the rationals. Hence, by Theorem 28, the problem has degree 3. ∎

With analogous reasoning as in the proof of Theorem 29, the following lower bounds can be established.

Theorem 30 *A (strong) checker for d-dimensional convex polytopes has degree at least d.*

Theorem 31 *A (strong) checker for* 2*-dimensional planar subdivision has degree at least* 2.

6.2 Degree of Checkers for Convex Polytopes

The geometric primitive that determines the sign of a determinant of the type Δ_{1234} (see the proof of Theorem 29) is called an *orientation test*. Referring to Figure 6, we observe that testing the condition of local convexity at edge (p_1, p_2) corresponds