An Extension of Popov Criterion to Multivariable Time-Varying Nonlinear Systems. Application to Criterion for Existence of Stable Limit Cycles

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N ° 3512

Octobre 1998

THÈME 4

Rapport de recherche

ISSN 0249-6399

ROCQUENCOURT

An extension of Popov criterion to multivariable time-varying nonlinear systems. Application to criterion for existence of stable limit cycles

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Thème 4 — Simulation et optimisation
de systèmes complexes
Projet Sosso

Rapport de recherche n˚3512 — Octobre 1998 — 16 pages

Abstract: This report deals with sufficient conditions for absolute stability of multivariable control systems. The proposed conditions extend in a simple way the classical Popov criterion to time-varying memoryless nonlinearities. They are expressed in terms of some Linear Matrix Inequalities (LMIs). A weaker frequency domain criterion is deduced, leading to a simple graphical interpretation. The results ensure in general local stability, however, the stability is global for example for linear time-varying systems. As an application, a Popov-like criterion for existence of stable periodic solutions for periodic systems is proposed, using previous results by the authors.

Key-words: Absolute stability, Popov criterion, nonlinear systems, multivariable systems, time-varying nonlinearities, linear matrix inequalities, frequency domain, robustness margin, forced oscillations, limit cycles stability.

(Résumé : tsvp)

A.M. Krasnosel'skii was partially supported by Grants 97-01-00692 and 96-15-96048 of Russian Foundation of Fundamental Researchs. Work partly done during visits of A.M. Krasnosel'skii to I.N.R.I.A.

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Une extension du critère de Popov aux systèmes multivariables non-linéaires instationnaires. Application à un critère d'existence de cycles-limites stables

Résumé : Ce rapport est consacré à l'étude de conditions suffisantes de stabilité absolue pour des systèmes multivariables. Les conditions proposées étendent de manière simple le critère de Popov à des non-linéarités instationnaires sans mémoire. Elles s'expriment sous forme d'inégalités linéaires matricielles (*Linear Matrix Inequalities (LMIs)* en anglais). Un critère fréquentiel plus faible en est déduit, qui mène à un critère graphique généralisant simplement celui de Popov. Les résultats assurent la stabilité locale, néanmoins, la stabilité est globale par exemple dans le cas de systèmes linéaires instationnaires. Un critère d'existence de solutions périodiques stables pour des systèmes périodiques est présenté comme application, utilisant des résultats antérieurs des auteurs.

Mots-clé : Stabilité absolue, critère de Popov, systèmes non-linéaires, systèmes multivariables, systèmes instationnaires, inégalités linéaires matricielles, domaine fréquentiel, marge de robustesse, oscillations forcées, stabilité des cycles-limites.

1 Introduction

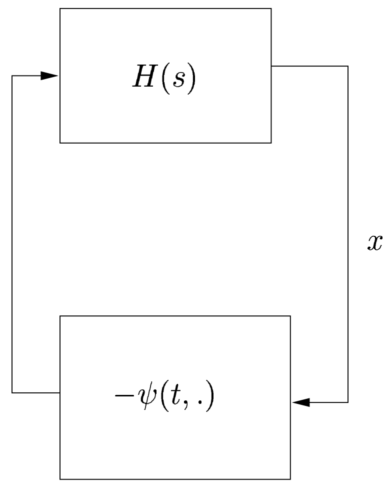

Figure 1: The system under study

Consider the multivariable control system given in Figure 1, where H is a strictly proper rational transfer matrix and ψ a time-dependent nonlinearity. This system verifies the following ordinary differential equation:

$$L\left(\frac{d}{dt}\right) y = -M\left(\frac{d}{dt}\right)(\psi(t,y)) \ , \tag{1}$$

where L, M are two coprime real polynomial matrices such that $H(s) = L^{-1}(s)M(s)$. Choosing a minimal representation (A, B, C) of the transfer matrix H, one writes [8]:

$$\dot{x} = Ax + Bu, \quad u = -\psi(t,y), \quad y = Cx \ , \tag{2}$$

where $n, p \in \mathbb{N} \setminus \{0\}$, $x \in \mathbb{R}^n$, $y \in \mathbb{R}^p$, $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times p}$, $C \in \mathbb{R}^{p \times n}$.

In the case where ψ is decentralized (that is [9]: $\forall i, i' \in \{1, \dots p\}$, $i \neq i' \Rightarrow \frac{\partial \psi_i}{\partial y_{i'}} \equiv 0$) and time-invariant, and verifies

$$\psi(0) = 0 \text{ and } \forall y, \psi(y)^T(\psi(y) - Ky) \leq 0 \ ,$$

for a certain nonnegative diagonal matrix K, the Popov criterion (see e.g. [15, 9, 10]) ensures that the system (1) is asymptotically stable in the large if the roots of L have negative real part and if there exists a constant $\eta \in \mathbb{R}$ such that

$$I + (I + \eta s)KH(s) \quad \text{is strictly positive real (SPR)} \ . \tag{3}$$

This result has been extended by Yakubovich to some hysteresis nonlinearities ψ [16]. This requires a condition on the sign of η, and the integrals $\int \psi(y(t))\dot{y}(t)dt$ to be bounded from below independently of the trajectory.

Some attempts have been made to generalize the Popov criterion to time-varying systems. Anderson *et al.* [2] provide a criterion for systems with nonstationnary linear part and time-independent nonlinearities (the results proposed in the case of time-varying nonlinearities indeed reduce to circle criterion). Narendra *et al.* [11, especially Chapter VI] have obtained conditions for global stability. They involve two parts: the Popov condition plus a differential (in the case of a separate nonlinearity $\psi(t,y) = k(t)f(y)$) or integrodifferential inequality linking $\frac{\partial \psi}{\partial t}$ and ψ.

In the present paper, one gives an extension of the Popov criterion to nonautonomous systems. More precisely, one provides conditions ensuring local stability of the origin. These conditions are expressed in terms of Linear Matrix Inequalities (LMIs) in Theorem 1, as a frequency condition in Corollary 3, as a graphical condition in the Popov plane in Corollary 5. A computational advantage of the LMIs is the fact that they now constitute a standard class of problems, for which sound numerical methods have been developed [5]. Both three results ensure local exponential stability (contrary to the Popov criterion in the autonomous case, where the stability is global). However, in some cases — in particular in the case of a linear time-varying operator ψ —, the stability is global. Compared with Popov criterion, the proposed criteria use essentially the second derivatives $\frac{\partial^2 \psi_i}{\partial y_i \partial t}(t,0)$ as a supplementary ingredient.

The criteria are given in Section 2, their proofs in Section 4. The method of demonstration and the results presented herein are closely related to Narendra *et al.*'s work [11, Chapter VI].

As an application, we consider in Section 3 systems with periodic nonlinearities (0 is not anymore an equilibrium). In [3, 4], one was able to give a criterion for existence of at least one periodic solution, under conditions similar to the Popov criterion. Studying the behavior of the solutions in the vicinity of this limit cycle, one applies the previous results, and states a criterion for existence of stable periodic solutions. Proof is given in Section 4.

In all the paper, $\|.\|$ denotes the euclidian norm or the induced matrix norm, I_r denotes the $r \times r$ identity matrix.

Acknowledgment. The authors wish to thank their colleagues S. Gaubert, R. Nikoukhah and M. Sorine for valuable remarks.

2 An extension of Popov stability criterion to time-varying systems

In all the sequel, we assume that there exist global solutions of (1), that is, by definition: for all $x_0 \in \mathbb{R}^n$, there exists an absolutely continuous function x defined on $[0,+\infty)$ s.t. $x(0) = x_0$ and (2) is fulfilled almost everywhere on $[0,+\infty)$.

Theorem 1. *Assume that there exists a neighborhood $\mathcal{O}$ of 0 in $\mathbb{R}^p$ such that*

(H1) *the nonlinearity ψ is decentralized and that there exists a diagonal nonnegative matrix $K \stackrel{\text{def}}{=} \text{diag}\{K_i\} \in \mathbb{R}^{p\times p}$ such that*

$$\forall (t,y) \in \mathbb{R}^+ \times \mathcal{O}, \quad \psi(t,y)^T(\psi(t,y) - Ky) \leq 0 \ , \tag{4}$$

(H2) *$\psi(t,y)$ is differentiable wrt t for any $(t,y) \in \mathbb{R}^+ \times \mathcal{O}$, that $\frac{\partial \psi}{\partial t}(t,y)$ is differentiable wrt y in 0 for any $t \in \mathbb{R}^+$, and that there exists $\gamma : \mathbb{R}^+ \to \mathbb{R}^+$ with $\lim_{z\to 0} \gamma(z) = 0$, such that*

$$\forall (t,y) \in \mathbb{R}^+ \times \mathcal{O}, \quad \left\| \frac{\partial \psi}{\partial t}(t,y) - \nabla_y \frac{\partial \psi}{\partial t}(t,0) \cdot y \right\| \leq \|y\| \ \gamma(\|y\|) \ .$$

If the following LMI is feasible:

$$P > 0, \quad \eta \stackrel{\text{def}}{=} \text{diag}\{\eta_i\} \geq 0 \ ,$$
$$\begin{pmatrix} A^TP + PA + C^TK^{\frac{1}{2}} \text{ diag}\left\{\operatorname*{sup\ ess}_{t\geq 0} \ \eta_i \dfrac{\partial^2 \psi_i}{\partial y_i \partial t}(t,0)\right\} K^{\frac{1}{2}}C & -PB + C^TK + A^TC^TK\eta \\ -B^TP + KC + \eta KCA & -2I - \eta KCB - B^TC^TK\eta \end{pmatrix} < 0 \ , \tag{5}$$

where the symmetric matrix $P \in \mathbb{R}^{n\times n}$ and the numbers η_i, $i = 1, \dots, p$, are the variables, then, the origin of system (1) *is locally exponentially stable. Moreover, if $\frac{\partial \psi}{\partial t}$ is linear wrt y and $\mathcal{O} = \mathbb{R}^p$, then the origin is globally exponentially stable.*

Remark 2. • Remark that ψ does not have to be continuous[1] wrt y. However, when ψ is $\mathcal{C}^2$, $\frac{\partial^2 \psi_i}{\partial y_i \partial t} = \frac{\partial^2 \psi_i}{\partial t \partial y_i}$: what has been done is equivalent to a linearization of the system around $x = 0$. In this case, there exist functions $k_i(t, y_i)$ such that $\psi_i(t, y_i) = k_i(t, y_i) y_i$, $i = 1, \dots, p$, and it may be fruitful to express the results in terms of the k_i. Indeed, (H1) requires that $0 \leq k_i(t, y_i) \leq K_i$, and (H2) expresses the continuity of $\frac{\partial k_i}{\partial t}(t, y_i)$ in $y_i = 0$, as $\frac{\partial^2 \psi_i}{\partial y_i \partial t}(t,0) = \frac{\partial k_i}{\partial t}(t, 0)$. • If (5) is feasible, then the derivative $\frac{\partial^2 \psi_i}{\partial y_i \partial t}(t,0)$ is (essentially) bounded from above (resp. from below) if $\eta_i \geq 0$ (resp. $\eta_i \leq 0$); this derivative may be replaced in (5) by an upper bound (resp. lower bound). • Equation (5) is fulfilled with $\eta = 0$ if and only if the following LMI is feasible:

$$P > 0, \quad \begin{pmatrix} A^TP + PA & -PB + C^TK \\ -B^TP + KC & -2I \end{pmatrix} < 0 \ .$$

When the zeros of L have negative real part, this condition may be interpreted, by application of Kalman-Yakubovich-Popov (KYP) Lemma [9], as a frequency condition, yielding circle criterion:

$$I + KH(s) \quad \text{is SPR} \ .$$

Recall that this result ensures (global) stability, even in the case of time-varying nonlinearities [6, 9, 10, 15]. • In the case where $\frac{\partial^2 \psi_i}{\partial y_i \partial t}(t,0) \leq 0$ t-a.e., condition (5) is fulfilled if and only if the following LMI is feasible:

$$P > 0, \quad \eta = \text{ diag}\{\eta_i\} \geq 0, \quad \begin{pmatrix} A^TP + PA & -PB + C^TK + A^TC^TK\eta \\ -B^TP + KC + \eta KCA & -2I - \eta KCB - B^TC^TK\eta \end{pmatrix} < 0 \ .$$

When the zeros of L have negative real part, the frequency interpretation of this condition yields the Popov criterion. • When the solutions of (2) are continuous wrt the initial conditions, one may consider in (5) the essential supremum [14] on $[t_0, +\infty)$, for any $t_0 \geq 0$. Indeed, due to the strict inequality involved, one may even use the upper limit [14] of this expression when $t_0 \to +\infty$. ♡

When (5) fails, one may apply to system (2) the transformation proposed by Rekasius and Gibson [12]. It consists in applying the following change of variable, which provides another representation of (1) (compare with (2)):

$$\dot{x} = (A - BKC)x - Bu, \quad u = -\varphi(t,y) \stackrel{\text{def}}{=} \psi(t,y) - Ky, \quad y = Cx \ . \tag{6}$$

[1]Extension to multivalued operators could be done as well.

Moreover, if ψ verifies Hypotheses (H1), (H2), then the same holds for φ, as (4) implies:

$$\forall (t,y) \in \mathbb{R}^+ \times \mathcal{O}, \quad \varphi(t,y)^T(\varphi(t,y) - Ky) \leq 0 \ .$$

More generally, one may choose instead of ψ (up to a reordering of the components) the input $((\varphi_i)^T_{1\leq i\leq r},\ (\psi_i)^T_{r+1\leq i\leq p})^T$ for a certain $r \in \{1,\dots,p\}$. One may hence replace condition (5) in Theorem 1 by any of a total of 2^p LMI problems of the type:

$$P > 0, \quad \eta = \operatorname{diag}\{\eta_i\} \geq 0, \quad \begin{pmatrix} \mathcal{A}^T P + P\mathcal{A} + C^T K^{\frac{1}{2}} D K^{\frac{1}{2}} C & -P\mathcal{B} + C^T K + \mathcal{A}^T C^T K \eta \\ -\mathcal{B}^T P + KC + \eta K C \mathcal{A} & -2I - \eta K C \mathcal{B} - \mathcal{B}^T C^T K \eta \end{pmatrix} < 0 \ , \tag{7}$$

where (using the identity $\frac{\partial^2 \varphi_i}{\partial y_i \partial t} = -\frac{\partial^2 \psi_i}{\partial y_i \partial t}$),

$$\mathcal{A} = A - B \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} KC, \ \mathcal{B} = B \begin{pmatrix} -I_r & 0 \\ 0 & I_{p-r} \end{pmatrix} \ , \tag{8a}$$

$$D = \operatorname{diag}\left\{ \operatorname{diag}\left\{ -\eta_i \operatorname*{inf\,ess}_{t\geq 0} \frac{\partial^2 \psi_i}{\partial y_i \partial t}(t,0) \right\}_{1\leq i\leq r},\ \operatorname{diag}\left\{ \eta_i \operatorname*{sup\,ess}_{t\geq 0} \frac{\partial^2 \psi_i}{\partial y_i \partial t}(t,0) \right\}_{r+1\leq i\leq p} \right\} \ . \tag{8b}$$

In the case of autonomous systems, $\frac{\partial^2 \psi_i}{\partial y_i \partial t}(t,0) \equiv 0$, and the previous transformation leads to the statement of Popov criterion with negative values of the Popov slope η, see [1] for details[2]. We did not manage to express the LMIs (7) in such a unified way; in Corollary 3 however, this unification is done. Remark that in the autonomous case, the assumptions made in Theorem 1 are not stronger than those made in Popov criterion.

For $z \in \mathbb{R}$, let us denote

$$|z|_+ \stackrel{\text{def}}{=} \max\{z, 0\}, \quad |z|_- \stackrel{\text{def}}{=} \max\{-z, 0\} \ .$$

Corollary 3 (A frequential criterion). *Assume that Hypotheses* (H1), (H2) *hold, together with*

(H3) *the roots of L have negative real part.*

If there exists $\eta \in \mathbb{R}^p$ s.t. the transfer matrix

$$I + (I + \eta s) K H(s) - H^*(s) K^{\frac{1}{2}} \operatorname{diag}\left\{ \operatorname*{sup\,ess}_{t\geq 0} \left| \eta_i \frac{\partial^2 \psi_i}{\partial y_i \partial t}(t,0) \right|_+ \right\} K^{\frac{1}{2}} H(s) \quad \textit{is SPR} \ , \tag{9}$$

then the conclusions of Theorem 1 *hold.*

Remark 4. • The additional term in (9) (compare with Popov criterion (3)) is nonnegative. • Expression (9) is piecewise linear wrt the η_i, as

$$\operatorname*{sup\,ess}_{t\geq 0} \left| \eta_i \frac{\partial^2 \psi_i}{\partial y_i \partial t}(t,0) \right|_+ = |\eta_i|_+ \operatorname*{sup\,ess}_{t\geq 0} \left| \frac{\partial^2 \psi_i}{\partial y_i \partial t}(t,0) \right|_+ + |\eta_i|_- \operatorname*{inf\,ess}_{t\geq 0} \left| \frac{\partial^2 \psi_i}{\partial y_i \partial t}(t,0) \right|_- \ .$$

♡

[2] Strangely enough, some western classical references on nonlinear systems provide only the result for nonnegative η, see [6, 9].

Denoting the H_∞-norm of H by $\|H(s)\|_\infty \stackrel{\text{def}}{=} \sup\{\|H(s)\| \ : \ \operatorname{Re} s > 0\}$ (which is equal to $\sup\{\|H(j\omega)\| \ : \ \omega \in \mathbb{R}\}$ when H is stable and proper), one deduces easily the following

Corollary 5 (A graphical criterion in the SISO case). *Assume that system* (2) *is SISO* ($p = 1$)*, and that Hypotheses* (H1), (H2), (H3) *hold. If there exists* $\eta \in \mathbb{R}$ *s.t. the transfer matrix*

$$1 + (1+\eta s)KH(s) - K\|H(s)\|_\infty^2 \sup_{t\geq 0} \text{ess} \left| \eta \frac{\partial^2 \psi}{\partial y \partial t}(t,0) \right|_+ \quad \textit{is SPR} , \tag{10}$$

then the conclusions of Theorem 1 *hold.*

When $K > 0$, formula (10) is equivalent to (11a) OR (11b), where

$$\exists \eta \geq 0,\ \forall \omega \in \mathbb{R},\ \frac{1}{K} + \operatorname{Re} H(j\omega) - \eta \left(\omega \operatorname{Im} H(j\omega) + \|H(s)\|_\infty^2 \sup_{t\geq 0} \text{ess} \left| \frac{\partial^2 \psi}{\partial y \partial t}(t,0) \right|_+ \right) \geq 0 , \tag{11a}$$

$$\exists \eta \leq 0,\ \forall \omega \in \mathbb{R},\ \frac{1}{K} + \operatorname{Re} H(j\omega) - \eta \left(\omega \operatorname{Im} H(j\omega) - \|H(s)\|_\infty^2 \sup_{t\geq 0} \text{ess} \left| \frac{\partial^2 \psi}{\partial y \partial t}(t,0) \right|_- \right) \geq 0 , \tag{11b}$$

and this has a clear interpretation:

> *If, apart from the sector, regularity and stability conditions* (H1), (H2), (H3)*, a Popov line lies above (resp. below) the Popov locus and may be translated vertically towards the locus by a distance*
>
> $$\|H(s)\|_\infty^2 \sup_{t\geq 0} \text{ess} \left| \frac{\partial^2 \psi}{\partial y \partial t}(t,0) \right|_+ \quad \textit{(resp. } \|H(s)\|_\infty^2 \sup_{t\geq 0} \text{ess} \left| \frac{\partial^2 \psi}{\partial y \partial t}(t,0) \right|_- \textit{)}$$
>
> *without intersecting it, then the local stability property holds. If* $\frac{\partial \psi}{\partial t}$ *is linear wrt* y*, then the stability is global.*

This is illustrated in Figure 2: in the left (resp. right) diagram, (11a) (resp. (11b)) holds if

$$\sup_{t\geq 0} \text{ess} \frac{\partial^2 \psi}{\partial y \partial t}(t,0) < \frac{d}{\|H(s)\|_\infty^2} \quad \text{(resp. } \sup_{t\geq 0} \text{ess} \frac{\partial^2 \psi}{\partial y \partial t}(t,0) > -\frac{d}{\|H(s)\|_\infty^2}\text{)} .$$

In the configurations shown in Figure 2, the quantity d involved is indeed *the least* $z > 0$ *s.t. one of the point* $(-1/K, \pm z)$ *belongs to the convex hull of the Popov locus* $\mathcal{P}$. To show this, remark that e.g. for the right diagram,

$$d = -\inf_{\eta\in\mathbb{R}} \sup_{\omega\in\mathbb{R}} \left(\eta \left(\operatorname{Re} H(j\omega) + \frac{1}{K} \right) - \omega \operatorname{Im} H(j\omega) \right) .$$

Now, $\sup_{\omega\in\mathbb{R}} \eta(\operatorname{Re} H(j\omega) + 1/K) - \omega \operatorname{Im} H(j\omega)$ may be seen as the value of the support function of the set $\mathcal{P}+1/K$ applied to the vector $(\eta, -1)$ [13]. One may hence replace the set $\mathcal{P}$ by its convex hull $\operatorname{conv} \mathcal{P}$, and then reverse the order of inf and sup. One gets:

$$d = -\sup \left\{ \inf_{\eta\in\mathbb{R}} \eta(z_1 + 1/K) - z_2 \ : \ (z_1, z_2) \in \operatorname{conv} \mathcal{P} \right\} = \inf \{ z_2 \ : \ (-1/K, z_2) \in \operatorname{conv} \mathcal{P} \} .$$

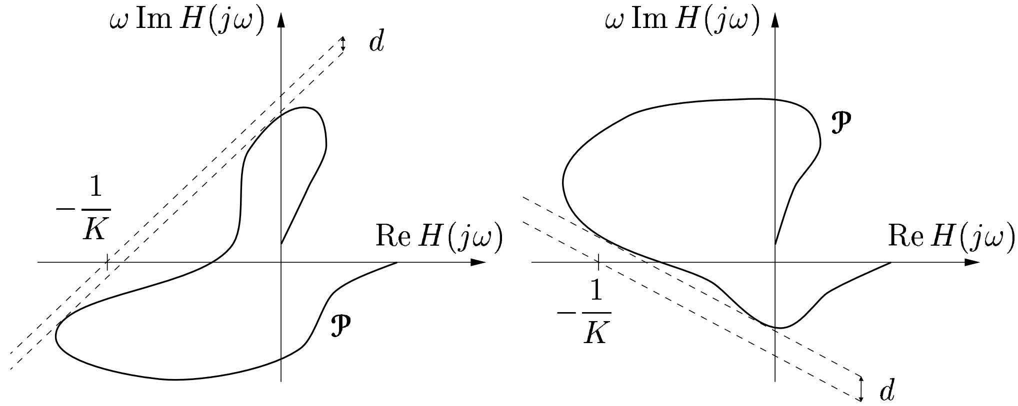

Figure 2: Graphical stability criterion

Remark 6. • From the three preceding results, one may propose measures of the stability robustness for nominal stationnary systems subject to time-varying perturbations: the problem of estimating (from below) the maximal value of $\sup\operatorname{ess}_{t\geq 0} |\frac{\partial^2\psi}{\partial y \partial t}(t,0)|$ preserving stability may be solved as a generalized eigenvalue problem [5, p. 10] (Theorem 1, Corollary 3), or graphically (Corollary 5). • Apart from the regularity conditions, only one condition has to be checked. In [11], two conditions must be verified: the Popov condition plus a differential (in the case of a separate nonlinearity $\psi(t,y) = k(t)f(y)$) or integrodifferential inequality linking $\frac{\partial\psi}{\partial t}$ and ψ. ♡

3 Application: criterion for existence of stable forced periodic solutions

Theorem 7. *Let $T > 0$, $K > 0$, let $\delta : \mathbb{R}^+ \to \mathbb{R}^+$ be s.t.*

$$\lim_{z\to+\infty} \delta(z) = 0 \ .$$

Let ψ be a scalar nonlinearity s.t.

(h0) $\forall (t,y) \in \mathbb{R}^+ \times \mathbb{R},\ \psi(t+T,y) = \psi(t,y)$,

(h1) $\forall (t,y) \in [0,T] \times \mathbb{R}$, *there exist* $\dfrac{\partial\psi}{\partial t}(t,y)$, $\dfrac{\partial\psi}{\partial y}(t,y)$ *and both are continuous,*

(h2) $\forall (t,y) \in [0,T] \times \mathbb{R}\setminus\{0\},\ 0 \leq \dfrac{\psi(t,y)}{y} \leq K$,

(h3) *condition* (3) *holds,*

(h4) $\forall (t,y) \in [0,T] \times \mathbb{R},\ \left|\displaystyle\int_0^y \frac{\partial\psi}{\partial t}(t,z)\ dz\right| \leq y^2\delta(|y|)$.

Then, there exist at least one T-periodic solution of (1)*, and two numbers $\alpha, \alpha' > 0$ such that $\|Cx^*\|_{\mathcal{C}} \leq \alpha$, $\|C\dot{x}^*\|_{\mathcal{C}} \leq \alpha'$ for any T-periodic solution x^* of* (1).

Suppose moreover that

(h1') $\forall (t,y) \in [0,T] \times \mathbb{R}$, *there exist* $\dfrac{\partial^2 \psi}{\partial y \partial t}(t,y)$, $\dfrac{\partial^2 \psi}{\partial y^2}(t,y)$, *and, uniformly in* $(t,y) \in [0,T] \times \mathbb{R}$,
$$\lim_{y'\to y} \frac{\partial \psi}{\partial t}(t,y') - \frac{\partial \psi}{\partial t}(t,y) - \frac{\partial^2 \psi}{\partial y \partial t}(t,y)\cdot(y'-y) = \lim_{y'\to y} \frac{\partial \psi}{\partial y}(t,y') - \frac{\partial \psi}{\partial y}(t,y) - \frac{\partial^2 \psi}{\partial y^2}(t,y)\cdot(y'-y) = 0,$$

(h2') $\forall (t,y,y') \in [0,T] \times \mathbb{R} \times \mathbb{R}, y \neq y'$, $0 \leq \dfrac{\psi(t,y)-\psi(t,y')}{y-y'} \leq K$,

(h3') *LMI* (5) *is feasible, with*
$$\sup_{t\geq 0} \text{ess } \eta \frac{\partial^2 \psi}{\partial y \partial t}(t,0) \quad \textit{replaced by} \quad |\eta| \left(\left\| \frac{\partial^2 \psi}{\partial y \partial t} \right\|_{L^\infty([0,T]\times[-\alpha,\alpha])} + \alpha' \left\| \frac{\partial^2 \psi}{\partial y^2} \right\|_{L^\infty([0,T]\times[-\alpha,\alpha])} \right) .$$

Then, any T-periodic solution of (1) *is locally exponentially stable. Moreover, if ψ is affine wrt y, there exists exactly one T-periodic solution, and it is globally exponentially stable.*

The existence part of the statement is taken from [3, 4], see those papers for computation of α, α'. The proof of this part uses topological methods. See also some related results for delay systems in [7].

4 Proofs

4.1 Proof of Theorem 1

- One shall prove that $V : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}$ defined below is a Lyapunov function[3]:

$$V(t,x) \stackrel{\text{def}}{=} x^T P x + 2 \sum_{i=1}^p \eta_i K_i \int_0^{(Cx)_i} \psi_i(t,\sigma_i) d\sigma_i \ .$$

One has $V(t,x) > 0$ for $x \in \mathbb{R}^n \setminus \{0\}$, and $V(t,0) \equiv 0$. Moreover, for any solution x of (1),

$$\frac{d}{dt} V(t,x(t)) = \dot{x}^T P x + x^T P \dot{x} + 2 \sum_{i=1}^p \eta_i K_i \psi_i(t,y_i(t)) \dot{y}_i(t) + 2 \sum_{i=1}^p \eta_i K_i \int_0^{y_i(t)} \frac{\partial \psi_i}{\partial t}(t,\sigma_i) d\sigma_i \quad t\text{-a.e.}$$

Now, $\frac{\partial \psi}{\partial t}$ being differentiable wrt y in 0,

$$\frac{\partial \psi_i}{\partial t}(t,y_i) = \frac{\partial \psi_i}{\partial t}(t,0) + \frac{\partial^2 \psi_i}{\partial y_i \partial t}(t,0) y_i + o_t(y_i) \ ,$$

where by definition, for any $t \geq 0$, $o_t(y_i)/y_i$ goes to 0 with y_i (in the case where $\frac{\partial \psi}{\partial t}$ is linear wrt y, one has $o_t(y_i) = 0$[4]). Hence, using Hypothesis (H2) and the fact that $\psi(t,0) \equiv 0$, one deduces that, $\forall i \in \{1,\ldots,p\}, \forall (t,y) \in \mathbb{R}^+ \times \mathcal{O}$,

$$\left| \frac{\partial \psi_i}{\partial t}(t,y_i) - \frac{\partial^2 \psi_i}{\partial y_i \partial t}(t,0) y_i \right| \leq |y_i| \ \gamma(\|y\|) \ .$$

[3]Compare with [9] in the autonomous case. See also [11] for global stability results.

[4]To eliminate the rest here, it is indeed sufficient to have: $\forall t \geq 0, \forall y \in \mathcal{O}, \forall i,\ y_i \neq 0 \Rightarrow \frac{\partial \psi_i}{\partial t}(t,y_i)/y_i \leq \frac{\partial^2 \psi_i}{\partial y_i \partial t}(t,0)$.

Therefore, if $Cx(t) \in \mathcal{O}$, this yields t-almost everywhere

$$\begin{aligned}
\frac{d}{dt}V(t,x(t)) \;&\leq\; (x^T A^T - \psi(t,y)^T B^T)Px + x^T P(Ax - B\psi(t,y)) + 2\sum_{i=1}^{p} \eta_i K_i \psi_i(t,y_i(t))\dot{y}_i(t) \\
&\quad +2\sum_{i=1}^{p} \eta_i K_i \int_0^{y_i(t)} \left(\frac{\partial^2 \psi_i}{\partial y_i \partial t}(t,0) + \gamma(\|y(t)\|) \right) \sigma_i d\sigma_i \\
&= (x^T A^T - \psi(t,y)^T B^T)Px + x^T P(Ax - B\psi(t,y)) \\
&\quad +2\sum_{i=1}^{p} \eta_i K_i \psi_i(t,y_i(t)) C(Ax - B\psi(t,y)) \\
&\quad +\sum_{i=1}^{p} \eta_i K_i \left(\frac{\partial^2 \psi_i}{\partial y_i \partial t}(t,0) + \gamma(\|y(t)\|) \right) y_i^2(t) \\
&\leq (x^T A^T - \psi(t,y)^T B^T)Px + x^T P(Ax - B\psi(t,y)) \\
&\quad +2\sum_{i=1}^{p} \eta_i K_i \psi_i(t,y_i(t)) C(Ax - B\psi(t,y)) + x^T C^T K^{\frac{1}{2}} (Q + \gamma(\|y(t)\|)\eta) K^{\frac{1}{2}} Cx \ ,
\end{aligned}$$

where the fact that $K \geq 0$ has been used, and the symmetric diagonal matrix Q is defined by

$$Q \stackrel{\text{def}}{=} \operatorname{diag}\left\{ \sup_{t \geq 0} \operatorname{ess} \eta_i \frac{\partial^2 \psi_i}{\partial y_i \partial t}(t,0) \right\} \geq 0 \ .$$

Using Hypothesis (H1), one deduces that, if $Cx(t) \in \mathcal{O}$, for almost any $t \geq 0$,

$$\begin{aligned}
&\frac{d}{dt}V(t,x(t)) \\
&\leq\; x^T (A^T P + PA + C^T K^{\frac{1}{2}} Q K^{\frac{1}{2}} C)x - 2x^T PB\psi(t,y) \\
&\quad +2\psi(t,y)^T \eta K C(Ax - B\psi(t,y)) - 2\psi(t,y)^T(\psi(t,y) - KCx) + o\left(\|y(t)\|^2\right) \\
&= \begin{pmatrix} x \\ \psi(t,y) \end{pmatrix}^T \begin{pmatrix} A^T P + PA + C^T K^{\frac{1}{2}} Q K^{\frac{1}{2}} C & -PB + C^T K + A^T C^T K \eta \\ -B^T P + KC + \eta K C A & -2I - \eta K C B - B^T C^T K \eta \end{pmatrix} \begin{pmatrix} x \\ \psi(t,y) \end{pmatrix} \\
&\quad +o\left(\|y(t)\|^2\right) \ ,
\end{aligned}$$

where the last term, of higher order, may be omitted when $\frac{\partial \psi}{\partial t}$ is linear wrt y and $\mathcal{O} = \mathbb{R}^p$.

- Now, due to the feasibility of (5), there exists $\varepsilon > 0$ such that the matrix involved in (5) is smaller than $-2\varepsilon I_{n+p}$. Due to Hypothesis (H2), there exists an open neighborhood $\mathcal{V}$ of 0 in $\mathbb{R}^n$ such that $C\mathcal{V} \subset \mathcal{O}$ and

$$\forall x \in \mathcal{V}, \quad \frac{d}{dt}V(t,x) \leq -\varepsilon \left\| \begin{pmatrix} x \\ \psi(t,Cx) \end{pmatrix} \right\|^2 \quad t\text{-a.e.} \ ,$$

and one may choose $\mathcal{V} = \mathbb{R}^n$ in the case where ψ is linear wrt y. Also, due to (4),

$$\sum_{i=1}^{p} \eta_i K_i \int_0^{(Cx)_i} \psi_i(t,\sigma_i) d\sigma_i \leq \frac{1}{2} \sum_{i=1}^{p} \eta_i K_i^2 (Cx)_i^2 \ .$$

Therefore, the symmetric positive definite matrix $P_1 \stackrel{\text{def}}{=} P + \frac{1}{2} C^T K \eta K C$ is such that

$$\forall (t,x) \in \mathbb{R}^+ \times \mathbb{R}^n, \quad V(t,x) \leq x^T P_1 x \ .$$

On the other hand, $\mathcal{V}$ being open and P positive, there exists $\varepsilon_1 > 0$ for which

$$\mathcal{W} \stackrel{\text{def}}{=} \{x \in \mathbb{R}^n \ : \ x^T P x < \varepsilon_1\} \subset \mathcal{V} \ .$$

Choose now $x(0)$ in $\{x \in \mathbb{R}^n \ : \ x^T P_1 x < \varepsilon_1\} \subset \mathcal{W}$. As long as $x(t) \in \mathcal{V}$, one has $Cx(t) \in \mathcal{O}$ and

$$\frac{d}{dt} V(t, x(t)) \leq -\varepsilon \|x\|^2 \leq -\frac{\varepsilon}{\rho(P_1)} x^T P_1 x \leq -\frac{\varepsilon}{\rho(P_1)} V(t, x(t)) \quad t\text{-a.e.} \ ,$$

where ρ denotes the spectral radius, so

$$x^T(t) P x(t) \leq V(t, x(t)) \leq V(0, x(0)) e^{-\frac{\varepsilon}{\rho(P_1)} t} \leq V(0, x(0)) \leq x(0)^T P_1 x(0) < \varepsilon_1 \ .$$

This implies that $\forall t \geq 0$, $x(t) \in \mathcal{W}$, and finally:

$$x(0) \in \left\{x \in \mathbb{R}^n \ : \ x^T P_1 x < \varepsilon_1\right\} \Rightarrow \forall t \geq 0, \quad x^T(t) P x(t) \leq V(t, x(t)) \leq V(0, x(0)) e^{-\frac{\varepsilon}{\rho(P_1)} t} \ ,$$

which proves the local exponential stability of the equilibrium and achieves the proof of Theorem 1.

4.2 Proof of Corollary 3

The hypotheses imply the existence of a certain open set $\mathcal{E} \subset \mathbb{R}^p$ s.t. (9) is fulfilled for any $\eta \in \mathcal{E}$. The proof consists in showing that in this case, there exists a certain $\hat{\eta} \in \mathcal{E}$ for which (5) holds. Theorem 1 is then applied with $\eta = \hat{\eta}$.

- Let us first suppose that the roots of L are simple, and that $\mathcal{E} \cap (\mathbb{R}^+ \setminus \{0\})^p \neq \emptyset$.

For $\eta \in \mathcal{E} \cap (\mathbb{R}^+ \setminus \{0\})^p$, the matrix A being Hurwitz, there exists a symmetric nonpositive matrix W satisfying the Lyapunov equation

$$A^T W + W A = C^T K^{\frac{1}{2}} \operatorname{diag} \left\{ \operatorname*{sup\,ess}_{t \geq 0} \eta_i \left| \frac{\partial^2 \psi_i}{\partial y_i \partial t}(t, 0) \right|_+ \right\} K^{\frac{1}{2}} C \ .$$

Due to the use of the positive part $|\cdot|_+$ and the fact that $\eta \geq 0$, (5) is feasible if the following LMI is feasible:

$$P > 0, \quad \eta = \operatorname{diag}\{\eta_i\} \geq 0, \quad \begin{pmatrix} A^T(P+W) + (P+W)A & -PB + C^T K + A^T C^T K \eta \\ -B^T P + KC + \eta KCA & -2I - \eta KCB - B^T C^T K \eta \end{pmatrix} < 0 \ ,$$

which is itself equivalent to:

$$\begin{gathered} P > W, \quad \eta = \operatorname{diag}\{\eta_i\} \geq 0 \ , \\ \begin{pmatrix} A^T P + PA & -(P-W)B + C^T K + A^T C^T K \eta \\ -B^T (P-W) + KC + \eta KCA & -2I - \eta KCB - B^T C^T K \eta \end{pmatrix} < 0 \ . \end{gathered}$$

As W is nonpositive, the latter condition is in turn feasible if the following LMI is feasible:

$$\begin{gathered} P > 0, \quad \eta = \operatorname{diag}\{\eta_i\} \geq 0 \ , \\ \begin{pmatrix} A^T P + PA & -(P-W)B + C^T K + A^T C^T K \eta \\ -B^T (P-W) + KC + \eta KCA & -2I - \eta KCB - B^T C^T K \eta \end{pmatrix} < 0 \ . \end{gathered} \tag{12}$$

Now, in order to apply KYP Lemma, one must check some controllability and observability conditions. The pair $\{A, B\}$ is controllable, due to the minimality of the representation (A, B, C).

Let us show that there exists a value $\hat{\eta} \in \mathcal{E} \cap (\mathbb{R}^+ \setminus \{0\})^p$ of η for which the pair $\{B^T W + KC + \eta KCA, A\}$ is observable. Otherwise, due to the Popov-Belevitch-Hautus (PBH) eigenvector test [8],

$$\forall \eta \in \mathcal{E} \cap (\mathbb{R}^+ \setminus \{0\})^p,\ \exists (v, \lambda) \in \mathbb{R}^n \setminus \{0\} \times \mathbb{R},\ Av = \lambda v, (B^T W + KC + \eta KCA)v = 0\ .$$

Now, as the roots of L are simple, the eigenvalues of A are also simple, and one may deduce that

$$\exists (v, \lambda) \in \mathbb{R}^n \setminus \{0\} \times \mathbb{R},\ \forall m \in \mathbb{N},\ \exists \eta_m \in \mathcal{E} \cap (\mathbb{R}^+ \setminus \{0\})^p \quad \text{s.t. } Av = \lambda v, (B^T W_m + KC + \eta_m KCA)v = 0\ ,$$

where the symmetric nonpositive matrix W_m verifies

$$A^T W_m + W_m A = C^T K^{\frac{1}{2}} \operatorname{diag}\left\{ \operatorname*{sup\ ess}_{t \geq 0}\ \eta_{m,i} \left| \frac{\partial^2 \psi_i}{\partial y_i \partial t}(t, 0) \right|_+ \right\} K^{\frac{1}{2}} C\ .$$

The set $\mathcal{E} \cap (\mathbb{R}^+ \setminus \{0\})^p$ being open, one may even suppose that $\forall m \neq m' \in \mathbb{N}$, $\forall i \in \{1, \dots, p\}$, $\eta_{m,i} \neq \eta_{m',i}$. One may hence "differentiate" wrt η_m the (locally) *affine* equality $(B^T W_m + KC + \eta_m KCA)v = 0$, which yields $KCv = 0$. From the fact that the eigenvalue λ is nonzero, due to (H3), one then deduces that $KCv = KCAv = \dots = KCA^{n-1}v = 0$. Now, one may suppose that K is invertible, as a zero term does not intervene in the criteria. Hence, $Cv = CAv = \dots = CA^{n-1}v = 0$, which contradicts the observability of the pair $\{C, A\}$.

One may now apply KYP Lemma with $\eta = \hat{\eta}$. One gets that (12) is equivalent to the frequency condition

$$\hat{G}(s) \stackrel{\text{def}}{=} (B^T \hat{W} + KC + \hat{\eta} KCA)(sI_n - A)^{-1} B + I_p - \frac{1}{2}(\hat{\eta} KCB + B^T C^T K \hat{\eta}) \quad \text{is SPR}\ ,$$

where the nonpositive symmetric matrix $\hat{W}$ verifies

$$A^T \hat{W} + \hat{W} A = C^T K^{\frac{1}{2}} \operatorname{diag}\left\{ \operatorname*{sup\ ess}_{t \geq 0}\ \hat{\eta}_i \left| \frac{\partial^2 \psi_i}{\partial y_i \partial t}(t, 0) \right|_+ \right\} K^{\frac{1}{2}} C\ .$$

Now, putting in display the fragment $G(s) \stackrel{\text{def}}{=} (KC + \hat{\eta} KCA)(sI_n - A)^{-1} B + I_p - \frac{1}{2}(\hat{\eta} KCB + B^T C^T K \hat{\eta})$ appearing in the usual form of Popov criterion, one gets

$$\begin{aligned}
\hat{G}^T(\bar{s}) + \hat{G}(s) &= G^T(\bar{s}) + G(s) + B^T(\bar{s}I - A^T)^{-1} \hat{W} B + B^T \hat{W} (sI - A)^{-1} B \\
&= G^T(\bar{s}) + G(s) + B^T(\bar{s}I - A^T)^{-1}((\bar{s}I - A^T)\hat{W} + \hat{W}(sI - A))(sI - A)^{-1} B \\
&= G^T(\bar{s}) + G(s) + (s + \bar{s}) B^T(\bar{s}I - A^T)^{-1} \hat{W} (sI - A)^{-1} B \\
&\quad - B^T(\bar{s}I - A^T)^{-1} C^T K^{\frac{1}{2}} \operatorname{diag}\left\{ \operatorname*{sup\ ess}_{t \geq 0} \hat{\eta}_i \left| \frac{\partial^2 \psi_i}{\partial y_i \partial t}(t, 0) \right|_+ \right\} K^{\frac{1}{2}} C (sI - A)^{-1} B\ ,
\end{aligned}$$

where the definition of $\hat{W}$ has been used. On the other hand, $G^T(\bar{s}) + G(s) = 2I + H^T(\bar{s}) K (I + \bar{s}\hat{\eta}) + (I + s\hat{\eta}) K H(s)$. We hence proved that, if $\mathcal{E} \cap (\mathbb{R}^+ \setminus \{0\})^p \neq \emptyset$ and (9) holds for any $\eta \in \mathcal{E} \cap (\mathbb{R}^+ \setminus \{0\})^p$, then (5) holds for a certain $\hat{\eta} \in \mathcal{E}$.

- We now remove the assumption that $\mathcal{E} \cap (\mathbb{R}^+)^p \neq 0$.

Due to the fact that $\mathcal{E}$ is open, its intersection with at least one open quadrant of the space $\mathbb{R}^p$ is nonvoid. Let us hence suppose, without loss of generality, that $r \in \{1, \dots, p\}$ verifies $\mathcal{E} \cap (\mathbb{R}^- \setminus \{0\})^r \times (\mathbb{R}^+ \setminus \{0\})^{p-r} \neq \emptyset$. By assumption, for any $\eta \in \mathcal{E} \cap (\mathbb{R}^- \setminus \{0\})^r \times (\mathbb{R}^+ \setminus \{0\})^{p-r}$, one has:

$$I + (I + \eta s) K H(s) - H^*(s) K^{\frac{1}{2}} D K^{\frac{1}{2}} H(s) \quad \text{is SPR}\ , \tag{13}$$

where D is defined in (8b).

One now applies Rekasius and Gibson's transformation, as exposed after Theorem 1. The transformed system has transfer matrix $C(sI-\mathcal{A})^{-1}\mathcal{B}$ and input $((\varphi_i)^T_{1\leq i\leq r},\ (\psi_i)^T_{r+1\leq i\leq p})^T$, where $\mathcal{A},\mathcal{B}$ are defined in (8a), and φ by (6).

Lemma 8. *Let us denote*

$$\mathcal{I}_r \stackrel{\text{def}}{=} \begin{pmatrix} -I_r & 0 \\ 0 & I_{p-r} \end{pmatrix},\quad \mathcal{J}_r \stackrel{\text{def}}{=} \frac{I_p-\mathcal{I}_r}{2} = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix},$$
$$\hat{H}(s) \stackrel{\text{def}}{=} C(sI-\mathcal{A})^{-1}\mathcal{B} = C\left(sI-A+B\mathcal{J}_rKC\right)^{-1}B\mathcal{I}_r\ .$$

Then,

$$\hat{H}(s) = (I+H(s)\mathcal{J}_rK)^{-1}H(s)\mathcal{I}_r = H(s)(I+\mathcal{J}_rKH(s))^{-1}\mathcal{I}_r\ .$$

Proof. Lemma 8 is a consequence of the feedback structure involved when changing the input of the linear plant. Indeed, denoting $\hat{\psi} \stackrel{\text{def}}{=} \mathcal{J}_r\varphi + (I-\mathcal{J}_r)\psi$ the new input, one has:

$$\begin{aligned} y = -H\psi &= -H\left(\mathcal{J}_r\psi + (I-\mathcal{J}_r)\psi\right) = -H\left(\mathcal{J}_r(Ky-\varphi)+(I-\mathcal{J}_r)\psi\right) \\ &= -H\left(\mathcal{J}_rKy - \mathcal{J}_r\hat{\psi} + (I-\mathcal{J}_r)\hat{\psi}\right) = -H\left(\mathcal{I}_r\hat{\psi}+\mathcal{J}_rKy\right)\ , \end{aligned}$$

as $\mathcal{J}_r^2=\mathcal{J}_r$ and $I_p-2\mathcal{J}_r=\mathcal{I}_r$. Finally:

$$(I+H\mathcal{J}_rK)y = -H\mathcal{I}_r\hat{\psi}\ ,$$

which gives the 1st equality. Deduction of the 2nd equality is straightforward. ♠

One now shows that the modified system verifies a condition analog to (9), with the *nonnegative* coefficients $|\eta_i|$, that is

$$I + (I+\eta\mathcal{I}_rs)\,K\hat{H}(s) - \hat{H}^*(s)K^{\frac{1}{2}}DK^{\frac{1}{2}}\hat{H}(s) \quad \text{is SPR}\ , \tag{14}$$

where D is defined in (8b) and the notations of Lemma 8 are used: the first part of the proof may then be applied to the transformed system, as $(|\eta_i|)\in\mathcal{I}_r[\mathcal{E}\cap(\mathbb{R}^-\setminus\{0\})^r\times(\mathbb{R}^+\setminus\{0\})^{p-r}] = \mathcal{I}_r\mathcal{E}\cap(\mathbb{R}^+\setminus\{0\})^p\neq\emptyset$. Indeed, applying Lemma 8 one gets

$$\begin{aligned} &2I + (I+\eta\mathcal{I}_rs)\,K\hat{H}(s) + K\hat{H}^T(\bar{s})\,(I+\eta\mathcal{I}_r\bar{s}) - 2\hat{H}^*(s)K^{\frac{1}{2}}DK^{\frac{1}{2}}\hat{H}(s) \\ &= 2I + (I+\eta\mathcal{I}_rs)\,KH(s)(I+\mathcal{J}_rKH(s))^{-1}\mathcal{I}_r + \mathcal{I}_r(I+H^T(\bar{s})K\mathcal{J}_r)^{-1}H^T(\bar{s})K\,(I+\eta\mathcal{I}_r\bar{s}) \\ &\quad -2\mathcal{I}_r(I+H^T(\bar{s})K\mathcal{J}_r)^{-1}H^T(\bar{s})K^{\frac{1}{2}}DK^{\frac{1}{2}}H(s)(I+\mathcal{J}_rKH(s))^{-1}\mathcal{I}_r \\ &= \mathcal{I}_r(I+H^T(\bar{s})K\mathcal{J}_r)^{-1}\Big[2(I+H^T(\bar{s})K\mathcal{J}_r)(I+\mathcal{J}_rKH(s)) \\ &\quad +(I+H^T(\bar{s})K\mathcal{J}_r)\mathcal{I}_r\,(I+\eta\mathcal{I}_rs)\,KH(s) + H^T(\bar{s})K\,(I+\eta\mathcal{I}_r\bar{s})\,\mathcal{I}_r(I+\mathcal{J}_rKH(s)) \\ &\quad -2H^T(\bar{s})K^{\frac{1}{2}}DK^{\frac{1}{2}}H(s)\Big]\,(I+\mathcal{J}_rKH(s))^{-1}\mathcal{I}_r \\ &= \mathcal{I}_r(I+H^T(\bar{s})K\mathcal{J}_r)^{-1}\Big[2I+(I+\eta s)KH(s)+H^T(\bar{s})K(I+\eta\bar{s}) \\ &\quad -2H^T(\bar{s})K^{\frac{1}{2}}DK^{\frac{1}{2}}H(s) + (s+\bar{s})H^T(\bar{s})K\eta\mathcal{J}_rKH(s)\Big]\,(I+\mathcal{J}_rKH(s))^{-1}\mathcal{I}_r\ , \end{aligned}$$

using the fact that the (diagonal) matrices $K, \eta, \mathcal{I}_r, \mathcal{J}_r$ commute, and that $(\mathcal{J}_r + \mathcal{I}_r)\mathcal{J}_r = 0$, $2\mathcal{J}_r + \mathcal{I}_r = I_p$, $\mathcal{I}_r^2 = I_p$. We then get that (13) and (14) are equivalent, which achieves the proof of Corollary 3 in the case where the roots of L are simple.

• Let us now face the case where the roots of L are not simple. We shall reduce this case to the previous one by adequate pole-shifting.

Let J be a $p \times n$ nonzero matrix. Write the differential equation in system (2) as $\dot{x} = (A + BJ)x - B(\psi(t,y) + Jx)$. Using the Kronecker symbol, define J_i, $i = 1, \dots, n$, by$(J_i)_{j,k} = \delta_i^k J_{j,k}$, in such a way that $J = J_1 + \dots + J_n$ ("column decomposition"), and

$$\hat{B} \stackrel{\text{def}}{=} \begin{pmatrix} B & \sqrt{\|J\|}B & \dots & \sqrt{\|J\|}B \end{pmatrix}, \ \hat{C} \stackrel{\text{def}}{=} \begin{pmatrix} C \\ \frac{1}{\sqrt{\|J\|}} J_1 \\ \vdots \\ \frac{1}{\sqrt{\|J\|}} J_n \end{pmatrix},$$

$$\hat{\psi}(t,\hat{y}) \stackrel{\text{def}}{=} \begin{pmatrix} \psi(t, \begin{pmatrix} I_p & 0_{p\times pn} \end{pmatrix} \hat{y}) \\ \begin{pmatrix} 0_{pn\times p} & I_{pn} \end{pmatrix} \hat{y} \end{pmatrix} \text{ for } \hat{y} \in \mathbb{R}^{p(n+1)} .$$

Then

$$\hat{\psi}(t,\hat{y}) = \begin{pmatrix} \psi(t,y) \\ \frac{1}{\sqrt{\|J\|}} J_1 x \\ \vdots \\ \frac{1}{\sqrt{\|J\|}} J_n x \end{pmatrix} \quad \text{for } \hat{y} = \hat{C}x ,$$

and one may rewrite system (2) under the form

$$\dot{x} = (A + BJ)x - \hat{B}\hat{\psi}(t,\hat{y}), \quad \hat{y} = \hat{C}x .$$

Remark that the nonlinearity $\hat{\psi}$ is decentralized and verifies:

$$\forall (t,\hat{y}) \in \mathbb{R}^+ \times \mathcal{O} \times \mathbb{R}^{pn}, \quad \hat{\psi}(t,y)^T(\hat{\psi}(t,\hat{y}) - \hat{K}\hat{y}) \leq 0 \text{ with } \hat{K} \stackrel{\text{def}}{=} \operatorname{diag}\{K, I_{pn}\} ,$$

and that the transfer matrix $\hat{H}(s) \stackrel{\text{def}}{=} \hat{C}(sI - A - BJ)^{-1}\hat{B}$ writes

$$\hat{H}(s) = \begin{pmatrix} H(s) & \sqrt{\|J\|}H(s) & \dots & \sqrt{\|J\|}H(s) \\ \frac{1}{\sqrt{\|J\|}} J_1(sI - A - BJ)^{-1}B & J_1(sI - A - BJ)^{-1}B & \dots & J_1(sI - A - BJ)^{-1}B \\ \vdots & \vdots & & \vdots \\ \frac{1}{\sqrt{\|J\|}} J_n(sI - A - BJ)^{-1}B & J_n(sI - A - BJ)^{-1}B & \dots & J_n(sI - A - BJ)^{-1}B \end{pmatrix}, \tag{15}$$

in such a way that

$$\lim_{J\to 0} \hat{H}(s) = \begin{pmatrix} H(s) & 0_{p\times pn} \\ 0_{pn\times p} & 0_{pn\times pn} \end{pmatrix} .$$

Now, one wants to exhibit a matrix J for which 1. $A + BJ$ is stable with simple eigenvalues and 2. there exists $\hat{\eta} \in \mathbb{R}^{p(n+1)}$ s.t.

$$\hat{G}(s) \stackrel{\text{def}}{=} I + (I + \hat{\eta}s)\hat{K}\hat{H}(s) - \hat{H}^*(s)\hat{K}^{\frac{1}{2}} \operatorname{diag}\left\{ \sup_{t\geq 0} \text{ess} \left| \hat{\eta}_i \frac{\partial^2 \hat{\psi}_i}{\partial \hat{y}_i \partial t}(t,0) \right|_+ \right\} \hat{K}^{\frac{1}{2}}\hat{H}(s) \quad \text{is SPR} .$$

This being done, it will then suffice to apply the results previously demonstrated to get the stability property and to achieve the proof of Corollary 5. Indeed, let $\varepsilon \in (0, \frac{1}{2})$ such that $\mathcal{E}_p \neq \emptyset$, where

$$\mathcal{E}_p \stackrel{\text{def}}{=} \Bigg\{ \eta \in \mathcal{E} \;:\; (1-\varepsilon)I + (I+\eta s)KH(s) \\ -H^*(s)K^{\frac{1}{2}} \operatorname{diag}\left\{ \sup_{t\geq 0} \operatorname{ess} \left| \eta_i \frac{\partial^2 \psi_i}{\partial y_i \partial t}(t,0) \right|_+ \right\} K^{\frac{1}{2}} H(s) \quad \text{is SPR} \Bigg\} \ .$$

The pair (A, B) being controllable, one may choose the eigenvalues of $A+BJ$ by adequate choice of J. These eigenvalues depend continuously upon J. It is hence possible to choose J ("small") such that 1. the eigenvalues of $A+BJ$ are simple with negative real part, 2. the lower right square block of size pn of matrix $((1-2\varepsilon)I + \hat{K}\hat{H})$ is SPR, and 3. the following inequality holds:

$$\sqrt{\|J\|}\ \max\left\{\|H(s)\|_\infty,\ \operatorname{diam}(\mathcal{E}_p)\ \|sH(s)\|_\infty, \|(sI-A-BJ)^{-1}B\|_\infty\right\} < \frac{\varepsilon}{2} \ .$$

Define $\mathcal{E}_{pn}$ as the open ball around 0 in $\mathbb{R}^{pn}$ of radius $\varepsilon / \max\{\|\hat{K}\|\ \|s\hat{H}(s)\|_\infty, 2\sqrt{\|J\|}\ \|(sI - A - BJ)^{-1}B\|_\infty\}$. Then, for $\hat{\eta} \in \mathcal{E}_p \times \mathcal{E}_{pn}$, for any $z = \begin{pmatrix} z_p^T & z_{pn}^T \end{pmatrix}^T \in \mathbb{R}^{p(n+1)} \setminus \{0\}$, one has, using (15) and the fact that $\frac{\partial^2 \hat{\psi}_i}{\partial \hat{y}_i \partial t} \equiv 0$ for $i = p+1, \ldots, p(n+1)$:

$$\frac{1}{2} z^T(\hat{G}(s) + \hat{G}^T(\bar{s}))z > \varepsilon(\|z_p\|^2 + \|z_{pn}\|^2 - 2\|z_p\|\ \|z_{pn}\|) \geq 0 \ .$$

This achieves the Proof of Corollary 5.

4.3 Proof of Theorem 7

One first shows the existence of a T-periodic solution x^*. This is done applying Theorem 1 from [3, 4], whose application is allowed by Hypotheses (h0) to (h4). One additionnally obtains a priori upper bounds on $\|x^*\|_{\mathcal{C}}$ and $\|\dot{x}^*\|_{\mathcal{C}}$, depending upon H, T, K, η and the function δ.

Let x be a solution of (1), and define $X \stackrel{\text{def}}{=} x - x^*$. Then,

$$\dot{X} = AX + BU, \quad U = -\Psi(t, Y), \quad Y = CX \ ,$$

where $\Psi(t, Y) \stackrel{\text{def}}{=} \psi(t, Y + Cx^*(t)) - \psi(t, Cx^*(t))$. Now, one wishes to apply Theorem 1 to this system. Hypotheses (h1') to (h3') permit to do this ((H2) is implied by (h1')), as

$$\frac{\partial^2 \Psi}{\partial Y \partial t}(t,0) = \frac{\partial^2 \psi}{\partial y \partial t}(t, Cx^*(t)) + C\dot{x}^*(t)\frac{\partial^2 \psi}{\partial y^2}(t, Cx^*(t)) \ .$$

Suppose $\frac{\partial \psi}{\partial t}$ and $\frac{\partial \psi}{\partial y}$ are affine wrt y (and indeed constant, due to (h2) and (h4), so ψ is affine wrt y), then $\frac{\partial \Psi}{\partial t}$ is linear wrt Y, and the asymptotic stability is global, so the T-periodic solution is unique.

References

[1] M.A. Aizerman, F.R. Gantmacher, *Absolute stability of regulator systems*, Holden-Day Inc., 1964

[2] B.D.O. Anderson, J.B. Moore, Construction of Lyapunov functions for nonstationary systems containing memoryless nonlinearities, *Automat. Remote Control*, 33, n. 5, part 1, 716-722, 1972; translated from *Avtomat. i Telemekh.* no. 5, 14-21, 1972 [Russian]

[3] P.-A. Bliman, A.M. Krasnosel'skii, Popov criterion in the problem of forced oscillations in control systems, *Avtomatika i telemekhanika*, 4, 3-14, 1998 [Russian]

[4] P.-A. Bliman, A.M. Krasnosel'skii, Popov-like frequency criterion for existence of forced periodic oscillations, *Proc. of 37th IEEE-CDC*, Tampa (Florida), Dec. 16-18, 1998

[5] S. Boyd, L. El Ghaoui, E. Feron, V. Balakrishnan, *Linear matrix inequalities in system and control theory*, SIAM Studies in Applied Mathematics vol. 15, 1994

[6] C.A. Desoer, M. Vidyasagar, *Feedback systems: input-output properties*, Academic Press, 1975

[7] A. Halanay, Almost periodic solutions for a class of nonlinear systems with time lag, *Rev. Roumaine Math. Pures Appl.*, **14**, 1269-1276, 1969

[8] T. Kailath, *Linear systems*, Prentice-Hall Inc., 1980

[9] H.K. Khalil, *Nonlinear systems*, Macmillan Publishing Company, 1992

[10] G.A. Leonov, I.M. Burkin, A.I. Shepeljavyi, *Frequency methods in oscillation theory*, Kluwer, 1996

[11] K.S. Narendra, J.H. Taylor, *Frequency domain criteria for absolute stability*, Academic Press, New-York London, 1973

[12] Z.V. Rekasius, J.E. Gibson, Stability analysis of nonlinear control systems by the second method of Lyapunov, *IRE Trans. Automatic Control*, **AC-7**, no 1, 3-15, 1962

[13] R.T. Rockafellar, *Convex analysis*, Princeton Mathematical Series, No. 28, Princeton University Press, 1970

[14] W. Rudin, *Real and complex analysis*, 3rd edition, McGraw-Hill, 1987

[15] J.L. Willems, *Stability Theory of Dynamical Systems*, Thomas Nelson ans Sons Ltd, 1970

[16] V.A. Yakubovich, The method of matrix inequalities in the theory of stability of non-linear controlled systems. III. Absolute stability of systems with hysteresis non-linearities, *Autom. Remote Control* 26, 753–763, 1965

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Éditeur
INRIA, Domaine de Voluceau, Rocquencourt, BP 105, 78153 LE CHESNAY Cedex (France)
http://www.inria.fr
ISSN 0249-6399