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Abstract: We consider the asymptotics of the Perron eigenvalue and eigenvector of irreducible nonnegative matrices whose entries have a geometric dependance in a large parameter. The first term of the asymptotic expansion of these spectral elements is solution of a spectral problem in a semifield of jets, which generalizes the max-algebra. We state a "Perron-Frobenius theorem" in this semifield, which allows us to characterize the first term of this expansion in some non-singular cases. The general case involves an aggregation procedure à la Wentzell–Freidlin.

Key-words: Perron-Frobenius Theorem, Max-algebra, Perturbation of eigenvalues, Perturbation of linear operators, Asymptotics, Freidlin-Wentzell theory, Large deviations

(Résumé : tsvp)

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Asymptotiques de la valeur propre et du vecteur propre de Perron via l'algèbre max-plus

Résumé : On s'intéresse à l'asymptotique de la valeur propre et du vecteur propre de Perron de matrices à coefficients positifs ou nuls, dépendant géométriquement d'un grand paramètre. Le premier terme du développement asymptotique de ces élements spectraux est solution d'un problème spectral sur un semi-corps de jets, qui généralise le semi-corps max-plus. Nous établissons un "théorème de Perron-Frobenius" pour les jets, qui nous permet de caractériser le premier terme de ce développement dans des cas non-singuliers. Le cas général requiert une procédure d'agrégation à la Wentzell–Freidlin.

Mots-clé : Théorème de Perron-Frobenius, Algèbre max-plus, Perturbation de valeurs propres, Perturbation d'opérateurs linéaires, Asymptotiques, Théorie de Freidlin-Wentzell, Grandes déviations

1 Introduction

Let $\mathcal{A}_p$ denote a $n \times n$ nonnegative matrix, depending on a large real parameter p. We consider the nonnegative spectral problem:

$$\mathcal{A}_p \mathcal{U}_p = \mathcal{L}_p \mathcal{U}_p, \qquad \mathcal{U}_p \in (\mathbb{R}^+)^n \setminus 0, \quad \mathcal{L}_p \in \mathbb{R}^+ \;, \tag{1}$$

where $\mathbb{R}^+$ denotes the set of nonnegative real numbers. When $\mathcal{A}_p$ is irreducible, $\mathcal{L}_p$ is unique, and it is called the *Perron eigenvalue* of $\mathcal{A}_p$ (see e.g. [4, Ch. 2]). We call *normalized Perron eigenvector* the unique $\mathcal{U}_p$ that satisfies $\sum_i (\mathcal{U}_p)_i = 1$. In this note, we adress the following problem: *can we determine the asymptotic behavior of $\mathcal{L}_p$ and $\mathcal{U}_p$ from that of $\mathcal{A}_p$*?

We begin with an elementary large deviation type result, which extends the result given in [10] for $\mathcal{A}_p = (A_{ij}^p)$.

THEOREM 1 (LARGE DEVIATION OF $\mathcal{L}_p$). *If the limits*

$$A_{ij} \stackrel{\text{def}}{=} \lim_{p\to\infty} (\mathcal{A}_p)_{ij}^{\frac{1}{p}} \tag{2}$$

exist for $i, j = 1, \ldots, n$, and if $A = (A_{ij})$ is irreducible, then

$$\lim_{p\to\infty} (\mathcal{L}_p)^{\frac{1}{p}} = \max_{1\le k\le n} \max_{i_1\ldots i_k} (A_{i_1 i_2} A_{i_2 i_3} \ldots A_{i_k i_1})^{\frac{1}{k}} \;. \tag{3}$$

Indeed, $0 \le (\mathcal{U}_p)_i \le \sum_j (\mathcal{U}_p)_j = 1$. Hence, $(\mathcal{U}_p)_i^{\frac{1}{p}}$, which is bounded, has a limit point $0 \le U_i \le 1$, and $\max_j U_j = 1$. It follows from (1) that $(\mathcal{L}_p)^{\frac{1}{p}}$ also has a limit point Λ, which satisfies

$$\max_j A_{ij} U_j = \Lambda U_i, \quad \text{for } i = 1, \ldots, n \;. \tag{4}$$

Now, it is convenient to introduce the *max-times semifield*[1] $\mathbb{R}_{\max} = (\mathbb{R}^+, \max, \times, 0, 1)$. We recognize in (4) a spectral problem for the matrix A in

[1] A *semiring* $(S, \oplus, \otimes, 0, 1)$ is a set S equipped with two laws $(a,b) \mapsto a \oplus b$, $(a,b) \mapsto a \otimes b$, called addition and multiplication, respectively, such that $(S, \oplus, 0)$ is a commutative monoid, $(S, \otimes, 1)$ is a monoid, the multiplication distributes over the addition, and the zero element 0 is absorbing for multiplication. A *semifield* is a semiring whose non zero elements have an inverse. In any semiring, we can define the matrix multiplication as usual (e.g. in $\mathbb{R}_{\max}$, $(AU)_i = \bigoplus_j A_{ij} \otimes U_j = \max_j A_{ij} U_j$).

the semifield $\mathbb{R}_{\max}$. The $\mathbb{R}_{\max}$ analogue of the Perron-Frobenius theorem states that an irreducible matrix A has a unique eigenvalue, given by the *maximal circuit mean* $\rho_{\max}(A)$, which, by definition, is the right hand side of (3) (see e.g. [2, Th. 3.100],[6, §VI],[11, §3.7]). Thus, $\Lambda = \rho_{\max}(A)$ holds for all limit points Λ of $(\mathcal{L}_p)^{\frac{1}{p}}$. This proves Theorem 1.

The above argument does not guarantee the convergence of $(\mathcal{U}_p)^{\frac{1}{p}}$, except when all the eigenvectors of A are proportional: this simple case is dealt with in §2. In §3, we show that if the non-zero entries of $\mathcal{A}_p$ have asymptotic expansions of the form

$$(\mathcal{A}_p)_{ij} \sim a_{ij} A_{ij}^p \quad , \tag{5}$$

then $\mathcal{L}_p$ has an asymptotic expansion of the same form. This expansion is the unique eigenvalue of $(a_{ij}A_{ij}^p)$, seen as a matrix with entries in a semifield of *jets*. When all the eigenvectors of the later matrix are proportional, the entries of $\mathcal{U}_p$ also have asymptotic expansions of the form (5). However, in general, (5) need not imply the existence of the limits $U_i \stackrel{\text{def}}{=} \lim_{p\to\infty}(\mathcal{U}_p)_i^{\frac{1}{p}}$, as shown by the following counter example:

$$\mathcal{A}_p = \begin{bmatrix} 1+\cos(p)e^{-p} & e^{-2p} \\ e^{-2p} & 1 \end{bmatrix} \quad ,$$

$$\liminf_{p\to\infty} \left(\frac{(\mathcal{U}_p)_2}{(\mathcal{U}_p)_1}\right)^{\frac{1}{p}} = e^{-1} < \limsup_{p\to\infty} \left(\frac{(\mathcal{U}_p)_2}{(\mathcal{U}_p)_1}\right)^{\frac{1}{p}} = e \quad .$$

In [1], we prove via an extension of the Puiseux expansion theorem that when the entries of $\mathcal{A}_p$ have *Dirichlet series* expansions (see [13],[14, Ch. VI]), $\mathcal{L}_p$ and the entries of $\mathcal{U}_p$ also have Dirichlet series expansions. Then, a fortiori, the limit $U = (U_i)$ exists. It can be computed using an aggregation procedure. In §4, we only present the first step of this procedure, which is enough to determine U in some non-singular cases.

The problem of computing the limits Λ and U arises in particular in Statistical Mechanics, when using the transfer operator methods at small temperatures $T = 1/p$ (see e.g. [3],[5]). Some of the results given below can be seen as partial extensions to the case of nonnegative matrices of the classical Freidlin-Wentzell singular perturbation results [9, Ch. 6] which deal with the special case of Markov matrices $\mathcal{A}_p$. Other max-algebra related (W.K.B. type) asymptotic results have been obtained in [7].

The proofs of the results presented here will be detailed in [1].

2 When max-times spectral theory determines the asymptotics

Let $(S, \oplus, \otimes, 0, 1)$ denote an arbitrary semiring. With a $n \times n$ matrix A with entries in S, we associate (as in conventional Perron-Frobenius theory) a digraph $\mathsf{G}(A)$ with nodes $1, \ldots, n$, and set of arcs $\{(i,j) \mid A_{ij} \neq 0\}$. We say that A is *irreducible* if $\mathsf{G}(A)$ is strongly connected.

When the reflexive and transitive relation $\preceq$, defined by $a \preceq b \iff \exists c,\ b = a \oplus c$, is an order relation, and in particular, when $S = \mathbb{R}_{\max}$, we define the *Kleene star* a^* as the least upper bound of the monotone sequence $(\bigoplus_{1\leq k\leq K} a^k)_{K\geq 1}$, when it exists.

When S is the $\mathbb{R}_{\max}$ semiring, we say that a circuit $c = (i_1, \ldots, i_k)$ is *critical* if its mean geometric weight $(A_{i_1 i_2} A_{i_2 i_3} \ldots A_{i_k i_1})^{\frac{1}{k}}$ attains the maximum in the right hand side of (3). The *critical graph* $\mathsf{CG}(A)$ is the subgraph of $\mathsf{G}(A)$, composed uniquely of the nodes and arcs in critical circuits. The strongly connected components of the critical graph are called *critical classes*. We set $\tilde{A} = \rho_{\max}(A)^{-1} A$. Then, $(\tilde{A})^*$ exists. A column of $(\tilde{A})^*$, whose index lies in a critical class, is called *critical*. The max-times spectral theorem (see [2, Th. 3.100], [6, Th. VI.10], [11, §3.7] and the references therein) states that if we select (arbitrarily) one critical column per critical class, we obtain a minimal generating set of the eigenspace of an irreducible matrix A. As an application of this result, we obtain:

THEOREM 2 (LARGE DEVIATION OF $\mathcal{U}_p$). *If $\mathcal{A}_p$ satisfies the assumptions of Theorem 1, and if A has a unique critical class, then*

$$\lim_{p\to\infty} (\mathcal{U}_p)_i^{\frac{1}{p}} = \frac{(\tilde{A})^*_{ij}}{\bigoplus_k (\tilde{A})^*_{kj}}, \quad \text{for } i = 1, \ldots, n \ ,$$

where j is an arbitrary node of this critical class.

Recall that $(\tilde{A})^*$ is equal to $\bigoplus_{0\leq k\leq n-1}(\tilde{A})^k$, and that it can be computed in $O(n^3)$ time using semiring versions of Gauss algorithm (see [2, Th. 3.20] and [12, Ch. 3,Algo. 3], respectively).

3 When the spectral theory of max-jets determines the asymptotics

We denote by $\mathbb{J}_{\max}$ the semifield with set of elements $\{(b,B) \mid b>0,\ B>0\}\cup\{(0,0)\}$, equipped with the two laws

$$(b,B)\oplus(c,C)=\begin{cases}(b,B) & \text{if } B>C\\ (c,C) & \text{if } B<C\\ (b+c,B) & \text{if } B=C\ ,\end{cases} \tag{6}$$

$$(b,B)\otimes(c,C)=(bc,BC)\ .$$

The zero element $(0,0)$ and the unit $(1,1)$ will be denoted by $0,1$, respectively. This semifield was introduced in [8]. It is isomorphic to the semifield of asymptotic expansions of the form $bB^p+o(B^p)$ around $p=\infty$, equiped with the usual addition and multiplication.

We will say that a nonnegative real valued function f of a large parameter p has a *first max-jet* (b,B), and we will write $f(p)\sim(b,B)$, if $f(p)=bB^p+o(B^p)$ around $p=\infty$ (when $(b,B)=0$, this means that $f(p)=0$ for p parge enough). The above definition and notation will be extended to matrices and vectors (entrywise).

If $\mathcal{A}_p$ and $\mathcal{U}_p$ have first max-jets $\mathcal{A}\in(\mathbb{J}_{\max})^{n\times n}$ and $\mathcal{U}\in(\mathbb{J}_{\max})^n$ respectively, it follows from (1) that $\mathcal{L}_p$ has also a first max-jet $\mathcal{L}\in\mathbb{J}_{\max}$, which satisfies $\mathcal{A}\mathcal{U}=\mathcal{L}\mathcal{U}$. Thus, the max-jet $\mathcal{U}$ of $\mathcal{U}_p$, if it exists, will be characterized in the particular cases when all the eigenvectors of $\mathcal{A}$ are proportional. We next state a $\mathbb{J}_{\max}$ analogue of the Perron-Frobenius theorem.

For any subgraph C of the digraph associated with a matrix A with entries in any semiring, we denote by A^{C} the matrix with entries $A^{\mathsf{C}}_{ij}=A_{ij}$ if (i,j) is an arc of C, and $A^{\mathsf{C}}_{ij}=0$ otherwise. Given an eigenvector $U\in(\mathbb{R}_{\max})^n$ of a matrix $A\in(\mathbb{R}_{\max})^{n\times n}$, the *saturation graph* $\mathsf{S}(A,U)$ is the subgraph of $\mathsf{G}(A)$ with set of arcs $\{(i,j)\mid A_{ij}U_j=\rho_{\max}(A)U_i\}$.

Let $\mathcal{A} = (a, A) \in (\mathbb{J}_{\max})^{n\times n}$. Clearly, $\mathcal{A}$ has an eigenvector $\mathcal{U} = (u, U) \in (\mathbb{J}_{\max})^n$, with eigenvalue $\mathcal{L} = (\lambda, \Lambda)$, iff

$$AU = \Lambda U, \qquad a^{\mathsf{S}(A,U)}u = \lambda u \tag{7}$$

(the first identity is a spectral problem in $\mathbb{R}_{\max}$, the second identity is an ordinary nonnegative spectral problem). The saturation graph in general depends on the particular choice of U, but when A is irreducible, for all eigenvectors U of A, $\mathsf{CG}(A) \subset \mathsf{S}(A, U)$, and any circuit of $\mathsf{S}(A, U)$ is contained in $\mathsf{CG}(A)$. The matrix $a^{\mathsf{CG}(A)}$ is block diagonal, the blocks being exactly the critical classes. We call *basic classes* of $\mathcal{A} = (a, A)$ the basic classes of $a^{\mathsf{CG}(A)}$ in the usual sense, i.e. the classes with maximal Perron eigenvalue. We denote by $\rho(b)$ the usual Perron eigenvalue of a matrix b.

THEOREM 3 ("PERRON-FROBENIUS THEOREM" FOR MAX-JETS). *An irreducible matrix $\mathcal{A} = (a, A) \in (\mathbb{J}_{\max})^{n\times n}$ admits the unique eigenvalue*

$$\rho_{\mathbb{J}}(\mathcal{A}) \stackrel{\text{def}}{=} (\rho(a^{\mathsf{CG}(A)}), \rho_{\max}(A)) \ . \tag{8}$$

The characterization of the eigenspace is more subttle in $\mathbb{J}_{\max}$ than in $\mathbb{R}_{\max}$. We will only need the following simple result.

THEOREM 4 (GEOMETRIC SIMPLICITY OF THE EIGENVALUE). *An irreducible matrix $\mathcal{A} = (a, A) \in (\mathbb{J}_{\max})^{n\times n}$ has a unique eigenvector (up to a proportionality factor) iff it has a unique basic class. An eigenvector $\mathcal{U} = (u, U)$ is obtained as follows: U is a column of $(\tilde{A})^*$, whose index belongs to the basic class; u is a positive eigenvector of $a^{\mathsf{S}(A,U)}$.*

As a consequence of Theorems 3 and 4, we obtain:

THEOREM 5 (FIRST ORDER ASYMPTOTICS). *If $\mathcal{A}_p$ has a first max-jet $\mathcal{A} \in (\mathbb{J}_{\max})^{n\times n}$, then*

$$\mathcal{L}_p \sim \rho_{\mathbb{J}}(\mathcal{A}) \ . \tag{9}$$

Moreover, if $\mathcal{A}$ has a unique basic class, then $\mathcal{U}_p$ has a first max-jet, which is the unique eigenvector $\mathcal{U}$ of $\mathcal{A}$ with sum 1.

4 Aggregated matrix

When the matrix $\mathcal{A}$ has several basic classes, the determination of the limit eigenvector relies on an aggregation procedure, the first step of which we next describe.

We denote by $\mathsf{B}_1,\dots,\mathsf{B}_s$ the basic classes of $\mathcal{A}$. We set $\mathsf{B} = \cup_{1\leq i\leq s}\mathsf{B}_i$ and $\mathsf{N} = \{1,\dots,n\}\setminus\mathsf{B}$. Let $\mathcal{V}_1,\dots,\mathcal{V}_s$ be eigenvectors of the submatrices $\mathcal{A}_{\mathsf{B}_1\mathsf{B}_1},\dots,\mathcal{A}_{\mathsf{B}_s\mathsf{B}_s}$, respectively (for all subsets $J,K\subset\{1,\dots,n\}$, $\mathcal{A}_{JK}$ denotes the $J\times K$ submatrix of $\mathcal{A}$). The following key lemma is a consequence of the fact that $au\leq\rho(a)u$ implies $au=\rho(a)u$, for all irreducible nonnegative matrices a and nonnegative vectors u (see e.g. [4, Ch. 1,Th. 3.35]).

LEMMA 6. *Any eigenvector $\mathcal{U}$ of an irreducible matrix $\mathcal{A}\in(\mathbb{J}_{\max})^{n\times n}$ is of the form $\mathcal{U}=\mathcal{V}\mathcal{U}'$, for some $\mathcal{U}'\in(\mathbb{J}_{\max})^s$, where*

$$\mathcal{V} \stackrel{\text{def}}{=} \begin{bmatrix} I \\ (\rho_{\mathbb{J}}(\mathcal{A})^{-1}\mathcal{A}_{\mathsf{NN}})^*\rho_{\mathbb{J}}(\mathcal{A})^{-1}\mathcal{A}_{\mathsf{NB}} \end{bmatrix} \mathrm{diag}(\mathcal{V}_1,\dots,\mathcal{V}_s) \ . \tag{10}$$

Here, I is the $\mathsf{B}\times\mathsf{B}$ identity matrix, and for all (possibly rectangular) matrices $X_1,\dots,X_k$, diag$(X_1,\dots,X_k)$ denotes the (possibly rectangular) block diagonal matrix with diagonal blocks $X_1,\dots,X_k$.

In the sequel, we will identify the max-jet (resp. the matrix of max-jets) (b,B) to the function $p\mapsto bB^p$ (resp. $p\mapsto(b_{ij}B_{ij}^p)$). This allows us to write $\mathcal{A}_p = \mathcal{A}^{\mathsf{BG}(\mathcal{A})}+\mathcal{R}_p$, where $\mathsf{BG}(\mathcal{A})$ denotes the subgraph of $\mathsf{CG}(\mathcal{A})$ with set of nodes B. In general, the remainder matrix $\mathcal{R}_p$ has negative entries.

Let $\mathcal{M}_1,\dots,\mathcal{M}_s$ denote the left eigenvectors of the submatrices $\mathcal{A}_{\mathsf{B}_1\mathsf{B}_1},\dots,\mathcal{A}_{\mathsf{B}_s\mathsf{B}_s}$, respectively, normalized by the condition $\mathcal{M}_i\mathcal{V}_i=1$, for $i=1,\dots,s$. We set $\mathcal{M}=\mathrm{diag}(\mathcal{M}_1,\dots,\mathcal{M}_s)\begin{bmatrix} I & 0\end{bmatrix}$ (0 is the $\mathsf{B}\times\mathsf{N}$ zero matrix). Left multiplying $\mathcal{A}_p\mathcal{U}_p=\mathcal{L}_p\mathcal{U}_p$ by $\mathcal{M}$, we obtain

$$\mathcal{M}\mathcal{R}_p\mathcal{U}_p = (\mathcal{L}_p-\rho_{\mathbb{J}}(\mathcal{A}))\mathcal{M}\mathcal{U}_p \ .$$

Using Lemma 6, we obtain the following result.

THEOREM 7 (SECOND ORDER ASYMPTOTICS). *If $\mathcal{A}_p$ and $\mathcal{R}_p$ have first max-jets $\mathcal{A}$ and $\mathcal{R}$, respectively, then*

$$\mathcal{L}_p = \rho_{\mathbb{J}}(\mathcal{A})+\rho_{\mathbb{J}}(\mathcal{A}')+o(\rho_{\mathbb{J}}(\mathcal{A}')) \ , \tag{11}$$

where $\mathcal{A}' \stackrel{\text{def}}{=} \mathcal{MRV} \in (\mathbb{J}_{\max})^{s\times s}$. *Moreover, if* $\mathcal{A}'$ *has a unique basic class, then* $\mathcal{U}_p$ *has a first max-jet, which is the unique vector with sum* 1 *of the form* $\mathcal{VU}'$,< $\mathcal{U}'$ *being an eigenvector of* $\mathcal{A}'$, *and* $\mathcal{V}$ *being defined in* (10).

Example 8. Consider the transfer matrix of the simplest one dimensional Ising model [3, Ch. 2]

$$\mathcal{A}_{1/T} = \begin{bmatrix} \exp((J+H)/T) & \exp(-J/T) \\ \exp(-J/T) & \exp((J-H)/T) \end{bmatrix} \ , \text{ with } J>0, H \in \mathbb{R} \ .$$

Setting $K = \exp(J) > 1$, $L = \exp(H) > 0$, $p = 1/T$, we can write the first max-jet of $\mathcal{A}_p$ as $\mathcal{A} = (a, A)$ with $a = \left[\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right]$ and $A = \left[\begin{smallmatrix} KL & K^{-1} \\ K^{-1} & KL^{-1} \end{smallmatrix}\right]$. We have $\rho_{\mathbb{J}}(\mathcal{A}) = (1, \max(KL, KL^{-1}))$. Thus, $\mathcal{L}_p \sim (\max(KL, KL^{-1}))^p$. When $H > 0$, there is a unique critical class, $\mathsf{C}_1 = \{1\}$, and $\tilde{A} = \left[\begin{smallmatrix} 1 & K^{-2}L^{-1} \\ K^{-2}L^{-1} & L^{-2} \end{smallmatrix}\right]$, $(\tilde{A})^* = \left[\begin{smallmatrix} 1 & K^{-2}L^{-1} \\ K^{-2}L^{-1} & 1 \end{smallmatrix}\right]$. By Theorem 5, $\mathcal{U}_p \sim \left[\begin{smallmatrix} 1 & (K^{-2}L^{-1})^p \end{smallmatrix}\right]^T$. By symmetry, if $H < 0$, then $\mathcal{U}_p \sim \left[\begin{smallmatrix} (K^{-2}L^{-1})^p & 1 \end{smallmatrix}\right]^T$: the limit eigenvector at zero temperature is selected by the sign of the external magnetic field H. When $H = 0$, $\mathcal{A}$ has two distinct critical classes $\mathsf{C}_1 = \{1\}, \mathsf{C}_2 = \{2\}$, that are both basic. Theorem 7 allows us to determine the equivalent of $\mathcal{U}_p$. Indeed, $\mathcal{V} = \mathcal{M} = I$ (the 2×2 identity matrix), and $\mathcal{A}^{\mathsf{BG}(\mathcal{A})} = \left[\begin{smallmatrix} (1,K) & 0 \\ 0 & (1,K) \end{smallmatrix}\right]$. We obtain $\mathcal{R}_p = \mathcal{R} = \mathcal{A}' = \left[\begin{smallmatrix} 0 & (1,K^{-1}) \\ (1,K^{-1}) & 0 \end{smallmatrix}\right]$. Thus, $\rho_{\mathbb{J}}(\mathcal{A}') = (1, K^{-1})$, $\mathcal{L}_p = K^p + K^{-p} + o(K^{-p})$, and $\mathcal{U}_p \sim \left[\begin{matrix} \frac{1}{2} & \frac{1}{2} \end{matrix} \right]^T$.

Version française abrégée

Soit $\mathcal{A}_p$ une matrice $n \times n$ à coefficients réels positifs ou nuls, définie au voisinage de $p = +\infty$. On considère le problème spectral (1) dans le cas où $\mathcal{A}_p$ est irréductible : $\mathcal{L}_p$ est unique et il existe un unique $\mathcal{U}_p$ vérifiant $\sum_i (\mathcal{U}_p)_i = 1$ (voir par exemple [4, Ch. 2]). On cherche à déterminer les asymptotiques de $\mathcal{L}_p$ et $\mathcal{U}_p$ à partir de celles de $\mathcal{A}_p$.

En utilisant les résultats analogues au théorème de Perron-Frobenius dans le semi-corps $\mathbb{R}_{\max} = (\mathbb{R}^+, \max, \times, 0, 1)$, isomorphe au semi-corps max-plus (voir par exemple [2, Th. 3.100],[6, §VI],[11, §3.7]), on obtient les asymptotiques de type grandes déviations de $\mathcal{L}_p$, et dans certains cas celles de $\mathcal{U}_p$.

THÉORÈME 1. *Si les limites* (2) *existent et si* $A = (A_{ij})$ *est irréductible, alors* $\lim_{p\to\infty}(\mathcal{L}_p)^{\frac{1}{p}}$ *existe. Elle est égale à la valeur propre de* A *dans* $\mathbb{R}_{\max}$*, notée* $\rho_{\max}(A)$*, donnée par le second membre de* (3)*.*

Un circuit $c = (i_1, \ldots, i_k)$ est dit *critique* si $(A_{i_1 i_2} A_{i_2 i_3} \ldots A_{i_k i_1})^{\frac{1}{k}}$ réalise le maximum dans (3). On appelle *graphe critique* le graphe orienté formé des nœuds et arcs des circuits critiques. On appelle *classe critique* une composante fortement connexe du graphe critique. On pose $\tilde{A} = \rho_{\max}(A)^{-1}A$, et on note $(\tilde{A})^* = \bigoplus_{k=0}^{\infty}(\tilde{A})^k$ l'étoile de Kleene de $\tilde{A}$ (la somme et les puissances sont dans $\mathbb{R}_{\max}$).

THÉORÈME 2. *Si* $\mathcal{A}_p$ *satisfait aux hypothèses du théorème 1, et si* A *n'a qu'une classe critique, alors* $\lim_{p\to\infty}(\mathcal{U}_p)_i^{\frac{1}{p}} = U_i$ *où* $U = (U_i)$ *est l'unique vecteur propre de* A *dans* $\mathbb{R}_{\max}$ *vérifiant* $\max_i U_i = 1$*. Celui-ci est proportionnel à n'importe quelle colonne de* $(\tilde{A})^*$ *d'indice critique.*

Afin d'obtenir des asymptotiques plus précises, on utilise le semi-corps de *jets* $\mathbb{J}_{\max}$ (introduit dans [8]) composé de l'ensemble des couples (b, B), avec $b, B > 0$ ou $b = B = 0$, muni des lois (6). On dit que la fonction f de p admet un *premier max-jet* si $f(p) = bB^p + o(B^p)$ autour de $p = +\infty$. On note alors $f(p) \sim (b, B)$. On étend cette notation aux vecteurs et matrices (coordonnée par coordonnée).

THÉORÈME 3. *Si* $\mathcal{A}_p \sim \mathcal{A} = (a, A) \in (\mathbb{J}_{\max})^{n\times n}$*, avec* $\mathcal{A}$ *irréductible, alors* $\mathcal{L}_p \sim \rho_{\mathbb{J}}(\mathcal{A})$ *où* $\rho_{\mathbb{J}}(\mathcal{A}) = (\rho(a^{\mathsf{CG}(A)}), \rho_{\max}(A))$ *est la valeur propre de* $\mathcal{A}$ *dans* $\mathbb{J}_{\max}$ *,* $a^{\mathsf{CG}(A)}$ *est la matrice obtenue à partir de* a *en annulant les coefficients* a_{ij} *tels que l'arc* (i, j) *n'est pas dans le graphe critique, et où* $\rho(\cdot)$ *désigne la valeur propre de Perron.*

Si $a^{\mathsf{CG}(A)}$ *n'a qu'une classe basique, alors* $\mathcal{U}_p \sim \mathcal{U}$*, l'unique vecteur propre de* $\mathcal{A}$ *dans* $\mathbb{J}_{\max}$ *de somme* 1*. Celui-ci est de la forme* (u, U)*, où* U *est une colonne de* $(\tilde{A})^*$ *d'indice basique et* u *est un vecteur propre positif de la matrice* $a^{\mathsf{S}(A,U)}$*, obtenue en annulant les* a_{ij} *tels que* $A_{ij}U_j < \rho_{\max}(A)U_i$*.*

On appellera *classes basiques* de $\mathcal{A}$ les classes basiques de $a^{\mathsf{CG}(A)}$. Si $\mathcal{A}$ admet les classes basiques $\mathsf{B}_1, \ldots, \mathsf{B}_s$, alors tout vecteur propre $\mathcal{U}$ de $\mathcal{A}$ dans $\mathbb{J}_{\max}$ s'écrit $\mathcal{U} = \mathcal{V}\mathcal{U}'$, où $\mathcal{V}$ est donné par (10). Dans (10), $\mathcal{V}_1, \ldots, \mathcal{V}_s$ sont des vecteurs propres de $\mathcal{A}_{\mathsf{B}_1\mathsf{B}_1}, \ldots, \mathcal{A}_{\mathsf{B}_s\mathsf{B}_s}$, respectivement, $\mathcal{A}_{JK}$ désigne la $J \times K$ sous-matrice de $\mathcal{A}$,

$\mathsf{B} = \cup_{1\leq i\leq s}\mathsf{B}_i$, $\mathsf{N} = \{1,\dots,n\} \setminus \mathsf{B}$, et l'étoile dans $\mathbb{J}_{\max}$ est définie par la même formule que dans $\mathbb{R}_{\max}$. Soit $\mathcal{M} = \mathrm{diag}(\mathcal{M}_1,\dots,\mathcal{M}_s) \begin{bmatrix} I & 0 \end{bmatrix}$, où $\mathcal{M}_1,\dots,\mathcal{M}_s$ désignent les vecteurs propres à gauche de $\mathcal{A}_{\mathsf{B}_1\mathsf{B}_1},\dots,\mathcal{A}_{\mathsf{B}_s\mathsf{B}_s}$, respectivement, vérifiant $\mathcal{M}_i\mathcal{V}_i = 1$.

THÉORÈME 4. *Si* $\mathcal{A}_p \sim \mathcal{A} \in (\mathbb{J}_{\max})^{n\times n}$ *et* $\mathcal{R}_p = \mathcal{A}_p - \mathcal{A} \sim \mathcal{R} \in (\mathbb{J}_{\max})^{n\times n}$*, alors* $\mathcal{L}_p$ *admet le développement asymptotique* (11)*, où* $\mathcal{A}' = \mathcal{M}\mathcal{R}\mathcal{V} \in (\mathbb{J}_{\max})^{s\times s}$.

Si $\mathcal{A}'$ *n'a qu'une classe basique, alors* $\mathcal{U}_p \sim \mathcal{U}$*, l'unique élément de* $(\mathbb{J}_{\max})^n$ *de somme* 1*, de la forme* $\mathcal{V}\mathcal{U}'$*, où* $\mathcal{U}'$ *est un vecteur propre de* $\mathcal{A}'$*, et* $\mathcal{V}$ *est donné par* (10)*.*

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