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***Griffith formulae for elasticity systems with  
unilateral conditions***

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## Griffith formulae for elasticity systems with unilateral conditions

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**Abstract:** In the paper we consider the elasticity equations in nonsmooth domains in  $R^n$ ,  $n = 2, 3$ . The domains have a crack whose length may change. At the crack faces, inequality type boundary conditions describing a mutual nonpenetration of the crack faces are prescribed. The derivative of the energy functional with respect to the crack length is obtained. The Griffith formulae are derived in 2D and 3D cases and the other properties of the solutions are established. In two-dimensional case the Rice–Cherepanov’s integral over a closed curve is constructed. The path independence of the Rice–Cherepanov’s integral is shown.

**Key-words:** Griffith formula, Rice–Cherepanov’s integral, unilateral conditions on crack faces

*(Résumé : tsvp)*

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# La formule de Griffith et l'intégrale de Rice–Cherepanov pour des équations elliptiques avec conditions unilatérales dans des domaines non réguliers

**Résumé :** La formule de Griffith est obtenue dans le cas de conditions unilatérales sur la fissure. L'indépendance par rapport au chemin de l'intégrale de Rice–Cherepanov est prouvée. L'indépendance par rapport au chemin de l'intégrale de Rice–Cherepanov était précédemment prouvée en élasticité pour les conditions aux limites linéaires  $\sigma_{22} = 0, \sigma_{12} = 0$  sur  $\Xi_l^\pm$  dans le cas bidimensionnel. La formule de Griffith est aussi obtenue dans le cas tridimensionnel. Dans ce cas, la fissure bidimensionnelle est incluse dans un domaine tridimensionnel.

**Mots-clé :** formule de Griffith, fissure, intégrale de Rice–Cherepanov, élasticité

### 1. Two-dimensional case.

Let  $D \subset \mathbb{R}^2$  be a bounded domain with a smooth boundary  $\Gamma$ , and  $\Xi_{l+\delta}$  be the set  $\{(x_1, x_2) \mid 0 < x_1 < l + \delta, x_2 = 0\}$ . We assume that this set belongs to the domain  $D$  for all sufficiently small  $\delta$ , and  $l > 0$ .

The domains with cracks  $\Xi_{l+\delta}, \Xi_l$  are denoted by  $\Omega_\delta = D \setminus \overline{\Xi_{l+\delta}}$ ,  $\Omega = D \setminus \overline{\Xi_l}$ , respectively.

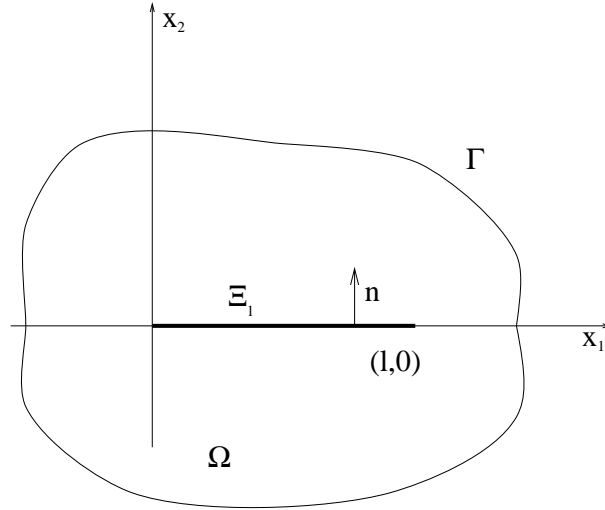


Figure 1: Domain  $\Omega$ .

The elasticity system analysed in the paper can be formulated as follows. We want to find a function  $W = (u, v)$  such that

$$-\sigma_{ij,j} = f_i, \quad i = 1, 2, \quad \text{in } \Omega, \quad (1.1)$$

$$W = 0 \quad \text{on } \Gamma, \quad (1.2)$$

$$[W]n \geq 0 \quad \text{on } \Xi_l. \quad (1.3)$$

Here  $\sigma_{ij} = \sigma_{ij}(W)$  are the stress tensor components, the summation convention is used over the repeated indices  $j = 1, 2$ ,  $n = (0, 1)$  is a normal vector to  $\Xi_l$ , and  $[W] = W^+ - W^-$  is the jump of  $W$  across  $\Xi_l$ . The signs  $\pm$  correspond to the positive and negative directions of  $n$ . Hooke's law is assumed to be fulfilled,

$$\sigma_{ij} = 2\mu\varepsilon_{ij} + \lambda \operatorname{div} W \delta_j^i, \quad i, j = 1, 2. \quad (1.4)$$

By  $\lambda \geq 0$ ,  $\mu > 0$  we denote the Lamé parameters,  $\varepsilon_{ij} = \varepsilon_{ij}(W)$ ,  $\delta_j^i$  is the Kronecker symbol,

$$\varepsilon_{11} = u_{x_1}, \quad \varepsilon_{22} = v_{x_2}, \quad \varepsilon_{12} = 1/2(u_{x_2} + v_{x_1}). \quad (1.5)$$

We assume that  $f = (f_1, f_2) \in C^1(\bar{D})$ . The formulation of the problem (1.1) – (1.3) is not complete. Actually, we define a variational solution by the minimisation of the functional

$$I(\Omega; U) = \frac{1}{2} \int_{\Omega} \sigma_{ij}(U) \varepsilon_{ij}(U) - \int_{\Omega} fU, \quad U = (u, v), \quad (1.6)$$

over the set

$$K_0 = \{(u, v) \in H^1(\Omega) \mid u = v = 0 \text{ on } \Gamma, \quad [v] \geq 0 \text{ on } \Xi_l\}. \quad (1.7)$$

In this case the solution  $W$  of the minimisation problem satisfies (1.1) – (1.3) and some additional boundary conditions prescribed on  $\Xi_l$ . These conditions are analysed below (see (1.28)).

The perturbed problem corresponding to (1.1) – (1.3) is defined as follows. We want to find a function  $W^\delta = (u^\delta, v^\delta)$  such that

$$-\sigma_{ij,j} = f_i, \quad i = 1, 2, \quad \text{in } \Omega_\delta, \quad (1.8)$$

$$W^\delta = 0 \quad \text{on } \Gamma, \quad (1.9)$$

$$[W^\delta]n \geq 0 \quad \text{on } \Xi_{l+\delta}. \quad (1.10)$$

Here  $\sigma_{ij} = \sigma_{ij}(W^\delta)$ ,  $\varepsilon_{ij} = \varepsilon_{ij}(W^\delta)$ , and  $\sigma_{ij}$ ,  $\varepsilon_{ij}$  satisfy Hooke's law (1.4).

Similar to (1.1) – (1.3), the complete formulation of the problem (1.8) – (1.10) is variational i.e., we minimise the functional

$$I(\Omega_\delta; U) = \frac{1}{2} \int_{\Omega_\delta} \sigma_{ij}(U) \varepsilon_{ij}(U) - \int_{\Omega_\delta} fU \quad (1.11)$$

over the set

$$K_\delta = \{(u, v) \in H^1(\Omega_\delta) \mid u = v = 0 \text{ on } \Gamma, \quad [v] \geq 0 \text{ on } \Xi_{l+\delta}\}. \quad (1.12)$$

The inequality (1.10) is, in fact, only a part of the complete system of boundary conditions prescribed on  $\Xi_{l+\delta}$ .

We are going to establish the existence of the derivative of the energy functional

$$\lim_{\delta \rightarrow 0} \frac{I(\Omega_\delta; W^\delta) - I(\Omega; W)}{\delta}, \quad (1.13)$$

where  $W^\delta, W$  are the solutions of (1.8) – (1.10) and (1.1) – (1.3), respectively. In this case the limit (1.13) is equal to the derivative

$$\left. \frac{dJ(\Omega_\delta)}{d\delta} \right|_{\delta=0}, \quad (1.14)$$

where  $J(\Omega_\delta) = I(\Omega_\delta; W^\delta)$ .

To find the derivative (1.14), we transform the domain  $\Omega_\delta$  onto  $\Omega$  in the following way. Consider a function  $\theta \in C_0^\infty(D)$  such that  $\theta = 1$  in a neighbourhood of the point  $x_l = (l, 0)$ . To simplify the arguments the function  $\theta$  is assumed to be equal to zero in a neighbourhood of the point  $(0, 0)$ . Introduce the transformation of independent variables

$$y_1 = x_1 - \delta\theta(x_1, x_2), \quad y_2 = x_2. \quad (1.15)$$

Here  $(y_1, y_2) \in \Omega$ ,  $(x_1, x_2) \in \Omega_\delta$ . The transformation (1.15) maps  $\Omega_\delta$  onto  $\Omega$ , and it is one-to-one. The Jacobian  $q_\delta$  of the transformation is positive for small  $\delta$ ,

$$q_\delta = \left| \frac{\partial(y_1, y_2)}{\partial(x_1, x_2)} \right| = 1 - \delta\theta_{x_1}.$$

Let  $x = x(y, \delta)$  correspond to the transformation (1.15), and  $W^\delta(x)$  be the solution of (1.8) – (1.10). Then  $W^\delta(x) = W_\delta(y)$ ,  $y \in \Omega$ . Also, let  $W$  be the solution of (1.1) – (1.3). The following statement holds.

**Lemma.** *We have, as  $\delta \rightarrow 0$*

$$\|W_\delta - W\|_{H^1(\Omega)} \rightarrow 0.$$

The proof is omitted here. The lemma can be proved by using the same arguments as those of Lemma 1 in (Khludnev, Sokolowski, 1998). We only



indicate that  $W^\delta$  and  $W$  are the solutions of the following variational inequalities

$$W^\delta \in K_\delta : \int_{\Omega_\delta} \sigma_{ij}(W^\delta)(\varepsilon_{ij}(V) - \varepsilon_{ij}(W^\delta)) \geq \int_{\Omega_\delta} f(V - W^\delta) \quad \forall V \in K_\delta, \quad (1.16)$$

$$W \in K_0 : \int_{\Omega} \sigma_{ij}(W)(\varepsilon_{ij}(V) - \varepsilon_{ij}(W)) \geq \int_{\Omega} f(V - W) \quad \forall V \in K_0. \quad (1.17)$$

Moreover, similar to (Khludnev, Sokolowski, 1998) we can prove that there exists a constant  $c > 0$  such that

$$\|W_\delta - W\|_{H^1(\Omega)} \leq c\delta.$$

Using the transformation (1.15) we obtain

$$\int_{\Omega_\delta} f_i w^\delta dx = \int_{\Omega} f_i^\delta(y) \delta(y) dy, \quad f_i^\delta(y) = \frac{f_i(x(y, \delta))}{1 - \delta\theta_{x_1}}, \quad i = 1, 2, \quad w^\delta(x) = w_\delta(y).$$

It is possible to find the derivatives

$$f_i'(y) = \lim_{\delta \rightarrow 0} \frac{f_i^\delta(y) - f_i^0(y)}{\delta} = \left. \frac{df_i^\delta}{d\delta} \right|_{\delta=0}.$$

In fact, assuming that  $y, \delta$  are independent variables in (1.15), we have  $x = x(y, \delta)$ . Differentiation of (1.15) with respect to  $\delta$  yields

$$0 = \frac{dx_1}{d\delta} - \theta - \delta\theta_{x_1} \frac{dx_1}{d\delta},$$

whence

$$\frac{dx_1}{d\delta} = \frac{\theta}{1 - \delta\theta_{x_1}}, \quad \frac{dx_2}{d\delta} = 0.$$

Consequently,

$$\left. \frac{\partial f_i(x(y, \delta))}{\partial \delta} \right|_{\delta=0} = f_{ix_1} \frac{dx_1}{d\delta} \Big|_{\delta=0} + f_{ix_2} \frac{dx_2}{d\delta} \Big|_{\delta=0} = f_{iy_1} \theta, \quad i = 1, 2. \quad (1.18)$$

Now we are in a position to find the derivative  $f_i'(y)$ . Indeed, by (1.18), we have

$$f_i'(y) = \lim_{\delta \rightarrow 0} \left( \frac{f_i(x(y, \delta))}{1 - \delta\theta_{x_1}} - f_i(y) \right) \frac{1}{\delta} = \lim_{\delta \rightarrow 0} \frac{f_i(x(y, \delta)) - f_i(y)}{\delta} +$$

$$+\theta_{x_1} f_i(y)|_{\delta=0} = f_{iy_1} \theta + \theta_{y_1} f_i = \frac{\partial}{\partial y_1} (\theta f_i),$$

i.e.

$$f'_i(y) = (\theta f_i)_{y_1}(y), \quad i = 1, 2. \quad (1.19)$$

By  $f_i \in C^1(\bar{\Omega})$ , we can see that as  $\delta \rightarrow 0$

$$\frac{f_i^\delta(y) - f_i^0(y)}{\delta} \rightarrow f'_i(y) \quad \text{in } L^\infty(\Omega), \quad i = 1, 2. \quad (1.20)$$

Convergence (1.20) is used in the sequel to find the derivative of the energy functional.

Let  $W^\delta = (u^\delta, v^\delta)$  be the solution of the problem (1.8) – (1.10). Denote  $u^\delta(x) = \tilde{u}(y)$ ,  $v^\delta(x) = \tilde{v}(y)$ ,  $x \in \Omega_\delta$ ,  $y \in \Omega$ ,  $x = x(y, \delta)$ . Here, we use the tilde and omit  $\delta$  for convenience.

Using (1.15), the following formulae are derived

$$\begin{cases} u_{x_1}^\delta = \tilde{u}_{y_1}(1 - \delta\theta_{x_1}) \\ u_{x_2}^\delta = \tilde{u}_{y_1}(-\delta\theta_{x_2}) + \tilde{u}_{y_2}, \end{cases} \quad \begin{cases} v_{x_1}^\delta = \tilde{v}_{y_1}(1 - \delta\theta_{x_1}) \\ v_{x_2}^\delta = \tilde{v}_{y_1}(-\delta\theta_{x_2}) + \tilde{v}_{y_2}. \end{cases} \quad (1.21)$$

Since

$$\begin{aligned} \int_{\Omega_\delta} \sigma_{ij}(W^\delta) \varepsilon_{ij}(W^\delta) &= \int_{\Omega_\delta} \left( (2\mu + \lambda)(\varepsilon_{11}^2(W^\delta) + \right. \\ &\left. + \varepsilon_{22}^2(W^\delta)) + 2\lambda \varepsilon_{11}(W^\delta) \varepsilon_{22}(W^\delta) + 4\mu \varepsilon_{12}^2(W^\delta) \right), \end{aligned}$$

in view of (1.21), we can perform the change of variables and replace the domain of integration  $\Omega_\delta$  by  $\Omega$ . This leads to

$$\begin{aligned} \frac{1}{2} \int_{\Omega_\delta} \sigma_{ij}(W^\delta) \varepsilon_{ij}(W^\delta) dx - \int_{\Omega_\delta} f W^\delta dx &= \frac{1}{2} \int_{\Omega} \frac{1}{q_\delta} \left( \tilde{u}_{y_1}^2 ((2\mu + \lambda)(1 - \delta\theta_{x_1})^2 - \mu\delta^2\theta_{x_2}^2) + \right. \\ &+ \mu\tilde{u}_{y_2}^2 + \tilde{v}_{y_1}^2 ((2\mu + \lambda)\delta^2\theta_{x_2}^2 + \mu(1 - \delta\theta_{x_1})^2) + (2\mu + \lambda)\tilde{v}_{y_2}^2 + 2\mu\tilde{u}_{y_1}\tilde{u}_{y_2}(-\delta\theta_{x_2}) + \\ &+ 2(\lambda + \mu)\tilde{u}_{y_1}\tilde{v}_{y_1}(-\delta\theta_{x_2})(1 - \delta\theta_{x_1}) + 2\lambda\tilde{u}_{y_1}\tilde{v}_{y_2}(1 - \delta\theta_{x_1}) + \\ &\left. + 2\mu\tilde{u}_{y_2}\tilde{v}_{y_1}(1 - \delta\theta_{x_1}) + 2(2\mu + \lambda)\tilde{v}_{y_1}\tilde{v}_{y_2}(-\delta\theta_{x_2}) \right) dy - \int_{\Omega} f^\delta \tilde{W} dy. \end{aligned} \quad (1.22)$$

Denote by  $I_\delta(\Omega; \tilde{W})$  the right-hand side of (1.22),  $\tilde{W} = (\tilde{u}, \tilde{v})$ . In this case, formula (1.22), actually, provides the transformation of the energy functional

$$I(\Omega_\delta; W^\delta) = I_\delta(\Omega; W_\delta), \quad (1.23)$$

where  $W_\delta = \tilde{W}$ . Again, let  $W^\delta(x) = W_\delta(y)$ ,  $x = x(y, \delta)$ . Then  $W_\delta \in K_0$  provided that  $W^\delta \in K_\delta$  and conversely, the inclusion  $W_\delta \in K_0$  implies  $W^\delta \in K_\delta$ . Thus we obtain a one-to-one mapping between  $K_\delta$  and  $K_0$ . In particular, this implies

$$\min_{U \in K_\delta} I(\Omega_\delta; U) = \min_{U \in K_0} I_\delta(\Omega; U). \quad (1.24)$$

By (1.23), (1.24), we have

$$\begin{aligned} \frac{J(\Omega_\delta) - J(\Omega)}{\delta} &= \frac{I(\Omega_\delta; W^\delta) - I(\Omega; W)}{\delta} = \frac{I_\delta(\Omega; W_\delta) - I(\Omega; W)}{\delta} \leq \\ &\leq \frac{I_\delta(\Omega; W) - I(\Omega; W)}{\delta}, \end{aligned}$$

whence

$$\limsup_{\delta \rightarrow 0} \frac{J(\Omega_\delta) - J(\Omega)}{\delta} \leq \limsup_{\delta \rightarrow 0} \frac{I_\delta(\Omega; W) - I(\Omega; W)}{\delta}. \quad (1.25)$$

On the other hand

$$\liminf_{\delta \rightarrow 0} \frac{J(\Omega_\delta) - J(\Omega)}{\delta} \geq \liminf_{\delta \rightarrow 0} \frac{I_\delta(\Omega; W_\delta) - I(\Omega; W_\delta)}{\delta}. \quad (1.26)$$

Taking into account (1.22), (1.20) and Lemma, we can show that the right-hand sides of (1.25) and (1.26) coincide. Consequently, there exists a limit

$$\lim_{\delta \rightarrow 0} \frac{J(\Omega_\delta) - J(\Omega)}{\delta}$$

which proves the existence of the derivative (1.14). Direct calculation of the right-hand sides of (1.25), (1.26) implies that

$$\left. \frac{dJ(\Omega_\delta)}{d\delta} \right|_{\delta=0} = \frac{1}{2} \int_{\Omega} \left( (2\mu + \lambda)u_{y_1}^2 (-\theta_{y_1}) + \mu u_{y_2}^2 \theta_{y_1} + 2\mu u_{y_1} u_{y_2} (-\theta_{y_2}) + \right.$$

$$\begin{aligned}
& +\mu v_{y_1}^2(-\theta_{y_1}) + (2\mu + \lambda)v_{y_2}^2\theta_{y_1} + 2(\lambda + \mu)u_{y_1}v_{y_1}(-\theta_{y_2}) + \\
& + 2(2\mu + \lambda)v_{y_1}v_{y_2}(-\theta_{y_2}) \Big) - \int_{\Omega} (\theta f_1)_{y_1} u - \int_{\Omega} (\theta f_2)_{y_1} v.
\end{aligned} \tag{1.27}$$

As a result, the Griffith formula (1.27) is derived. The formula represents the derivative of the energy functional with respect to the crack length for two-dimensional elasticity with nonlinear boundary conditions (1.28) below. Derivatives of the energy functional for the Poisson equation and the linear elasticity equations with linear boundary conditions prescribed on  $\Xi_l$  have been studied in (Mazja and Nazarov, 1987), (Grisvard, 1992), (see also (Bui and Ehrlacher, 1997), (Ohtsuka, 1994), (Blat and Morel, 1988), (Destuynder and Jaoua, 1981), (Bonnet, 1994)). Regularity of solutions to elliptic equations in nonsmooth domains are analysed in (Khludnev, 1995, 1996a, 1996b), (Kondratiev *et al.*, 1982). Other aspects of elliptic problems in domains with nonsmooth boundaries can be found in (Khludnev and Sokolowski, 1997), (Grisvard, 1992), (Dauge, 1988), (Nazarov and Plamenevskii, 1991). First, we analyse boundary conditions prescribed on  $\Xi_l$  for the problem (1.1) – (1.3). According to (Khludnev and Sokolowski, 1997), the following conditions hold at  $\Xi_l$

$$[v] \geq 0, \quad \sigma_{22} \leq 0, \quad [\sigma_{22}] = 0, \quad \sigma_{12} = 0, \quad [v]\sigma_{22} = 0. \tag{1.28}$$

Moreover, by (Khludnev, 1996a), the solution of the problem (1.1) – (1.3) (i.e., in fact, the problem (1.17)) has additional regularity properties up to the crack faces. Namely, for any  $x \in \Xi_l$  there exists a neighbourhood  $V$  of the point  $x$  such that  $W \in H^2(V \setminus \Xi_l)$ . Consequently, the solution  $W$  is continuous up to the crack faces, and the conditions (1.28) hold almost everywhere at  $\Xi_l$ . Note that

$$\sigma_{22} = (2\mu + \lambda)v_{y_2} + \lambda u_{y_1}, \quad \sigma_{12} = \mu(u_{y_2} + v_{y_1}). \tag{1.29}$$

In addition to (1.28) we can prove that

$$[\sigma_{22}v_{y_1}] = \sigma_{22}[v_{y_1}] = 0 \quad \text{a.e. on } \Xi_l. \tag{1.30}$$

Indeed, since  $v$  is continuous the set

$$M = \{y \in \Xi_l \mid [v(y)] > 0\}$$

is open. At any point  $y \in M$  we have  $[v(y)] > 0$ , therefore, by the last equality in (1.28),  $\sigma_{22}(y) = 0$ , and consequently  $\sigma_{22}[v_{y_1}] = 0$  a.e. on  $M$ . We have  $[v] = 0$  on the set  $\Xi_l \setminus M$ , whence  $[v_{y_1}] = 0$  (see Kinderlehrer and Stampacchia, 1980, Chapter 2, Theorem A.1) which implies  $\sigma_{22}[v_{y_1}] = 0$ . The equality (1.30) is proved.

Now we prove that the right-hand side of (1.27) is independent of  $\theta$ . Consider two right-hand sides corresponding to two functions  $\theta_1, \theta_2$  with the required properties. Denote by  $\Lambda$  the difference between the right-hand sides of (1.27). We have

$$\begin{aligned} \Lambda = & \frac{1}{2} \int_{\Omega} \left( (2\mu + \lambda)u_{y_1}^2(-\theta_{y_1}) + \mu u_{y_2}^2 \theta_{y_1} + 2\mu u_{y_1} u_{y_2}(-\theta_{y_2}) + \right. \\ & + \mu v_{y_1}^2(-\theta_{y_1}) + (2\mu + \lambda)v_{y_2}^2 \theta_{y_1} + 2(\lambda + \mu)u_{y_1} v_{y_1}(-\theta_{y_2}) + \\ & \left. + 2(2\mu + \lambda)v_{y_1} v_{y_2}(-\theta_{y_2}) \right) - \int_{\Omega} (\theta f_1)_{y_1} u - \int_{\Omega} (\theta f_2)_{y_1} v, \end{aligned} \quad (1.31)$$

where  $\theta = \theta_1 - \theta_2$ . The functions  $\theta_1, \theta_2$  are equal to 1 in neighbourhoods of the point  $x_l$ , consequently, the integration in (1.31) is actually fulfilled over  $\Omega \setminus B_{x_l}$ , where  $B_{x_l}$  is a ball of small radius centred at  $x_l$ . Integrating by parts in (1.31) leads to

$$\begin{aligned} \Lambda = & \int_{\Omega \setminus B_{x_l}} \theta \left( (2\mu + \lambda)u_{y_1} u_{y_1 y_1} + \mu u_{y_1} u_{y_2 y_2} + \mu v_{y_1} v_{y_1 y_1} + (\lambda + \mu)v_{y_1} u_{y_1 y_2} + \right. \\ & \left. + (\lambda + \mu)u_{y_1} v_{y_1 y_2} + (2\mu + \lambda)v_{y_1} v_{y_2 y_2} + f_1 u_{y_1} + f_2 v_{y_1} \right) + \\ & + \int_{\Xi_l} \theta \left( \mu[u_{y_1} u_{y_2}] + (\lambda + \mu)[u_{y_1} v_{y_1}] + (2\mu + \lambda)[v_{y_1} v_{y_2}] \right). \end{aligned} \quad (1.32)$$

Note that the equilibrium equations (1.1) can be written as

$$\begin{aligned} (2\mu + \lambda)u_{y_1 y_1} + \mu u_{y_2 y_2} + (\lambda + \mu)v_{y_1 y_2} &= -f_1, \\ (2\mu + \lambda)v_{y_2 y_2} + (\lambda + \mu)u_{y_1 y_2} + \mu v_{y_1 y_1} &= -f_2, \end{aligned} \quad (1.33)$$

and

$$\mu[u_{y_1} u_{y_2}] + (\lambda + \mu)[u_{y_1} v_{y_1}] + (2\mu + \lambda)[v_{y_1} v_{y_2}] = [\sigma_{12} u_{y_1}] + [\sigma_{22} v_{y_1}]. \quad (1.34)$$

By (1.28) – (1.30), (1.33), the right-hand side of (1.32) is equal to zero which proves the independence of the right-hand side of (1.27) on  $\theta$ . Note that the independence, actually, follows by simpler arguments. In fact, we have

$$\liminf_{\delta \rightarrow 0} \frac{J(\Omega_\delta) - J(\Omega)}{\delta} = \limsup_{\delta \rightarrow 0} \frac{J(\Omega_\delta) - J(\Omega)}{\delta} \quad (1.35)$$

which proves the existence  $dJ(\Omega_\delta)/d\delta|_{\delta=0}$ . Both sides of (1.35) are independent of  $\theta$ , hence  $\lim_{\delta \rightarrow 0} (J(\Omega_\delta) - J(\Omega))\delta^{-1}$  exists and is independent of  $\theta$ .

In some particular cases we can prove an additional regularity of the solution. For instance, assume that the solution  $W$  of the problem (1.1) – (1.3) has the property

$$[W] = 0 \quad \text{on } B_{x_l} \cap \Xi_l,$$

where  $B_{x_l}$  is any ball centred at  $x_l$ . Then the arguments used in (Khludnev, 1995) to prove  $C^\infty$ -regularity of the solution can be used to show that the equations

$$-\sigma_{ij,j}(W) = f_i, \quad i = 1, 2,$$

are satisfied in  $B_{x_l}$  in the sense of distributions. Consequently,  $W \in H_{loc}^3(B_{x_l})$ . In addition, by the inclusion  $f \in H^1(D)$ , we have  $W \in H_{loc}^3(\Omega)$ . In this case

$$\frac{dJ(\Omega_l)}{dl} = 0, \quad (1.36)$$

where we denote the domain  $\Omega$  by  $\Omega_l$ . In fact, we can integrate by parts in the right-hand side of (1.27) which gives

$$\begin{aligned} \frac{dJ(\Omega_l)}{dl} &= \int_{\Omega} \theta ((\sigma_{11,1} + \sigma_{12,2})u_{y_1} + (\sigma_{21,1} + \sigma_{22,2})v_{y_1} + f_1 u_{y_1} + f_2 v_{y_1}) + \quad (1.37) \\ &\quad + \int_{\Xi_l} \theta (\sigma_{22}[v_{y_1}] + [\sigma_{12}u_{y_1}]). \end{aligned}$$

By (1.1), (1.28), (1.30), the right-hand side of (1.37) is equal to zero and (1.36) follows.

The regularity  $W \in H^2(B_{x_l} \setminus \Xi_l)$  is in fact sufficient to justify (1.36). In this case we can repeat the above arguments and obtain that the right-hand side of (1.37) is equal to zero.

The formula (1.27) can be written in the form which does not contain the function  $\theta$ . To show this we choose a ball  $B_{x_l}(r)$  of radius  $r$  with the boundary  $\Gamma(r)$  such that  $\theta = 1$  on  $B_{x_l}(r)$ . In this case the integration by parts in (1.27) yields

$$\begin{aligned} \frac{dJ(\Omega_l)}{dl} = & \int_{B_{x_l}(r) \setminus \Xi_l} (f_1 u_{y_1} + f_2 v_{y_1}) + \frac{1}{2} \int_{\Gamma(r)} \nu_1 \left( (2\mu + \lambda)(u_{y_1}^2 - v_{y_2}^2) + \mu(v_{y_1}^2 - u_{y_2}^2) \right) \\ & + \int_{\Gamma(r)} \nu_2 \left( (2\mu + \lambda)v_{y_1} v_{y_2} + (\lambda + \mu)u_{y_1} v_{y_1} + \mu u_{y_1} u_{y_2} \right), \end{aligned} \quad (1.38)$$

where  $(\nu_1, \nu_2)$  is the unit normal exterior vector to  $\Gamma(r)$ .

Now assume that  $f = 0$  in some neighbourhood  $V$  of the point  $x_l$ . For sufficiently small  $r$ ,  $0 < r < r_0$ , we have  $B_{x_l}(r) \subset V$ , and the formula (1.38) implies

$$\begin{aligned} \frac{dJ(\Omega_l)}{dl} = & \frac{1}{2} \int_{\Gamma(r)} \nu_1 \left( (2\mu + \lambda)(u_{y_1}^2 - v_{y_2}^2) + \mu(v_{y_1}^2 - u_{y_2}^2) \right) + \\ & + \int_{\Gamma(r)} \nu_2 \left( (2\mu + \lambda)v_{y_1} v_{y_2} + (\lambda + \mu)u_{y_1} v_{y_1} + \mu u_{y_1} u_{y_2} \right). \end{aligned} \quad (1.39)$$

The right-hand side of (1.39) does not depend on  $r$ , consequently, we have the following property. Let  $W$  be the solution of the problem (1.1)– (1.3), and  $f$  be equal to zero in some neighbourhood of the point  $x_l$ . Then the integral

$$\begin{aligned} I = & \frac{1}{2} \int_{\Gamma(r)} \nu_1 \left( (2\mu + \lambda)(u_{y_1}^2 - v_{y_2}^2) + \mu(v_{y_1}^2 - u_{y_2}^2) \right) + \\ & + \int_{\Gamma(r)} \nu_2 \left( (2\mu + \lambda)v_{y_1} v_{y_2} + (\lambda + \mu)u_{y_1} v_{y_1} + \mu u_{y_1} u_{y_2} \right) \end{aligned}$$

is independent of  $r > 0$  for all sufficiently small  $r$ . Moreover, the above arguments show that the integral

$$I = \frac{1}{2} \int_C \nu_1 \left( (2\mu + \lambda)(u_{y_1}^2 - v_{y_2}^2) + \mu(v_{y_1}^2 - u_{y_2}^2) \right) + \quad (1.40)$$

$$+ \int_C \nu_2 \left( (2\mu + \lambda)v_{y_1}v_{y_2} + (\lambda + \mu)u_{y_1}v_{y_1} + \mu u_{y_1}u_{y_2} \right)$$

is path independent for any closed curve  $C$  surrounding the point  $x_l$ . In this case  $\nu = (\nu_1, \nu_2)$  is the normal unit vector to the curve  $C$ . Some part of this curve denoted by  $\Xi = \Xi_l \cap C$  may belong to  $\Xi_l$ . In this case, taking into account (1.34), (1.30), (1.28), we can integrate over  $\Xi^+$  or  $\Xi^-$  in (1.40) and the same value of the integral is obtained. Of course, the above independence takes place provided that  $f$  is equal to zero in a domain with the boundary  $C$ .

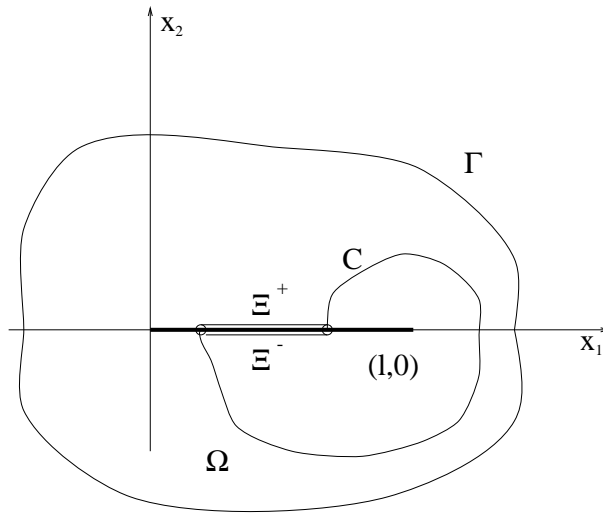


Figure 2: Curve  $C$ .

The integral of the form (1.40) is called the Rice–Cherepanov’s integral. We have to note that the result is obtained for nonlinear boundary conditions (1.28). The well-known path independence of the Rice–Cherepanov’s integral was previously proved in elasticity theory for linear boundary conditions  $\sigma_{22} = 0, \sigma_{12} = 0$  holding on  $\Xi_l^\pm$  (see (Parton and Morozov, 1985)).

## 2. Three-dimensional case.



Let  $D \subset R^3$  be a bounded domain with a smooth boundary  $\Gamma$ . The crack is defined in the form of the two-dimensional surface,

$$\Xi_{l+\delta} = \{(x_1, x_2, x_3) \mid x_3 = 0, \quad -h < x_2 < h, \quad 0 < x_1 < l + \delta\}.$$

Here,  $h > 0$ ,  $l > 0$ ,  $\delta$  is a small parameter which converges to zero. Assume that  $\Xi_{l+\delta} \subset D$  and denote  $\Omega_\delta = D \setminus \Xi_{l+\delta}$ ,  $\Omega = D \setminus \Xi_l$ .

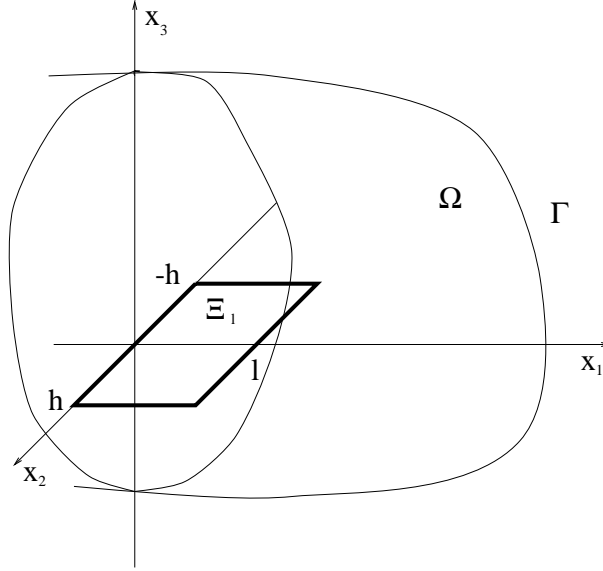


Figure 3: Domain  $\Omega$  and crack  $\Xi_l$  in  $R^3$ .

The equilibrium problem for an elastic body occupying the domain  $\Omega_\delta$  can be formulated as follows. We want to find a function  $W = (u, v, w)$  such that

$$-\sigma_{ij,j} = f_i, \quad i = 1, 2, 3, \quad \text{in } \Omega, \quad (2.1)$$

$$W = 0 \quad \text{on } \Gamma, \quad (2.2)$$

$$[W]n \geq 0 \quad \text{on } \Xi_l. \quad (2.3)$$

Here  $\sigma_{ij} = \sigma_{ij}(W)$  are stress tensor components,  $n = (0, 0, 1)$  is a normal vector to the surface  $\Xi_{l+\delta}$ . We assume that Hooke's law is fulfilled,

$$\sigma_{ij} = 2\mu\varepsilon_{ij} + \lambda \operatorname{div} W \delta_j^i, \quad i, j = 1, 2, 3, \quad (2.4)$$

$$\varepsilon_{ij}(W) = \frac{1}{2}(w_{,j}^i + w_{,i}^j), \quad (w^1, w^2, w^3) \equiv (u, v, w),$$

where  $\lambda, \mu$  are the Lamé parameters,  $\delta_j^i$  is the Kronecker symbol.

Actually, by considering the problem (2.1) – (2.3) we have in mind the minimisation of the functional

$$I(\Omega; U) = \frac{1}{2} \int_{\Omega} \sigma_{ij}(U) \varepsilon_{ij}(U) - \int_{\Omega} fU, \quad U = (u, v, w), \quad (2.5)$$

over the set

$$K_0 = \{(u, v, w) \in H^1(\Omega) \mid u = v = w = 0 \text{ on } \Gamma, [w] \geq 0 \text{ on } \Xi_l\}, \quad (2.6)$$

where  $f = (f_1, f_2, f_3) \in C^1(\bar{D})$ . In this case the conditions (2.1) – (2.3) are satisfied. Moreover, a system of equations and inequalities holds on  $\Xi_l$ , and (2.3) is a part of this system (see (2.22) below).

The perturbed problem corresponding to (2.1) – (2.3) is defined as follows. We want to find a function  $W^\delta = (u^\delta, v^\delta, w^\delta)$  such that

$$-\sigma_{ij,j} = f_i, \quad i = 1, 2, 3, \quad \text{in } \Omega_\delta, \quad (2.7)$$

$$W^\delta = 0 \quad \text{on } \Gamma, \quad (2.8)$$

$$[W^\delta]n \geq 0 \quad \text{on } \Xi_{l+\delta}. \quad (2.9)$$

Here  $\sigma_{ij} = \sigma_{ij}(W^\delta)$ , and  $\sigma_{ij}(W^\delta), \varepsilon_{ij}(W^\delta)$  satisfy Hooke's law (2.4). Analogously, by considering the problem (2.7) – (2.9) we, in fact, consider the minimisation of the energy functional

$$I(\Omega_\delta; U) = \frac{1}{2} \int_{\Omega_\delta} \sigma_{ij}(U) \varepsilon_{ij}(U) - \int_{\Omega_\delta} fU \quad (2.10)$$

over the set

$$K_\delta = \{(u, v, w) \in H^1(\Omega_\delta) \mid u = v = w = 0 \text{ on } \Gamma, [w] \geq 0 \text{ on } \Xi_{l+\delta}\}. \quad (2.11)$$

Denote by  $J(\Omega_\delta) = I(\Omega_\delta; W^\delta)$ ,  $J(\Omega) = I(\Omega; W)$  the energy functionals, where  $W^\delta, W$  are the solutions of (2.1) – (2.3) and (2.7) – (2.9), respectively. In this section, we determine the derivative

$$\lim_{\delta \rightarrow 0} \frac{J(\Omega_\delta) - J(\Omega)}{\delta}. \quad (2.12)$$

Consider a function  $\theta \in C_0^\infty(D)$ ,  $\theta = 1$  in a neighbourhood of the set  $L$ ,

$$L = \{(x_1, x_2, x_3) \mid x_1 = l, -h < x_2 < h, x_3 = 0\}.$$

The transformation of the variables

$$y_1 = x_1 - \delta\theta(x_1, x_2, x_3), \quad y_2 = x_2, \quad y_3 = x_3 \quad (2.13)$$

maps  $\Omega_\delta$  on  $\Omega$ . The Jacobian  $q_\delta$  is positive for all sufficiently small  $\delta$ ,

$$q_\delta = \left| \frac{\partial(y_1, y_2, y_3)}{\partial(x_1, x_2, x_3)} \right| = 1 - \delta\theta_{x_1}, \quad (2.14)$$

hence the transformation is one-to-one. For simplicity, the function  $\theta$  is assumed to be equal to zero in a neighbourhood of the set  $\{(x_1, x_2, x_3) \mid x_1 = 0, -h < x_2 < h, x_3 = 0\}$ . Let  $(u^\delta, v^\delta, w^\delta)$  be the solution of the problem (2.7) – (2.9). Denote  $(u^\delta(x), v^\delta(x), w^\delta(x)) = (\tilde{u}(y), \tilde{v}(y), \tilde{w}(y))$ . Let  $x = x(y, \delta)$  be the inverse transformation for the transformation (2.13). We use the tilde instead of  $\delta$  to simplify the formulae below. By (2.13), the following relations hold

$$u_{x_1}^\delta = \tilde{u}_{y_1}(1 - \delta\theta_{x_1}), \quad u_{x_2}^\delta = \tilde{u}_{y_1}(-\delta\theta_{x_2}) + \tilde{u}_{y_2}, \quad u_{x_3}^\delta = \tilde{u}_{y_1}(-\delta\theta_{x_3}) + \tilde{u}_{y_3}. \quad (2.15)$$

The similar formulae are valid for the functions  $v^\delta(x)$ ,  $w^\delta(x)$ .

Let  $\sigma_{ij} = \sigma_{ij}(W^\delta)$ ,  $\varepsilon_{ij} = \varepsilon_{ij}(W^\delta)$ . In this case, by (2.4),

$$\sigma_{ij}\varepsilon_{ij} = (2\mu + \lambda)(\varepsilon_{11}^2 + \varepsilon_{22}^2 + \varepsilon_{33}^2) + 4\mu(\varepsilon_{12}^2 + \varepsilon_{13}^2 + \varepsilon_{23}^2) + 2\lambda(\varepsilon_{11}\varepsilon_{22} + \varepsilon_{22}\varepsilon_{33} + \varepsilon_{11}\varepsilon_{33}).$$

This allows us to obtain the transformation of the energy functional,

$$\begin{aligned} \frac{1}{2} \int_{\Omega_\delta} \sigma_{ij}\varepsilon_{ij} - \int_{\Omega_\delta} fW^\delta &= \frac{1}{2} \int_{\Omega} \frac{1}{q_\delta} \left( (2\mu + \lambda)((1 - \delta\theta_{x_1})^2 \tilde{u}_{y_1}^2 + (\tilde{v}_{y_1}(-\delta\theta_{x_2}) + \tilde{v}_{y_2})^2 + \right. \\ &+ (\tilde{w}_{y_1}(-\delta\theta_{x_3}) + \tilde{w}_{y_3})^2) + \mu((\tilde{u}_{y_1}(-\delta\theta_{x_2}) + \tilde{u}_{y_2} + \tilde{v}_{y_1}(1 - \delta\theta_{x_1}))^2 + (\tilde{u}_{y_1}(-\delta\theta_{x_3}) + \\ &+ \tilde{u}_{y_3} + \tilde{w}_{y_1}(1 - \delta\theta_{x_1}))^2 + (\tilde{v}_{y_1}(-\delta\theta_{x_3}) + \tilde{v}_{y_3} + \tilde{w}_{y_1}(-\delta\theta_{x_2}) + \tilde{w}_{y_2})^2) + \\ &+ 2\lambda(\tilde{u}_{y_1}(1 - \delta\theta_{x_1})(\tilde{v}_{y_1}(-\delta\theta_{x_2}) + \tilde{v}_{y_2}) + (\tilde{v}_{y_1}(-\delta\theta_{x_2}) + \tilde{v}_{y_2})(\tilde{w}_{y_1}(-\delta\theta_{x_3}) + \end{aligned}$$

$$+\tilde{w}_{y_3}) + \tilde{u}_{y_1}(1 - \delta\theta_{x_1})(\tilde{w}_{y_1}(-\delta\theta_{x_3}) + \tilde{w}_{y_3})) \Big) - \int_{\Omega} f^{\delta} \tilde{W}, \quad (2.16)$$

where

$$f^{\delta}(y) = \frac{f(x(y, \delta))}{1 - \delta\theta_{x_1}}, \quad \tilde{W} = (\tilde{u}, \tilde{v}, \tilde{w}) = W_{\delta}.$$

Consequently, we have the following equality

$$I(\Omega_{\delta}; W^{\delta}) = I_{\delta}(\Omega; W_{\delta}). \quad (2.17)$$

Now we can proceed in the same way as in the previous section, which gives

$$\limsup_{\delta \rightarrow 0} \frac{J(\Omega_{\delta}) - J(\Omega)}{\delta} \leq \limsup_{\delta \rightarrow 0} \frac{I_{\delta}(\Omega; W) - I(\Omega; W)}{\delta}, \quad (2.18)$$

$$\liminf_{\delta \rightarrow 0} \frac{J(\Omega_{\delta}) - J(\Omega)}{\delta} \geq \liminf_{\delta \rightarrow 0} \frac{I_{\delta}(\Omega; W_{\delta}) - I(\Omega; W_{\delta})}{\delta}. \quad (2.19)$$

Since the right-hand sides of (2.18), (2.19) coincide, we obtain the existence of limit (2.12) in the form of the following Griffith formula

$$\begin{aligned} \frac{dJ(\Omega_{\delta})}{d\delta} \Big|_{\delta=0} &= \frac{1}{2} \int_{\Omega} \left( (2\mu + \lambda)(u_{y_1}^2(-\theta_{y_1}) + v_{y_2}^2\theta_{y_1} + w_{y_3}^2\theta_{y_1} + 2v_{y_1}v_{y_2}(-\theta_{y_2}) + \right. \\ &+ 2w_{y_1}w_{y_3}(-\theta_{y_3})) + \mu(u_{y_2}^2\theta_{y_1} + v_{y_1}^2(-\theta_{y_1}) + u_{y_3}^2\theta_{y_1} + w_{y_1}^2(-\theta_{y_1}) + v_{y_3}^2\theta_{y_1} + \\ &+ w_{y_2}^2\theta_{y_1} + 2u_{y_1}u_{y_3}(-\theta_{y_3}) + 2u_{y_1}w_{y_1}(-\theta_{y_3}) + 2v_{y_1}v_{y_3}(-\theta_{y_3}) + 2v_{y_1}w_{y_2}(-\theta_{y_3}) + \\ &+ 2v_{y_3}w_{y_1}(-\theta_{y_2}) + 2w_{y_1}w_{y_2}(-\theta_{y_2}) + 2u_{y_1}u_{y_2}(-\theta_{y_2}) + 2v_{y_1}u_{y_1}(-\theta_{y_2}) + 2v_{y_3}w_{y_2}\theta_{y_1}) + \\ &\left. + 2\lambda(v_{y_2}w_{y_3}\theta_{y_1} + u_{y_1}w_{y_1}(-\theta_{y_3}) + v_{y_2}w_{y_1}(-\theta_{y_3}) + v_{y_1}w_{y_3}(-\theta_{y_2}) + u_{y_1}v_{y_1}(-\theta_{y_2})) \right) - \\ &\quad - \int_{\Omega} (\theta f_1)_{y_1} u - \int_{\Omega} (\theta f_2)_{y_1} v - \int_{\Omega} (\theta f_3)_{y_1} w. \end{aligned} \quad (2.20)$$

It is not difficult to show that the right-hand side of (2.20) does not depend on  $\theta$ . To prove this, consider the difference between right-hand sides of (2.20) corresponding to any two functions  $\theta_1, \theta_2$ . Let  $\theta = \theta_1 - \theta_2$ . We integrate by

parts in (2.20) which implies that the difference denoted by  $\Lambda$  between the right-hand sides of (2.20) evaluated for  $\theta_1, \theta_2$ , respectively, is equal to

$$\begin{aligned} \Lambda = & \int_{\Omega \setminus S_L} \theta ((\sigma_{1,j,j} + f_1)u_{,1} + (\sigma_{2,j,j} + f_2)v_{,1} + (\sigma_{3,j,j} + f_3)w_{,1}) + \\ & + \int_{\Xi_l \setminus S_L} \theta \left( (2\mu + \lambda)[w_{,1}w_{,3}] + \mu[u_{,1}u_{,3}] + \mu[u_{,1}w_{,1}] + \right. \\ & \left. + \mu[v_{,1}v_{,3}] + \mu[v_{,1}w_{,2}] + \lambda[u_{,1}w_{,1}] + \lambda[v_{,2}w_{,1}] \right). \end{aligned} \quad (2.21)$$

We should remind at this point that  $\theta = \theta_1 - \theta_2 = 0$  in some neighbourhood of  $L$ . This neighbourhood is denoted by  $S_L$ . It is known that the solution of the problem (2.1) – (2.3) has an additional regularity up to the crack faces (see (Khludnev and Sokolowski, 1997)). For any point  $x \in \Xi_l$  there exists a neighbourhood  $V$  of the point  $x$  such that

$$W \in H^2(V \setminus \Xi_l).$$

In particular, by the Sobolev imbedding theorem,  $W$  is continuous up to the crack faces. As it was shown in (Khludnev and Sokolowski, 1997), the solution  $W$  satisfies the following boundary condition on  $\Xi_l$

$$[w] \geq 0, \quad [\sigma_{33}] = 0, \quad \sigma_{33} \leq 0, \quad [w]\sigma_{33} = 0, \quad \sigma_{13} = 0, \quad \sigma_{23} = 0. \quad (2.22)$$

It is easy to see that

$$\sigma_{33} = 2\mu w_{,3} + \lambda(u_{,1} + v_{,2} + w_{,3}), \quad \sigma_{13} = \mu(u_{,3} + w_{,1}), \quad \sigma_{23} = \mu(v_{,3} + w_{,2}). \quad (2.23)$$

From (2.21), in view of (2.1), (2.23), it follows that

$$\Lambda = \int_{\Xi_l \setminus S_L} \theta ([\sigma_{33}w_{,1}] + [\sigma_{13}u_{,1}] + [\sigma_{23}v_{,1}]) \quad (2.24)$$

and, consequently, by (2.22),

$$\Lambda = \int_{\Xi_l \setminus S_L} \theta \sigma_{33}[w_{,1}].$$

Let us prove that

$$\sigma_{33}[w_{,1}] = 0 \quad \text{a.e. on} \quad \Xi_l \setminus S_L. \quad (2.25)$$

Using (2.22) we have  $\sigma_{33}(y) = 0$  on the set

$$M = \{y \in \Xi_l \setminus S_L \mid [w(y)] > 0\}$$

and, consequently,  $\sigma_{33}(y)[w_{,1}(y)] = 0$ . The complement  $(\Xi_l \setminus S_L) \setminus M$  is characterised by the condition

$$[w(y)] = 0,$$

hence  $[w_{,1}(y)] = 0$  a.e. on  $(\Xi_l \setminus S_L) \setminus M$  (see Kinderlehrer and Stampacchia, 1980, Chapter 2, Theorem A.1). Thus  $\sigma_{33}[w_{,1}] = 0$  a.e. on  $(\Xi_l \setminus S_L) \setminus M$ . As a consequence, we obtain (2.25), and hence  $\Lambda = 0$  which shows the independence of the right-hand side in (2.20) on  $\theta$ .

To conclude the section we write the formula (2.20) in the form which does not contain the function  $\theta$ . To this end, consider a neighbourhood  $S_L$  of the set  $L$  with a smooth boundary  $\Gamma_L$  and assume that  $\theta = 1$  on  $S_L$ . Denote by  $(\nu_1, \nu_2, \nu_3)$  the unit external normal vector to  $\Gamma_L$ . Integrating by parts in (2.20) we obtain

$$\begin{aligned} \frac{dJ(\Omega_\delta)}{d\delta} \Big|_{\delta=0} &= \int_{S_L} (f_1 u_{y_1} + f_2 v_{y_1} + f_3 w_{y_1}) + \frac{1}{2} \int_{\Gamma_L} \nu_1 \left( (2\mu + \lambda)(u_{y_1}^2 - v_{y_2}^2 - w_{y_3}^2) + \right. \\ &\quad \left. + \mu(v_{y_1}^2 - u_{y_2}^2 - u_{y_3}^2 + w_{y_1}^2 - v_{y_3}^2 - w_{y_2}^2 - 2v_{y_3} w_{y_2}) - 2\lambda v_{y_2} w_{y_3} \right) + \\ &\quad + \int_{\Gamma_L} \nu_2 \left( (2\mu + \lambda)v_{y_1} v_{y_2} + \mu(v_{y_3} w_{y_1} + w_{y_1} w_{y_2} + u_{y_1} u_{y_2} + v_{y_1} u_{y_1}) + \right. \\ &\quad \left. + \lambda(v_{y_1} w_{y_3} + u_{y_1} v_{y_1}) \right) + \int_{\Gamma_L} \nu_3 \left( (2\mu + \lambda)w_{y_1} w_{y_3} + \right. \\ &\quad \left. + \mu(u_{y_1} u_{y_3} + u_{y_1} w_{y_1} + v_{y_1} v_{y_3} + v_{y_1} w_{y_2}) + \lambda(u_{y_1} w_{y_1} + v_{y_2} w_{y_1}) \right). \end{aligned} \quad (2.26)$$

Denoting by  $k(l, h, f)$  the functional defined by the right-hand side of (2.26) we have

$$\frac{dJ(\Omega_\delta)}{d\delta} \Big|_{\delta=0} = k(l, h, f).$$

Therefore,

$$J(\Omega_\delta) = J(\Omega) + k(l, h, f)\delta + \alpha(\delta)\delta,$$

where  $\alpha(\delta) \rightarrow 0$  as  $\delta \rightarrow 0$ . Note that  $k(l, h, f)$  is independent of  $S_L$ .

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## References

- [1] Blat J., Morel J.M., 1988, Elliptic problems in image segmentation and their relation to fracture theory. In *Recent advances in nonlinear elliptic and parabolic problems. Proceedings of an international conference*, Eds.: P.Benilan, M.Chipot, L.C.Evans, M.Pierre. Longman Scientific and Technical, 1988.
- [2] Bonnet M., 1994, Equations intégral variationnelles pour le problème en vitesse de propagation de fissures en élasticité linéaire *C.R. Acad. Sci. Paris, série II*, v. 318, pp. 429-434.
- [3] Bui H.D., Ehrlacher, A., 1997, *Developments of fracture mechanics in France in the last decades* In *Fracture Research in Retrospect*, H.P. Rossmannith (Ed.), A.A.Balkema, Rotterdam, pp. 369-387.
- [4] Cherepanov G.P., 1974, *Mechanics of brittle fracture*, Moscow, Nauka, (in Russian).
- [5] Dauge M., 1988, *Elliptic boundary value problems on corner domains*, Lecture Notes in Mathematics, 1341, Springer-Verlag.
- [6] Destuynder P., Jaoua M., 1981, Sur une Interprétation Mathématique de l'Intégrale de Rice en Théorie de la Rupture Fragile. *Math. Meth. in the Appl. Sci.*, v. 3, pp. 70-87.
- [7] Grisvard P., 1992, *Singularities in Boundary Value Problems*, Masson, Springer-Verlag.
- [8] Khludnev A.M., 1995, The contact problem for a shallow shell with a crack. *J.Appl. Maths. Mechs.*, v. 59, N.2, pp. 299-306.
- [9] Khludnev A.M., 1996a, On a Signorini problem for inclusions in shells. *Europ. J. of Appl. Math.*, v.7, pp. 499-510.
- [10] Khludnev A.M., 1996b, Contact problem for a plate having a crack of minimal opening. *Control and Cybernetics*, v. 25, N. 3, pp. 605-620.



- [11] Khludnev A.M., Sokolowski J., 1997, *Modelling and Control in Solid Mechanics*, Birkhauser. Bazel, Boston, Berlin.
- [12] Khludnev A.M., Sokolowski J., 1998, Griffith formula and Rice integral for an elliptic equation with unilateral conditions in nonsmooth domains, Les prépublication de l'Institut Élie Cartan, N. 7.
- [13] Kinderlehrer D., Stampacchia G., 1980, *An introduction to variational inequalities and their applications*, Academic Press, New York.
- [14] Kondratiev V.A., Kopačhek J., Oleinik O.A., 1982, On behaviour of solutions to the second order equations and elasticity equations in a neighbourhood of boundary points. In *Proceedings of Petrovsky's seminar*, Moscow, Moscow University Publishers, v.8, pp. 135-152 (in Russian).
- [15] Mazja V.G., Nazarov S.A., 1987, Asymptotics of energy integrals for small perturbation of a boundary near corner and conical points. *Proceedings of Moscow math. society*, Moscow State Univ., 1987, pp.79-129 (in Russian).
- [16] Nazarov S.A., Plamenevskii B.A., 1991, *Elliptic problems in domains with piecewise smooth boundaries*, Moscow, Nauka, (in Russian).
- [17] Ohtsuka K., 1994, Mathematical aspects of fracture mechanics, *Lecture Notes in Numer. Appl. Anal.*, v. 13, pp. 39-59.
- [18] Parton V.Z., Morozov E.M., 1985, *Mechanics of elastoplastic fracture*, Moscow, Nauka, (in Russian).
- [19] Parton V.Z., Perlin P.I., 1981, *Methods of mathematical theory of elasticity*, Moscow, Nauka, (in Russian).
- [20] Sokolowski J., Zolesio J.P., 1992, *Introduction to shape optimisation. Shape sensitivity analysis*, Springer-Verlag.



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