The Planar Motion with Two Bounded Controls - the Acceleration and the Derivative of the Curvature

Elena Degtiariova-Kostova

INSTITUT NATIONAL DE RECHERCHE EN INFORMATIQUE ET EN AUTOMATIQUE

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The planar motion with two bounded controls – the acceleration and the derivative of the curvature *

Elena Degtiariova-Kostova †

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Abstract: We study the minimum time problem to go from one given point on the plane to another with given initial and final tangent angles, curvatures and absolute values of speed, the paths joining these given points being C^1 and along them the derivative of the curvature and acceleration remaining bounded by two constants B and A respectively (we denote by u_1 (by u_2) the control of acceleration (of the derivative of the curvature respectively)).

After the application of the Maximum Principal of Pontryagin and after the study of all possible forms of concatenation of arcs of curves of any extremal path we obtain the following results:

1) any general optimal path is a C^1-jonction of line segments in one and the same direction φ ($u_2 \equiv 0$; $\varphi \in [0, 2\pi]$ and it is defined by the initial and final conditions) and of arcs of curves with linear curvature ($u_2 \equiv \pm B$);

2) along any general optimal path the point moves with piecewise-linear absolute value of the speed ($u_1 \equiv \pm A$);

3) any optimal path contains at most one line segment;

4) if for some optimal path the point moves along the line segment in the direction $\varphi + \pi \pmod{2\pi}$, then this optimal path contains an infinite number of concatenated arcs of curves with linear curvature ($u_2 \equiv \pm B$) which accumulate towards each endpoint of the line segment.

Key-words: car-like robot, (sub)optimal path, Maximum Principle of Pontryagin

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† I N R I A, Sophia Antipolis, 2004, Route des Lucioles, B.P.93, 06902, SOPHIA ANTIPOLIS, Cedex(France)

Le mouvement plan avec deux contrôles bornés – l'accélération et la dérivée de la courbure

Résumé : Nous considérons le problème de temps minimal pour aller d'un point donné à un autre sur le plan, les angles tangents, les courbures et les valeurs absolues de vitesse étant donnés, les courbes joignant ces points étant C^1 et le long d'eux la dérivée de la courbure et l'accélération étant bornées par deux constantes B et A respectivement (on désigne par u_1 (par u_2) la fonction de contrôle de l'accélération (celle de la dérivée de la courbure respectivement)).

Après l'application du Principe de Maximum de Pontryagine et après l'étude des formes possibles de jonction d'arcs de courbe d'une trajectoire extrémale, on obtient les résultats suivants:

> 1) une trajectoire optimale générale est une C^1-jonction de segments de droite de la même direction φ ($u_2 \equiv 0$; $\varphi \in [0, 2\pi]$ et φ est défini par les conditions initiales et finales) et d'arcs de courbe avec la courbure linéaire ($u_2 \equiv \pm B$);
>
> 2) le long de chaque trajectoire optimale générale le point bouge avec la vitesse linéaire par morceaux ($u_1 \equiv \pm A$);
>
> 3) la courbe optimale contient au plus un segment de droite;
>
> 4) si pour une coubre optimale le point bouge le long du segment de droite vers la direction $\varphi + \pi \pmod{2\pi}$, alors cette trajectoire optimale contient un nombre infini d'arcs joints de courbe avec la courbure linéaire ($u_2 \equiv \pm B$) qui s'accumulent vers chaque bout du segment de droite.

Mots-clés : robot mobile, chemin (sous)optimal, principe du maximum de Pontryagine

1 Introduction

We study the minimum time problem to go from one given point on the plane to another with given initial and final tangent angles, curvatures and absolute values of the speed, the paths joining these given points being C^1 and along them the derivative of the curvature and the acceleration remaining bounded by two constants B and A respectively. So, we have two bounded controls – the acceleration and the derivative of the curvature.

The real background of the problem is to solve the minimum time problem for a car-like robot to go from one given point to another with the above mentioned initial and final conditions. One can turn the wheels of a car with a bounded speed. Hence, the speed of changing the curvature of the path of a real car is bounded. Evidently, the acceleration of a real car is also bounded.

Some similar problems have been the object of several efforts recently. Dubins in [8] considers the problem of constructing the optimal trajectory between two given points with given tangent angles and with bounded curvature (cusps are not allowed). He proves that there exists a unique optimal trajectory which is a concatenation of at most three pieces; every piece is either a straight line segment or an arc of a circle of fixed radius. The same model is considered by Cockayne and Hall in [7] but from another point of view: they provide the classes of trajectories by which a moving "oriented point" can reach a given point in a given direction and they obtain the set of all the points reachable at a fixed time.

Reeds and Shepp in [18] solve a similar problem, when cusps are allowed. They obtain the list of all possible optimal trajectories. This list contains forty eight types of trajectories. Each of them is a finite concatenation of pieces each of which is either a straight line or an arc of a circle.

Laumond and Souères in [15] obtain a complete synthesis for the Reeds-Shepp model in the case without obstacles.

A complete synthesis for the Dubins model in the case without obstacles is obtained by Boissonnat, Bui, Laumond and Souères (1994, see [4] and [5]).

All these authors use very particular methods in their proofs. It seems very difficult to generalize them. That is why the same problem is solved by Sussman and Tang in [19] and by Boissonnat, Cérézo and Leblond in [1] by means of simpler arguments based on the Maximum Principle of Pontryagin.

Using these arguments allows to treat more difficult models as the one considered in [3] by Boissonnat, Cérézo and Leblond and in [14] by Kostov and Degtiariova-Kostova. They study the problem to find the shortest path connecting two given points of $\mathbf{R}^2$ with given initial and final tangent angles and curvatures and with bounded derivative of the curvature (cusps are not allowed). In [3] Boissonnat, Cérézo and Leblond prove (using the Maximum Principle of Pontryagin) that any extremal path is a C^2 concatenation of line segments in one and the same direction and of arcs of clothoid, all of finite length. They study the possible variants of concatenation of arcs of clothoid and line segments and obtain that if an extremal path contains but is not reduced to a line segment, then it contains an infinite number of concatenated arcs of clothoids which accumulate towards each endpoint of the segment which is a switching point. In [14] Kostov and Degtiariova-Kostova prove that if the distance between the initial and the final points is greater than $320\sqrt{\pi}$, then, in the generic case, optimal paths have an infinite number of switching points.

In the present paper we consider a similar problem but now with two bounded controls – the acceleration and the derivative of the curvature (cusps are not allowed). We denote by u_1 (by u_2) the control of acceleration (of the derivative of the curvature respectively).

After the application of the Maximum Principal of Pontryagin we obtain the following result: any extremal path is a C^1-jonction of line segments in one and the same direction φ ($u_2 \equiv 0$; $\varphi \in [0, 2\pi]$ and it is defined by the initial and final conditions) and of arcs of curves with linear curvature ($u_2 \equiv \pm B$) and along any optimal path the point moves with piecewise-linear absolute value of the speed ($u_1 \equiv \pm A$).

We study all possible forms of concatenation of arcs of curves with linear curvature and of line segments and we obtain that if an optimal path contains but is not reduced to a line segment, then:

1) along any optimal path the point moves with piecewise-linear absolute value of the speed ($u_1 \equiv \pm A$),

2) any optimal path contains at most one line segment,

3) if for some optimal path the point mouve along the line segment in the direction $\varphi + \pi \pmod{2\pi}$, then this optimal path contains an infinite number of concatenated arcs of curves with linear curvature ($u_2 \equiv \pm B$) which accumulate towards each endpoint of the line segment.

In Section 2 we formulate the problem and we study the controllability of the system and the existence of an optimal solution. In Section 3 we apply the Maximum Principle of Pontryagin to the problem and we formulate the obtained results in Section 4. We study all possible variants of concatenation of different pieces of extremals in Section 5 and in Section 6 we give the main result of the paper (see Lemma 13).

2 Statement of the problem, controllability of the system and existence of an optimal solution

Consider the minimum time problem to go from one given point on the plane to another with given initial and final tangent angles, curvatures and absolute values of speed, the paths joining these given points being C^1 and along them the derivative of the curvature and acceleration remaining bounded by two constants B and A respectively.

Denote by $x(t)$ and $y(t)$ the planar coordinates of a point, by $\kappa(t)$ its curvature, by $\alpha(t)$ its tangent angle and by $v(t)$ the absolute value of its speed.

For $\dot{x}(t)$, $\dot{y}(t)$ we have the following equations:

$$\begin{cases} \dot{x}(t) = v(t)\cos\alpha(t) \\ \dot{y}(t) = v(t)\sin\alpha(t) \end{cases}$$

Obtain now the equation for $\dot{\alpha}(t)$.

$$\kappa(t) = \frac{\dot{x}(t)\ddot{y}(t) - \dot{y}(t)\ddot{x}(t)}{\left(\dot{x}^2(t) + \dot{y}^2(t)\right)^{3/2}} =$$

$$= [v(t)\cos\alpha(t)(\dot{v}(t)\sin\alpha(t) + v(t)\dot{\alpha}(t)\cos\alpha(t)) -$$

$$- v(t)\sin\alpha(t)(\dot{v}(t)\cos\alpha(t) - v(t)\dot{\alpha}(t)\sin\alpha(t))] / v^3(t) =$$

$$= \left[v^2(t)\dot{\alpha}(t)\cos^2\alpha(t) + v^2(t)\dot{\alpha}(t)\sin^2\alpha(t)\right] / v^3(t) = \dot{\alpha}(t)/v(t) \ .$$

Hence, $\dot{\alpha}(t) = v(t)\kappa(t)$. So, we have the following system:

$$\dot{X}(t) = \begin{cases} \dot{x}(t) = v(t)\cos\alpha(t) & \\ \dot{y}(t) = v(t)\sin\alpha(t) & \\ \dot{v}(t) = u_1(t) & |u_1(t)| \leq A \\ \dot{\alpha}(t) = v(t)\kappa(t) & \\ \dot{\kappa}(t) = u_2(t) & |u_2(t)| \leq B \end{cases} \tag{1}$$

with initial and final conditions

$$X(0) = (x^0, y^0, v^0, \alpha^0, \kappa^0) \ , \qquad X(T) = (x^T, y^T, v^T, \alpha^T, \kappa^T) \ . \tag{2}$$

We control the acceleration by $u_1(t)$ and we control the derivative of the curvature by $u_2(t)$ (they are measurable real-valued functions). The vector-function $u(t) = (u_1(t), u_2(t))$ belongs to the set $U = [-A, A] \times [-B, B]$.

We want to find functions $x(t)$, $y(t)$, $v(t)$, $\alpha(t)$, $\kappa(t)$ satisfying (1) and (2) and such that the associated control function $u(t) = ((u_1(t), u_2(t))$ should minimize the time:

$$J(u_1, u_2) = T = \int_0^T dt \ . \tag{3}$$

We can prove the complete controllability of system (1), (2) using the results of the treating of one more simple problem with only one control function $u_2(t)$ (see [13]).

In [13] we set $v(t) \equiv 1$, we control the derivative of the curvature and we prove the complete controllability of the system. This means that for any points $(x^0, y^0, \alpha^0, \kappa^0) \in \mathbf{R}^4$ and $(x^T, y^T, \alpha^T, \kappa^T) \in \mathbf{R}^4$ we can construct a path connecting these points. Denote this path by $\mathcal{R}$. Now in order to satisfy the initial and final conditions for the variable $v(t)$ we assume (if $v^0 < v^T$) that the point move along the path $\mathcal{R}$ with the absolute value of the speed $v(t) = At + v^0$ for $t \in [0, (v^T - v^0)/A]$ and $v(t) \equiv v^T$ for $t \in [(v^T - v^0)/A, T]$. If $v^0 > v^T$, then we assume that the point move along the path $\mathcal{R}$ with the absolute value of the speed $v(t) = -At + v^0$ for $t \in [0, (v^0 - v^T)/A]$ and $v(t) \equiv v^T$ for $t \in [(v^0 - v^T)/A, T]$. So, we have proved the complete controllability of system (1), (2).

In order to prove the existence of an optimal solution we can use Filippov's existence theorem, see [6], th.5.1.ii. So, rewrite system (1) in the form

$$\dot{X}(t) = F(X(t), u(t)) \ , \qquad X(t) \in \mathbf{R}^2 \times [0, +\infty) \times \mathbf{R}^2 \ , \qquad u(t) \in U \ .$$

All requirements of the theorem of Filippov are satisfied: all functions $F(X(t), u(t))$ are continuous together with their partial derivatives; the function under the sign of the integral in (3) is continuous; the control functions $u_1(t)$, $u_2(t)$ are bounded and the range of control is convex; $X(t) \in \mathbf{R}^2 \times [0, +\infty) \times \mathbf{R}^2$ ($\mathbf{R}^2 \times [0, +\infty) \times \mathbf{R}^2$ is closed); the initial and final points $(X(0), X(T))$ are fixed; one can verify that there exists a constant $C > 0$ such that for every $X(t) \in \mathbf{R}^2 \times [0, +\infty) \times \mathbf{R}^2$ and $u \in U$ the following inequality is satisfied: $XF(X) \leq C(|X|^2 + 1)$. Thus we can assume the existence of an optimal solution and an optimal control for problem (1), (2), (3).

3 Application of the Maximum Principle of Pontryagin to the problem

3.1 Formulation of the Maximum Principle of Pontryagin

System (1), (2) is autonomous with fixed endpoints. Apply now the Maximum Principle of Pontryagin for this type of systems. Rewrite system (1), (2) and integral (3) as the following system:

$$\left\{ \begin{array}{llll} \dot{x}_0(t) = 1 & x_0(0) = 0 & & \\ \dot{x}(t) = v(t)\cos\alpha(t) & x(0) = x^0 & x(T) = x^T & \\ \dot{y}(t) = v(t)\sin\alpha(t) & y(0) = y^0 & y(T) = y^T & \\ \dot{v}(t) = u_1(t) & v(0) = v^0 & v(T) = v^T & |u_1(t)| \leq A \\ \dot{\alpha}(t) = v(t)\kappa(t) & \alpha(0) = \alpha^0 & \alpha(T) = \alpha^T & \\ \dot{\kappa}(t) = u_2(t) & \kappa(0) = \kappa^0 & \kappa(T) = \kappa^T & |u_2(t)| \leq B \end{array} \right.$$

Denote by $\Psi(t) = (\psi_0(t), \psi_1(t), \psi_2(t), \psi_3(t), \psi_4(t), \psi_5(t))$ the vector of "dual" variables; the Hamiltonian H is defined for every $t \in [0, T]$ by

$$\begin{array}{c} H(X(t), \Psi(t), u_1(t), u_2(t)) = \psi_0(t) + \psi_1(t)v(t)\cos\alpha(t) + \\ + \psi_2(t)v(t)\sin\alpha(t) + \psi_3(t)u_1(t) + \psi_4(t)v(t)\kappa(t) + \psi_5(t)u_2(t)\ . \end{array} \tag{4}$$

We have the following adjoint system for every $t \in [0, T]$:

$$\dot{\Psi}(t) = \left\{ \begin{array}{l} \dot{\psi}_0(t) = 0 \\ \dot{\psi}_1(t) = 0 \\ \dot{\psi}_2(t) = 0 \\ \dot{\psi}_3(t) = -\psi_1(t)\cos\alpha(t) - \psi_2(t)\sin\alpha(t) - \psi_4(t)\kappa(t) \\ \dot{\psi}_4(t) = \psi_1(t)v(t)\sin\alpha(t) - \psi_2(t)v(t)\cos\alpha(t) \\ \dot{\psi}_5(t) = -\psi_4(t)v(t) \end{array} \right. \tag{5}$$

So $\psi_0(t)$, $\psi_1(t)$, $\psi_2(t)$ are constant on $[0, T]$. Hence there exist $\lambda \geq 0$ and $\varphi \in [0, 2\pi]$ such that $\psi_1 = \lambda\cos\varphi$, $\psi_2 = \lambda\sin\varphi$ (here $\lambda = \sqrt{\psi_1^2 + \psi_2^2} \geq 0$ and for $\lambda > 0$ we have

$\cos\varphi = \psi_1/\sqrt{\psi_1^2+\psi_2^2}$, $\sin\varphi = \psi_2/\sqrt{\psi_1^2+\psi_2^2}$). Then we can rewrite the adjoint system (5) and the Hamiltonian (4) as follows:

$$\left\{\begin{array}{l} \psi_0(t) \equiv \psi_0 \\ \psi_1(t) \equiv \lambda\cos\varphi \\ \psi_2(t) \equiv \lambda\sin\varphi \\ \dot\psi_3(t) = -\lambda\cos(\alpha(t)-\varphi) - \psi_4(t)\kappa(t) \\ \dot\psi_4(t) = \lambda v(t)\sin(\alpha(t)-\varphi) \\ \dot\psi_5(t) = -\psi_4(t)v(t) \end{array}\right. \tag{6}$$

$$\begin{aligned} H(X(t),\Psi(t),u_1(t),u_2(t)) &= \lambda v(t)\cos(\alpha(t)-\varphi) + \psi_3(t)u_1(t) + \\ &+\psi_4(t)v(t)\kappa(t) + \psi_5(t)u_2(t) + \psi_0 = \\ &= -\dot\psi_3(t)v(t) + \psi_3(t)\dot v(t) + \psi_5(t)u_2(t) + \psi_0 \ . \end{aligned} \tag{7}$$

Define

$$M(X(t),\Psi(t)) = \min_{\substack{u_1(t)\in[-A,A] \\ u_2(t)\in[-B,B]}} H(X(t),\Psi(t),u_1(t),u_2(t))$$

where $\Psi(t)$, $X(t)$, $u_1(t)$ and $u_2(t)$ are considered as independent variables.

The Maximum Principle of Pontryagin ([6] Th. 5.1i) asserts that if $(u_1^*(t),\ u_2^*(t))$ is an optimal control, then

(a) there exists an absolutely continuous non-zero vector-function $\Psi(t)$ which is a continuous solution to (6);

(b) for almost every fixed $t\in[0,T]$ the function $H(X(t),\Psi(t),u_1(t),u_2(t))$ of the variables $u_1(t)\in[-A,+A]$ and $u_2(t)\in\ [-B,+B]$ only attains its minimum at the point $u_1(t)=u_1^*(t)$, $u_2(t)=u_2^*(t)$:

$$M(X(t),\Psi(t)) = H(X(t),\Psi(t),u_1^*(t),u_2^*(t)) \quad , \quad t\in[0,T];$$

(c) the function $M(t) = M(X(t),\Psi(t))$ is absolutely continuous in $[0,T]$ and

$$\frac{dM}{dt}(X(t),\Psi(t)) = \frac{\partial H}{\partial t}(X(t),\Psi(t),u_1(t),u_2(t));$$

(d) at any time $t\in[0,T]$ the relations $\psi_0\geq 0$ and $M(X(t),\Psi(t))=0$ are satisfied.

A measurable control $(u_1(t),u_2(t))$ and the associated trajectory $X(t)$ of (1) satisfying all conditions of the Maximum Principle of Pontryagin will be called *extremal control* and *extremal trajectory*. A point $X(t_s)$ of an extremal trajectory will be called a *switching point* if at $t=t_s$ the control function $u(t)$ has a discontinuity; the time t_s will be called a *switching time*.

From condition (d) and from (7) we obtain that for an extremal trajectory the following equality holds:

$$-\dot{\psi}_3(t)v(t) + \psi_3(t)\dot{v}(t) + \psi_5(t)u_2(t) + \psi_0 = 0 \tag{8}$$

We calculate now $\dot{H}(t)$.

$$\dot{H}(t) = -\ddot{\psi}_3(t)v(t) - \dot{\psi}_3(t)\dot{v}(t) + \dot{\psi}_3(t)\dot{v}(t) + \psi_3(t)\ddot{v}(t) + \dot{\psi}_5(t)u_2(t) + \psi_5(t)\dot{u}_2(t) =$$

$$= [-\ddot{\psi}_3(t)v(t) + \dot{\psi}_5(t)u_2(t)] + \psi_3(t)\dot{u}_1(t) + \psi_5(t)\dot{u}_2(t) \ .$$

The expression in the brackets is equal to zero. Really,

$$-\ddot{\psi}_3(t)v(t) + \dot{\psi}_5(t)u_2(t) = [\lambda \sin(\alpha(t) - \varphi)\dot{\alpha}(t) - \dot{\psi}_4(t)\kappa(t) - \psi_4(t)\dot{\kappa}(t)]v(t) +$$

$$+\dot{\psi}_5(t)u_2(t) = \lambda\kappa(t)v^2(t)\sin(\alpha(t) - \varphi) - \lambda\kappa(t)v^2(t)\sin(\alpha(t) - \varphi) -$$

$$-\psi_4(t)v(t)u_2 + \psi_4(t)v(t)u_2 = 0 \ .$$

For any extremal trajectory we have $\dot{H}(t) \equiv 0$, i.e the following equality holds:

$$\psi_3(t)\dot{u}_1(t) + \psi_5(t)\dot{u}_2(t) = 0 \ . \tag{9}$$

3.2 Treatement of condition (b) of the Maximum Principle of Pontryagin in the interior of U

We treat condition (b) in the interior of U, i.e. we consider the case when there exists an interval $[t_*, t_{**}] \subset [0, T]$ such that $\partial H/\partial u_1 \equiv 0$ and $\partial H/\partial u_2 \equiv 0$ for $t \in [t_*, t_{**}]$.

From equality $\partial H/\partial u_1 \equiv 0$ we obtain from (7) $\psi_3(t) \equiv 0$, hence $\dot{\psi}_3(t) \equiv 0$, i.e. (from (6))

$$\lambda \cos(\alpha(t) - \varphi) + \psi_4(t)\kappa(t) = 0 \ . \tag{10}$$

From equality $\partial H/\partial u_2 \equiv 0$ we obtain from (7) $\psi_5(t) \equiv 0$, i.e. $\dot{\psi}_5(t) \equiv 0$, hence (from (6)) $\psi_4(t) \equiv 0$, i.e. $\dot{\psi}_4(t) \equiv 0$, hence (from (6))

$$\lambda \sin(\alpha(t) - \varphi) = 0 \ . \tag{11}$$

Now, using equalities $\psi_3(t) \equiv 0$ and $\psi_5(t) \equiv 0$, we obtain from (8)

$$\psi_0 = 0 \ .$$

We can rewrite (10) (using equality $\psi_4(t) \equiv 0$) as follows:

$$\lambda \cos(\alpha(t) - \varphi) = 0 \ . \tag{12}$$

Comparing (11) and (12) we obtain $\lambda = 0$. Hence (from (6)) $\psi_1(t) = \psi_2(t) \equiv 0$ and we obtain a contradiction with condition (a) from the Maximum Principle of Pontryagin (because we obtain $\psi_1(t) = \psi_2(t) = \psi_3(t) = \psi_4(t) = \psi_5(t) \equiv \psi_0 = 0$).

Conclusion: there are no extremal trajectories containing a piece corresponding to some $u_1(t)$, $u_2(t)$ from the interior of U.

3.3 Treatement of condition (b) of the Maximum Principle of Pontryagin on the bound of U

We treat condition (b) on the bound of U. We must consider three cases:

1) the case when there exists some interval $[t_*, t_{**}] \subset [0, T]$ such that $\partial H/\partial u_1 \not\equiv 0$ and $\partial H/\partial u_2 \not\equiv 0$ for $t \in (t_*, t_{**})$,

2) the case when there exists some interval $[t_*, t_{**}] \subset [0, T]$ such that $\partial H/\partial u_1 \not\equiv 0$ and $\partial H/\partial u_2 \equiv 0$ for $t \in (t_*, t_{**})$,

3) the case when there exists some interval $[t_*, t_{**}] \subset [0, T]$ such that $\partial H/\partial u_1 \equiv 0$ and $\partial H/\partial u_2 \not\equiv 0$ for $t \in (t_*, t_{**})$.

Consider the case 1).

Using $\partial H/\partial u_1 \neq 0$, we obtain from (7) $u_1(t) = -A\text{sign}(\psi_3(t))$. Hence $\dot{v}(t) = -A\text{sign}(\psi_3(t))$ for $t \in (t_*, t_{**}) \subset [0, T]$. Using $\partial H/\partial u_2 \neq 0$, we obtain from (7) $u_2(t) = -B\text{sign}(\psi_5(t))$. Hence $\dot{\kappa}(t) = -B\text{sign}(\psi_5(t))$ for $t \in (t_*, t_{**}) \subset [0, T]$.

Conclusion: in the case 1) we obtain the result that the part of the extremal path corresponding to the interval (t_*, t_{**}) is some curve with the piecewise-linear curvature $\dot{\kappa}(t) = -B\text{sign}(\psi_5(t))$ and with the piecewise-linear absolute value of speed $\dot{v}(t) = -A\text{sign}(\psi_3(t))$ ($\psi_0 \geq 0$, $\lambda \geq 0$).

Consider the case 2).

Using $\partial H/\partial u_1 \neq 0$, we obtain from (7) $u_1(t) = -A\text{sign}(\psi_3(t))$. Hence $\dot{v}(t) = -A\text{sign}(\psi_3(t))$ for $t \in (t_*, t_{**}) \subset [0, T]$.

Using $\partial H/\partial u_2 \equiv 0$, we obtain from (7) $\psi_5(t) \equiv 0$, i.e. $\dot{\psi}_5(t) \equiv 0$, hence (from (6)) $\psi_4(t) \equiv 0$, i.e. $\dot{\psi}_4(t) \equiv 0$, hence (from (6))

$$\lambda \sin(\alpha(t) - \varphi) = 0 \ . \tag{13}$$

From (8) we obtain

$$-\dot{\psi}_3(t)v(t) + \psi_3(t)\dot{v}(t) + \psi_0 = 0 \ . \tag{14}$$

From (13) follows that we must consider two possibilities:

a) $\sin(\alpha(t) - \varphi) = 0$,
b) $\lambda = 0$.

In the subcase a) we have $\sin(\alpha(t) - \varphi) = 0$, i.e. $\alpha = \varphi (\text{mod}\pi)$ for $t \in (t_*, t_{**}) \subset [0, T]$.

Thus, in the subcase a) we obtain a line segment in the direction φ and with the piecewise-linear absolute value of speed $\dot{v}(t) = -A\text{sign}(\psi_3(t))$ ($\psi_0 \geq 0$, $\lambda > 0$).

In the subcase b) $\lambda = 0$, then we obtain from (6) that $\psi_1(t) = \psi_2(t) = \psi_4(t) = \psi_5(t) \equiv 0$, $\dot{\psi}_3(t) \equiv 0$. Hence $\psi_3(t) \equiv \psi_3(t_*) = const$. From (14) we obtain $-A\psi_3(t_*)\text{sign}(\psi_3(t_*)) + \psi_0 = 0$, i.e. $-A|\psi_3(t_*)| + \psi_0 = 0$, i.e. $|\psi_3(t_*)| = \psi_0/A \neq 0$ if $\psi_0 > 0$. Hence, for $\psi_0 > 0$ there is no contradiction with condition (a) from the Maximum Principle of Pontryagin (for $\psi_0 = 0$

we obtain $\psi_1(t) = \psi_2(t) = \psi_3(t) = \psi_4(t) = \psi_5(t) \equiv 0$ – a contradiction with condition (a) from the Maximum Principle of Pontryagin).

Remark that if $\lambda = 0$ for some extremal path, then it follows from (6) that $\dot{\psi}_4(t) \equiv 0$, i.e. $\psi_4(t) = const = \psi_{40}$ along the whole extremal path. Hence, as in the subcase b) we have $\psi_4(t) \equiv 0$ along the part of the extremal path corresponding to the interval (t_*, t_{**}), then $\psi_4(t) \equiv 0$ along the whole extremal path. But if $\lambda = 0$ and $\psi_4(t) \equiv 0$ along some extremal path, then it follows from (6) that $\dot{\psi}_3(t) \equiv 0$, i.e. $\psi_3(t) = const = \psi_{30}$ along the whole extremal path. Hence, $\psi_3(t_*) = \psi_{30}$ and we obtain that along the whole extremal path the speed is either increasing or decreasing function (it depends on the initial and final conditions), i.e. $\dot{v}(t) = -A\text{sign}(\psi_{30})$ and $v(t) = -At\text{sign}(\psi_{30}) + v^0$. So, the extremal path is run in time $T = (v^T - v^0)/(-A\text{sign}(\psi_{30})) = |v^T - v^0|/A$.

So, in the subcase b) and if $\psi_0 > 0$ we obtain the result that optimal paths are all admissible curves with the linear absolute value of speed $v(t) = -At\text{sign}(\psi_{30}) + v^0$ and which are run in time $T = |v^T - v^0|/A$.

Conclusion: in the case 2) we obtain the result that if $\lambda > 0$, $\psi_0 \geq 0$, then the part of the extremal path corresponding to the interval (t_*, t_{**}) is a line segment in the direction φ and with the piecewise-linear absolute value of speed $\dot{v}(t) = -A\text{sign}(\psi_3(t))$; if $\lambda = 0$, $\psi_0 > 0$, then optimal paths are all admissible curves with the linear absolute value of speed $v(t) = -At\text{sign}(\psi_{30}) + v^0$, they are run in time $T = |v^T - v^0|/A$.

Consider the case 3).

Using $\partial H/\partial u_1 \equiv 0$, we obtain from (7) $\psi_3(t) \equiv 0$, hence $\dot{\psi}_3(t) \equiv 0$, i.e. (from (6))

$$\lambda \cos(\alpha(t) - \varphi) + \psi_4(t)\kappa(t) = 0 \ . \tag{15}$$

From (8) we obtain

$$\psi_5(t)u_2(t) + \psi_0 = 0 \ . \tag{16}$$

Using $\partial H/\partial u_2 \neq 0$, we obtain from (7) $u_2(t) = -B\text{sign}(\psi_5(t))$. Hence $\dot{\kappa}(t) = -B\text{sign}(\psi_5(t))$ for $t \in (t_*, t_{**}) \subset [0, T]$.

From (16) follows that we must consider two possibilities:

a) $\psi_0 = 0$,
b) $\psi_0 > 0$.

Consider the subcase a). We have $\psi_0 = 0$, hence, from (16) we obtain $\psi_5(t)u_2(t) = 0$. Then $\psi_5(t) \equiv 0$ (because $u_2(t) \neq 0$), i.e. $\dot{\psi}_5(t) \equiv 0$, hence (from (6)) $\psi_4(t) \equiv 0$, i.e. $\dot{\psi}_4(t) \equiv 0$, hence (from (6)) $\lambda \sin(\alpha(t) - \varphi) = 0$. From (15) we obtain $\lambda \cos(\alpha(t) - \varphi) = 0$. Thus, we have $\lambda = 0$ and $\psi_1(t) = \psi_2(t) = \psi_3(t) = \psi_4(t) = \psi_5(t) \equiv \psi_0 = 0$ – a contradiction with condition (a) from the Maximum Principle of Pontryagin.

So, in the subcase a) no extremal curve can contain a piece such that $\partial H/\partial u_1 \neq 0$ and $\partial H/\partial u_2 \equiv 0$ along this piece.

Consider the subcase b). We have $\psi_0 > 0$ and $u_2(t) = -B\text{sign}(\psi_5(t))$, hence, from (16) we obtain $|\psi_5(t)| = \psi_0/B$. Hence, $\dot{\psi}_5(t) \equiv 0$ and (from (6)) $\psi_4(t) \equiv 0$, i.e. $\dot{\psi}_4(t) \equiv 0$, hence (from (6)) $\lambda \sin(\alpha(t) - \varphi) = 0$. From (15) we obtain $\lambda \cos(\alpha(t) - \varphi) = 0$. Thus, we have

$\lambda = 0$ and $\psi_1(t) = \psi_2(t) = \psi_3(t) = \psi_4(t) \equiv 0$, $\psi_5(t) \not\equiv 0$, $\psi_0 > 0$ – there is no contradiction with condition (a) from the Maximum Principle of Pontryagin.

Remark that if $\lambda = 0$ for some extremal path, then it follows from (6) that $\dot{\psi}_4(t) \equiv 0$, i.e. $\psi_4(t) = const = \psi_{40}$ along the whole extremal path. Hence, as in the subcase b) we have $\psi_4(t) \equiv 0$ along the part of the extremal path corresponding to the interval (t_*, t_{**}), then $\psi_4(t) \equiv 0$ along the whole extremal path. But if $\lambda = 0$ and $\psi_4(t) \equiv 0$ along some extremal path, then it follows from (6) that $\dot{\psi}_5(t) \equiv 0$, i.e. $\psi_5(t) = const = \psi_{50}$ along the whole extremal path. Hence, in the subcase b) $\psi_{50} = \psi_0/B$ and we obtain that along the whole extremal path the curvature is either increasing or decreasing function (it depends on the initial and final conditions), i.e. $\dot{\kappa}(t) = -B\text{sign}(\psi_{50})$ and $\kappa(t) = -Bt\text{sign}(\psi_{50}) + \kappa^0$. So, the extremal path is run in time $T = (\kappa^T - \kappa^0)/(-B\text{sign}(\psi_{50})) = |\kappa^T - \kappa^0|/B$.

So, in the subcase b) we obtain the result that $\lambda = 0$ and that optimal paths are curves with the linear curvature $\kappa(t) = -Bt\text{sign}(\psi_{50}) + \kappa^0$, with some admissible absolute value of speed and which are run in time $T = |\kappa^T - \kappa^0|/B$.

Conclusion: in the case 3) we obtain the result that if $\lambda = 0$, $\psi_0 > 0$, then optimal paths are all curves with the linear curvature $\kappa(t) = -Bt\text{sign}(\psi_{50}) + \kappa^0$ and with some admissible absolute value of speed, they are run in time $T = |\kappa^T - \kappa^0|/B$; if either $\psi_0 = 0$, $\lambda \geq 0$ or $\psi_0 > 0$, $\lambda > 0$, then no extremal curve can contain a piece such that $\partial H/\partial u_1 \neq 0$ and $\partial H/\partial u_2 \equiv 0$ along this piece.

4 Conclusions made after the application of the Maximum Principal of Pontryagin to the problem

Summarizing the results obtained in the Section 3 we can make the following conclusions:

Lemma 1 *If for some extremal path the control functions have finitely many points of discontinuity and if $\lambda > 0$, then this path is a C^1-jonction of arcs of the curves with the piecewise-linear curvature $\dot{\kappa}(t) = -B\text{sign}(\psi_5(t))$ and line segments in one and the same direction φ and it is run with the piecewise-linear absolute value of speed $\dot{v}(t) = -A\text{sign}(\psi_3(t))$ ($\psi_0 \geq 0$).*

If for some extremal path the control functions have finitely many points of discontinuity, if $\lambda = 0$ and $\psi_0 \geq 0$, then this path is a C^1-jonction the arcs of the curves with the piecewise-linear curvature $\dot{\kappa}(t) = -B\text{sign}(\psi_5(t))$ and it is run with the piecewise-linear absolute value of speed $\dot{v}(t) = -A\text{sign}(\psi_3(t))$.

For $\lambda = 0$ and $\psi_0 > 0$ there are two special cases:

1) optimal paths are all admissible curves with the linear absolute value of speed $v(t) = -At\text{sign}(\psi_{30}) + v^0$ and they are run in time $T = |v^T - v^0|/A$,

2) optimal paths are all curves with the linear curvature $\kappa(t) = -Bt\text{sign}(\psi_{50}) + \kappa^0$, with some admissible absolute value of speed and they are run in time $T = |\kappa^T - \kappa^0|/B$.

5 Concatenation of different pieces of extremals

From now on we use the following notation for arcs of curves with $u_2(t) = \pm B$ and line segments:

1) "Cl^+" – an arc of curve with $u_2(t) \equiv B$, $\psi_5(t) < 0$,
2) "Cl^-" – an arc of curve with $u_2(t) \equiv -B$, $\psi_5(t) > 0$,
3) "S^φ" – a line segment in the direction φ ($u_2(t) \equiv 0$, $\psi_5(t) \equiv 0$),
4) "$Cl_\nu^{+(-)}$" – an arc of curve of length ν with $u_2(t) \equiv B$ ($u_2(t) \equiv -B$),
5) "$*^{v+}$" – a piece of an extremal path (a line segment or an arc of curve with $u_2(t) = \pm B$) with an increasing linear speed ($u_1(t) \equiv A$, $\psi_3(t) < 0$),
6) "$*^{v-}$" – a piece of an extremal path (a line segment or an arc of curve with $u_2(t) = \pm B$) with a decreasing linear speed ($u_1(t) \equiv -A$, $\psi_3(t) > 0$),
7) "." – a switching point.

Remark 2 *If $v(t) \equiv 0$ on some interval $[t^*, t^{**}]$, then we cannot correctly define the piece of the path corresponding to $[t^*, t^{**}]$ using the initial system (1) and, hence, extremal paths have no intervals of zero speed.*

Remark 3 *As $v(t) \in [0, +\infty)$ and as extremal paths can have only some points of zero speed and not intervals, we obtain that the point $\tilde{t}$ (where $v(\tilde{t}) = 0$) is a switching point of the control $u_1(t)$ at which the variable $\psi_3(t)$ change the sign from "$-$" to "$+$".*

To characterize extremal paths we consider the following problem: how these arcs of curves corresponding to $u_2(t) = -B\text{sign}(\psi_5(t))$ and line segments are arranged along an extremal trajectory? To theat this problem we use the method introduced by Boissonnat, Cérézo and Leblond in [3].

At first we prove some common properties for the extremal trajectories in the two cases (i.e. for $\lambda = 0$ and for $\lambda > 0$).

Proposition 4 *At any switching point of the control $u_1(t)$ ($*^{v+}.*^{v-}$ or $*^{v-}.*^{v+}$) $\psi_3(t) = 0$.*

Proof

At a switching point $*^{v+}.*^{v-}$ ($*^{v-}.*^{v+}$) the signs of $u_1(t)$ and $\psi_3(t)$ change, hence $\psi_3(t) = 0$ at this point.

The proposition is proved. □

Proposition 5 *At any switching point of the control $u_2(t)$ ($Cl.Cl$, $Cl.S^\varphi$ or $S^\varphi.Cl$) $\psi_5(t) = 0$.*

Proof

On S^φ the continuous function $\psi_5(t) \equiv 0$, hence, $\psi_5(t) = 0$ at a switching point $Cl.S^\varphi$ or $S^\varphi.Cl$. At a switching point $Cl.Cl$ the signs of $u_2(t)$ and $\psi_5(t)$ change, hence $\psi_5(t) = 0$ at this point.

The proposition is proved. □

Now we consider the case $\lambda > 0$.

5.1 Concatenation of different pieces of extremals – the case $\lambda > 0$

Proposition 6 *If an extremal path contains a line segment S^φ (it is run during the time interval $[t_1, t_2]$), then*

$$\psi_3(t)u_1(t) + \psi_0 < 0 \quad \text{if} \quad \alpha(t) \equiv \varphi(\text{mod} 2\pi) \quad \text{and} \quad v(t) \neq 0 \quad \text{on} \quad S^\varphi \ ,$$

$$\psi_3(t)u_1(t) + \psi_0 > 0 \quad \text{if} \quad \alpha(t) \equiv \varphi + \pi(\text{mod} 2\pi) \quad \text{and} \quad v(t) \neq 0 \quad \text{on} \quad S^\varphi$$

and

$$\psi_3(t)u_1(t) < 0 \quad \text{if} \quad v(t) \neq 0 \quad \text{on} \quad S^\varphi \ .$$

Proof

If an extremal path contains a line segment S^φ, then on S^φ we have $\psi_4(t) = \dot{\psi}_4(t) = \psi_5(t) = \dot{\psi}_5(t) \equiv 0$. Hence, using (8) and (6), we obtain

$$\lambda v(t)\cos(\alpha(t) - \varphi) + \psi_3(t)u_1(t) + \psi_0 = 0 \tag{17}$$

Remind, that $\alpha(t) = \varphi(\text{mod}\pi)$ on S^φ. Consider two possibilities:

a) $\alpha(t) \equiv \varphi(\text{mod} 2\pi)$ on S^φ ,
b) $\alpha(t) \equiv \varphi + \pi(\text{mod} 2\pi)$ on S^φ .

In the case a) and if $v(t) \neq 0$ on S^φ we have the following inequality: $\lambda v(t)\cos(\alpha(t) - \varphi) = \lambda v(t) > 0$ (because $\lambda > 0$ and $v(t) > 0$). Hence, from (17) we obtain

$$\psi_3(t)u_1(t) + \psi_0 < 0 \ .$$

In the case b) and if $v(t) \neq 0$ on S^φ we have the following inequality: $\lambda v(t)\cos(\alpha(t) - \varphi) = -\lambda v(t) < 0$ (because $\lambda > 0$ and $v(t) > 0$). Hence, from (17) we obtain

$$\psi_3(t)u_1(t) + \psi_0 > 0 \ .$$

Remind that the point move along S^φ with the piecewise-linear absolute value of speed $\dot{v}(t) = -A\text{sign}(\psi_3(t))$, hence,

$$\psi_3(t)u_1(t) = \psi_3(t)\dot{v}(t) = -A\psi_3(t)\text{sign}(\psi_3(t)) = -A|\psi_3(t)| < 0$$

(if $v(t) \neq 0$ on S^φ).

The proposition is proved. □

Proposition 7 *For the case under consideration (i.e. $\lambda > 0$) the expression $\psi_4(t) - \psi_1(t)y(t) + \psi_2(t)x(t)$ is constant along any extremal path. For any $c \in \mathbf{R}$, all the points of an extremal path where $\psi_4(t) = c$ lie on the same straight line D_c^φ of direction φ $(\mathrm{mod}\,\pi)$.*

Proof

From (5) we have the following equation:

$$\dot{\psi}_4(t) = \psi_1(t)v(t)\sin\alpha(t) - \psi_2(t)v(t)\cos\alpha(t) \ ,$$

hence, using (1),

$$\dot{\psi}_4(t) = \psi_1(t)\dot{y}(t) - \psi_2(t)\dot{x}(t) \ , \tag{18}$$

where $\psi_1(t)$ and $\psi_2(t)$ are constants ($\psi_1(t) \equiv \lambda\cos\varphi$, $\psi_2(t) \equiv \lambda\sin\varphi$).

So, it follows from (18) that there exists a constant $c_0 \in \mathbf{R}$ such that $\psi_4(t) + c_0 = \lambda\cos\varphi y(t) - \lambda\sin\varphi x(t)$ (the first statement of Proposition 7 is proved).

We consider the case $\lambda > 0$, hence, $\lambda\cos\varphi$ and $\lambda\sin\varphi$ cannot be both equal to zero, so $c + c_0 = \lambda\cos\varphi y(t) - \lambda\sin\varphi x(t)$ is the equation of a line of direction φ $(\mathrm{mod}\pi)$ $(\tan\varphi = \psi_2/\psi_1)$.

The proposition is proved. □

Corollary 8 *Any line segment S^φ of an extremal path is contained in the straight line D_0^φ and, hence, the extremal path contain only one line segment.*

Proof

On S^φ we have $\psi_4(t) \equiv 0$. Hence, it follows from Proposition 7 that S^φ is contained in the straight line D_0^φ of direction φ.

The corollary is proved. □

Proposition 9 *For any extremal path there exists a coordinate system Oxy such that in this coordinate system the mean values of $y(t)v(t)$ on every interval between two consecutive switching points of the control $u_2(t)$ are equal to zero.*

Proof

Consider some extremal path. It follows from Proposition 7 that $\psi_4(t) = \psi_1(t)y(t) - \psi_2(t)x(t) - c_0$ along any extremal path ($\psi_1(t) \equiv \lambda\cos\varphi$, $\psi_2(t) \equiv \lambda\sin\varphi$). Hence, we can rotate the given coordinate system Oxy at some angle $\tilde{\alpha}$ such that in the new coordinate system $\psi_4(t) = y(t)$ along the extremal path.

So, using (6), we obtain

$$\dot{\psi}_5(t) = -y(t)v(t) \ . \tag{19}$$

This equation holds along the extremal path. Consider some interval $[t_1, t_2] \in [0, T]$ between two consecutive switching points of the control $u_2(t)$. So, it follows from (19) that

$$\psi_5(t_2) = -\int_{t_1}^{t_2} y(t)v(t)dt + \psi_5(t_1) \ . \tag{20}$$

But $\psi_5(t_1) = \psi_5(t_2) = 0$ (it follows from Proposition 5). Hence, we obtain $\int_{t_1}^{t_2} y(t)v(t)dt = 0$ for any interval between two consecutive switching points of the control $u_2(t)$, i.e. the mean values of $y(t)v(t)$ on every interval between two consecutive switching points of the control $u_2(t)$ are equal to zero.

The proposition is proved. □

Proposition 10 *For the case $\lambda > 0$ each open arc of clothoid Cl_ν ($\nu > 0$) of an extremal path (except possibly the initial and the final ones) intersect D_0^φ at least once or has a point with zero absolute value of speed.*

Proof

Consider an extremal path and consider some arc Cl_ν belonging to this path, $\nu > 0$ (Cl_ν is neither the initial nor the final one). So, both endpoints of such arc are switching points. Denote by $]t_3, t_4[$ the time interval during which Cl_ν is run. By Proposition 5 we have $\psi_5(t_3) = \psi_5(t_4) = 0$. As $t_4 - t_3 = \nu > 0$, there exists at least one $t \in]t_3, t_4[$ (denote it by t_5) such that $\dot{\psi}_5(t_5) = 0$. Hence, from (6), we obtain $\psi_4(t_5)v(t_5) = 0$. So, either $\psi_4(t_5) = 0$, or $v(t_5) = 0$. If $\psi_4(t_5) = 0$, then (from Proposition 7) we obtain that the point of Cl_ν corresponding to $t = t_5$ belongs to D_0^φ.

The proposition is proved. □

Proposition 11 *An extremal path (in the case $\lambda > 0$) containes no portion of type $S^\varphi.Cl_\nu$ (or $Cl_\nu.S^\varphi$) with $\nu > 0$ if $\alpha(t) \equiv \varphi + \pi \pmod{2\pi}$ along S^φ.*

Proof

Assume that there is a piece of an extremal path of type $S^\varphi.Cl_\nu$ with $\nu > 0$. Let t_1 is a switching point between S^φ and Cl_ν. Using (1) and (6) we obtain the following expressions for the four first derivatives of $\psi_5(t)$ (valid on S^φ and Cl_ν):

$$\dot{\psi}_5(t) = -\psi_4(t)v(t)\ ,$$

$$\ddot{\psi}_5(t) = -\dot{\psi}_4(t)v(t) - \psi_4(t)u_1(t) = -\lambda v^2(t)\sin(\alpha(t)-\varphi) - \psi_4(t)u_1(t)\ , \tag{21}$$

$$\begin{aligned}\dddot{\psi}_5(t) &= -2\lambda v(t)u_1(t)\sin(\alpha(t)-\varphi) - \lambda v^2(t)\cos(\lambda(t)-\varphi)\dot{\alpha} - \dot{\psi}_4(t)u_1(t) = \\ &= -3\lambda v(t)u_1(t)\sin(\alpha(t)-\varphi) - \lambda\kappa(t)v^3(t)\cos(\alpha(t)-\varphi)\ ,\end{aligned} \tag{22}$$

$$\begin{aligned}\ddddot{\psi}_5(t) &= -3\lambda u_1(t)\left[u_1(t)\sin(\alpha(t)-\varphi) + v(t)\dot{\alpha}(t)\cos(\alpha(t)-\varphi)\right] - \\ &\quad -\lambda u_2(t)v^3(t)\cos(\alpha(t)-\varphi) - \\ &\quad -\lambda\kappa(t)\left[3v^2(t)u_1(t)\cos(\alpha(t)-\varphi) - v^3(t)\dot{\alpha}(t)\sin(\alpha(t)-\varphi)\right] = \\ &= -3\lambda u_1^2(t)\sin(\alpha(t)-\varphi) - 3\lambda u_1(t)v^2(t)\kappa(t)\cos(\alpha(t)-\varphi) - \\ &\quad -\lambda u_2(t)v^3(t)\cos(\alpha(t)-\varphi) - 3\lambda\kappa(t)v^2(t)u_1(t)\cos(\alpha(t)-\varphi) + \\ &\quad +\lambda\kappa^2(t)v^4(t)\sin(\alpha(t)-\varphi) = (\lambda v^4(t)\kappa^2(t) - 3\lambda u_1^2(t))\sin(\alpha(t)-\varphi) -\end{aligned}$$

$$-(6\lambda\kappa(t)u_1(t)v^2(t)+\lambda u_2(t)v^3(t))\cos(\alpha(t)-\varphi)=$$
$$=\lambda(v^4(t)\kappa^2(t)-3u_1^2(t))\sin(\alpha(t)-\varphi))-$$
$$-\lambda v(t)(6\kappa(t)u_1(t)v(t)+u_2(t)v^2(t))\cos(\alpha(t)-\varphi)\ .$$

From (8) and (6) we have

$$H(X(t),\Psi(t),u_1(t),u_2(t))=$$
$$=\lambda v(t)\cos(\alpha(t)-\varphi)+\psi_3(t)u_1(t)+\psi_4(t)v(t)\kappa(t)+\psi_5(t)u_2(t)+\psi_0=0$$

Hence,

$$-\lambda v(t)\cos(\alpha(t)-\varphi)=\psi_3(t)u_1(t)+\psi_4(t)v(t)\kappa(t)+\psi_5(t)u_2(t)+\psi_0\ ,$$

and

$$\ddddot{\psi}_5(t)=\lambda(v^4(t)\kappa^2(t)-3u_1^2(t))\sin(\alpha(t)-\varphi))+(6\kappa(t)u_1(t)v(t)+$$
$$+u_2(t)v^2(t))(\psi_3(t)u_1(t)+\psi_4(t)v(t)\kappa(t)+\psi_5(t)u_2(t)+\psi_0)\ . \tag{23}$$

Hence, the variable $\psi_5(t)$ is of class $\mathbf{C}^3$ in the neighbourhood of t_1. Remind that on S^φ the following equalities hold:

$$\dot\psi_5(t)=\psi_5(t)=\dot\psi_4(t)=\psi_4(t)\equiv 0\ ,\quad \alpha(t)\equiv\varphi(\mathrm{mod}\pi)\ ,\quad \kappa(t)\equiv 0\ .$$

Hence, we obtain from equations (21)–(23) that on S^φ (and by continuity at t_1) the following equalities hold:

$$\ddot\psi_5(t)=\dddot{\psi}_5(t)\equiv 0\ ,\qquad \ddddot{\psi}_5(t)=u_2(t)v^2(t)(\psi_3(t)u_1(t)+\psi_0)\ .$$

So, there exists an ε, $0<\varepsilon<\nu$, such that for $t\in[t_1,t_1+\varepsilon[$ we have:

$$\psi_5(t)=u_2(t)v^2(t)(\psi_3(t)u_1(t)+\psi_0)\frac{(t-t_1)^4}{4!}+O((t-t_1)^5)\ .$$

We know from Proposition 6 that

$$\psi_3(t)u_1(t)+\psi_0<0\ \text{ if }\ \alpha(t)\equiv\varphi(\mathrm{mod}2\pi)\ \text{ and }\ v(t)\neq 0\ \text{ on }\ S^\varphi$$

and

$$\psi_3(t)u_1(t)+\psi_0>0\ \text{ if }\ \alpha(t)\equiv\varphi+\pi(\mathrm{mod}2\pi)\ \text{ and }\ v(t)\neq 0\ \text{ on }\ S^\varphi\ .$$

Evidently, that there exists a subinterval $[t_1^*,t_2^*]\in[t_1,t_1+\varepsilon[$ such that $v(t)\neq 0$ on $[t_1^*,t_2^*]$. Hence, in the case $\alpha(t)\equiv\varphi+\pi$ $(\mathrm{mod}2\pi)$ on S^φ we obtain that $v^2(t)(\psi_3(t)u_1(t)+\psi_0)>0$, hence, $\psi_5(t)$ and $u_2(t)$ have the same sign on $[t_1^*,t_2^*]$ – a contradiction with condition (b) of the Maximum Principle of Pontryagin.

The proposition is proved. □

Conclusion: in the case $\lambda>0$ to characterize extremal paths we can say:

1) along any optimal path the point mouve with the piecewise-linear absolute value of speed $\dot{v}(t) = -A\text{sign}(\psi_3(t))$,

2) the extremal path contains only one line segment S^φ which is contained in the straight line D_0^φ,

3) each open arc of clothoid Cl_ν ($\nu > 0$) of an extremal path (except possibly the initial and the final ones) intersect D_0^φ at least once or has a point with zero absolute value of speed,

4) an extremal path containes no portion of type $S^\varphi.Cl_\nu$ (or $Cl_\nu.S^\varphi$) with $\nu > 0$ if $\alpha(t) \equiv \varphi + \pi \pmod{2\pi}$ along S^φ.

5.2 Concatenation of different pieces of extremals – the case $\lambda = 0$

Consider extremal paths for which $\lambda = 0$ and $\psi_0 \geq 0$ (except two special cases mentioned in Lemma 1 for $\lambda = 0$ and $\psi_0 > 0$). They are constructed from arcs of curves with the piecewise-linear curvature $\dot{\kappa}(t) = -B\text{sign}(\psi_5(t))$ and they are run with the piecewise-linear absolute value of speed $\dot{v}(t) = -A\text{sign}(\psi_3(t))$.

Proposition 12 *If for some extremal path $\lambda = 0$ and $\psi_0 \geq 0$, then it is either of type Cl and it is run with the linear absolute value of speed $v(t) = -At\text{sign}(\psi_{30}) + v^0$, or it consists of arcs of the curves with the piecewise-linear curvature $\dot{\kappa}(t) = -B\text{sign}(\psi_5(t))$ which are run with the piecewise-linear absolute value of speed $\dot{v}(t) = -A\text{sign}(\psi_3(t))$: between two consecutive switching points of the control u_1 the extremal path is of the type Cl, $Cl.Cl$ or $Cl.Cl.Cl$ and between two consecutive switching points of the control u_2 the extremal path is an arc which is run with the piecewise-linear absolute value of speed which can change the sign of its acceleration at most two times.*

Proof

Using $\lambda = 0$ we obtain (from (6)) $\dot{\psi}_4 = 0$ and, hence, $\psi_4(t) \equiv const = \psi_{40}$.

If $\psi_{40} = 0$, then (from (6)) $\dot{\psi}_3(t) = 0$ and $\dot{\psi}_5(t) = 0$, hence, $\psi_3(t) \equiv const = \psi_{30}$ and $\psi_5(t) \equiv const = \psi_{50}$.

Any extremal path consists of some arcs Cl, hence, $\psi_{50} \neq 0$, because for every Cl the function $\psi_5(t)$ is positive or negative. Hence, it follows from Proposition 5 that such extremal curve doesn't contain any switching point of the control $u_2(t)$. Thus, this curve is an arc Cl. For any extremal path we have $\dot{v}(t) = -A\text{sign}(\psi_{30})$, hence $\psi_{30} \neq 0$ and it follows from Proposition 4 that such extremal curve doesn't contain any switching point of the control $u_1(t)$. So, this extremal path is an arc Cl which is run with the linear absolute value of speed $v(t) = -At\text{sign}(\psi_{30}) + v^0$ and it is run in time $T = |v^T - v^0|/A$.

If $\psi_{40} \neq 0$, then (from (6)) $\dot{\psi}_3 = -\psi_{40}\kappa(t)$ and $\dot{\psi}_5 = -\psi_{40}v(t)$ where $\kappa(t)$ and $v(t)$ are some piecewise-linear functions.

Between two consecutive switching points of the control $u_1(t)$ the absolute value of speed is a linear function of t (increasing or decreasing). Hence, $\psi_5(t)$ is a quadratic function of t. So, this function has at most two zero. Hence, between two consecutive switching points of the control $u_1(t)$ the extremal path is of the type Cl, $Cl.Cl$ or $Cl.Cl.Cl$ (it follows from Proposition 5).

Between two consecutive switching points of the control $u_2(t)$ the curvature is some linear function of t (increasing or decreasing). Hence, $\psi_3(t)$ is some quadratic function of t. So, this function has at most two zero. Hence, between two consecutive switching points of the control $u_2(t)$ the extremal path is an arc with $u_2(t) = -B\mathrm{sign}(\psi_5(t))$ and this arc is run with the piecewise-linear absolute value of speed which can change the sign of its acceleration at most two times (it follows from Proposition 4).

The proposition is proved. □

6 Conclusions

After the studing of the concatenation of different pieces of extremals we can make the following conclusions.

Lemma 13 *If for some extremal trajectory $\lambda > 0$, $\psi_0 \geq 0$, to it characterize we can say:*

1) along any optimal path the point mouve with the piecewise-linear absolute value of speed $\dot{v}(t) = -A\mathrm{sign}(\psi_3(t))$,

2) the extremal path contains only one line segment S^φ which is contained in the straight line D_0^φ,

3) each open arc of clothoid Cl_ν ($\nu > 0$) of an extremal path (except possibly the initial and the final ones) intersect D_0^φ at least once or has a point with zero absolute value of speed,

4) an extremal path containes no portion of type $S^\varphi.Cl_\nu$ (or $Cl_\nu.S^\varphi$) with $\nu > 0$ if $\alpha(t) \equiv \varphi + \pi \pmod{2\pi}$ along S^φ.

If for some extremal trajectory $\lambda = 0$, $\psi_0 \geq 0$, to it characterize we can say:

1) either it is of type Cl and it is run with the linear absolute value of speed $v(t) = -At\mathrm{sign}(\psi_{30}) + v^0$,

2) or it consists of arcs of the curves with the piecewise-linear curvature $\dot{\kappa}(t) = -B\mathrm{sign}(\psi_5(t))$ and they are run with the piecewise-linear absolute value of speed $\dot{v}(t) = -A\mathrm{sign}(\psi_3(t))$: between two consecutive switching points of the control u_1 the extremal path is of the type Cl, $Cl.Cl$ or $Cl.Cl.Cl$ and between two consecutive switching points of the control u_2 the extremal path is an arc which is run with the piecewise-linear absolute value of speed which can change the sign of its acceleration at most two times.

For $\lambda = 0$, $\psi_0 > 0$ there are two special cases:

1) optimal paths are all admissible curves with the linear absolute value of speed $v(t) = -At\mathrm{sign}(\psi_{30}) + v^0$ and they are run in time $T = |v^T - v^0|/A$, *2) optimal paths are all curves with the linear curvature* $\kappa(t) = -Bt\mathrm{sign}(\psi_{50}) + \kappa^0$*, with some admissible absolute value of speed and they are run in time* $T = |\kappa^T - \kappa^0|/B$*.*

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