An Algebra for Queueing Networks with Time Varying Service and its Application to the Analysis of Integrated Service Networks

Rajeev Agrawal, François Baccelli, Rajendran Rajan

INSTITUT NATIONAL DE RECHERCHE EN INFORMATIQUE ET EN AUTOMATIQUE

An Algebra for Queueing Networks with Time Varying Service

and its application to the analysis of integrated service networks

Rajeev Agrawal — François Baccelli — Rajendran Rajan

N° 3435

Mai 1998

THÈME 1

Rapport de recherche

ISSN 0249-6399

An Algebra for Queueing Networks with Time Varying Service

and its application to the analysis of integrated service networks

Rajeev Agrawal * , François Baccelli † , Rajendran Rajan ‡

Thème 1 — Réseaux et systèmes
Projet Mistral

Rapport de recherche n° 3435 — Mai 1998 — 36 pages

Abstract: We introduce a queueing network model that allows us to capture the time-varying service delivered to a traffic stream due to the presence of random perturbations (e.g. cross-traffic in a communication network).

We first present the model for a single queue and then describe how such queues may be interconnected using the operations of *fork (or in-synchronization)* and *join (or out-synchronization)*. Such networks may be seen as a generalization of stochastic event graphs, and of the class of fork-join networks. The departure processes in such a network satisfy a system of equations along with the exogenous arrival processes and the service processes of the various queues. This system can be seen as a general time-varying linear system in the $(\min, +)$ semi-field.

We obtain an explicit representation of the departure process in terms of the exogenous arrival processes and the service processes. Sufficient conditions are derived for this system to have a unique solution. We also study liveness and absence of explosion in this class of networks. Under appropriate stationarity and ergodicity assumptions, we establish stability theorems for such networks. To this end we first obtain a rate result for the departure process in terms of the rates of the exogenous arrival process and the throughput of a saturated system. We then use this result to show that in a network with a single exogenous arrival, all queue lengths are finite with probability one if the arrival rate is less than the throughput of the saturated system, and to give a representation of the queue length process.

This class of networks allows for a detailed description of controlled sessions in integrated service networks. We also show that it contains several earlier discrete event models of the literature pertaining to stochastic Petri nets, service curves and fork-join networks, and show how the present model unifies them in a single algebraic structure. This structure is that of a semi–ring of functions of two real variables, where the addition is the pointwise minimum and the multiplication a generalization of inf–convolution.

Key-words: Time-varying service process, burstiness constraint, service curve, fluid network, fork-join network, Petri-net, event graph, $(\max, +)$–algebra, inf–convolution, Skorohod's reflection map, saturation throughput, stability.

* Department of Electrical and Computer Engineering, University of Wisconsin, Madison, WI 53706, USA, agrawal@engr.wisc.edu. The work of this author was funded in part by the National Science Foundation under grant NCR-93-05018.

† INRIA B.P. 93 06902 Sophia Antipolis Cedex, France baccelli@sophia.inria.fr. The work of this author was funded in part by the European project TMR–Alapedes.

‡ IBM T.J. Watson Research Center 30 Saw Mill River Road Hawthorne, NY, USA, raju@watson.ibm.com

Une algèbre pour les réseaux de files d'attente avec capacité de service variant dans le temps

Applications à l'analyse de réseaux à intégration de service

Résumé : Nous introduisons un modèle de réseau permettant de représenter les variations aléatoires de la capacité de service offerte à un flux de trafic donné, résultant de la présence d'autres flux dans ce réseau.

Le modèle est d'abord décrit dans le cas d'une seule file d'attente. Nous montrons ensuite comment interconnecter de telles files au moyen d'opérations de synchronisation. Les réseaux ainsi obtenus peuvent être vus comme des généralisations des graphes d'événements stochastiques ou de certaines classes de réseaux synchronisés. Les processus de départ vérifient un système d'équations qui est une généralisation inhomogène des systèmes d'équations $(\min, +)$ linéaires classiques.

Nous donnons une représentation explicite de ces processus de départ comme fonctionnelles des processus de service et des processus d'arrivée exogènes, ainsi que des conditions pour que ces processus soient définis de manière unique. Nous étudions aussi les phénomènes d'interblocage et d'explosion, ainsi que les conditions de stabilité, sous des hypothèses de stationarité et d'ergodicié. Dans ce but, nous établissons une loi forte des grands nombres pour les processus de départ, et des relations entre les débits asymptotiques des diverses files, les débits des sources et ceux du réseau saturé. Nous en déduisons en particulier une représentation du processus de congestion et des conditions pour que ce processus converge vers une limite finie.

Cette classe de réseaux permet un étude détaillée du contrôle de session dans les réseaux à intégration de services. Nous montrons aussi que le modèle proposé fournit un cadre algébrique qui contient plusieurs classes de réseaux étudiées dans le passé, dont les graphes d'événements, certaines classes de réseaux synchronisés ou encore les réseaux de stations définies par des courbes de service. La structure algébrique en question est un semi anneau de fonctions de deux variables réelles, où l'addition est le minimum terme à terme, et le produit une généralisation de l'inf–convolution.

Mots-clés : Processus de service variable dans le temps, contrôle des rafales, courbe de service, réseau fluide, réseau synchronisé, réseau de Petri, graphe d'événement, algèbre $(\max, +)$, inf–convolution, opérateur de reflexion de Skorokhod, débit de saturation, stabilité.

1 Introduction

In terms of mathematical contents, this paper focuses on the class of continuous time, time varying $(\min, +)$–linear systems.

This class of systems may receive a very natural interpretation in terms of networks of queues with time varying service capacity. One of our main initial motivations is actually the representation of different types of network elements – devices called *regulators* used to smooth session traffic to a predefined traffic profile, *service curve schedulers* which abstract the operation of a link scheduler, all used in communication networks, which will be exemplified in §2. As we will see, the model proposed here goes beyond this and also allows one to represent various apparently unrelated constructs such as Skorohod's reflection map, or synchronization primitives of Petri net theory etc.

The basic time-varying queue with service process b, describes the relationship between an arrival process a into this queue, and the departure process x. If $a(s,t)$ describes the cumulative arrival process into the queue between times s and t, and $b(s,t)$ quantifies the amount of service obtained between these times, then the cumulative volume of departures $x(s,t)$ (when assuming that there are no arrivals prior to time s) is given by

$$x(s,t) = \inf_{s \le u \le t} \{a(s,u) + b(u,t)\}, \qquad s \le t \in \mathbb{R}. \tag{1}$$

The amount of service allocated will depend on the service capacity of the server, the amount of cross-traffic and the scheduling algorithm used to share the service capacity amongst the various traffic streams.

In order to interconnect queues of the above type we consider the *join* operation. The output x from a network node that joins the two inputs a and b is simply given by

$$x(s,t) = a(s,t) \wedge b(s,t), \qquad s \le t \in \mathbb{R}.$$

For instance, if u were to describe an exogenous arrival process seeking admission into the network, constrained by a window flow control mechanism to await feedback in the form acknowledgements described by the process b. In fact, as we shall see, for a session with exogenous arrival process u traversing a network node with service process a, the actual arrival process x into the time-varying queue satisfies the equation

$$x(s,t) = \{\inf_{s \le u \le t} x(s,u) + a(u,t)\} \wedge u(s,t), \qquad s \le t \in \mathbb{R}. \tag{2}$$

This equation immediately raises the questions of existence and uniqueness of solution, which we address in this paper. We shall also address the following natural questions on the qualitative behavior of the queue: when does the $x(s,t)$ function tend to ∞ when t tends to ∞ (*liveness*)? When can we have $x(s,t)$ infinite for some finite t (*explosion*)? When do long term pathwise rates exist for x, given that u and a have such rates? When is the queue size process asymptotically finite (*stability*)?

More generally we describe how such queues may be interconnected using the operations of *fork (or in-synchronization)* and *join (or out-synchronization)*. Such networks may be seen as a generalization of stochastic event graphs, and of the class of fork-join networks. The departure processes in such a network satisfy a system of equations along with the exogenous arrival processes and the service processes of the various queues. This system can be seen as a general dimension, time-varying linear system in a semi-ring, which we study in detail in the paper, with (2) as the 1-dimensional special case. This matrix algebra allows us to represent the dynamics of networks with multiple senders and receivers, as well as feedback. Using the algebraic theory for such structures, we show how this system of equations may be solved to yield an explicit representation of the departure process in terms of the exogenous arrival processes and the service processes. All the questions alluded to above (existence, uniqueness, liveness, explosion, existence of rates and stability) can be addressed within this framework for networks of any dimensions. In particular, we give a representation of the queue length process.

In Section 2, we begin with a more detailed presentation on the dynamics of the single server queue, and then discuss the evolution equations for larger networks with several such queues with and without feedback. The equations that we have considered thus far, describe network processes in terms of the amount of arrivals and departures as a function of time, and are known as *counter equations*. We also present *dater equations* which explore the relationship between the corresponding arrival and departure time processes. These too may be analyzed through the same algebra, and the results obtained in this paper hold for a very different class of queueing networks which are described through dater equations. In this section, we also discuss in detail the relation with the literature on stochastic Petri nets, on service curves and on fork join networks, and show how the present model unifies them in a single algebraic structure. A detailed model of flow control in an integrated service network is given in Subsection 2.4. We present the main network results of the paper in Section 3.

In Section 4 we first present the scalar dioid and its matrix extension. In Section 5 we introduce some useful properties such as monononicity and time-reversal. In Section 6 we first consider the existence and uniqueness of solutions to scalar affine equations of the form $x = x \otimes a \oplus b$, as well as its matrix counterpart $x = xA \oplus B$, and we also give conditions under which this solution is finite or bounded. We follow this with Section 7 where we obtain right and left rates for $a \oplus b$, $a \otimes b$ as well as A^*, in terms of the rates of a, b and A. In Section 8 we consider similar questions with mild stochastic constraints on a, b and A, and apply these results in Section 9 to obtain rates of the departure processes and on the representation of stationary queue lengths in networks.

2 Networking Motivations and Relations with earlier Models

In this section we describe the main network model and its motivations. We start with the basic model for an isolated "queue" and then describe how we can interconnect such queues to form networks. We also thought is useful to indicate in detail the relationship with earlier models of the literature, and show how these models can be seen as special cases of our general framework.

2.1 Isolated Queue

The model for an isolated queue is in terms of two primitive quantities – the *arrival process* a and the *service process* b. The *departure process* c is a derived quantity that is obtained as a functional of the arrival and service processes. In our model, all processes are functions of two time variables $s, t \in \mathbb{R}$, $s \leq t$, taking values in $\overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty\} \cup \{-\infty\}$. We interpret $a(s,t)$ (resp. $c(s,t)$) as the amount of arrivals (resp. departures) that take place over the interval $(s,t]$ when only arrivals to the system taking place after time s are considered. The departure process is obtained in terms of the arrival and service processes as follows:

$$c(s,t) = \inf_{s \leq u \leq t} \{a(s,u) + b(u,t)\} =: (a \otimes b)(s,t), \qquad s \leq t \in \mathbb{R}. \tag{3}$$

Note that if the arrival process a is equal to e, where

$$e(s,t) := \begin{cases} 0, & \text{for } s = t, \\ \infty, & \text{for } s < t, \end{cases} \tag{4}$$

then the departure process $c(s,t)$ equals the service process $b(s,t)$ for $s \leq t$. Thus, we can interpret the quantity $b(s,t)$ as the amount of service that takes place over the interval $(s,t]$ if the queue is saturated with customers after time s. In particular, $b(t,t)$ is the amount stored in the queue that is instantaneously available at the output at time t. Note that it makes sense to define the *queue length process* as the difference of the arrival process and the departure processes, i.e.,

$$q(s,t) := a(s,t) - c(s,t) \qquad s \leq t \in \mathbb{R}. \tag{5}$$

When a is additive, i.e., $a(r,t) = a(r,s) + a(s,t)$ for all $r \leq s \leq t$, the above reduces to

$$q(s,t) = \sup_{s \leq u \leq t} \{a(u,t) - b(u,t)\} \qquad s \leq t \in \mathbb{R}. \tag{6}$$

If $b(t,t) = 0$ then $c(s,t) \leq a(s,t)$ for all $s \leq t \in \mathbb{R}$, and the queue length is always non-negative. More generally, $q(s,t) + b(t,t)$ is always non-negative.

The above model generalizes a number of models of the literature.

In particular, if we take the arrival and service processes to be of the form $a(s,t) = A(t) - A(s)$ and $b(s,t) = B(t) - B(s)$ for some non-decreasing functions A and B with $A(t) = B(t) = 0$ for $t < 0$, then for all $r < 0 \leq t$, $c(r,t)$ defined in (3) is such that

$$c(r,t) = C(t) := \inf_{u \leq t} \{A(u) + B(t) - B(u)\}, \qquad t \geq 0. \tag{7}$$

Note also that for this case $A(t) = a(r,t)$ and $B(t) = b(r,t)$ for all $r < 0 \leq t$. The model (7) is essentially Skorohod's reflection map, considered within the framework of queueing networks by Harrison and Reiman (1979) [12] (see also [17], [1]). This model captures queues with intrinsically time-varying service capacity that results from the presence of cross traffic. A popular model for such time-varying service and arrival processes are *modulated* fluid or Poisson processes. The modulating process can be Markovian or perhaps even exhibit long range dependence.

If we take the arrival process to be as above, the service process to be such that $b(s,t) = B(t-s)$, where $B(t), t \geq 0$, is a non-negative, non-decreasing function, and if we define

$$C(t) := \inf_{s \leq t} \{A(s) + B(t-s)\}, \qquad t \geq 0, \tag{8}$$

then $C(t) = c(r,t)$ for all $r < 0 \leq t$. The model (8) was used in the modeling of regulators and service-curve-schedulers in Agrawal and Rajan (1996) [2], Cruz and Okino (1996) [11], and Le Boudec (1996) [14]. For instance, the popular (σ, ρ) regulator is modeled with $B(t) = \sigma + \rho t$ for $t > 0$ which allows an instantaneous burst of size σ and a long term average rate of ρ (see [10]). With $B(t) = r(t-\theta)^+$, (8) provides a lower bound on the departures from a *latency-rate* service curve scheduler with latency θ and rate r (see [18], [2]). For the particular case $B(t-s) = t-s$, (6) gives the classical workload representation formulas of queueing theory for a constant rate server. (See Section 2.4 for more discussion on regulators and schedulers.)

A similar discrete-time model is given by

$$C(n) := \min_{0 \leq m \leq n} \{A(m) + B(n-m)\} \qquad n \geq 0. \tag{9}$$

If we set $a(0,t) = A(\lfloor t \rfloor), t \geq 0$ and $b(s,t) = B(\lfloor t \rfloor - \lfloor s \rfloor), 0 \leq s \leq t$, then the departure process in (3) is related to the above by $c(0,t) = C(\lfloor t \rfloor), t \geq 0$. This model has been considered by Chang (1997) in [8].

This discrete time model is also the counter evolution equation for the synchronization problem considered in Baccelli, Cohen, Olsder and Quadrat (1992). In terms of timed event graphs, if $B(p) \geq 0$ for all $0 \leq p \leq M$ and $B(p) = +\infty$ for all $p > M$, then (9) is the evolution equation for the number of firings of a transition under the following assumptions: this transition is connected to some input via M places; place p has $B(p)$ initial tokens and a lag time of p, $0 \leq p \leq M$; the input counter is given by the A function.

An important remark is in order. In (3) where b is interpreted as a service process, it makes sense to assume that this function is additive, or at least non-decreasing, whereas in the above event graph interpretation, the B function has no reason to be increasing or additive. It is why we shall consider later a general framework where none of these assumptions will be made.

To summarize, in this elementary 1 dimensional case,

1. the mapping (3) may be considered a generalization of the reflection map (7) to non-additive service processes, a generalization of the service curve model (8) to time-varying service curves, a generalization of the counter equation for discrete time synchronization (9) to fluid and time-varying lag times;

2. while the model presented here is ostensibly in continuous-time, the discrete-time case can in fact be handled as a special case. Furthermore, since all the amounts (of arrivals, service, and departures) are allowed to be real-valued, we have a general fluid model, with pure discrete, pure fluid, or mixed flows as special cases.

2.2 Networks

We now consider networks obtained by interconnecting queues modeled by (3) by using the two operations of *fork* (or *out-synchronization*) and *join* (or *in-synchronization*).

Let u_l, $l = 1, \dots, m$ denote the exogenous arrival processes and let $x_i, i = 1, \dots, n$ denote certain derived departure processes (precisely how these are derived is described below). Each arrival and departure process is *forked* or *out-synchronized* to generate n copies. The j-th copy of the arrival process u_l is fed into a queue with service process $B_{l,j}$ and the j-th copy of the departure process x_i is fed into a queue with service process $A_{i,j}$. The departure process from all the queues fed by the j-th copies of the arrival and departure processes are joined or *in-synchronized*, i.e., taken the minimum of, to obtain the departure process x_j. Thus,

$$\begin{aligned} x_j(s,t) &= \wedge_{i=1}^{n} \inf_{s \le u \le t} \{x_i(s,u) + A_{i,j}(u,t)\} \\ & \quad \wedge \wedge_{l=1}^{m} \inf_{s \le u \le t} \{u_l(s,u) + B_{l,j}(u,t)\}, \quad s \le t \in \mathbb{R},\ j = 1, \dots, n, \end{aligned} \tag{10}$$

where $x \wedge y := \min\{x, y\}$. For $a, b \in D$ define the operation $a \oplus b(s,t) = a(s,t) \wedge b(s,t), s \le t \in \mathbb{R}$. The set D along with the two operations $\oplus$ and $\otimes$ is shown to be a *dioid* [4] in §4. Using these operations we can rewrite (10) as

$$x_k = \bigoplus_{1 \le j \le n} x_j \otimes A_{j,k} \oplus \bigoplus_{1 \le i \le m} u_i \otimes B_{i,k}, \quad 1 \le k \le n.$$

Let $x = (x_1, \dots, x_n)$, $u = (u_1, \dots, u_m)$ be n and m dimensional row vectors, respectively, and $A = (A_{j,k} : 1 \le j, k \le n)$, $B = (B_{i,j} : 1 \le i \le m, 1 \le j \le n)$ be $n \times n$ and $m \times n$ dimensional matrices, respectively. We can extend the operations $\oplus$ and $\otimes$ to matrices in the usual way. For two matrices C and D of the same dimension, $C \oplus D$ is the matrix with entries $C \oplus D_{i,j} = C_{i,j} \oplus D_{i,j}$. For two matrices C and D of dimensions $m \times n$ and $n \times p$, respectively, $C \otimes D$ is the $m \times p$ matrix with entries

$$C \otimes D_{i,j} = \bigoplus_{k=1}^{n} C_{i,k} \otimes D_{k,j}.$$

Using this notation, we can write the above in matrix form as

$$x = x \otimes A \oplus u \otimes B. \tag{11}$$

Note that the primitive quantities are the exogenous arrival processes u_l and the internal service processes $A_{i,j}, B_{l,j}$ of the various queues in the network. The departure processes x_j are derived from these primitive quantities. However, for the general fork-join network described above, this derivation is not explicit but rather involves a system of n implicit equations.

The way to build networks with these two operations has been considered under a variety of different names and instanciations in the literature: timed event graphs in [4], fork-join networks in [17], [2]. In the language of timed event graphs, the arrival process u_l counts the amount of firings of the input *transition* l, x_j counts the amount of firings from the internal *transition* j. Consider for instance the following discrete time special case of (10), which can also be seen as a vectorial generalization of (9):

$$\begin{aligned} x_j(p) &= \wedge_{i=1}^{n} \inf_{q \le p} \{x_i(q) + A_{i,j}(p-q)\} \wedge \\ & \quad \wedge_{l=1}^{m} \inf_{q \le p} \{u_l(q) + B_{l,j}(p-q)\}, \ 0 \le p \in \mathbb{N}, \ j = 1, \dots, n. \end{aligned} \tag{12}$$

This class of equations was first introduced and studied by Cohen, Dubois, Quadrat and Viot (1983) [9] in the particular case when the entries of the matrices $A(.)$ and $B(.)$ are all integer valued and have *finite support*, namely all entries of $A(r)$ and $B(r)$ are equal to $+\infty$, for all $r > K$.

The interpretation is then as follows: if $A_{i,j}(r) = h < \infty$, there is an arc from transition i to transition j with place with h initial tokens, and this place has a delay of r units. Such a delay in place p means that a token produced by x_i at time q only becomes available for enabling x_i at time $q+r$. The interpretation is the same for finite entries of $B_{l,j}(.)$. Since the total number of firings that transition x_j initiates by time p is the minimum, over the set of places that precede x_j, of the number of tokens that have become available by time p, we get (12).

To summarize, this class of network models can be seen as a generalization of

1. the fork-join network model considered in [17], [2] to non-additive service processes;
2. the models considered in [1] and [7] to time-varying service processes. The focus in [1] and [8] was on worst-case performance bounds;
3. the counter equations for timed event graphs considered in [9] and [4] to the time-varying, infinite support and continuous time case.

2.3 Dater Equations

In this section, we describe another class of models. The relationship with the above class is discussed in the last subsection.

2.3.1 Isolated Queue

Both the isolated queue and network equations given in Sections 2.1 and 2.2 involved counters, i.e., the quantity of arrivals, service, or departures over different intervals of time. These equations used the operations min and $+$. Analogous equations using max and $+$ have been considered in the literature to describe the behavior of daters (epochs at which certain events occur). More precisely consider the mapping

$$c(s,t) = \sup_{s \le u \le t} \{a(s,u) + b(u,t)\}, \qquad s \le t \in \mathbb{R}, \tag{13}$$

In a queueing like interpretation, $a(s,u)$ represents the time that elapses between the arrival of the s-th unit and u-th unit, whereas $b(u,t)$ represents the delay imposed by the u-th unit onto the t-th, and $c(s,t)$ the time that elapses between the initiation of the system, when this one starts with the service of s-th unit, and the departure of t-th.

The mapping (13) was considered in [4] as a composition rule for impulse responses. Special instances of this mapping arise in various branches of mathematics. If $a(s,t) = f(t-s)$ and $b(s,t) = g(t-s)$, then $x(s,t) = h(t-s)$ and

$$h(t) = \sup_{0 \le u \le t} f(u) + g(t-u),$$

a mapping which is often referred to as a sup-convolution. If $a(s,t) = z.(t-s)$, we get

$$x(s,t) - z.(t-s) = \sup_{0 \le u \le t} b(u,t) - z.(t-u).$$

In the last mapping, we recognize the basic workload equation of a single server queue with offered workload between time s and t given by the function $b(s,t)$, and with a single server working with a speed equal to z. In this interpretation, it makes sense to assume that b is additive and non-negative, and that $z > 0$. The Legendre transform of convex analysis:

$$x(z) = \sup_{0 \leq u} y(u) - z.u$$

is also a special asymptotic case of the last mapping.

2.3.2 Networks

Next consider the $(\max, +)$ analog of the network equations (10)

$$\begin{aligned} x_j(s,t) &= \vee_{i=1}^{n} \sup_{s \leq u \leq t} \{x_i(s,u) + A_{i,j}(u,t)\} \\ & \quad \vee \vee_{l=1}^{m} \sup_{s \leq u \leq t} \{u_l(s,u) + B_{l,j}(u,t)\}, \quad s \leq t \in \mathbf{R},\ j = 1, \dots, n, \end{aligned} \tag{14}$$

where $x \vee y := \max\{x, y\}$. Here u_l may be considered to be the daters corresponding to the exogenous arrival process l and x_i the daters corresponding to the departure process i. Note that the join or in-synchronization operation in the dater context corresponds to max. Thus the above system of equations is precisely what would result when considering a fork-join network based on isolated element with daters satisfying (13).

The discrete index, finite range, special case of (14)

$$x_j(l,k) = \vee_{i=1}^{n} \max_{l \leq p \leq k} \{x_i(l,p) + A_{i,j}(p,k)\} \vee \vee_{i=1}^{m} \max_{l \leq p \leq k} \{u_i(l,p) + B_{i,j}(p,k)\}, \quad l < k, \tag{15}$$

has been studied in Baccelli and Liu (1992) [5] and in [4], Chapter 7, where ergodic theorems studying rates and stationary waiting times where derived, whereas the infinite support extension of (15) was analyzed in [15]. In these references, the functions A and B are such that the sequences $\{A(t-r,t)\}_r$ and $\{B(t-r,t)\}_r$ are stationary and ergodic w.r.t. t, and u is additive, with stationary and ergodic increments.

The last sections of the present paper will give an extension of these ergodic theorems to the counter $(\min, +)$ model (10), and also a generalization of the above $(\max, +)$ results to the continuous index case.

2.3.3 Counters versus Daters

Note that in general the counter $(\min, +)$ equation and the dater $(\max, +)$ equation describe different queueing mechanisms. Typically, the counter $(\min, +)$ equation corresponds to the *synchronous* case ([17], [2]), which captures queues with intrinsically *time-varying* service capacity, whereas the dater $(\max, +)$ equation corresponds to the *asynchronous* case ([3]), which captures queues with *customer-varying* service times. In certain special cases (e.g. fixed delay, or constant rate fluid server) the queueing mechanism may admit both descriptions. However, this is not possible in general. Nevertheless, we can convert the dater $(\max, +)$ equations (13) and (14) into (3) and (10), respectively, by simply taking negatives of all quantities. These quantities do not have the same physical meaning as counters, but remarkably lead to the same type of algebraic equations. This allows the study of both the counter and dater models in the same algebraic framework.

When using these two formalisms, one should nevertheless be very careful with the mathematical translation of physical concepts and constraints. For instance, we may assume all quantities in the counter (min,+) as well as in the dater (max, +) equations to be non-negative and non-decreasing. However, by taking negatives on both sides to convert the dater (max,+) equations into (min,+) ones, we get the exact opposite conditions.

2.3.4 Special Topologies

In a general network, each departure process is allowed to constrain any other departure process through the service process $A_{i,j}$. However, by choosing $A_{i,j} = \varepsilon$ where

$$\varepsilon(s,t) = \infty, \qquad s \le t \in \mathbb{R}, \tag{16}$$

we can render this constraint ineffective since $x \otimes \varepsilon = \varepsilon$ and $x \oplus \varepsilon = x$. Based on this we can define a directed graph on the set of nodes $\mathcal{V} = \{1, \dots, n\}$. There is arc from i to j if the departure process x_i constrains the departure process x_j, i.e., the set of arcs is given by $\mathcal{E} = \{(i,j) \in \mathcal{V}^2 : A_{i,j} \neq \varepsilon\}$. For any node $j \in \mathcal{V}$, define the set of predecessor nodes $\pi(j) = \{i \in \mathcal{V} : (i,j) \in \mathcal{E}\}$ and the set of successor nodes $\sigma(j) = \{k \in \mathcal{V} : (j,k) \in \mathcal{E}\}$.

As alluded to above, in the discrete time case, this graph is refined as follows to get the graphical Petri net description: if in (12) there is an entry $A_{i,j}(r)$ which has a non $-\infty$ value $h \in \mathbb{N}$, then there is a place with h tokens between transition i and transition j, with a lag time of h.

Here are a few special cases of network topologies:

- A network is said to be *acyclic* if the corresponding directed graph $(\mathcal{V}, \mathcal{E})$ is acyclic. An acyclic directed graph can be partitioned into $p \le n$ *levels*, such that a node belongs to level q iff all its predecessor nodes belong to levels less than q and all its successor nodes belong to levels greater than q. These level sets $\mathcal{L}_q$ can be defined inductively starting with $q = 1$. In doing so we include at least one node at each new level set. Hence the number of levels p is less than or equal to n, the number of nodes. Let $\mathcal{U}_q = \cup_{q' \le q} \mathcal{L}_{q'}$. Then note that for an acyclic network (10) reduces to

$$x_j \quad = \quad \bigoplus_{i \in \mathcal{U}_{q-1}} x_i \otimes A_{i,j} \oplus \bigoplus_{l=1}^{m} u_l \otimes B_{l,j}, \quad j \in \mathcal{L}_q. \tag{17}$$

  Thus, in an acyclic network, any departure process at level q is constrained only by departure processes at strictly lower levels and hence we solve explicitly for these processes by induction on the levels.

- A special case of an acyclic network is the acyclic *tandem*. In such a network we may order the departure processes so that $\mathcal{E} = \{(i, i+1) : 1 \le i < n\}$. It is easy to check that in this case, $p = n$ and $\mathcal{L}_q = \{q\}, 1 \le q \le p$. Further, there is a single exogenous arrival process ($m = 1$) which constrains only the departure process x_1, i.e., $B_{1,j} = \varepsilon$, $j \neq 1$. Redefine $a_0 := u := u_1$, $a_1 := B_{1,1}$, $a_i := A_{i-1,i}$, $1 < i \le n$. Then (17) reduces to

$$\begin{aligned} x_1 &= u \otimes a_1, \\ x_j &= x_{j-1} \otimes a_j, \quad 1 < j \le n. \end{aligned} \tag{18}$$

- If the network is not acyclic, then the equations involving the departure processes are implicit. The simplest instance of the implicit equation corresponds to the scalar case $n = 1$. In this case the single departure process x is fed into a single queue with service process a which is then joined with an exogenous arrival process u to get back the departure process x. Note that we restrict attention to a single exogenous arrival process which is directly joined to the output of the queue without loss of generality (since we can always take the entire second line in (10) to be that process). This yields the following equation

$$x = x \otimes a \oplus u, \tag{19}$$

  corresponding to a single queue with feedback.

- Several other examples of topologies stemming from engineering problems in manufacturing (assembly or production lines) or in scheduling (jobshops), can also be found in [4].

2.4 Integrated Services Networks

In order to understand packet dynamics in an integrated services network, it is necessary to characterize the transformation effected on packet streams by their passage through network elements such as routers and access control devices, by interactions with other packet streams, as well as the effect of flow control protocols that may restrict the amount of traffic that is injected into the network. We take an approach focusing on a single session by incorporating the effect of cross-traffic and the scheduling mechanism in the service process experienced by the session.

To illustrate the use of time-varying queues in such modeling, consider Figure 2.4 which depicts a single session where packets are transmitted from a file server to an end host accessing information across an internet. Packets originating at the server are subject to access control restrictions before entering the wide area network. This is enforced through a network access router which provides per-flow services, in that it can reshape and schedule flows individually. The backbone router on the wide area network, on the other hand, serves too many flows to provide such flow level discrimination, and hence serves all arriving packets in a FIFO manner. Packets reaching the network access router on the host end are transmitted to the host. Acknowledgements from the host to the server are transmitted on the reverse path.

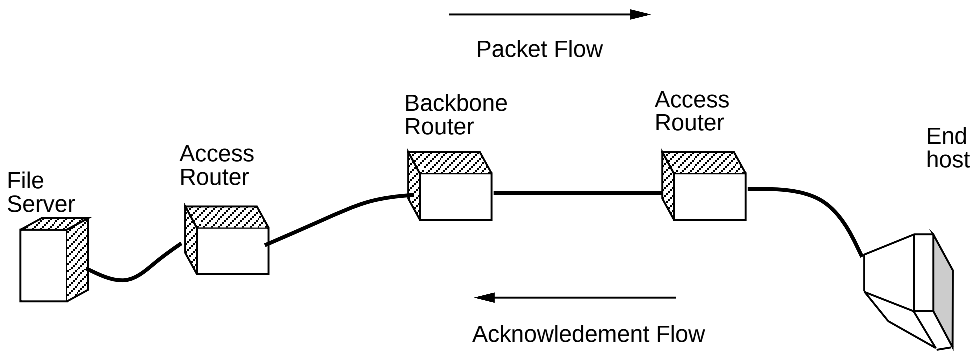

Figure 1: A file transfer session.

There are a number of different network elements that have to be modeled even with this simple example, as seen in Figure 2.4. The first access router functions as a regulator [10, 2, 14, 8]. and the wide area links are constant rate links with a fixed propagation delay across them. The backbone router, as seen by the traffic stream, has time-varying service capacity resulting from the presence of cross-traffic and its effect on the traffic stream may be modeled through a reflection map. The access router to the host LAN schedules flows using some fair queueing mechanism and may be modeled as a service curve scheduler [2, 11, 14, 8]. In Figure 2.4, acknowledgements are assumed to have the same volume as the forward traffic and return instantaneously, though both restrictions may easily be relaxed. The acknowledgements are joined with the forward traffic, and act to restrict the volume of packets outstanding in the network at any given time.

The resulting network is modeled through a number of time-varying servers, as well as a join operation. If a denotes the packet generation process at the file server, and a_8 the process of acknowledgements returning to it, then the process a_1 describes the insertion of packets into the network, with the relation $a_1 = a \oplus a_8$. The network elements $1, \ldots, 7$ are described through time-varying service processes $b_1, \ldots, b_7$, with the departure process a_{i+1} from element i being obtained as $a_{i+1} = a_i \otimes b_i$. Each of the processes b_i, $1 \leq i \leq 6$, has a particular form depending on the nature of the device being modeled. The regulator b_1 has the form $b_1(s,t) = B(t-s)$ where $B(t), t \geq 0$, is a non-negative, non-decreasing function, and the departure process from it is given by $a_2 = a_1 \otimes b_1$. The constant rate link elements b_3 and b_5 are described by $b_3(s,t) = C \cdot (t-s)$ and $b_5(s,t) = C' \cdot (t-s)$ where C and C' are the capacities of the respective links. The delay element b_4 with a propagation delay of δ, is modeled by the process $b_4(s,t) = 0$ if $s = t - \delta$, and ∞ otherwise. The delay element b_6 with propagation delay of δ' may be modeled similarly. The backbone router is modeled as in (7) using the process b_4, with $b_4(s,t) = S(t) - S(s)$, where S is some possibly random service process, e.g. a Markov modulation of the service rate as seen by the packets of the session. Finally, the host access router is described as in (8) through the process b_7, with the relation $a_8 \geq a_7 \otimes b_7$ connecting the acknowledgement process to the forward traffic process.

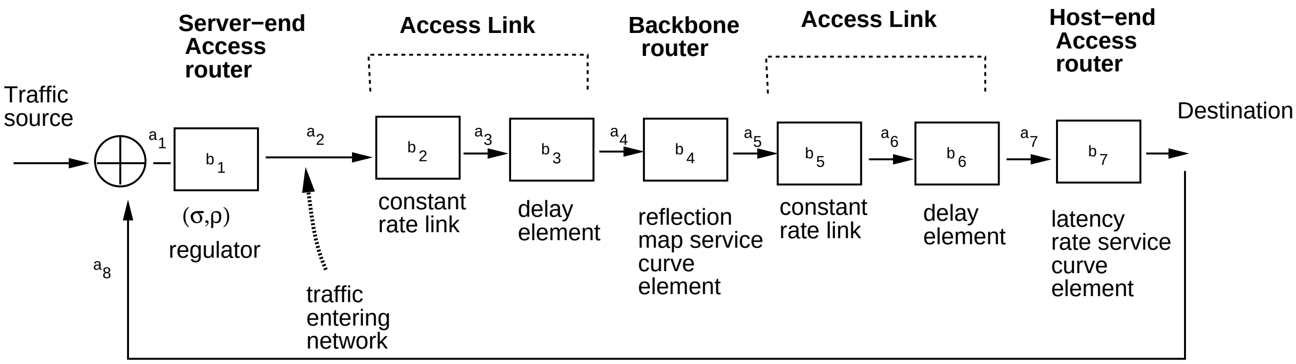

Figure 2: Modeling the file transfer session through time varying queues.

3 Main Network Results

In this section we present the main results as applied to the network models introduced above. Our main aim consists in obtaining logical and asymptotic properties for such networks. We first introduce some basic notions for functions a of two variables, which are needed to present these results.

- We say that a is *non-negative*, if for all $s \leq t \in \mathbb{R}$, $a(s,t) \geq 0$. We say it is *non-decreasing*, if for all $s' \leq s \leq t \leq t' \in \mathbb{R}$, $a(s',t') \geq a(s,t)$.
- We say it has *constant initial condition*, if for all $s \in \mathbb{R}$, $a(s,s) = a(0,0)$.
- It has *finite initial condition*, if for all $s \in \mathbb{R}$, $a(s,s) > -\infty$.
- A function a is *subadditive* if for all $s \leq u \leq t$, $a(s,t) \leq a(s,u) + a(u,t)$. It is said to be *additive* if we have equality.
- A function a has a *forward-rate* $\alpha \in \overline{\mathbb{R}}$ if

$$\exists \lim_{t\to\infty} \frac{a(s,t)}{t-s} = \alpha, \quad \forall s \in \mathbb{R}. \tag{20}$$

  The function a has a backward-rate α if

$$\exists \lim_{s\to-\infty} \frac{a(s,t)}{t-s} = \alpha, \quad \forall t \in \mathbb{R}. \tag{21}$$

- In many cases, a function a will be assumed to satisfy the following condition: for all $-\infty < s \leq t < +\infty$,

$$\inf_{v:\ s\leq v\leq t} a(s,v) > -\infty. \tag{22}$$

In a network of dimension n, described by an equation like (10), or equivalently (11), the primitive functions will of course be the arrival processes u_l and the service processes $A_{i,j}$ and $B_{l,j}$. The following result is a summary of lemmas proved in Section 6.

Theorem 1 *Let A have non-negative entries and satisfy the condition*

$$\min_{i=1,\dots,n} \min_{1\leq m\leq n} \inf_{u,v:s\leq u\leq v\leq t} A^m_{i,i}(u,v) =: \langle A\rangle(s,t) > 0, \qquad \forall s\leq t, \tag{23}$$

where $A^1 = A$ and $A^m = A\otimes A^{m-1}$. Define

$$A^* := \bigoplus_{m\geq 0} A^m.$$

If u has entries which satisfy Condition (22) and if in addition for all l, j, and for all $s\leq t$,

$$\inf_{u,v:\ s\leq u\leq v\leq t} B_{l,j}(u,v) > -\infty,$$

then:

1. *The function $x = u\otimes B\otimes A^*$ is the unique solution of this system of equations which satisfies (22).*
2. *This solution is free of explosion, namely $x_j(s,t) < \infty$ for all j and all finite $s\leq t$, provided for all k, there exists an index j such that $A^*_{j,k}(t,t) > -\infty$ and the function $u\otimes B_j$ has no such explosion.*
3. *This solution is live, namely for all j and s, $\lim_{t\to\infty} x_j(s,t) = \infty$, whenever all the entries of u are live and either (a) or (b) below hold:*
   - *(a) For each of the entries of A and B seen as a function f of the two variables s and t, we have $f(t,t) = Constant$, $f(s,t)\leq f(s',t')$ if $s'\leq s\leq t\leq t'$, and $\lim_{t\to\infty} f(s,t) = \infty$ for all s.*
   - *(b) Each entry of B satisfies the same properties as in (a), and A satisfies the following two properties:*

     $$\lim_{t\to\infty} \min_{i,j} \inf_{u,v:s\leq u\leq v\leq t} A_{i,j}(u,v) = K(s) > 0, \qquad \forall s,$$

     and

     $$\forall i,\ \exists\ \eta>0,\ p>0 \text{ and } L<\infty \text{ such that } (A^p)_{i,i}(s,t)\leq L, \quad \forall s\leq t\leq s+\eta.$$

Property 1 follows from Lemmas 13 and 4. Property 2 is established in Lemma 15. Finally, Properties 3a and 3b are proved in Lemmas 16 and 17 respectively.

Remark 1 *Condition (23) may be interpreted as requiring that each cycle in the network has a positive amount of customers. When the entries of A are non-decreasing, this is equivalent to*

$$\min_{i=1,\dots,n} \min_{1\leq m\leq n} A^m_{i,i}(0,0) =: \langle A\rangle > 0. \tag{24}$$

If this condition is violated, then the queues are cyclically locked, *i.e., there is at least one cycle of queues where none of them can have a departure without some departures from another queue in the cycle.*

Remark 2 *The two sufficient conditions (a) and (b) in point 3 above are meant to cover different situations; in case of regulators like in §2.4, it makes sense to assume that the service functions satisfy the non-decreasingness assumptions of (a), whereas in the event graph setting (e.g. pure delay systems or more general cases as considered in §2.1), these functions have no reasons to satisfy this assumption; (b) then gives a simple sufficient condition for liveness to hold in case of finite support service functions.*

The matrix A^* has the following key properties, which are proved in Section 7.1.

Theorem 2 *Each diagonal entry of A^* is subadditive. For all $i \neq j$, let $\mathcal{P}_{i,j}$ denote the set of all sequences $\pi = (i_0 = i, i_1, \ldots, i_m = j)$ of elements of $\{1, \ldots, n\}$ such that $i_l \neq i_k$ for $l \neq k$, and with arbitrary length $m < n$. Then*

$$A^*_{i,j} = \bigoplus_{\pi \in \mathcal{P}_{i,j}} A^*_{i_0,i_0} \otimes A_{i_0,i_1} \otimes A^*_{i_1,i_1} \otimes A_{i_1,i_2} \otimes \ldots \otimes A_{i_{m-1},i_m} \otimes A^*_{i_m,i_m}. \tag{25}$$

Remark 3 *The off-diagonal elements $A^*_{i,j}, i \neq j$ are not subadditive in general. However, the above theorem allows us to represent them with a finite number of $\oplus$ and $\otimes$ operations involving the diagonal elements $A^*_{i,i}$ and the off-diagonal elements $A_{i,j}$ of the matrix A only.*

For the results below, we make the following additional assumptions on the primitive functions that are not equal to e or ε (defined in (4) and (16), respectively):

- They are non-negative, non-decreasing, subadditive, and with finite, constant initial condition.
- They are random functions defined on a common probability space, all integrable, jointly stationary (stationarity means here that $\{a(s+r, t+r)\}_{s,t}$ has a law which does not depend on r) and ergodic.
- All arrival processes are additive.

Note that the assumption of additivity on an exogenous arrival process u implies that it has a constant initial condition $u(t,t) = 0$. The constant initial condition $a(t,t) = a^0$ of a service process a may be interpreted as the initial volume of fluid a^0 available at the output of the queue at time the system is started. In the context of Petri nets, this would be the initial marking of the corresponding place.

By Corollary 4 in §7.1, the above assumptions imply that each primitive process has almost sure backward and forward rates which are equal and constant.

Let us start with the 1-dimensional case, which is the analogue of Loynes' theorem for single server queues [16] (see Section 9).

Theorem 3 *For a single queue with equation $x = a \otimes u$, the departure process x has the almost sure backward and forward rate $\xi = v \wedge \alpha$, where v and α are the respective almost sure rates of the arrival process u and the service process a. Further, $q(s,t) = u(s,t) - x(s,t)$ converges in distribution to $q(-\infty, 0)$ as $t \to \infty$ for all $s \in \mathbb{R}$, where*

$$q(-\infty, 0) := \lim_{s \to -\infty} q(s, 0) = \inf_{r \leq 0} a(r, 0) - u(r, 0).$$

Moreover, $\lim_{t\to\infty} P(q(s,t) < \infty) = 1$ *if* $v < \alpha$ *and* $\lim_{t\to\infty} P(q(s,t) < \infty) = 0$ *if* $v > \alpha$.

In the general dimension network case, we denote by v_i, $\beta_{i,j}$, and $\alpha_{j,k}$, the constant backward and forward rates of the processes u_i, $B_{i,j}$, $A_{j,k}$, respectively.

Let ε be the function defined in (16). For any given $1 \leq k \leq n$, let $J_k := \{(i,j) : 1 \leq i \leq m, 1 \leq j \leq n, B_{i,j} \neq \varepsilon, A^*_{j,k} \neq \varepsilon\}$ be the set of ordered pairs (i,j) such that input i constrains departure j

in one hop and departure j constrains departure k in zero, one, or more hops. Let i,k be such that $(i,j) \in J_k$, for some j. Define the queue length process $q_{i,k}$ as the difference between the arrival process u_i and the departure process x_k, i.e.,

$$q_{i,k} := u_i - x_k,$$

assuming that $u_i < \infty$.

Theorem 4 *Assume that A satisfies (24). The processes $A^*_{j,k}$ have almost sure backward and forward rates $\alpha^*_{j,k}$. The non-diagonal rates are obtained from the diagonal ones via the formula;*

$$\alpha^*_{j,k} = \bigwedge_{\pi=(i_0,\ldots,i_p)\in\mathcal{P}_{j,k}} \alpha^*_{i_0,i_0} \wedge \alpha_{i_0,i_1} \wedge \cdots \wedge \alpha_{i_{p-1},i_p} \wedge \alpha^*_{i_p,i_p}, \quad j \neq k. \tag{26}$$

The departure process x_k has an almost sure backward and forward rate ξ_k, where

$$\xi_k = \bigwedge_{(i,j)\in J_k} (v_i \wedge \beta_{i,j} \wedge \alpha^*_{j,k}), \quad 1 \le k \le n. \tag{27}$$

Let i,k be such that $(i,j) \in J_k$ for some j and let

$$\gamma_{i,k} := \min_{(l,j)\in J_k, l\neq i} (v_l \wedge \beta_{l,j} \wedge \alpha^*_{j,k}) \wedge (\min_{j:(i,j)\in J_k} \beta_{i,j} \wedge \alpha^*_{j,k}).$$

If $v_i < \gamma_{i,k}$, then, for all $s \in \mathbb{R}$,

$$\lim_{t\to\infty} P(q_{i,k}(s,t) < \infty) = P(\lim_{s\to-\infty} q_{i,k}(s,t') < \infty) = 1$$

with

$$q_{i,k}(-\infty,t) := \sup_{u\le t}\{u_i(u,t) - (B \otimes A^*)_{i,k}(u,t)\}.$$

If $v_i > \gamma_{i,k}$, then $\lim_{t\to\infty} P(q_{i,k}(s,t) < \infty) = P(\lim_{s\to-\infty} q_{i,k}(s,0) < \infty) = 0$.

Proof. See Section 9.

The rate results of Theorem 4 may be seen as extensions of the first order ergodic theorems in Chapter 7 of [4] for the discrete index dater $(\max,+)$ recurrence model (15) considered in Subsection 2.3.2. In particular, the Lyapunov exponents of the sequence $\{A(k)\}$ correspond to the formulas giving the existence of rates for A^* in Theorem 4. The first order ergodic theorems on the growth rate of $x_{0,k}$ correspond to Equation (27) above. Similarly, the second order ergodic theorems of [4] concerning the stationary waiting time process $w(-\infty,0) = x(-\infty,0) - u(-\infty,0)$, corresponds to the stationary queue length formula in Theorem 4.

4 The Dioid

The network model(s) described in the previous section involve processes (arrival, service, and departure) that take value in the set D of $\overline{\mathbb{R}}$-valued functions of two real arguments s and t such that $s \le t$. We will denote $a(s,t)$ the value of $a \in D$ at point (s,t). The dynamics of such queues are described in terms of two binary operations on D. When considering the counter case, these are the pointwise minimum and the "inf-convolution" defined in (3), which we call addition and multiplication, respectively. Thus we need to study the set D endowed with two internal operations:

- an addition denoted $\oplus$, where the value of $a \oplus b$ at point (s,t) is given by the formula

$$a \oplus b(s,t) = \min(a(s,t), b(s,t)) = a(s,t) \wedge b(s,t), \quad s \le t \in \mathbb{R};$$

- a multiplication denoted $\otimes$ and defined by the formula

$$a \otimes b(s,t) = \inf_{s \le u \le t} \{a(s,u) + b(u,t)\}, \quad s \le t \in \mathbb{R},$$

with the convention $+\infty + (-\infty) = +\infty$. The functions ε defined in (16) and e defined in (4) turn out to be the $\oplus$ and $\otimes$-neutral elements, respectively. The set D endowed with $(\oplus, \otimes)$ forms a dioid (see [4]). In particular, addition is associative and commutative; multiplication is associative; multiplication distributes over addition; there is a zero element which is absorbing for multiplication; there is an identity element; and addition is idempotent. The only non-trivial properties to check are the distributivity of multiplication over addition and the associativity of multiplication. The left-distributivity is obtained as follows

$$\begin{aligned} a \otimes (b \oplus c)(s,t) &= \inf_{s \le u \le t} \{a(s,u) + (b(u,t) \wedge c(u,t))\} \\ &= \inf_{s \le u \le t} \{(a(s,u) + b(u,t)) \wedge (a(s,u) + c(u,t))\} \\ &= \inf_{s \le u \le t} \{a(s,u) + b(u,t)\} \wedge \inf_{s \le u \le t} \{a(s,u) + c(u,t))\} \\ &= a \otimes b \oplus a \otimes c(s,t), \end{aligned}$$

where the third equality follows from the bounds:

$$\begin{aligned} \inf_{s \le u \le t} \{(a(s,u) + b(u,t)) \wedge (a(s,u) + c(u,t))\} &\le \inf_{s \le u \le t} \{a(s,u) + b(u,t)\} \\ \inf_{s \le u \le t} \{(a(s,u) + b(u,t)) \wedge (a(s,u) + c(u,t))\} &\le \inf_{s \le u \le t} \{a(s,u) + c(u,t)\} \end{aligned}$$

and

$$\begin{aligned} &\inf_{s \le u \le t} \{(a(s,u) + b(u,t)) \wedge (a(s,u) + c(u,t))\} \\ \ge\ & \inf_{s \le u \le t} \left\{ \inf_{s \le u \le t} \{(a(s,u) + b(u,t))\} \wedge (a(s,u) + c(u,t)) \right\} \\ =\ & \inf_{s \le u \le t} \{a(s,u) + b(u,t)\} \wedge \inf_{s \le u \le t} \{a(s,u) + c(u,t)\}. \end{aligned}$$

The proof for right-distributivity is similar. As for the associativity of multiplication,

$$\begin{aligned} a \otimes (b \otimes c)(s,t) &= \inf_{s \le u \le t} \{a(s,u) + \inf_{u \le v \le t} \{b(u,v) + c(v,t)\}\} \\ &= \inf_{s \le u \le t} \inf_{u \le v \le t} \{a(s,u) + b(u,v) + c(v,t)\} \\ &= \inf_{s \le v \le t} \inf_{s \le u \le v} a(s,u) + b(u,v) + c(v,t) \\ &= \inf_{s \le v \le t} \{ \inf_{s \le u \le v} \{a(s,u) + b(u,v)\} + c(v,t)\} \\ &= \inf_{s \le v \le t} \{a \otimes b(s,v) + c(v,t)\} \\ &= (a \otimes b) \otimes c. \end{aligned}$$

Note that

- multiplication is not commutative in general;
- multiplication has no inverse in general; this specific dioid is a semi-ring, not a semi-field;
- the function $T(s,t) = -\infty$ is the top element of the dioid.

We prove that the dioid $(D, \oplus, \otimes)$ is complete. For this we have to show that

1. D is closed for infinite $\oplus$-sums;
2. $\otimes$ distributes over infinite $\oplus$-sums.

The first property follows from the fact that T belongs to D. As for the second one, we get by arguments similar to those used above that

$$\begin{aligned} a \otimes (\bigoplus_i b)(s,t) &= \inf_{s \le u \le t} \{a(s,u) + (\inf_i b_i(u,t))\} \\ &= \inf_{s \le u \le t} \inf_i \{a(s,u) + b_i(u,t)\} \\ &= \inf_i \inf_{s \le u \le t} \{a(s,u) + b_i(u,t)\} \\ &= \bigoplus_i a \otimes b_i(s,t). \end{aligned}$$

The proof for left-distributivity is similar.

4.1 Matrix Extensions

We consider the following matrix extension of D that we will denote $D^{m,n}$: an element A of $D^{m,n}$ is a $m \times n$ matrix with entries $A_{i,j}$ in D.

- For A and B in $D^{m,n}$, $A \oplus B$ is the matrix with entries $A \oplus B_{i,j} = A_{i,j} \oplus B_{i,j}$ (as in the conventional algebra, we use the same notation $\oplus$ to denote addition of scalars and matrices. The context should allow one to say what is meant).
- For $A \in D^{m,n}$ and $B \in D^{n,p}$, $A \otimes B$ is the $m \times p$ matrix with entries

$$A \otimes B_{i,j} = \bigoplus_{k=1}^{n} A_{i,k} \otimes B_{k,j}.$$

It is easy to check that $(D^{n,n}, \oplus, \otimes)$ forms a complete dioid, with $\oplus$-neutral element $\mathcal{E}$, the square matrix with all its entries equal to ε, and with $\otimes$-neutral element the identity matrix E, with diagonal elements equal to e and non-diagonal elements equal to ε.
A remark is in order: using the mapping $a(s,t) \to -a(s,t)$, it is easy to check that the dioid for counters is isomorphic to that for daters, which is based on an addition defined as the pointwise max, and on a multiplication defined by (13). Of course, one should be careful with such concepts as nullity, ordering, finiteness or boundedness, when moving from one diod to the other. With this caveat in mind, all results obtained in one domain will have a direct isomorphic counterpart in the other. It is why we concentrate on the counter dioid in this paper.

5 Preliminary Algebraic Results

In this section we introduce some additional properties and prove some preliminary results on how these properties are affected by the operations $\oplus$ and $\otimes$. The terminology which is used is the natural one within this algebraic structure. There will be a couple of exceptions though, which we will stress in due course.

- A function $a \in D$ will be said to be of *full support* if $a(s,t) < +\infty$, for all $-\infty < s \le t < +\infty$.
- A function $a \in D$ will be said to be *asymptotically null* if the following limit holds for all s:

$$\lim_{t \to \infty} a(s,t) = +\infty.$$

- A function $a \in D$ will be said to have *bounded support* if $a(s,t) = +\infty$, for $t - s > K$, where K is a finite constant.
- A function $a \in D$ will be said to be *finite* if for all $-\infty < s \leq t < +\infty$, $a(s,t) > -\infty$. Note that the function $a = \varepsilon \equiv +\infty$ is finite.
- A function $a \in D$ will be said to be *right locally bounded* if for all $-\infty < s \leq t < +\infty$,

$$\inf_{v:\ s \leq v \leq t} a(s,v) > -\infty.$$

  It is easy to check that this property is preserved by finite sums.
- A function $a \in D$ will be said to be *locally bounded* if for all $-\infty < s \leq t < +\infty$,

$$\inf_{u,v:\ s \leq u \leq v \leq t} a(u,v) > -\infty.$$

  It is easy to check that this property is preserved by finite sums and products.
- A function $a \in D$ will be said to be *bounded* if

$$\inf_{s \leq t} a(s,t) > -\infty.$$

We will say that $a \geq b$ if for all $s \leq t$, $a(s,t) \leq b(s,t)$. Note that this is one exception to our rule since the natural ordering of this specific dioid is just the other way around. If $a \geq b$ with b constant and equal to 0, we say that a is non-negative.

Remark (Networking interpretation) When using the counter framework for networks, the fact that a function a, counting the number of departures from a node, is asymptotically null means that an infinite amount of service takes place there when time grows, which means that this node is *live*. The fact that this function has full support, means that this node has no accumulation point of departures in any finite time, namely no *explosion*.

Lemma 1 *If a and b are non-negative, then so are $a \oplus b$ and $a \otimes b$. If a and b are non-decreasing and have constant initial condition, then so are $a \oplus b$ and $a \otimes b$.*

Proof. The first part is obvious. To establish the second note that for all $s' \leq s \leq t \leq t' \in \mathbb{R}$,

$$\begin{aligned}
a \otimes b(s',t') &= \inf_{s' \leq u \leq t'} \{a(s',u) + b(u,t')\} \\
&= \inf_{s' \leq u \leq s} \{a(s',u) + b(u,t')\} \wedge \inf_{s \leq u \leq t} \{a(s',u) + b(u,t')\} \\
&\quad \wedge \inf_{t \leq u \leq t'} \{a(s',u) + b(u,t')\} \\
&\geq \{a(s',s') + b(s,t)\} \wedge \inf_{s \leq u \leq t} \{a(s,u) + b(u,t)\} \wedge \{a(s,t) + b(t',t')\} \\
&= \{a(s,s) + b(s,t)\} \wedge \inf_{s \leq u \leq t} \{a(s,u) + b(u,t)\} \wedge \{a(s,t) + b(t,t)\} \\
&= \inf_{s \leq u \leq t} \{a(s,u) + b(u,t)\} \\
&= a \otimes b(s,t).
\end{aligned}$$

□

It is easy to verify the following:

$$a \otimes b(s,t) \leq \{a(s,t) + b(t,t)\} \wedge \{a(s,s) + b(s,t)\}.$$

The following lemma follows easily from this.

Lemma 2 *If b has the constant initial condition $b(t,t)=0$, then $a \otimes b \leq a$. Similarly if a has the constant initial condition $a(s,s)=0$, then $a \otimes b \leq b$.*

The following lemma is obvious.

Lemma 3 *If a has full support and for all $s \leq t$ there is a real number $u : s \leq u \leq t$ such that $b(u,t) < \infty$, then $a \otimes b$ has full support.*

Lemma 4 *If a is right locally bounded and b is locally bounded, then $a \otimes b$ is right locally bounded. Conversely, if $-a$ and $a \otimes b$ are right locally bounded, then b is locally bounded.*

Proof. For the first assertion:

$$\begin{aligned}\inf_{v:s\leq v\leq t} a \otimes b(s,v) &= \inf_{v:s\leq v\leq t} \inf_{u:s\leq u\leq v} \{a(s,u)+b(u,v)\} \\ &\geq \inf_{u:s\leq u\leq t} a(s,u) + \inf_{u,v:s\leq u\leq v\leq t} b(u,v) > -\infty, \quad \forall s \leq t.\end{aligned}$$

For the second one, the assumption that $a \otimes b$ is right locally bounded implies that for all $-\infty < s \leq t < \infty$, there exists K such that

$$\inf_{v:s\leq v\leq t} \inf_{u:s\leq u\leq v} \{a(s,u)+b(u,v)\} = K > -\infty.$$

This in turn implies that for all sequences (u_n, v_n) with $s \leq u_n \leq v_n \leq t$,

$$a(s,u_n) + b(u_n,v_n) \geq K.$$

Similarly, the assumption that $-a$ is right locally bounded implies that for all s, there exists a L such that

$$\inf_{v:s\leq v\leq t} -a(s,v) = L > -\infty.$$

Assume now that b is not locally bounded. Then there exists a sequence (u_n, v_n) as above and such that $b(u_n, v_n)$ tends to $-\infty$ when n tends to ∞. But since

$$b(u_n,v_n) \geq K - a(s,u_n) \geq K + L > -\infty,$$

we reach a contradiction.

□

Lemma 5 *If a and b are bounded, non-decreasing and asymptotically null, then so are $a \oplus b$ and $a \otimes b$.*

Proof. The only non-trivial property is that $a \otimes b$ is asymptotically null. For showing this, we prove that for all $K \in \mathbf{R}^+$, there exists a V such that $\inf_{t\geq V} a \otimes b(s,t) \geq K$. Since a, b are assumed to be bounded, $\underline{a} := \inf_{s\leq t} a(s,t) > -\infty$ and $\underline{b} := \inf_{s\leq t} b(s,t) > -\infty$. Then, for all functions $f(s)$ such that $s \leq f(s) \leq V$,

$$\begin{aligned}\inf_{t\geq V} a \otimes b(s,t) &= \inf_{t\geq V} \inf_{s\leq u\leq t} a(s,u) + b(u,t) \\ &= \left(\inf_{t\geq V} \inf_{s\leq u\leq f(s)} a(s,u)+b(u,t)\right) \wedge \left(\inf_{t\geq V} \inf_{f(s)\leq u\leq t} a(s,u)+b(u,t)\right) \\ &\geq (\underline{a} + b(f(s),V)) \wedge (a(s,f(s)) + \underline{b}),\end{aligned}$$

where the last inequality follows from the fact that a, b are non-decreasing. We first choose $f(s)$ such that $a(s,f(s)) + \underline{b}$ is larger than K, which is possible since a is asymptotically null, and then choose V such that $V \geq f(s)$ and $\underline{a} + b(f(s),V)$ is larger than K, which is possible since b is asymptotically null. This concludes the proof.

□

Lemma 6 *If a and b have bounded support, then so have $a \oplus b$ and $a \otimes b$.*

Proof. The only non-trivial property is for $a \otimes b$. Using the relation

$$a \otimes b(s,t) = \inf_{u:\ s \le u \le t} a(s,u) + b(u,t),$$

we see that if $t - s > K_a + K_b$, then for all u as above, either $u - s > K_a$ or $t - u > K_b$ and in both cases $a(s,u) + b(u,t) = +\infty$.

□

Next we introduce the time reversal of a process and show how it is affected by the operations $\oplus$ and $\otimes$. This turns out to be valuable in translating results about the behavior of the system forwards and backwards in time.

Definition 1 (Time Reversal) *For any $a \in D$, define its* time reversal *a^R to be*

$$a^R(s,t) := a(-t,-s), \quad s \le t \in \mathbb{R}. \tag{28}$$

Lemma 7 *Time reversal satisfies the following properties: for any $a, b \in D$,*

1. *$(a^R)^R = a$,*
2. *$(a \oplus b)^R = a^R \oplus b^R$, and*
3. *$(a \otimes b)^R = b^R \otimes a^R$.*

Proof. The first two results are obvious. To obtain the third, note that for all $s \le t \in \mathbb{R}$,

$$\begin{aligned}
(a\otimes b)^R(s,t) &= (a\otimes b)(-t,-s) \\
&= \inf_{-t \le r \le -s} \{a(-t,r) + b(r,-s)\} \\
&= \inf_{s \le r \le t} \{a(-t,-r) + b(-r,-s)\} \\
&= \inf_{s \le r \le t} \{a^R(r,t) + b^R(s,r))\} \\
&= (b^R \otimes a^R)(s,t).
\end{aligned}$$

□

6 Affine Equations

In this section we are interested in the affine equation $x = x \otimes A \oplus B$ and in its scalar version $x = x \otimes a \oplus b$. We first explore the scalar case and then the vector case.

6.1 Scalar Affine Equations

For any $a \in D$, we define

$$a^* := \bigoplus_{n \ge 0} a^n.$$

By the completeness of D, $a^* \in D$ (since D is closed for infinite $\oplus$-sums). For all a and $b \in D$, the largest (by largest, we mean here in the usual sense–it would be the smallest for the natural order of the dioid) solution of the affine equation $x = x \otimes a \oplus b$ is $x_0 = b \otimes a^*$ (Theorem 4.75 of [4]). Below we give a sufficient condition for the uniqueness of this solution.

Lemma 8 *Assume that a satisfies the following condition*

$$\inf_{u,v: s\leq u\leq v\leq t} a(u,v) =: \langle a\rangle(s,t) > 0, \qquad \forall s \leq t. \tag{29}$$

Then a^* is locally bounded. If b is right locally bounded, then $x_0 = b\otimes a^*$ is the unique right locally bounded solution of the affine equation $x = x\otimes a \oplus b$. Similarly, if b is left locally bounded, then, $x_0 = a^* \otimes b$ is the unique left locally bounded solution of the affine equation $x = a\otimes x \oplus b$.

Proof. That $x_0 = b\otimes a^*$ is right locally bounded follows from Lemma 4 if we show that a^* is locally bounded. This is immediate since by induction $a^n(u,v) \geq 0$ for all $n \geq 0$ and hence $a^*(u,v) \geq 0$ for all $u \leq v$.

Let y be another solution of this equation. By induction, for all $l \in \mathbb{N}^+$,

$$y = y \otimes a^l \oplus b(\bigoplus_{p=0}^{l-1} a^p). \tag{30}$$

Note that the second term in (30) admits the following pointwise limit

$$b\otimes (\bigoplus_{p=0}^{l-1} a^l)(s,t) \to_{l\to\infty} b\otimes a^*(s,t) = x_0(s,t), \qquad \forall s \leq t.$$

This follows from the monotone convergence theorem (see the Appendix). Let

$$|y\rangle(s,t) := \inf_{v:\ s\leq v\leq t} y(s,v).$$

Then the first term of (30) is such that

$$y\otimes a^p(s,t) \geq |y\rangle(s,t) + \langle a^p\rangle(s,t), \qquad \forall s \leq t.$$

But by induction, it follows from (29), that

$$\langle a^p\rangle(s,t) \geq l\langle a\rangle(s,t) \to_{l\to\infty} +\infty, \qquad \forall s\leq t. \tag{31}$$

Therefore, if y is right locally bounded, i.e., $|y\rangle(s,t) > -\infty$ for all $s \leq t$, then $y\otimes a^p(s,t) \to_{l\to\infty} +\infty$. By letting $l \to \infty$ on the right hand side of (30) we get that $y = x_0$.

The result for the affine equation $x = ax \oplus b$ follows by observing that by time-reversal this equation is equivalent to $x^R = x^R a^R \oplus b^R$, and the boundedness conditions on a and b are equivalent to corresponding ones on a^R and b^R.

□

Remark 4 *An immediate consequence of (31) is that under the assumptions of Lemma 8, $a^*(t,t) = 0$ for all t.*

Remark 5 *There can be multiple solutions to the affine equation. In particular the infinite function $x = T \equiv -\infty$ is always a solution provided $\langle a\rangle(s,t) < \infty$ for all $s \leq t$. The condition given in Lemma 8 is a sufficient condition ensuring the existence of a right locally bounded solution. The following lemma provides a necessary and sufficient condition on the existence of such a solution under the condition that both b and $-b$ are right locally bounded.*

Lemma 9 *Assume that both b and $-b$ are right locally bounded. Then the affine equation $x = x\otimes a\oplus b$ admits a right locally bounded solution iff a^* is locally bounded.*

Proof. Assume that a^* is locally bounded. Then from Lemma 4, and since b is assumed right locally bounded, $b \otimes a^*$ is right locally bounded. But this is always a solution of the equation.

Conversely, assume that the equation admits a right locally bounded solution. Then necessarily the solution $b \otimes a^*$ (which is the minimal solution for the natural ordering of the dioid) is right locally bounded. In view of Lemma 4, this and the assumption that $-b$ is right locally bounded imply that a^* is locally bounded.

□

Remark 6 *Here is an example where the affine equation has a right locally bounded solution although (29) is not satisfied: take $a(s,t) = -\alpha(t-s)$ with $\alpha > 0$. Indeed, $a^* = a$ is locally bounded, so that the affine equation has a right locally bounded solution when b is right locally bounded.*

Below are sufficient conditions for the solution of an affine equation to have full support (which corresponds to non-explosion), and to be asymptotically null (which corresponds to liveness).

Lemma 10 *Under the assumptions of Lemma 8, the solution of the affine equation $x = x \otimes a \oplus b$ has full support if b has full support.*

Proof. The result follows from Lemma 3 and the fact that $a^*(t,t) = 0$ for all t.

□

Lemma 11 *Under the assumptions of Lemma 8, each of the two following conditions is sufficient for the solution of the affine equation $x = x \otimes a \oplus b$ to be asymptotically null:*

1. *a and b have constant finite initial condition, are non-decreasing and are asymptotically null.*
2. *b has constant finite initial condition, is non-decreasing and asymptotically null; a has bounded support and is such that*

$$\lim_{t\to\infty} \langle a \rangle (s,t) = K(s) > 0,$$

(this limit exists because for all s, $\langle a \rangle (s,t)$ is monotone in t), and

$$\exists \eta > 0,\ L < \infty \text{ such that } \quad a(s,t) \le L, \quad \forall s \le t \le s+\eta.$$

Proof.

1. We first prove that under the assumptions of 1, a^* has constant finite initial condition, is non-decreasing and asymptotically null. It has already been noted that $a^*(t,t) = 0$. That it is non-decreasing follows easily from Lemma 1. It remains to show that it is asymptotically null. Since a is assumed to be non-decreasing with constant initial condition, (29) is equivalent to $a(0,0) > 0$. For all integers $L > 0$,

$$\begin{aligned} a^*(s,t) &= \bigoplus_{0\le l\le L} a^l(s,t) \oplus \bigoplus_{l \ge L} a^l(s,t) \\ &\ge \bigoplus_{0\le l\le L} a^l(s,t) \oplus La(0,0), \qquad \forall s \le t. \end{aligned}$$

We first choose L such that $La(0,0) > J$, and then t such that $\bigoplus_{0\le l\le L} a^l(s,t) \ge J$, which is possible in view of Lemma 5, which implies that this last function is asymptotically null. So a^* is asymptotically null.

The proof is now concluded using Lemma 5, which implies that $b \otimes a^*$ is asymptotically null.

2. We first prove that, under the assumptions of 2, a^* is also asymptotically null. For all $L > 0$,

$$\begin{aligned} a^*(s,t) &= \bigoplus_{0\le l\le L} a^l(s,t) \oplus \bigoplus_{l\ge L} a^l(s,t) \\ &\ge \bigoplus_{0\le l\le L} a^l(s,t) \oplus LK(s). \end{aligned}$$

We first choose L such that $LK(s) > J$, and then t such that $\bigoplus_{0\le l\le L} a^l(s,t) \ge J$, which is possible in view of Lemma 6, which implies that this last function is asymptotically null. Therefore a^* is asymptotically null.

To prove that $b \otimes a^*$ is asymptotically null, we use the assumption that b is non-decreasing and the same method as in the proof of Lemma 5 to establish that for all functions $f(s)$ such that $s \le f(s) \le t$,

$$b \otimes a^*(s,t) \ \ge \ \inf_{s\le u\le f(s)} (\underline{b} + a^*(u,t)) \wedge (b(s,f(s))),$$

Using now the subadditivity of a^* (see lemma 12 below), we get that

$$\inf_{s\le u\le f(s)} (\underline{b} + a^*(u,t)) \ge \inf_{s\le u\le f(s)} (\underline{b} + a^*(s,t) - a^*(s,u)).$$

We first choose $f(s)$ such that $b(s,f(s))$ is larger than K, which is possible since b is asymptotically null, and then choose t such that $t \ge f(s)$ and

$$\underline{b} + a^*(s,t) \ge K + \sup_{s\le u\le f(s)} a^*(s,u),$$

which is possible since a^* is asymptotically null and since

$$\sup_{s\le u\le f(s)} a^*(s,u) < \infty.$$

This last property follows from the assumption that $a(u,v) < L$, for all $v - u \le \eta$, which implies that $a^m(s,u) < mL$ for all $u - s \le m\eta$.

□

The following lemma will be the key of the ergodic theorems later on.

Lemma 12 *For all $a \in D$, a^* is subadditive.*

Proof. We have $a^* = a^*a^*$, which implies that

$$a^*(s,t) = \inf_{s\le u\le t} a^*(s,u) + a^*(u,t) \le a^*(s,u) + a^*(u,t), \quad \forall s \le u \le t.$$

□

6.2 Matrix Affine Equations

We start with a multidimensional generalization of Lemma 8.

Lemma 13 *Let A and B be matrices of $D^{n,n}$. Assume that the entries of A are non-negative and satisfy the condition*

$$\min_{i=1,\ldots,n} \min_{1\leq m\leq n} \inf_{u,v: s\leq u\leq v\leq t} A^m_{i,i}(u,v) =: \langle A\rangle(s,t) > 0, \qquad \forall s \leq t. \tag{32}$$

Then A^* is locally bounded. If B has right locally bounded entries, then $X_0 = B \otimes A^*$ is the unique solution with right locally bounded entries of the affine equation $X = X \otimes A \oplus B$, of unknown $X \in D^{n,n}$. Similarly, if B has left locally bounded entries, then $X_0 = A^* \otimes B$ is the unique solution with left locally bounded entries of the affine equation $X = A \otimes X \oplus B$.

Proof. That $X_0 = B \otimes A^*$ has right locally bounded entries follows from Lemma 4, if we show that A^* has locally bounded entries. But this is immediate since by induction A^l has non-negative entries.

Let Y be another solution of this equation. By induction, for all $l \in \mathbb{N}^+$,

$$Y = Y \otimes A^l \oplus B \otimes (\bigoplus_{p=0}^{l-1} A^l). \tag{33}$$

We prove as above that the second term in (33)

$$B \otimes (\bigoplus_{p=0}^{l-1} A^l)(s,t) \rightarrow_{l\rightarrow\infty} B \otimes A^*(s,t) = X_0, \qquad \forall s \leq t.$$

Let $s \leq t$ be fixed but arbitrary. Note that A^l involves products of the type

$$A_{i_0,i_1} \otimes A_{i_1,i_2} \otimes \ldots \otimes A_{i_{l-1},i_l}.$$

where $i_0, \ldots, i_l$ is some path of length l consisting of elements of $\{1, \ldots, n\}$. Let j_1 be the first integer $j \leq l$ such that $i_j = i_k$ for some $k < j$. If $l \geq n$ there exists such an integer, and $j_1 \leq n$. Then for all $r : s \leq r \leq t$,

$$\begin{aligned}
& A_{i_0,i_1} \otimes \ldots \otimes A_{i_{l-1},i_l}(r,t) \\
&= \inf_{u,v:\ r\leq u\leq v\leq t} A_{i_0,i_1} \otimes \ldots \otimes A_{i_{k-1},i_k}(r,u) + A_{i_k,i_{k+1}} \otimes \ldots \otimes A_{i_{j_1-1},i_{j_1}}(u,v) \\
&\qquad + A_{i_{j_1},i_{j_1+1}} \otimes \ldots \otimes A_{i_{l-1},i_l}(v,t) \\
&\geq \langle A\rangle(s,t) + \inf_{u,v:\ r\leq u\leq v\leq t} A_{i_0,i_1} \otimes \ldots \otimes A_{i_{k-1},i_k}(r,u) + A_{i_{j_1},i_{j_1+1}} \otimes \ldots \otimes A_{i_{l-1},i_l}(v,t) \\
&\geq \langle A\rangle(s,t) + \inf_{v:\ r\leq v\leq t} A_{i_{j_1},i_{j_1+1}} \otimes \ldots \otimes A_{i_{l-1},i_l}(v,t),
\end{aligned}$$

where we successively used (32) and the assumption that the entries of A are non-negative. Whenever $l \geq 2n$, by applying the same bounding technique to the sub-product

$$A_{i_{j_1},i_{j_1+1}} \otimes \ldots \otimes A_{i_{l-1},i_l},$$

we obtain the existence of an integer $j_2 \leq 2n$ such that

$$A_{i_{j_1},i_{j_1+1}} \otimes \ldots \otimes A_{i_{l-1},i_l}(v,t) \geq \langle A\rangle(s,t) + \inf_{w:\ v\leq w\leq t} A_{i_{j_2},i_{j_2+1}} \otimes \ldots \otimes A_{i_{l-1},i_l}(w,t)$$

and so on. Finally,

$$A_{i_0,i_1} \otimes \ldots \otimes A_{i_{l-1},i_l}(r,t) \quad \geq \quad \lfloor \frac{l}{n} \rfloor \langle A\rangle(s,t).$$

If Y has right locally bounded entries, then

$$|Y\rangle(s,t) := \min_{p,q=1,\dots,n} \inf_{v:\ s\le v\le t} Y_{p,q}(s,v) > -\infty.$$

Thus, for all p and q,

$$(Y \otimes A^l)_{p,q}(s,t) \geq |Y\rangle(s,t) + \lfloor \frac{l}{n} \rfloor \langle A \rangle(s,t) \to_{l\to\infty} +\infty, \qquad \forall s \leq t.$$

By letting $l \to \infty$ on the right hand side of (33) we get that $Y = X_0$.

The result for the affine equation $X = AX \oplus B$ follows by observing that by time-reversal, this equation is equivalent to $X^R = X^R A^R \oplus B^R$, and the boundedness conditions on A and B are equivalent to corresponding ones on A^R and B^R.

□

Remark 7 *Note that on the diagonal, $A^*_{j,j}(t,t) = 0$ for all j. Off the diagonal, $A^*_{j,l}(t,t) < \infty$ iff there exists a path $\pi = (i_0, i_1, \dots, i_q) \in \mathcal{P}_{jl}$ (see Theorem 2 for the definition of $\mathcal{P}_{jl}$) such that $A_{i_{p-1},i_p}(t,t) < \infty$ for all $p = 1, \dots, q$. This is an immediate consequence of Theorem 2.*

Lemma 9 admits the following generalization:

Lemma 14 *Assume that B has right locally bounded entries, and that for all i, j, there exists an index l such that $-B_{l,j}$ is right locally bounded. Then the affine equation $X = X \otimes A \oplus B$ has a right locally bounded solution iff A^* is locally bounded.*

Proof. The proof of the sufficient condition is the same as in the scalar case. For the necessary condition, we first prove as in the scalar case that $B \otimes A^*$ is right locally bounded. This in turn implies that for all i, j, l, $B_{l,i} \otimes A^*_{i,j}$ is right locally bounded, and the conclusion follows from Lemma 4.

□

We now give a sufficient condition for the absence of explosion:

Lemma 15 *The assumptions are those of Lemma 13. If for each k, l, there exists an index j such that $B_{k,j}$ is of full support and $A^*_{j,l}(t,t) < \infty$ $\forall t$, then the solution of the affine equation $X = X \otimes A \oplus B$ has all entries with full support.*

Proof. Follows easily from Lemmas 13 and 3.

□

Here are two sufficient conditions for multidimensional liveness:

Lemma 16 *If A and B have entries which have constant finite initial condition, are non-decreasing and asymptotically null, then under the assumptions of Lemma 13, the solution of the affine equation $X = XA \oplus B$ has asymptotically null entries.*

Proof. From Lemma 1, it follows that A^l has entries that have constant initial condition and are non-decreasing. Moreover due to the non-negativity assumption on the entries of A, the entries of A^l are also non-negative. Hence, A^* has entries which have constant finite initial condition and are non-decreasing. We now show that A^* is asymptotically null. Since A is assumed to be non-decreasing with constant initial condition, it follows that (32) is equivalent to (24). For all $L > 0$,

$$
\begin{aligned}
(A^*)_{i,j}(s,t) &= \bigoplus_{0\le l\le L} (A^l)_{i,j}(s,t) \oplus \bigoplus_{l\ge L} (A^l)_{i,j}(s,t) \\
&\ge \bigoplus_{0\le l\le L} (A^l)_{i,j}(s,t) \oplus \lfloor \frac{L}{n} \rfloor \langle A\rangle .
\end{aligned}
$$

We first choose L such that $\lfloor \frac{L}{n}\rfloor \langle A\rangle > K$, and then t such that $\bigoplus_{0\le l\le L}(A^l)_{i,j}(s,t) \ge K$, which is possible in view of Lemma 5, which implies that this last function is asymptotically null. The proof is now concluded using Lemma 5, which implies that $B \otimes A^*$ has asymptotically null entries.

$\square$

Lemma 17 *If the entries of B all have constant finite initial condition, are non-decreasing and asymptotically null, and if the entries of A all have bounded support and satisfy the following two properties*

$$
\lim_{t\to\infty} \min_{i,j} \langle A_{i,j}\rangle (s,t) = K(s) > 0, \qquad \forall s,
$$

and, for all l, there exists $\eta > 0$, $p > 0$ and $L < \infty$ such that

$$
(A^p)_{l,l}(s,t) \le L, \quad \forall s \le t \le s+\eta,
$$

then under the assumptions of Lemma 13, the solution of the affine equation $X = X \otimes A \oplus B$ has asymptotically null entries.

Proof. The structure of the proof is similar to that of point 2 in Lemma 11. We first prove in the same way that

$$
\bigoplus_{l\ge L} A^l(s,t) \ge LK(s)E,
$$

and this allows one to prove the A^* is asymptotically null. Using the same notations and approach as in the scalar case, we then obtain the following inequalities:

$$
\begin{aligned}
(B\otimes A^*)_{i,j}(s,t) &\ge \min_l \left\{ \inf_{s\le u\le f(s)} \underline{B}_{i,l} + A^*_{l,j}(u,t) \right\} \wedge B_{i,l}(s,f(s)) \\
&\ge \min_l \left\{ \inf_{s\le u\le f(s)} \underline{B}_{i,l} + A^*_{l,l}(u,t) \right\} \wedge B_{i,l}(s,f(s)),
\end{aligned}
$$

where the last inequality follows from the relation $A^*_{l,j} \ge A^*_{l,l}$, which in turn follows from (25) and the positiveness assumption on the entries of A. Since $A^*_{l,l}$ is subadditive, the proof can be concluded as in the scalar case.

$\square$

On a class of affine equations Let f be a fixed function of D. For all functions a of D, we denote by $\widetilde{a}$ the function

$$
\widetilde{a}(s,t) = a(s,t) - f(s,t), \quad s \le t.
$$

Note that

$$
\widetilde{(a \oplus b)} = \widetilde{a} \oplus \widetilde{b}
$$

and that when f is additive

$$\widetilde{(a \otimes b)} = \widetilde{a} \otimes \widetilde{b},$$

whereas

$$\widetilde{(a \otimes b)} \leq \ (\text{resp. } \geq) \ \widetilde{a} \otimes \widetilde{b},$$

when it is subadditive (resp. superaddtitive).

Consider the following affine equation:

$$X = X \otimes A \oplus U \otimes B, \tag{34}$$

where $A \in D^{n,n}$, $X \in D^{1,n}$, $U \in D^{1,m}$ and $B \in D^{m,n}$. Let $1 \leq \mu \leq m$ be fixed. Take $f = U_\mu$. The above notation is extended to vectors and matrices. For instance, $\widetilde{X}$ is the vector with entries

$$\widetilde{X}_i(s,t) \quad = \quad X_i(s,t) - U_\mu(s,t), \quad 1 \leq i \leq n$$

and $\widetilde{A}$ is the matrix with entries

$$\widetilde{A}_{i,j}(s,t) \quad = \quad A_{i,j}(s,t) - U_\mu(s,t), \quad 1 \leq i, j \leq n$$

From the above remarks, it is immediate that

Lemma 18 *If U_μ is additive, then $\widetilde{X}$ satisfies the affine equation*

$$\widetilde{X} = \widetilde{X} \otimes \widetilde{A} \oplus \widetilde{U} \otimes \widetilde{B}. \tag{35}$$

If U_μ is subadditive, then $\widetilde{X} \geq \widetilde{X} \otimes \widetilde{A} \oplus \widetilde{U} \otimes \widetilde{B}$, with the reversed inequality in case of a superadditive U_μ.

Consider the additive case. It follows from the completeness of the dioid that the largest solution of the affine equation (35) is $\widetilde{U}\widetilde{B}(\widetilde{A})^*$. Using the above calculus, it is easy to check that

$$(\widetilde{A})^* = \widetilde{(A^*)}$$

and that

$$\widetilde{U} \otimes \widetilde{B} \otimes \widetilde{(A^*)} = (U \otimes \widetilde{B \otimes} A^*).$$

In case (35) has a unique solution, we find back the relation defining $\widetilde{X}$, namely

$$\widetilde{X}(s,t) = U \otimes B \otimes A^*(s,t) - U_\mu(s,t).$$

7 Rate Results

In this section we show how rates may be obtained for $a \oplus b$, $a \otimes b$, and A^*, in terms of the rates of a, b, and A.

An element a of D is said to be

- *backward-sublinear* if there exists a function f such that $f(s) \geq s$, $\lim_{s \to -\infty} f(s)/s = 1$ and
$$\limsup_{s \to -\infty} \frac{a(s, f(s))}{-s} \leq 0;$$
- *forward-sublinear* if its time-reversal a^R is backward-sublinear.
- *backward non-null* if for all t there exists a $s \leq t$ such that $a(s,t) < \infty$.

- *forward non-null* if its time-reversal a^R is backward non-null.

It is easy to check that if a and b are backward-sublinear (resp. backward non-null), then so is $a \otimes b$.

Lemma 19 *Assume that the functions a and b of D have forward-rates α and β respectively. Then the function $a \oplus b$ has a forward-rate equal to $\gamma = \alpha \wedge \beta$.*

Proof. Property (20) for $a \oplus b$ follows from

$$\lim_{t\to\infty} \frac{a(s,t) \wedge b(s,t)}{t-s} = \lim_{t\to\infty} \frac{a(s,t)}{t-s} \wedge \frac{b(s,t)}{t-s} = \lim_{t\to\infty} \frac{a(s,t)}{t-s} \wedge \lim_{t\to\infty} \frac{b(s,t)}{t-s} = \gamma.$$

□

By time-reversal, a symmetrical statement holds for backward rates.

For all functions $f : \mathbb{R}_+ \to \overline{\mathbb{R}}$, let $\mathcal{L}f[z]$ be the convex conjugate or Legendre type transform defined by the relation

$$\mathcal{L}f[z] = \inf_{t\geq 0}\{f(t) - zt\}, \quad z \in \mathbb{R}. \tag{36}$$

The function $z \to \mathcal{L}f[z]$ is non-increasing in z. We will say that z_0 is within its domain of convergence if $\mathcal{L}f[z_0] > -\infty$. It is straightforward to verify the following: if f is finite and $z < \liminf_t \frac{f(t)}{t}$, then z is in the domain of convergence of $\mathcal{L}f$. If z is in the domain of convergence of $\mathcal{L}f$, then f is finite and $z \leq \liminf_t \frac{f(t)}{t}$.

Lemma 20 *If a and b have backward-rates α and β, a is subadditive and backward sublinear, and b is backward non-null, then $a \otimes b$ has a backward-rate equal to $\gamma = \alpha \wedge \beta$. If in addition $\alpha < \beta$ and $\mathcal{L}a(s-\cdot,s)[\alpha] > -\infty$, then $\mathcal{L}a \otimes b(s-\cdot,s)[\gamma] > -\infty$.*

Proof. Let $t \in \mathbb{R}$ be fixed but arbitrary. Let us first prove that

$$\limsup_{s\to-\infty} \frac{a \otimes b(s,t)}{t-s} \leq \gamma.$$

Let $f(s)$ be the function in the definition of backward sublinearity of a. Then,

$$\begin{aligned}
\limsup_{s\to-\infty} \frac{a \otimes b(s,t)}{t-s} &\leq \limsup_{s\to-\infty} \frac{a(s,f(s)) + b(f(s),t)}{t-s} \\
&\leq \limsup_{s\to-\infty} \frac{a(s,f(s))}{-s}\frac{-s}{t-s} + \frac{b(f(s),t)}{t-f(s)}\frac{t-f(s)}{t-s} \\
&\leq \beta.
\end{aligned}$$

Also, due to the backward non-nullity of b, there is a $t' \leq t$ such that $b(t',t) < \infty$. Therefore,

$$\limsup_{s\to-\infty} \frac{a \otimes b(s,t)}{t-s} \leq \limsup_{s\to-\infty} \frac{a(s,t') + b(t',t)}{t-s} = \alpha.$$

We now show that

$$\liminf_{s\to-\infty} \frac{a \otimes b(s,t)}{t-s} \geq \gamma. \tag{37}$$

Let $z = \gamma - \eta$ with $\eta > 0$. Using the assumption that a is subadditive, we get

$$\begin{aligned}
\mathcal{L}a \otimes b(t-\cdot,t)[z] &= \inf_{s:s\le t}\{\inf_{r:s\le r\le t}\{a(s,r)+b(r,t)\} - z(t-s)\} \\
&\ge \inf_{s:s\le t}\{\inf_{r:s\le r\le t}\{a(s,t)+b(r,t)-a(r,t)\} - z(t-s)\} \\
&= \inf_{s:s\le t}\{a(s,t) - (\alpha - \frac{\eta}{2})(t-s) \\
&\qquad + \inf_{r:s\le r\le t}\{b(r,t)-a(r,t)-(\gamma-\alpha-\frac{\eta}{2})(t-s)\}\} \\
&\ge \inf_{s:s\le t}\{a(s,t) - (\alpha - \frac{\eta}{2})(t-s) \\
&\qquad + \inf_{r:s\le r\le t}\{b(r,t)-a(r,t)-(\gamma-\alpha-\frac{\eta}{2})(t-r)\}\} \\
&\ge \inf_{s:s\le t}\{a(s,t) - (\alpha - \frac{\eta}{2})(t-s)\} \\
&\quad + \inf_{r:r\le t}\{b(r,t)-a(r,t)-(\gamma-\alpha-\frac{\eta}{2})(t-r)\} \\
&= \mathcal{L}a(t-\cdot,t)[\alpha-\frac{\eta}{2}] + \mathcal{L}(b-a)(t-\cdot,t)[\gamma-\alpha-\frac{\eta}{2}] \\
&> -\infty,
\end{aligned}$$

where the fourth inequality follows since $\gamma - \alpha - \frac{\eta}{2} = \alpha \wedge \beta - \alpha - \frac{\eta}{2} \le 0$, and the last inequality follows since $\alpha - \frac{\eta}{2}$ is in the domain of convergence of $\mathcal{L}a(t-\cdot,t)$ and $\gamma - \alpha - \frac{\eta}{2} < \beta - \alpha$ is in that of $\mathcal{L}(b-a)(t-\cdot,t)$. Therefore,

$$\liminf_{s\to-\infty} \frac{a\otimes b(s,t)}{t-s} \ge \gamma - \eta,$$

for all $\eta > 0$, which concludes the proof of (37). In order to show the second part of the lemma, simply observe that the above deductions hold also for $\eta = 0$ under the additional assumptions that $\alpha < \beta$ and $\mathcal{L}a(s-\cdot,s)[\alpha] > -\infty$.

□

By time-reversal, we have a corresponding result forwards in time.

Corollary 1 *If a and b have forward-rates α and β, b is subadditive and forward sublinear, and a is forward non-null, then $a \otimes b$ has a forward-rate equal to $\gamma = \alpha \wedge \beta$. If in addition $\alpha > \beta$ and $\mathcal{L}b(s, s+\cdot)[\beta] > -\infty$, then $\mathcal{L}a \otimes b(s, s+\cdot)[\gamma] > -\infty$.*

Corollary 2 *Let $A_{i,j}, 1 \le i \le n, 1 \le j \le m_i$ be subadditive, have backward (resp. forward) rates $\alpha_{i,j}$, and be backward (resp. forward) sublinear and non-null. Then $a := \oplus_{1\le i\le n} \otimes_{1\le j\le m_i} A_{i,j}$ has a backward (resp. forward) rate $\alpha := \wedge_{1\le i\le n} \wedge_{1\le j\le m_i} \alpha_{i,j}$.*

Proof. For each $1 \le i \le n$ and $1 \le l \le m_i$, we can show using backward induction and Lemma 20, that $\otimes_{l\le j\le m_i} A_{i,j}$ has the backward rate $\wedge_{l\le j\le m_i}\alpha_{i,j}$ Thus, by Lemma 19, it follows that a has the backward rate α. The result for forward rates follows by time-reversal.

□

7.1 Rate Results for A^*

Let $A \in D^{n,n}$. We begin this section by proving Theorem 2. We do so without any assumptions.

Proof of Theorem 2 Since $A^* = A^* \otimes A^*$,

$$\begin{aligned} A^*_{i,i}(s,t) &= \bigoplus_j A^*_{i,j} \otimes A^*_{j,i}(s,t) \\ &\leq A^*_{i,i} \otimes A^*_{i,i}(s,t) \\ &= \inf_{s \leq u \leq t} A^*_{i,i}(s,u) + A^*_{i,i}(u,t) \\ &\leq A^*_{i,i}(s,u) + A^*_{i,i}(u,t), \quad \forall\ s \leq u \leq t. \end{aligned}$$

In addition, off the main diagonal, $A^* = A^* \otimes A = A \otimes A^*$. Therefore, for $i \neq j$,

$$A^*_{i,j} = \bigoplus_k A^*_{i,k} \otimes A^*_{k,j} \leq A^*_{i,i} \otimes A^*_{i,j} = A^*_{i,i} \otimes (A \otimes A^*)_{i,j} \leq A^*_{i,i} \otimes A_{i,l} \otimes A^*_{l,j},$$

for all l. If we choose $l \neq j$, we can iterate this and prove that

$$A^*_{i,j} \leq A^*_{i,i} \otimes A_{i,l} \otimes A^*_{l,l} \otimes A_{l,k} \otimes A^*_{k,j},$$

for all k. Applying this recursively, we finally obtain that

$$A^*_{i,j} \leq \bigoplus_{\pi \in \mathcal{P}_{i,j}} A^*_{i_0,i_0} \otimes A_{i_0,i_1} \otimes A^*_{i_1,i_1} \otimes A_{i_1,i_2} \otimes \ldots \otimes A_{i_{m-1},i_m} \otimes A^*_{i_m,i_m}.$$

We now prove the converse inequality. Let $\Pi^p_{i,j}$ denote the set of all sequences

$$\pi = (j_0 = i, j_1, \ldots, j_p = j)$$

of elements of $\{1, \ldots, n\}$, with length $p \geq 1$. Then, by definition, for $i \neq j$,

$$A^*_{i,j} = \bigoplus_{p \geq 1} \bigoplus_{\pi \in \Pi^p_{i,j}} A_{j_0,j_1} \otimes \ldots \otimes A_{j_{p-1},j_p}.$$

For all sequences $\pi = (j_0, j_1, \ldots, j_p)$ of $\Pi^p_{i,j}$, let l_0 be the largest integer $l \geq 0$ such that $j_l = j_0$. We have

$$A_{j_0,j_1} \otimes \ldots \otimes A_{j_{p-1},j_p} \geq A^{l_0}_{j_0,j_0} \otimes A_{j_0,j_{l+1}} \ldots \otimes A_{j_{p-1},j_p},$$

where $j_{l+1} = i_1 \neq j_0 = i_0 = i$. Using the same argument recursively, we find a sequence $\pi' = (i_0 = i, i_1, \ldots, i_m = j)$ on $\mathcal{P}_{i,j}$ and integers $l_0, \ldots, l_m$ (which depend of course on π) such that

$$A^*_{i,j} \geq \bigoplus_{p \geq 1} \bigoplus_{\pi \in \Pi^p_{i,j}} A^{l_0(\pi)}_{i_0(\pi),i_0(\pi)} A_{i_0(\pi),i_1(\pi)} A^{l_1(\pi)}_{i_1(\pi),i_1(\pi)} A_{i_1(\pi),i_2(\pi)} \otimes \ldots \otimes A_{i_{m-1}(\pi),i_m(\pi)} A^{l_m(\pi)}_{i_m(\pi),i_m(\pi)}.$$

Therefore

$$A^*_{i,j} \geq \bigoplus_{p \geq 1} \bigoplus_{\pi \in \Pi^p_{i,j}} A^*_{i_0(\pi),i_0(\pi)} A_{i_0(\pi),i_1(\pi)} A^*_{i_1(\pi),i_1(\pi)} A_{i_1(\pi),i_2(\pi)} \otimes \ldots \otimes A_{i_{m-1}(\pi),i_m(\pi)} A^*_{i_m(\pi),i_m(\pi)}.$$

When aggregating in a single $\oplus$-sum all sequences π which lead to the same π', we finally get

$$A^*_{i,j} \geq \bigoplus_{\pi' \in \mathcal{P}_{i,j}} A^*_{i_0,i_0} A_{i_0,i_1} A^*_{i_1,i_1} A_{i_1,i_2} \otimes \ldots \otimes A_{i_{m-1},i_m} A^*_{i_m,i_m}.$$

$\square$

Remark 8 *Formula (25) actually holds in an arbitrary matrix dioid $D^{n,n}$ (see [4]). To see this, we use the partial order associated with the scalar dioid, namely*

$$a \prec b \;\Leftrightarrow\; \exists c \text{ s.t. } a = b \oplus c.$$

Note that for the dioid of interest here, we have in fact

$$a \prec b \;\Leftrightarrow\; a(s,t) \geq b(s,t) \;\forall s \leq t$$

Indeed, the properties $A^* = A^*A^*$ and $A^* = A^*A = AA^*$ (off the main diagonal) hold true in any matrix dioid. We then prove as in the first step of the proof of the above lemma that

$$A^*_{i,j} \succ \bigoplus_{\pi \in \mathcal{P}_{i,j}} A^*_{i_0,i_0} A_{i_0,i_1} A^*_{i_1,i_1} A_{i_1,i_2} \otimes \ldots \otimes A_{i_{m-1},i_m} A^*_{i_m,i_m}.$$

As for the second step, we have (using the same notations)

$$A_{j_0,j_1} \otimes \ldots A_{j_{p-1},j_p} \prec A^{l_0}_{j_0,j_0} \otimes A_{j_0,j_{l+1}} \ldots \otimes A_{j_{p-1},j_p},$$

from which we deduce that

$$A^*_{i,j} \prec \bigoplus_{\pi' \in \mathcal{P}_{i,j}} A^*_{i_0,i_0} A_{i_0,i_1} A^*_{i_1,i_1} A_{i_1,i_2} \otimes \ldots \otimes A_{i_{m-1},i_m} A^*_{i_m,i_m}.$$

Corollary 3 *Let $A \in D^{n,n}$ be such that*

- *all off-diagonal entries of A are subadditive, backward (resp. forward) sublinear and non-null and have backward (resp. forward) rates, say $\alpha_{i,j}$ for entry $A_{i,j}$;*
- *all diagonal entries of A^* have backward (resp. forward) rates, say $\alpha^*_{i,i}$ for entry $A^*_{i,i}$.*

Then all entries of A^* have backward (resp. forward) rates. The rate $\alpha^*_{i,j}$ of $A^*_{i,j}$, $i \neq j$, is given by the formula:

$$\alpha^*_{i,j} = \bigwedge_{\pi \in \mathcal{P}_{i,j}} \bigwedge_{i_k \in \pi} \alpha^*_{i_k,i_k} \bigwedge_{(i_k,i_{k+1}) \in \pi} \alpha_{i_k,i_{k+1}}.$$

Proof. Observe that the diagonal entries of A^* are backward (and forward) sublinear and non-null as $A^*_{i,i}(t,t) \leq 0$ for all t. The result now follows immediately from Theorem 2 and Corollary 2.

□

8 Stochastic Results

Let $(\Omega, \mathcal{F}, P)$ be a probability space with a group $\{\theta_r, r \in \mathbb{R}\}$ of measure preserving transformations with measure preserving inverse. All stochastic processes a in consideration will be defined on this probability space.

- A stochastic process a taking values in D will be said to *stationary* with respect to $\{\theta_r, r \in \mathbb{R}\}$ if $a(r+s, r+t) = a(s,t) \circ \theta_r$, $-\infty < s \leq t < \infty, r \in \mathbb{R}$.
- It is said to be *ergodic* if $\{\theta_r, r \in \mathbb{R}\}$ is ergodic.
- A collection of stochastic processes a_n can likewise be defined to be (jointly) stationary and ergodic with respect to $\{\theta_r, r \in \mathbb{R}\}$.

Throughout the section, we shall assume that any primitive process that is not equal to ε or e, satisfies the following properties:

P1 (Jointly) stationary and ergodic.

P2 Non-negative, non-decreasing, and constant finite initial condition.

P3 Integrable.

P4 Subadditive.

All exogenous arrival processes are assumed to satisfy the additional property

P5 Additive.

The key tool for obtaining the existence of rates is Kingman's subadditive ergodic theorem [13]. Below we give a slight modification of the continuous time extension from [6].

Lemma 21 *Let $\{a(s,t), -\infty < s \le t < \infty\}$ be stationary with respect to $\{\theta_r, r \in \mathbb{R}\}$, subadditive, and satisfy the following conditions*

1. *$a(s,t)$ is integrable for all $s \le t$.*
2. *The set $\{\frac{1}{t-s}E[a(s,t)]; s < t\}$ is bounded below.*
3. *There exists an integrable random variable M such that $|a(s,t)| \le M, \forall 0 \le s \le t \le 1$.*

Then $\frac{1}{t-s}a(s,t)$ converges almost surely and in expectation to an integrable invariant random variable α with $E[\alpha] = \inf\{\frac{1}{t-s}E[a(s,t)]; s < t\}$ as $s \to -\infty$. If in addition a is ergodic then α is constant almost surely.

Proof. The proof is the same as that of Corollary IV.1.3, [6].

□

It is easy to verify that the conditions of the above lemma are satisfied under the conditions **P1** – **P4**. Thus we obtain the following corollary.

Corollary 4 *If $a \in D$ satisfies properties* **P1** – **P4**, *then it has constant and equal backward and forward a.s. rates, α and*

$$\alpha = \inf\{\frac{1}{t-s}E[a(s,t)]; s < t\} = \inf\{\frac{1}{t}E[a(0,t)]; t > 0\}.$$

We will use the following notation:

$$\overline{a}(s,t) = E[a(s,t)].$$

Note that if a is stationary, then $E[a(s,t)] = E[a(0,t-s)]$, so that $\overline{a}(s,t) = \tilde{a}(t-s)$.

Lemma 22 *Let $a^1, a^2, \ldots, a, b$ satisfy properties* **P1** – **P4**. *Then, $\oplus_{i\ge 1} a_i$, $a \oplus b$ and $a \otimes b$ satisfy properties* **P1** – **P3**. *Further, $a \oplus b$ and $a \otimes b$ both have almost sure backward and forward rates equal to $\alpha \wedge \beta$, where α and β are the respective rates of a and b. Also, $\overline{a \oplus b} \le \overline{a} \oplus \overline{b}$, $\overline{\oplus_{i\ge 1} a_i} \le \oplus_{i\ge 1}\overline{a_i}$, and $\overline{a \otimes b} \le \overline{a} \otimes \overline{b}$.*

Proof. For any $a \in D$ and $r \in \mathbb{R}$, define a^r by $a^r(s,t) = a(r+s, r+t)$, $s \le t \in \mathbb{R}$. In order to establish stationarity, it suffices to show that for each $r \in \mathbb{R}$, $(\oplus_{i\ge1} a_i)^r = \oplus_{i\ge1} a_i^r$, $(a \oplus b)^r = a^r \oplus b^r$ and $(a \otimes b)^r = a^r \otimes b^r$. The first two are trivial. To check the third one, note that

$$\begin{aligned}
(a\otimes b)^r(s,t) &= (a\otimes b)(r+s, r+t) \\
&= \inf_{r+s\le r'\le r+t} \{a(r+s, r') + b(r', r+t)\} \\
&= \inf_{s\le r'\le t} \{a(r+s, r+r') + b(r+r', r+t)\} \\
&= \inf_{s\le r'\le t} \{a^r(s, r') + b^r(r', t)\} \\
&= a^r \otimes b^r(s,t), \quad s \le t \in \mathbb{R}.
\end{aligned}$$

This establishes **P1**. The only non-trivial part in establishing **P2** and **P3** is the non-decreasing property for $a\otimes b$. This follows from Lemma 1. The rate result for $a \oplus b$ and $a\otimes b$ follows from Lemmas 19 and 20, respectively. The conditions of those lemmas are met because a and b have backward and forward rates by Corollary 4, and because a, b are subadditive and are almost surely backward (and forward) sublinear and non-null by properties **P2**, **P3** and **P4**. To verify the last part observe that due to Jensen's inequality,

$$\overline{a \oplus b}(s,t) = E[a(s,t) \wedge b(s,t)] \le E[a(s,t)] \wedge E[b(s,t)] = \overline{a} \oplus \overline{b}(s,t).$$

From this it follows that $\overline{\oplus_{i\ge1} a_i} \le \oplus_{i\ge} \overline{a_i}$. Finally,

$$\begin{aligned}
\overline{a\otimes b}(s,t) &= E[a\otimes b(s,t)] = E[\inf_{s\le r\le t} \{a(s,r) + b(r,t)\}] \\
&\le \inf_{s\le r\le t} \{E[a(s,r)] + E[b(r,t)]\} = \overline{a}\otimes\overline{b}(s,t).
\end{aligned}$$

□

Corollary 5 *If a satisfies* **P1** – **P4***, then a^* satisfies* **P1** – **P4***. If a is stationary with respect to $\{\theta_r, r \in \mathbb{R}\}$, then so is a^*. Moreover, $\overline{a^*} \le \overline{a} \wedge e$ and a^* has a backward and forward rate $\alpha^* \le \alpha$ almost surely, where α is the backward and forward rate of a.*

Proof. Note that $a^* = \oplus_{n\ge0} a^n$. By the previous lemma, for all $n \ge 1$, a^n satisfies **P1** – **P3**, and hence so does a^*. By the previous lemma, it also follows that $\overline{a^*} \le e \oplus_{n\ge1} \overline{a}^n = e \oplus \overline{a}$ by the subadditivity of a. Further we know that a^* is subadditive and hence by Corollary 4 it follows that it has a backward and forward rate denoted α^*. By Lemma 22 it follows that for all $n \ge 1$, a^n has the same backward and forward rate α. Consequently, $\alpha^* \le \alpha$.

□

Lemma 23 *Assume that $\{A_{i,j} : 1 \le i,j \le n\}$ are either ε, e, or satisfy properties* **P1** – **P3***. Then $\{A^*_{i,j} : 1 \le i,j \le n\}$ also are either ε, e, or satisfy properties* **P1** – **P3***. Moreover, $\{A^*_{i,i} : 1 \le i \le n\}$ satisfy* **P4** *and consequently have constant backward and forward rates, say $\alpha^*_{i,i}$, almost surely. If in addition $\{A_{i,j} : 1 \le i \ne j \le n\}$ satisfy property* **P4***, then $\{A^*_{i,j} : 1 \le i \ne j \le n\}$ too have constant backward and forward rates, say $\alpha^*_{i,j}$, almost surely.*

Proof. Note that $A^* = \bigoplus_{k\ge0} A^k$. Moreover, each component in A^k is obtained by finite number of operations $\oplus, \otimes$ on the functions $A_{i,j}, 1 \le i,j \le n$. So by Lemma 22 $(A^m)_{i,j}, 1 \le i,j \le n, m \ge 0$ are either ε, e, or satisfy **P1** - **P3**. Therefore, by Lemma 22, $(A^*)_{i,j} = \oplus_{k\ge0}(A^k)_{i,j}$ also are either ε, e, or satisfy **P1** - **P3**. That $\{A^*_{i,i} : 1 \leq i \leq n\}$ satisfy **P4**, follows from Theorem 2. The existence of constant almost sure rates $\alpha^*_{i,i}$ is immediate from Corollary 4. The rate result for $\{A^*_{i,j} : 1 \leq i \neq j \leq n\}$ follows from Corollary 3.

□

9 Proof of Stochastic Results for Networks

In this section we obtain results on the rates of the departure processes and on the existence of stationary queue lengths in networks.

Proof of Theorem 3

The departure process is by definition $x = u \otimes a$. It follows from Lemma 22, that x has the backward and forward rate $\xi = v \wedge \alpha$ almost surely. By definition, for $s \leq t$,

$$\begin{aligned} q(s,t) &= u(s,t) - x(s,t) \\ &= u(s,t) - \inf_{s \leq r \leq t} \{u(s,r) + a(r,t)\} \\ &= - \inf_{s \leq r \leq t} \{a(r,t) - u(r,t)\} \\ &= - \inf_{s \leq r \leq t} \Delta(r,t), \end{aligned}$$

where $\Delta(r,t) := a(r,t) - u(r,t)$. The third equality above follows by the additivity assumption on u. It is clear that $q(s,t)$ is a non-increasing in s for $s \leq t$. Thus,

$$q(-\infty, t) := \lim_{s \to -\infty} q(s,t) = - \inf_{r \leq t} \Delta(r,t).$$

Moreover, by the assumption that a and u are jointly stationary it follows that $q(s,t)$ is stochastically equal to $q(s-t, 0)$ and hence, as $t \to \infty$, $q(s,t)$ converges in distribution to $q(-\infty, 0)$ for any $s \in \mathbb{R}$. Of course, for all t,

$$q(-\infty, t) = q(-\infty, 0) \circ \theta_t.$$

Thus $\lim_{t \to \infty} P(q(s,t) < \infty) = P(q(-\infty, t') < \infty)$. Now,

$$q(-\infty, t) = - \inf_{r \leq t} \Delta(r,t) = -\mathcal{L}\Delta(t - \cdot, t)[0].$$

Since Δ has the backward-rate $\alpha - v$ a.s., it follows from the discussion on the domain of convergence of Legendre transforms, that $q(-\infty, t) < \infty$ a.s. if $\alpha > v$ and $q(-\infty, t) = \infty$ a.s. if $\alpha < v$.

□

Proof of Theorem 4

The rate results for A^* follows from Lemma 23. That $x = uBA^*$ if A satisfies the condition (32), follows from Lemma 13. The rate result for the departure processes x_k follows from the expression $x = uBA^*$, the rate result for A^*, and Lemmas 19, 20, and 22, and from the assumption that the primitive processes are at least subadditive. Note that

$$\begin{aligned} x_k &= \bigoplus_{1 \leq i \leq m} \bigoplus_{1 \leq j \leq n} u_i \otimes B_{i,j} \otimes A^*_{j,k} \\ &= \bigoplus_{(i,j) \in J_k} u_i \otimes B_{i,j} \otimes A^*_{j,k}. \end{aligned}$$

Hence, we need to take a minimum of the rates of all components u_i, $B_{i,j}$, and $A^*_{j,k}$ that "constrain" the departure process x_k. If J_k is empty, then $\xi_k = \infty$.

Now, let i, k be such that $(i,j) \in J_k$ for some $1 \le j \le n$. Then,

$$\begin{aligned}
q_{i,k}(s,t) &= u_i(s,t) - x_k(s,t) \\
&= u_i(s,t) - \left(\bigoplus_{1\le l\le m} u_l \otimes (BA^*)_{l,k} \right)(s,t) \\
&= u_i(s,t) - \bigwedge_{1\le l\le m} \inf_{s\le u\le t} \{u_l(s,u) + (BA^*)_{l,k}(u,t)\} \\
&= \sup_{s\le u\le t} \{u_i(u,t) - (BA^*)_{i,k}(u,t)\} \vee \\
&\quad \bigvee_{l\ne i} \sup_{s\le u\le t} \{u_i(s,t) - u_l(s,u) - BA^*)_{l,k}(u,t)\} \\
&= \hat{q}_{i,k}(s,t) \vee \tilde{q}_{i,k}(s,t),
\end{aligned}$$

where

$$\begin{aligned}
\hat{q}_{i,k}(s,t) &:= \sup_{s\le u\le t} \{u_i(u,t) - (BA^*)_{i,k}(u,t)\} \\
\tilde{q}_{i,k}(s,t) &:= \bigvee_{l\ne i} (u_i(s,t) - u_l \otimes (BA^*)_{l,k}(s,t)).
\end{aligned}$$

As for the isolated queue case, it is clear that $\hat{q}_{i,k}(s,t)$ increases to $\hat{q}_{i,k}(-\infty,t)$ as $s \to -\infty$, where

$$\hat{q}_{i,k}(-\infty,t) := \sup_{u\le t}\{u_i(u,t) - (BA^*)_{i,k}(u,t)\}.$$

In case

$$v_i < (\min_{(l,j)\in J_k, l\ne i} (v_l \wedge \beta_{l,j} \wedge \alpha^*_{j,k}) \wedge (\min_{j:(i,j)\in J_k} \beta_{i,j} \wedge \alpha^*_{j,k}),$$

then $\tilde{q}_{i,k}(s,t) \to -\infty$ as $s \to -\infty$ and $\hat{q}_{i,k}(-\infty,t) < \infty$. In case

$$v_i > (\min_{(l,j)\in J_k, l\ne i} (v_l \wedge \beta_{l,j} \wedge \alpha^*_{j,k}) \wedge (\min_{j:(i,j)\in J_k} \beta_{i,j} \wedge \alpha^*_{j,k}),$$

then either $\tilde{q}_{i,k}(s,t) \to \infty$ as $s \to -\infty$ or $\hat{q}_{i,k}(-\infty,t) = \infty$. This combined with the stationarity assumption concludes the proof.

□

10 Appendix

Lemma 24 (monotone convergence theorem) *Let a_n, $n \ge 0$ and b be functions in D. Assume that the sequence a_n is pointwise monotone decreasing (remind that this actually means increasing w.r.t. the natural order of the algebra), with limit a_∞. Then pointwise*

$$\lim_{n\to\infty} a_n \otimes b = a_\infty \otimes b, \qquad \lim_{n\to\infty} b \otimes a_n = b \otimes a_\infty.$$

Proof. Using the assumption that $a_{n+1}(s,t) \le a_n(s,t)$, it is immediate that

$$\exists \lim_{n\to\infty} a_n \otimes b(s,t) \ge a_\infty \otimes b(s,t), \qquad \forall s \le t.$$

For all $s \le t$, there exists $s \le u_0 \le t$ such that

$$a_\infty \otimes b(s,t) = a_\infty(s,u_0) + b(u_0,t).$$

But it follows from the pointwise limit assumption on a_n that for all ε, there exists N such that for all $n \geq N$, $a_n(s, u_0) \leq a_\infty(s, u_0) + \varepsilon$. Therefore for all $n \geq N$,

$$a_\infty \otimes b(s,t) \leq a_n \otimes b(s,t) \leq a_n(s,u_0) + b(u_0,t) \leq a_\infty(s,u_0) + b(u_0,t) + \varepsilon = a_\infty \otimes b(s,t) + \varepsilon,$$

which concludes the proof of the first limit. The proof of the second one is similar.

□

References

[1] R. Agrawal and R. Rajan. A general framework for analyzing schedulers and regulators in integrated services networks. In *Proc. 34th Allerton Conference*, Allerton, IL, October 1996. invited paper.

[2] R. Agrawal and R. Rajan. Performance bounds for guaranteed and adaptive services. Technical Report RC 20649, IBM Research Division, 1996.

[3] F. Baccelli. Ergodic theory of stochastic Petri networks. *Annals of Prob.*, 20:375–396, 1992.

[4] F. Baccelli, G. Cohen, G. J. Olster, and B. Gaujal. *Synchronization and linearity : an algebra for discrete event systems*. Wiley Series in probability and mathematical statistics, 1992.

[5] F. Baccelli and Z. Liu. On a class of stochastic evolution equations. *The Annals of Probability*, 20:350–374, 1992.

[6] R. Carmona and J. Lacroix. *Spectral Theory of Random Schrödinger Operators*. Birkhäuser, 1990.

[7] C. S. Chang. Matrix extensions of filtering theory for traffic regulation and service guarantees. Preprint, March 1997.

[8] C. S. Chang. On deterministic traffic regulation and service guarantee: a systematic approach by filtering. In *Proc. IEEE INFOCOM'97*, 1997.

[9] G. Cohen, D. Dubois, J.P. Quadrat, and M. Viot. A linear system-theoretic view of discrete event processes and its use for performance evaluation in manufacturing. *IEEE Trans. Aut. Contr.*, 30:210–220, 1985.

[10] R. L. Cruz. A calculus for network delay, part I: Network elements in isolation. *IEEE Trans. Inform. Theory*, 37:114–131, 1991.

[11] R. L. Cruz and C. M. Okino. Service guarantees for a flow control problem. Preprint, December 1996.

[12] J. M. Harrison and M. I. Reiman. Reflected brownian motion on an orthant. *Annals of Probability*, 9(2):302–308, 1979.

[13] J. F. C. Kingman. Subadditive ergodic theory. *Annals of Probability*, 1:883–909, 1973.

[14] J. Y. Le Boudec. Network calculus made easy. Technical Report 96/218, EPFL-DI, December 1996.

[15] Z. Liu and F. Baccelli. Generalized precedence based queueing systems. *Mathematics of Operations Research*, 17:615–639, 1992.

[16] R. M. Loynes. The stability of queues with nonindependent inter-arrival and service times. *Proceedings of Cambridge Philosophical Society*, 58:497–520, 1962.

[17] R. Rajan and R. Agrawal. Synchronous constrained fluid systems. Technical Report RC 20313, IBM Research Division, 1995.

[18] D. Stiliadis and A. Varma. Latency-rate servers: A general model for analysis of traffic scheduling algorithms. In *Proc. IEEE INFOCOM'96*, pages 111–119, San Francisco, March 1996.

Unité de recherche INRIA Sophia Antipolis
2004, route des Lucioles - B.P. 93 - 06902 Sophia Antipolis Cedex (France)

Unité de recherche INRIA Lorraine : Technopôle de Nancy-Brabois - Campus scientifique
615, rue du Jardin Botanique - B.P. 101 - 54602 Villers lès Nancy Cedex (France)
Unité de recherche INRIA Rennes : IRISA, Campus universitaire de Beaulieu - 35042 Rennes Cedex (France)
Unité de recherche INRIA Rhône-Alpes : 655, avenue de l'Europe - 38330 Montbonnot St Martin (France)
Unité de recherche INRIA Rocquencourt : Domaine de Voluceau - Rocquencourt - B.P. 105 - 78153 Le Chesnay Cedex (France)

Éditeur
INRIA - Domaine de Voluceau - Rocquencourt, B.P. 105 - 78153 Le Chesnay Cedex (France)
http://www.inria.fr
ISSN 0249-6399