# A Variational Method and its Mathematical Study in Image Sequence Analysis

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**Abstract:** This article deals with the problem of restoring and motion segmenting noisy image sequences with a static background. Usually, motion segmentation and image restoration are tackled separately in image sequence restoration. Moreover, motion segmentation is often noise sensitive. In this article, the motion segmentation and the image restoration parts are performed in a coupled way, allowing the motion segmentation part to positively influence the restoration part and vice-versa. This is the key of our approach that allows to deal simultaneously with the problem of restoration and motion segmentation. To this end, we propose a theoretically justified optimization problem that permits to take into account both requirements. The model is theoretically justified. Existence and unicity are proved in the space of bounded variations. A suitable numerical scheme based on half quadratic minimization is then proposed and its convergence and stability demonstrated. Experimental results obtained on noisy synthetic data and real images will illustrate the capabilities of this original and promising approach.

**Key-words:** Sequence image restoration, motion segmentation, discontinuity preserving regularization, variational approaches.

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# Une Méthode Variationnelle et son Etude Mathématique pour l'Analyse de Séquences d'Images

**Résumé :** Cet article traite du problème de la restauration et de la segmentation du mouvement dans des séquences d'images bruitées à fond fixe. Habituellement, la segmentation du mouvement et la restauration sont traitées de manière séparées. Dans cet article, il s'agit de le faire de manière couplée, ce qui permettra à la restauration du mouvement d'influencer positivement les résultats de la restauration et inversement. C'est l'idée maîtresse de notre approche qui permet de traiter simultanément les deux problèmes. Pour ce faire, nous proposons un problème d'optimisation que nous justifions théoriquement. L'existence et l'unicité de la solution est prouvée dans l'espace des fonctions à variations bornées. Un schéma numérique adapté, basé sur la minimisation semi-quadratique est alors proposé. Nous démontrons sa convergence et sa stabilité. Des résultats expérimentaux sur des séquences synthétiques et réelles illustreront les capacités de cette méthode originale et prometteuse.

**Mots-clés :** Restauration de séquences d'images, segmentation du mouvement, régularisation avec préservation des discontinuités, approches variationnelles.

# 1 Introduction

Automatic image sequence restoration is clearly a very important problem. Applications areas include image surveillance, forensic image processing, image compression, digital video broadcasting, digital film restoration, medical image processing, remote sensing ... See, for example, the recent work done within the European projects, fully or in part, involved with this important problem : *AURORA*, *NOBLESSE*, *LIMELIGHT*, *IMPROOFS*,... Image sequence restoration is tightly coupled to motion segmentation. It requires to extract moving objects in order to separately restore the background and each moving region along its particular motion trajectory. Most of the work done to date mainly involves motion compensated temporal filtering techniques with appropriate 2D or 3D Wiener filter for noise suppression, 2D/3D median filtering or more appropriate morphological operators for removing impulsive noise [14, 34, 35, 29, 25, 46, 17, 15]. However, and due to the fact that image sequence restoration is an emerging domain compared to 2D image restoration, the literature is not so abundant than the one related to the problem of restoring just a single image. For example, numerous PDE based algorithms have been recently proposed to tackle the problems of noise removal, 2D image enhancement and 2D image restoration in real images with a particular emphasis on preserving the grey level discontinuities during the enhancement/restoration process. These methods, which have been proved to be very efficient, are based on evolving nonlinear partial differential equations (PDE's) (See the work of Alvarez *et al* [4], Aubert *et al.* [8], Chambolle & Lions [19], Chan [12, 61] Cohen [21], Cottet and Germain [22], Kornprobst & Deriche [39, 38, 37], Malladi & Sethian [41], Mumford & Shah [59, 47], Morel [3, 45], Nordström [48], Osher & Rudin [54], Perona & Malik [52], Proesman *et al.* [53], Sapiro *et al.* [18, 55, 56, 11, 57], Weickert [65, 66], You *et al.* [68], ...).

It is the aim of this article to tackle the important problem of image sequence restoration by applying this PDE based methodology, which has been proved to be very successful in anisotropically restoring images. Therefore, considering the case of an image sequence with some moving objects, we have to consider both motion segmentation and image restoration problems. Usually, these two problems are tackled separately in image sequence restoration. However, it is clear that these two problems must be tackled simultaneously in order to achieve better results. In this article, the motion segmentation and the image restoration parts are done in a coupled way, allowing the motion segmentation part to positively influence the restoration part and vice-versa. This is the key of our approach that allows to deal simultaneously with the problem of restoration and motion segmentation.

The organization of the article is as follows.

In Sect. 2, we make some precise recalls about one of our previous approach for denoising a single image [24, 8, 38]. The formalism and the methods introduced will be very useful in the sequel.

Section 3 is then devoted to the presentation of our new approach to deal with the case of noisy images sequence. We formulate the problem into an optimization problem.

The model is theoretically justified in Sect. 4 : we prove the existence and the unicity of the solution to our problem in the space of bounded variations.

A suitable algorithm is then proposed in Sect. 5 to approximate numerically the solution. We prove its convergence and its stability.

We propose in Sect. 6 some experimental results obtained on noisy synthetic and real data that will illustrate the capabilities of this new approach.

We conclude in Sect. 7 by recalling the specificities of that work and giving the future developments.

# 2 A Variational Method for Image Restoration

In Sect. 2.1, we recall a classical method in image restoration formulated as a minimization problem [24, 10, 8]. Section 2.2 presents a suitable algorithm called the half quadratic minimization which will also be used in the sequel.

## 2.1 A Classical Approach for Image Restoration

Let $N(x_1, x_2)$ be a given noisy image defined for $x = (x_1, x_2) \in \Omega \subset R^2$ which corresponds to the domain of the image. $\nabla.$ is the gradient operator. We search for the restored image $I(x_1, x_2)$ as the solution of the following minimization problem :

$$\inf_I \underbrace{\int_\Omega (I-N)^2 dx}_{\texttt{term 1}} + \alpha^{\mathrm{r}} \underbrace{\int_\Omega \phi(|\nabla I|) dx}_{\texttt{term 2}} \tag{1}$$

where $\alpha^{\mathrm{r}}$ is a constant and $\phi$ is a function still to be defined. Notice that if $\phi(t) = t^2$, we recognize the *Tikhonov* regularization term [62]. How can we interpret this minimization with this choice? In fact, we search for the function $I$ which will be simultaneously close to the initial image $N$ and smooth (since we want the gradient as small as possible). However, this method is well known to smooth the image isotropically without preserving discontinuities in intensity. The reason is that with the quadratic function, gradients are too much penalized. One solution to prevent the destruction of discontinuities but allows for isotropically smoothing uniform areas, is to change the above quadratic term. This point have been widely discussed [58, 60, 10, 8]. We refer to [24] for a review. The key idea is that for low gradients, isotropic smoothing is performed, and for high gradient, smoothing is only applied in the direction of the isophote and not across it. This condition can be mathematically formalized if we look at the Euler-Lagrange Equation (2), associated to energy (1) :

$$2(I-N) - \alpha^{\mathrm{r}} div \left( \frac{\phi'(\mid \nabla I \mid)}{\mid \nabla I \mid} \nabla I \right) = 0 \tag{2}$$

Notice that Neumann conditions are imposed on the boundaries. Let us concentrate on the regularization part associated to the `term 2` of (1). If we note $\eta = \frac{\nabla I}{|\nabla I|}$, and $\xi$ the normal vector to $\eta$, we can show that :

$$div\left(\frac{\phi'(\mid \nabla I \mid)}{\mid \nabla I \mid}\nabla I\right) = \underbrace{\frac{\phi'(\mid \nabla I \mid)}{\mid \nabla I \mid}}_{c_\xi} I_{\xi\xi} + \underbrace{\phi''(\mid \nabla I \mid)}_{c_\eta} I_{\eta\eta} \tag{3}$$

where $I_{\eta\eta}$ (respectively $I_{\xi\xi}$) denotes the second order derivate in the direction $\eta$ (respectively $\xi$). If we want a good restoration as described before, we would like to have the following properties :

$$\lim_{|\nabla I| \to 0} c_\eta = \lim_{|\nabla I| \to 0} c_\xi = a_0 > 0 \tag{4}$$

$$\lim_{|\nabla I| \to \infty} c_\eta = 0 \quad \text{and} \quad \lim_{|\nabla I| \to \infty} c_\xi = a_\infty > 0 \tag{5}$$

But it is clear that the two conditions in (5) are incompatible. So, we will only impose for high gradients [24, 10, 8] :

$$\lim_{|\nabla I| \to \infty} c_\eta = \lim_{|\nabla I| \to \infty} c_\xi = 0 \quad \text{and} \quad \lim_{|\nabla I| \to \infty} \left(\frac{c_\eta}{c_\xi}\right) = 0 \tag{6}$$

Many functions $\phi$ have been proposed in the literature that comply to the conditions (4) and (6) (see [24]). From now on, $\phi$ will be a convex function with linear growth at infinity which verifies conditions (4) and (6). For instance, a possible choice could be the hypersurface minimal function proposed by Aubert :

$$\phi(t) = \sqrt{1+t^2} - 1$$

In that case, existence and unicity of problem (1) has recently been shown in the Sobolev space $W^{1,1}(\Omega)$[10] (See also [63]).

## 2.2 The Half Quadratic Minimization

Solving directly the minimization problem (1) by solving directly its Euler Lagrange equation (2), is something hard because this equation is highly non linear.

To overcome the difficulty, the key idea is to introduce a new functional which, although defined over an extended domain, has the same minimum in $I$ and can be manipulated with linear algebraic methods. The method is based on the half quadratic minimization theorem, inspired from Geman and Reynolds [28]. The general idea is that under some hypotheses on $\phi$ (mainly $\phi(\sqrt{t})$ strictly concave), we can write it as an infimum :

$$\phi(t) = \inf_d (dt^2 + \psi(d)) \tag{7}$$

where $d$ will be called the *dual variable* associated to $x$, and where $\psi(\cdot)$ is a strictly convex and decreasing function. We refer to the Appendix A for more details. We can verify that the functions $\phi$ such that (4) (6) are true can be written as in (7). Consequently, the problem (1) is now to find $I$ and its dual variable $d_{\rm I}$ minimizing the functional $\mathcal{F}(I, d_{\rm I})$ defined by :

$$\mathcal{F}(I, d_{\rm I}) = \int_\Omega (I-N)^2 dx + \alpha^{\rm r} \int_\Omega (d_{\rm I}|\nabla I|^2 + \psi(d_{\rm I}))dx \tag{8}$$

It is easy to check that for a fixed $I$, the functional $\mathcal{F}$ is convex in $d_{\rm I}$ and for a fixed $d_{\rm I}$, it is convex in $I$. These properties are used to perform the algorithm which consists in minimizing alternatively in $I$ and $d_{\rm I}$ :

$$I^{\rm n+1} = \underset{I}{\operatorname{argmin}} \quad \mathcal{F}(I, d_{\rm I}^{\rm n}) \tag{9}$$

$$d_{\rm B}^{\rm n+1} = \underset{d_{\rm I}}{\operatorname{argmin}} \quad \mathcal{F}(I^{\rm n+1}, d_{\rm I}) \tag{10}$$

To perform each minimization, we simply solve the Euler-Lagrange equations, which can be written as :

$$I^{\rm n+1} - N - \alpha^{\rm r} div(d_{\rm I}^{\rm n} \nabla I^{\rm n+1}) = 0 \tag{11}$$

$$d_{\rm I}^{\rm n+1} = \frac{\phi'(|\nabla I^{\rm n+1}|)}{2|\nabla I^{\rm n+1}|} \tag{12}$$

with discretized Neumann conditions at the boundaries. Notice that (12) gives explicitly $d_{\rm I}^{\rm n+1}$ while for (11), for a fixed $d_{\rm I}^{\rm n}$, $I^{\rm n+1}$ is the solution of a linear equation. After discretizing in space, we have that $(I_{i,j}^{\rm n+1})_{(i,j)\in\Omega}$ is solution of a linear system which is solved iteratively by the Gauss-Seidel method for example. We refer to the Appendix B for more details about the discretization of the divergence operator.

# 3 Dealing with Noisy Images Sequences

Let $N(x_1, x_2, t)$ denotes the noisy images sequence for which the background is assumed to be static. A simple moving object detector can be obtained using a thresholding technique over the *inter-frame difference* between a so-called *reference image* and the image being observed. Decisions can be taken independently point by point [67]. More complex approaches can also be used [49, 51, 50, 1, 32, 40, 14, 34, 35, 29, 25, 46]. However, in our application, we are not just dealing with a motion segmentation problem neither just a restoration problem. In our case, the so-called *reference image* is built at the same time while observing the image sequence. Also, the motion segmentation and the restoration are done in a coupled way, allowing the motion segmentation part to positively influence the restoration part and vice-versa. This is the key of our approach that allows to deal simultaneously with the problem of restoration and motion segmentation.

We first consider that the data is continuous in time. This permits us to present the optimization problem we want to study (Sect. 3.1). In Sect. 3.2, we rewrite the problem when the sequence is given only by a finite set of images. This leads to the Problem 2.

### 3.1 An Optimization Problem

Let $N(x_1, x_2, t)$ denotes the noisy images sequence for which the background is assumed to be static. Let us describe the unknown functions and what we would like them ideally to be :

(i) $B(x_1, x_2)$, the restored background,

(ii) $C(x_1, x_2, t)$, the sequence which will indicate the moving regions. Typically, we would like that $C(x_1, x_2, t) = 0$ if the pixel $(x_1, x_2)$ belongs to a moving object at time $t$, and 1 otherwise.

Our aim is to find a functional depending on $B(x_1, x_2)$ and $C(x_1, x_2, t)$ so that the minimizers verify previous statements. We propose to solve the following problem :

**Problem 1.** Let $N(x_1, x_2, t)$ the given noisy image sequence. We search for the restored background $B(x_1, x_2)$ and the motion segmented sequence $C(x_1, x_2, t)$ as the solution of the following minimization problem :

$$\inf_{B,C} \Big( \underbrace{\int_t \int_\Omega C^2 (B-N)^2 dx dt}_{\texttt{term 1}} + \alpha_{\mathrm{c}} \underbrace{\int_t \int_\Omega (C-1)^2 dx dt}_{\texttt{term 2}} \\ + \underbrace{\alpha_{\mathrm{b}}^{\mathrm{r}} \int_\Omega \phi_1(|\nabla B|) dx + \alpha_{\mathrm{c}}^{\mathrm{r}} \int_t \int_\Omega \phi_2(|\nabla C|) dx dt}_{\texttt{term 3}} \Big) \tag{13}$$

where $\phi_1$ and $\phi_2$ are convex functions that comply conditions (4) and (6) , and $\alpha_{\mathrm{c}}, \alpha_{\mathrm{b}}^{\mathrm{r}}, \alpha_{\mathrm{c}}^{\mathrm{r}}$ are positive constants. We will specify later the spaces over which the minimization runs.

Getting the minimum of the functional means that we want each term to be small, having in mind the phenomena of the compensations.

The `term 3` is a regularization term. Notice that the functions $\phi_1$,$\phi_2$ have been chosen as in Sect. 2 so that discontinuities may be kept.

If we consider the `term 2`, this means that we want the function $C(x_1, x_2, t)$ to be close to one. In our interpretation, this means that we give a preference to the background. This is physically correct since the background is visible most of the time. However, if the data $N(x_1, x_2, t)$ is too far from the supposed background $B(x_1, x_2)$ at time $t$, then the difference $(B(x_1, x_2) - N(x_1, x_2, t))^2$ will be high, and to compensate this value, the minimization process will force $C(x_1, x_2, t)$ to be zero. Therefore, the function $C(x_1, x_2, t)$ can be interpretated as a motion detection function. Moreover, when searching for $B(x_1, x_2)$, we will not take into account $N(x_1, x_2, t)$ if $C(x_1, x_2, t)$ is small (`term 1`). This exactly means that $B(x_1, x_2)$ will be the restored image of the static background.

### 3.2 The Temporal Discretized Problem

In fact, we have only a finite set of images. Consequently, we are going to rewrite the Problem 1, taking into account that the sequence $N(x_1, x_2, t)$ is represented during a finite time by $T$ images noted $N_1(x_1, x_2), \ldots, N_{\mathrm{T}}(x_1, x_2)$. The Problem 1 becomes :

**Problem 2.** Let $N_1, \ldots, N_{\mathrm{T}}$ be the noisy sequence. We search for $B$ and $C_1, \ldots, C_{\mathrm{T}}$ as the solution of the following minimization problem :

$$\inf_{B, C_1, \ldots, C_{\mathrm{T}}} \Big( \underbrace{\sum_{h=1}^{T} \int_\Omega C_{\mathrm{h}}^2 (B - N_{\mathrm{h}})^2 dx}_{\texttt{term 1}} + \underbrace{\alpha_{\mathrm{c}} \sum_{h=1}^{T} \int_\Omega (C_{\mathrm{h}} - 1)^2 dx}_{\texttt{term 2}} \\ \underbrace{+ \alpha_{\mathrm{b}}^{\mathrm{r}} \int_\Omega \phi_1(|\nabla B|) dx + \alpha_{\mathrm{c}}^{\mathrm{r}} \sum_{h=1}^{T} \int_\Omega \phi_2(|\nabla C_{\mathrm{h}}|) dx}_{\texttt{term 3}} \Big) \tag{14}$$

Before going further, one may be interested in the link between this method and the variational method developed for image restoration in section 2. To this end, let us consider a sequence of the same noisy image. More generally, we can consider a sequence of noisy images where no moving objects are present. If we admit the interpretation of the functions $C_{\mathrm{h}}$, we will have $C_{\mathrm{h}} \equiv 1$. After few computations, (14) may be re-written :

$$\inf_B \Big( \int_\Omega (B - \frac{1}{T} \sum_{h=1}^{T} N_{\mathrm{h}})^2 dx + \frac{\alpha_{\mathrm{b}}^{\mathrm{r}}}{T} \int_\Omega \phi_1(|\nabla B|) dx \Big)$$

Consequently, if we observe the energy (1) proposed for the image restoration problem, we can consider $B$ as the restored version of the mean in time of the sequence. Notice that if the sequence has been obtained as a multiplication of the same image, both methods correspond exactly. Therefore, this model devoted to sequences of images can be considered as a natural extension of the previous one for single image restoration.

Now that we have justified the proposed model, let us prove that it is mathematically well posed. It is the purpose of the next section.

## 4 A Rigorously Justified Approach in The Space of Bounded Variations

Section 4.1 presents the mathematical background of our problem : the space of bounded variations which is suitable to most problems in vision [54, 20]. Roughly speaking, the idea is to generalize the classical Sobolev space $W^{1,1}(\Omega)$ so that discontinuities along hypersurfaces may be considered. After having precisely specified the problem in Sect. 4.2, we first prove the exitence of a solution in a constrained space (See Sect. 4.3. Using this result, we finally prove the existence and the unicity of a solution over the space in bounded variations in Sect. 4.4.

## 4.1 The Space $BV(\Omega)$ : a Short Overview

Let $\Omega$ be a bounded open set in $R^N$, we denote by $\mathcal{L}^{\rm n}$ or $dx$ the N-Lebesgue dimensional measure in $R^N$ and by $\mathcal{H}^\alpha$ the $\alpha-$dimensional Hausdorff measure. We also set $|E| = \mathcal{L}_N(E)$, the Lebesgue measure of a measurable set $E \subset R^N$. $\mathcal{B}(\Omega)$ denotes the family of the Borel subsets of $\Omega$. We will respectively denote the strong, the weak and the weak$\star$ convergences in a space $V(\Omega)$ by $\underset{V(\Omega)}{\longrightarrow}$, $\underset{V(\Omega)}{\rightharpoonup}$, $\underset{V(\Omega)}{\overset{\star}{\rightharpoonup}}$.

Let $\mathcal{C}_c^0(\Omega; R^{\rm n})$ be the space of continuous functions with compact support in $\Omega$. This space is a Banach space endowed with the norm $\|u\|_{\mathcal{C}_c^0(\Omega;R^{\rm n})} = \max_{x\in\Omega}|u(x)|$. The dual of $\mathcal{C}_c^0(\Omega; R^{\rm n})$ is the space of $R^N$ vector-valued Radon measures on $\Omega$ denoted $\mathcal{M}(\Omega)$. The total variation of $\mu = (\mu_1, \dots, \mu_N)$, for a Borel subset $A$, is denoted $|\mu|(A)$ and its derivative with respect to the measure $\nu$ is denoted $\dfrac{d\mu}{d\nu}(x)$.

As a natural extension of the space

$$W^{1,1}(\Omega) = \left\{u/u \in L^1(\Omega) \text{ and } Du \in (L^1(\Omega))^N\right\},$$

we usually propose the space of bounded variations, noted $BV(\Omega)$, the space of $L^1$-functions whose distributionnal derivatives belong to $\mathcal{M}(\Omega)$. We refer to [2, 26, 30, 27] for the complete theory.

The product topology of the strong topology of $L^1(\Omega)$ for $u$ and of the weak$\star$ topology of measures for $Du$ will be called the weak$\star$ topology of $BV$, and will be denoted by $BV - w*$.

$$u^n \underset{BV-w*}{\longrightarrow} u \iff \left\{ \begin{array}{ccc} u^n & \underset{L^1(\Omega)}{\longrightarrow} & u \\ |Du^n| & \rightarrow & |Du| \end{array} \right. \tag{15}$$

We recall that every bounded sequence in $BV(\Omega)$ admits a subsequence converging in $BV-w*$. Notice that we do not assert that $|Du^n - Du| \to 0$. Moreover,this sequence is also relatively compact in $L^p(\Omega)$ for $1 \le p < \frac{N}{N-1}$ and $N \ge 1$, and relatively weakly compact in $L^p(\Omega)$ for $p = \frac{N}{N-1}$ and $N \ge 2$ (Giusti [30], Acart-Vogel [2]). This is traduced by the following notations :

$$BV(\Omega) \underset{\text{strong}}{\hookrightarrow} L^p(\Omega) \quad \text{for} \quad 1 \le p < \frac{N}{N-1} \text{ and } \ N \ge 1 \tag{16}$$

$$BV(\Omega) \underset{\text{weak}}{\hookrightarrow} L^{\frac{N}{N-1}}(\Omega) \subset L^1(\Omega) \quad \text{for} \ \ N \ge 2 \tag{17}$$

We represent by $Du$ the distributional derivative of $u$ and by $\nabla u$ the density of the absolutely continuous part of $Du$ with respect to the Lebesgue measure.

We recall that the approximate limit of $u$ is defined by :

$$\lim_{r\to 0^+} r^{-N}\int_{B(x,r)} |u(y)-\tilde{u}(x)|dy = 0 \tag{18}$$

where $B(x,r)$ is the closed ball with center $x$ and radius $r$. The set of points where we can define $\tilde{u}$ is called the *Lebesgue set*. We denote by $S_u$ the complement of the Lebesgue set of $u$ which is Lebesgue negligible. For $x \in S_u$, it is possible to find $u^+(x), u^-(x) \in R$ unique, with $u^+(x) > u^-(x)$ and $n_u \in S^{N-1}$ such that :

$$\begin{aligned}
&\lim_{r\to 0^+} r^{-N}\int_{B^{+n_u}(x,r)} |u(y)-u^+(x)|dy = \\
&\lim_{r\to 0^+} r^{-N}\int_{B^{-n_u}(x,r)} |u(y)-u^-(x)|dy = 0
\end{aligned}$$

where $B^{+n_u}(x,r) = \{y \in B(x,r) : (y-x)\cdot n_u > 0\}$ and $B^{-n_u}(x,r) = \{y \in B(x,r) : (y-x)\cdot n_u < 0\}$. We suppose that the normal $n_u$ "points towards the largest value" of $u$. We recall the following decompositions :

$$Du = \nabla u \cdot \mathcal{L}_N + C_u + (u^+ - u^-)n_u \cdot \mathcal{H}^{N-1}_{|S_u} \tag{19}$$

$$|Du|(\Omega) = \int_\Omega |\nabla u|dx + \int_{\Omega\setminus S_u} |C_u| + \int_{S_u} (u^+ - u^-)d\mathcal{H}^{N-1} \tag{20}$$

where $d\mathcal{H}^{\mathrm{N}-1}$ is the Hausdorff measure of dimension $N-1$ and $C_u$ is the Cantor part of the distributional derivative $Du$ (see [5] for more details). We will sometimes write it by $C(u)$ for clearer notations when no doubt is possible.

We then recall the definition of a convex function of measures. We refer to the works of Goffman-Serrin [31] and Demengel-Temam [23] for more details. Let $\phi$ be convex and finite over $R$ with a linear growth to infinity. Let $\phi^\infty$ be the asymptote (or recession) function defined by :

$$\phi^\infty(z) := \lim_{t\to\infty} \frac{\phi(tz)}{t} \in [0; +\infty).$$

Then, for $u \in BV(\Omega)$, using a classical notation, we can define the notion of convex functions of measure by :

$$\begin{aligned}
\int_\Omega \phi(Du) = &\int_\Omega \phi(|\nabla u|)dx + \\
&\phi^\infty(1)\int_{S_u} (u^+ - u^-)d\mathcal{H}^{N-1} + \phi^\infty(1)\int_{\Omega\setminus S_u} |C_u|
\end{aligned} \tag{21}$$

We recall that $\displaystyle\int_\Omega \phi(Du)$ is lower semi-continuous for the $BV - w\star$ topology.

## 4.2 Setting the problem

Let us recall the problem. Notice that the derivatives will be now considered as distributional derivatives. Consequently, the problem is to minimize over $BV(\Omega)^{\mathrm{T}+1}$ the functional $E$ defined by :

$$\begin{aligned} E(B, C_1, .., C_{\mathrm{T}}) = & \sum_{h=1}^{T} \int_\Omega C_{\mathrm{h}}^2 (B - N_{\mathrm{h}})^2 dx + \alpha_{\mathrm{c}} \sum_{h=1}^{T} \int_\Omega (C_{\mathrm{h}} - 1)^2 dx \\ & + \alpha_{\mathrm{b}}^{\mathrm{r}} \int_\Omega \phi_1(DB) + \alpha_{\mathrm{c}}^{\mathrm{r}} \sum_{h=1}^{T} \int_\Omega \phi_2(DC_{\mathrm{h}}) \end{aligned} \tag{22}$$

where the precise hypotheses on the functions $(\phi_{\mathrm{j}})_{j=1,2}$ are :

$$\begin{aligned} & \phi_j : R \rightarrow R^+ \text{ is an even and strictly convex function, nondecreasing} \\ & \text{on } R^+ \text{ and there exist constants } c > 0 \text{ and } b \geq 0 \text{ such that} \\ & cx - b \leq \phi_j(x) \leq cx + b \text{ for all } x \in R^+ && (23) \\ & \phi(0) = 0, \quad \phi_j^\infty(1) = 1 && (24) \end{aligned}$$

We recall that the terms $\int_\Omega \phi(DB)$ and $\int_\Omega \phi(DC_{\mathrm{h}})$ are defined as convex functions of measure (see (21)). As for the data $(N_{\mathrm{h}})_{h=1..T}$, we will assume that :

$$N_{\mathrm{h}} \in BV(\Omega) \cap L^\infty(\Omega) \quad \forall h = 1..T \tag{25}$$

There exist two finite constants $m_{\mathrm{N}}$ and $M_{\mathrm{N}}$ defined by:

$$\left\{ \begin{array}{l} m_{\mathrm{N}} = \underset{h \in [0..T], (x_1, x_2) \in \Omega}{ess-inf} N_{\mathrm{h}}(x_1, x_2) \\ M_{\mathrm{N}} = \underset{h \in [0..T], (x_1, x_2) \in \Omega}{ess-sup} N_{\mathrm{h}}(x_1, x_2) \end{array} \right. \tag{26}$$

where $ess - inf$ (resp. $sup - ess$) is the essential infimum (resp. supremum).

## 4.3 Existence of a solution in a constrained space

The aim of this section is to show that the minimization problem :

$$\inf_{(B, C_1, .., C_{\mathrm{T}}) BV(\Omega)^{\mathrm{T}+1}} E(B, C_1, .., C_{\mathrm{T}}) \tag{27}$$

admits a solution in $BV(\Omega)^{\mathrm{T}+1}$. First of all, we are going to show that we can find a solution on a restricted space $\overline{\mathcal{E}}(\Omega)$ defined by :

$$\begin{aligned} \overline{\mathcal{E}}(\Omega) = \{ & (B, C_1, .., C_{\mathrm{T}}) \in BV(\Omega)^{\mathrm{T}+1} \text{ such that:} \\ & m_{\mathrm{N}} \leq B \leq M_{\mathrm{N}} \text{ a.e. and } 0 \leq C_{\mathrm{h}} \leq 1 \text{ a.e. } \forall h \} \end{aligned} \tag{28}$$

**Theorem 1** *Given a sequence of images $N_{\mathrm{h}}$ verifying (25)-(26), the minimization problem :*

$$\inf_{(B,C_1,..,C_{\mathrm{T}})\in\overline{\mathcal{E}}(\Omega)} E(B,C_1,..,C_{\mathrm{T}}) \tag{29}$$

*where $\phi_j$ verify (23)-(24), admits a solution in the set $\overline{\mathcal{E}}(\Omega)$.*

**Preuve** Let $(B^{\mathrm{n}},C_1^{\mathrm{n}},..,C_{\mathrm{T}}^{\mathrm{n}})\in\overline{\mathcal{E}}(\Omega)$ be a minimizing sequence of $E$. Since all the terms in $E$ are non negative, and thanks to (23), we have :

$$|DB^{\mathrm{n}}|\leq M \quad \text{and} \quad |DC_{\mathrm{h}}^{\mathrm{n}}|\leq M \quad \forall n \ \ \forall h$$

where $M$ is a constant which can be different from one line to another. Since the sequence belongs to $\overline{\mathcal{E}}(\Omega)$, we also have a $L^\infty$-majoration. Since $\Omega$ is bounded, the sequences $B^{\mathrm{n}}$ and $C_{\mathrm{h}}^{\mathrm{n}}$ are bounded in $L^p(\Omega)$ for every $p$ and especially $p=1,2,\infty$. Thanks to the compactness result (17), we can extract a subsequence again noted $(B^{\mathrm{n}},C_1^{\mathrm{n}},..,C_{\mathrm{T}}^{\mathrm{n}})$ converging to some $(\overline{B},\overline{C_1},..,\overline{C_{\mathrm{T}}})$ in $\overline{\mathcal{E}}(\Omega)$ for the topology $BV-w\star\times(BV-w\star L^2weak)^T$. Notice that the limit is effectively in $\overline{\mathcal{E}}(\Omega)$ because $\overline{\mathcal{E}}(\Omega)$ is weakly star closed. In other words, we can pass to the limit in the constraints of the space $\overline{\mathcal{E}}(\Omega)$. We also have :

$$\lim_{n\to\infty}\int_\Omega \phi_1(DB^{\mathrm{n}})\geq\int_\Omega\phi_1(D\overline{B}) \tag{30}$$

$$\lim_{n\to\infty}\int_\Omega \phi_2(DC_{\mathrm{h}}^{\mathrm{n}})\geq\int_\Omega\phi_2(D\overline{C_{\mathrm{h}}}) \tag{31}$$

$$\lim_{n\to\infty}\int_\Omega (C_{\mathrm{h}}^{\mathrm{n}}-1)^2dx\geq\int_\Omega(\overline{C_{\mathrm{h}}}-1)^2dx \tag{32}$$

But what can be said about :

$$\lim_{n\to\infty}\int_\Omega \underbrace{{C_{\mathrm{h}}^{\mathrm{n}}}^2(B^{\mathrm{n}}-N_{\mathrm{h}})^2}_{w_{\mathrm{h}}^{\mathrm{n}}}dx$$

Since we have the $L^1$-strong convergence for all the variables (17), we also have the pointwise convergence (up to a subsequence). So we have :

$$w_{\mathrm{h}}^{\mathrm{n}}\to\overline{C_{\mathrm{h}}}^2(\overline{B}-N_{\mathrm{h}})^2 \quad \text{a.e. on } \Omega$$

Moreover, thanks to the $L^\infty$ uniform bounds of $B^{\mathrm{n}}$ and $C_{\mathrm{h}}^{\mathrm{n}}$, we have in fact :

$$|w_{\mathrm{h}}^{\mathrm{n}}(x)|\leq M \quad \text{a.e.}$$

A direct application of the Lebesgue dominated convergence theorem permits us to pass to the limit :

$$\lim_{n\to\infty}\sum_{h=1}^{T}\int_\Omega {C_{\mathrm{h}}^{\mathrm{n}}}^2(B^{\mathrm{n}}-N_{\mathrm{h}})^2dx=\sum_{h=1}^{T}\int_\Omega\overline{C_{\mathrm{h}}}^2(\overline{B}-N_{\mathrm{h}})^2dx \tag{33}$$

So, using Equations (30)-(33) permits us to write :

$$\liminf_{n\to\infty} E(B^{\mathrm{n}}, C_1^{\mathrm{n}}, .., C_{\mathrm{T}}^{\mathrm{n}}) \geq E(\overline{B}, \overline{C_1}, .., \overline{C_{\mathrm{T}}})$$

This concludes the proof. ■

### 4.4 Existence and unicity of a solution over $BV(\Omega)$

The previous theorem establishes the existence of a solution on a restricted space. However, this result is not satisfying because working in a constrained space is not easy to handle because the optimality conditions are inequations and not equations. In fact, even if these constraints are natural (with regard to the interpretation of the variables), we would like to avoid them. This is the aim of Theorem 2 but we first need a preliminary result :

**Lemma 1** *Let $u \in BV(\Omega)$, $\phi$ a function verifying hypotheses (23)-(24), and $\varphi_{\alpha,\beta}$ the cut-off function defined by :*

$$\varphi_{\alpha,\beta}(x) = \begin{cases} \alpha & \textit{if} \quad x \leq \alpha \\ x & \textit{if} \quad \alpha \leq x \leq \beta \\ \beta & \textit{if} \quad x \geq \beta \end{cases} \tag{34}$$

*Then we have :*

$$\int_\Omega \phi(D\varphi_{\alpha,\beta}(u)) \leq \int_\Omega \phi(Du)$$

**Preuve** Let us first recall the Lebesgue decomposition of the measure $\phi(Du)$ :

$$\int_\Omega \phi(Du) = \underbrace{\int_\Omega \phi(|\nabla u|)dx}_{\texttt{term 1}} + \underbrace{\int_{S_u} |u^+ - u^-| d\mathcal{H}^{\mathrm{N}-1}}_{\texttt{term 2}} + \underbrace{\int_{\Omega/S_u} |C_u|}_{\texttt{term 3}}$$

We are going to show that cutting the fonction $u$ using the fonction $\varphi_{\alpha,\beta}$ permits to reduce each term. To simplify notations, we will sometimes use the notation $\hat{u}$ for the troncated function $\varphi_{\alpha,\beta}(u)$

`Term 1`: let $\Omega_{\mathrm{c}} = \{x \in \Omega / u(x) \leq \alpha \ \text{ or } \ u(x) \geq \beta\}$ and $\Omega_i = \Omega/\Omega_{\mathrm{c}}$. Thanks to [33], we have $\int_{\Omega_i} \phi(|\nabla \hat{u}|)dx = \int_{\Omega_i} \phi(|\nabla u|)dx$. Consequently :

$$\int_\Omega \phi(|\nabla \hat{u}|)dx = \int_{\Omega_i} \phi(|\nabla u|)dx + \int_{\Omega_{\mathrm{c}}} \underset{(\equiv 0)}{\phi(|\nabla \hat{u}|)}dx \leq \int_\Omega \phi(|\nabla u|)dx \tag{35}$$

`Term 2:` using results proved in [5], we know that :

$$\begin{aligned} &S_{\hat{u}} \subset S_u \\ &\hat{u}^+ = \varphi_{\alpha,\beta}(u^+) \quad \text{and} \quad \hat{u}^- = \varphi_{\alpha,\beta}(u^-) \end{aligned}$$

Thanks to these results, and since $\varphi_{\alpha,\beta}$ is Lipschitz continuous with a constant equals to 1, we have :

$$\int_{S_{\hat{u}}} |\hat{u}^+ - \hat{u}^-| d\mathcal{H}^{\mathrm{N}-1} \leq \int_{S_{\hat{u}}} |u^+ - u^-| d\mathcal{H}^{\mathrm{N}-1} \leq \int_{S_u} |u^+ - u^-| d\mathcal{H}^{\mathrm{N}-1} \tag{36}$$

`Term 3:` we need to understand how is the Cantor part of the distributional derivative of the composed function $\varphi_{\alpha,\beta}(u)$. Vol'pert [64] first proposed a chain rule formula for functions $v = \varphi(u)$ for $u \in BV(\Omega)$ and when $\varphi$ is continuously differentiable. Ambrosio and Dal Maso [6] gave extended results for functions $\varphi$ uniformely Lipschitz continuous. Since $u$ is scalar, it is demonstrated in [6] that we can write :

$$C(\varphi_{\alpha,\beta}(u)) = \varphi'_{\alpha,\beta}(\tilde{u})C(u) \quad |Du| \quad \text{a.e. on} \quad \Omega/S_u \tag{37}$$

where $\tilde{u}$ is the approximate limit of $u$ (see Eq. 18). Moreover, we have :

$$\int_{\Omega/S_{\hat{u}}} |C_{\widehat{u}}| = \int_{\Omega/S_u} |C_{\widehat{u}}| + \underbrace{\int_{S_u/S_{\hat{u}}} |C_{\widehat{u}}|}_{(\equiv 0)} \tag{38}$$

Notice that the second integral equals to zero because the Hausdorff dimension of the set $S_u/S_{\hat{u}}$ is at most $N-1$ and we know that for any $v \in BV(\Omega)$ and any set $S$ of Hausdorff dimension at most $N-1$, we have $C_v(S) = 0$. Then, using the chain rule formula (37), we have :

$$\int_{\Omega/S_{\hat{u}}} |C_{\widehat{u}}| \leq \underbrace{\|\varphi'_{\alpha,\beta}\|_{L^\infty}}_{(\leq 1)} \int_{\Omega/S_u} |C_u| \leq \int_{\Omega/S_u} |C_u| \tag{39}$$

Finally, using results (35), (36), (39) permits to write :

$$\int_\Omega \phi(D\varphi_{\alpha,\beta}(u)) \leq \int_\Omega \phi(Du)$$

This concludes the proof. ■

Now we are going to prove that the minimization problem (29) over $\overline{\mathcal{E}}(\Omega)$ is equivalent to the same problem posed over $BV(\Omega)^{\mathrm{T}+1}$, that is to say without any constraint. This remark will permit us to prove the existence of a solution. As for unicity, the difficulty comes from the apparent non convexity of the function :

$$(B, C_1, .., C_{\mathrm{T}}) \to \sum_{h=1}^{T} C_{\mathrm{h}}^2 (B - N_{\mathrm{h}})^2 + \alpha_{\mathrm{c}} \sum_{h=1}^{T} (C_{\mathrm{h}} - 1)^2$$

with respect to all variables (Notice that it is convex with respect to each variable). However, if $\alpha_{\mathrm{C}}$ is large enough, we will prove that this functional is convex over $\overline{\mathcal{E}}$.

**Theorem 2** *Under hypotheses (23)-(24)and (25)-(26), the minimization problem :*

$$\inf_{(B,C_1,..,C_{\mathrm{T}})\in BV(\Omega)^{\mathrm{T}+1}} E(B,C_1,..,C_{\mathrm{T}}) \tag{40}$$

*admits a solution in* $BV(\Omega)^{\mathrm{T}+1}$. *If moreover :*

$$\alpha_{\mathrm{C}} \geq 3(M_{\mathrm{N}}-m_{\mathrm{N}})^2 \tag{41}$$

*where the constants* $m_{\mathrm{N}}, M_{\mathrm{N}}$ *are defined by (26), then the solution is unique.*

**Preuve** Let us first show the existence.

`Existence :` Let $\mathbf{V}=(B,C_1,..,C_{\mathrm{T}})\in BV(\Omega)^{\mathrm{T}+1}$. The idea is to show that the vector valued cut-off function :

$$\varphi(\mathbf{V})\equiv(\varphi_{\mathrm{m_N},\mathrm{M_N}}(B),\varphi_{0,1}(C_1),..,\varphi_{0,1}(C_{\mathrm{T}}))$$

where $\varphi_{0,1}$ and $\varphi_{\mathrm{m_N},\mathrm{M_N}}$ are the cut-off functions defined as (34), belongs to $BV(\Omega)^{\mathrm{T}+1}$ and that :

$$E(\varphi(\mathbf{V}))\leq E(\mathbf{V}) \quad \forall \mathbf{V}\in BV(\Omega)^{\mathrm{T}+1} \tag{42}$$

This means that starting from any $\mathbf{V}$ and using cut-off functions, permits to diminish the energy $E$. As a consequence, the constraints proposed in the space $\overline{\mathcal{E}}$ are naturally checked. So we can conclude that the two minimization problems (29) and (40) are actually the same so there exists a solution to the problem (40).

Let us show inequality (42). Let $\mathbf{V}$ fixed. We are going to use cut-off fonctions, components per components. The first question is : what can we say about $E(\varphi_{\mathrm{m_N},\mathrm{M_N}}(B),C_1,..,C_{\mathrm{T}})$? Thanks to Lemma 1, we know that :

$$\int_\Omega \phi(D\varphi_{\mathrm{m_N},\mathrm{M_N}}(B))\leq\int_\Omega\phi(DB) \tag{43}$$

Noting $\left\{\begin{array}{l}\Omega^-=\{x/B(x)\leq m_{\mathrm{N}}\}\\ \Omega^+=\{x/B(x)\geq M_{\mathrm{N}}\}\\ \Omega^0=\{x/m_{\mathrm{N}}\leq B(x)\leq M_{\mathrm{N}}\}\end{array}\right.$ , we have :

$$\begin{aligned}\sum_{h=1}^T\int_\Omega & C_{\mathrm{h}}^2(\varphi_{\mathrm{m_N},\mathrm{M_N}}(B)-N_{\mathrm{h}})^2=\\ &=\sum_{h=1}^T\int_{\Omega^0}C_{\mathrm{h}}^2(B-N_{\mathrm{h}})^2dx+\sum_{h=1}^T\int_{\Omega^-}C_{\mathrm{h}}^2(m_{\mathrm{N}}-N_{\mathrm{h}})^2dx+\sum_{h=1}^T\int_{\Omega^+}C_{\mathrm{h}}^2(M_{\mathrm{N}}-N_{\mathrm{h}})^2dx\\ &\leq\sum_{h=1}^T\int_\Omega C_{\mathrm{h}}^2(B-N_{\mathrm{h}})^2dx\end{aligned} \tag{44}$$

Equations (43) and (44) permits to write :

$$E(\varphi_{\mathrm{m_N},\mathrm{M_N}}(B), C_1, .., C_{\mathrm{T}}) \leq E(B, C_1, .., C_{\mathrm{T}})$$

The effect of using the cut-off functions for variables $C_{\mathrm{h}}$ can be proved without any difficulty using same ideas. Consequently, (42) is proved and so the existence.

`Unicity :` Let $\overline{\mathbf{V}} = (\overline{B}, \overline{C_1}, .., \overline{C_{\mathrm{T}}})$ and $\hat{\mathbf{V}} = (\widehat{B}, \widehat{C_1}, .., \widehat{C_{\mathrm{T}}})$ be two distinct solutions of the minimization problem (40). We then consider the vector valued function :

$$\mathbf{V}^\theta = (\underbrace{\theta\overline{B} + (1-\theta)\widehat{B}}_{\equiv B^\theta}\ ,\ \underbrace{\theta\overline{C_1} + (1-\theta)\widehat{C_1}}_{\equiv C_1^\theta}\ , .., \ \underbrace{\theta\overline{C_{\mathrm{T}}} + (1-\theta)\widehat{C_{\mathrm{T}}}}_{\equiv C_{\mathrm{T}}^\theta})$$

Our aim is to prove that :

$$E(\mathbf{V}^\theta) < \theta E(\overline{\mathbf{V}}) + (1-\theta)E(\hat{\mathbf{V}}) \tag{45}$$

If we admit (45), then the result is demontrated because the right-hand side of (45) equals to $min(E)$ and then we have :

$$E(\mathbf{V}^\theta) < min(E)$$

which is impossible. Let us prove (45). To this end, we first split the functional $E$ in two parts :

$$E_1(B, C_1, .., C_{\mathrm{T}}) = \sum_{h=1}^{T} \int_\Omega C_{\mathrm{h}}^2 (B - N_{\mathrm{h}})^2 dx + \alpha_{\mathrm{c}} \sum_{h=1}^{T} \int_\Omega (C_{\mathrm{h}} - 1)^2 dx$$

$$E_2(B, C_1, .., C_{\mathrm{T}}) = \alpha_{\mathrm{b}}^{\mathrm{r}} \int_\Omega \phi_1(DB) + \alpha_{\mathrm{c}}^{\mathrm{r}} \sum_{h=1}^{T} \int_\Omega \phi_2(DC_{\mathrm{h}})$$

Looking first at the functional $E_2$, and using the strict convexity of the functions $\phi_j$, permits to write :

$$E_2(\mathbf{V}^\theta) < \theta E_2(\overline{\mathbf{V}}) + (1-\theta)E_2(\hat{\mathbf{V}}) \tag{46}$$

What can we say about $E_1(\mathbf{V}^\theta)$? Let $f^{\mathrm{h}} : \Omega \times R^2 \to R$ the function defined by :

$$f^{\mathrm{h}}(x, b, c) = c^2(b - N_{\mathrm{h}}(x))^2 + \alpha_{\mathrm{C}}(c-1)^2$$

and, for $x \in \Omega$ and $h \in [1..T]$, we introduce the function $l_{\mathrm{x}}^{\mathrm{h}} : R \to R$ defined by :

$$l_{\mathrm{x}}^{\mathrm{h}}(\theta) = f^{\mathrm{h}}(x, B^\theta, C_{\mathrm{h}}^\theta)$$

With these notations, we remark that if the function $l_{\mathrm{x}}^{\mathrm{h}}$ is convex for all $x$ in $\Omega$ and for all $h$, then we have :

$$E_1(\mathbf{V}^\theta) \leq \theta E_1(\overline{\mathbf{V}}) + (1-\theta)E_1(\hat{\mathbf{V}}) \tag{47}$$

Together with (46), this would lead to (45). The problem then becomes to study the convexity of the function $l_{\mathrm{x}}^{\mathrm{h}}$. Easy computations permit to obtain that :

$$\begin{aligned}\frac{d^2}{d\theta^2} l_{\mathrm{x}}^{\mathrm{h}}(\theta) =& 2(\overline{B}-\widehat{B})^2 {C_{\mathrm{h}}^{\theta}}^2 + 4(\overline{B}-\widehat{B})(\overline{C_{\mathrm{h}}}-\hat{C}_{\mathrm{h}})C_{\mathrm{h}}^{\theta}(B^{\theta}-N_{\mathrm{h}}) + \\ &+ 2(\overline{C_{\mathrm{h}}}-\hat{C}_{\mathrm{h}})^2(\alpha_{\mathrm{C}} + (B^{\theta}-N_{\mathrm{h}})^2) \\ =& 2\begin{pmatrix} \overline{B}-\widehat{B} \\ \overline{C_{\mathrm{h}}}-\hat{C}_{\mathrm{h}} \end{pmatrix}^t H \begin{pmatrix} \overline{B}-\widehat{B} \\ \overline{C_{\mathrm{h}}}-\hat{C}_{\mathrm{h}} \end{pmatrix}\end{aligned}$$

where :

$$H = \begin{pmatrix} {C_{\mathrm{h}}^{\theta}}^2 & C_{\mathrm{h}}^{\theta}(B^{\theta}-N_{\mathrm{h}}) \\ C_{\mathrm{h}}^{\theta}(B^{\theta}-N_{\mathrm{h}}) & (\alpha_{\mathrm{C}} + (B^{\theta}-N_{\mathrm{h}})^2) \end{pmatrix}$$

We recall that each functions are applied in $x$. If we want that the function $l_{\mathrm{x}}^{\mathrm{h}}$ be convex, a sufficient condition is that the matrix $H$ be definite positive. Saying that the determinant should be non negative permits to get the condition :

$$(B^{\theta}-N_{\mathrm{h}})^2 \leq \frac{\alpha_{\mathrm{C}}}{3} \tag{48}$$

As we know that the functions $B$ solution should belong to $[m_{\mathrm{N}}, M_{\mathrm{N}}]$ (see previous part of the proof), we also have :

$$\begin{aligned}(B^{\theta}-N_{\mathrm{h}})^2 &= |\theta(\overline{B}-N_{\mathrm{h}}) + (1-\theta)(\widehat{B}-N_{\mathrm{h}})|^2 \\ &\leq \theta|\overline{B}-N_{\mathrm{h}}|^2 + (1-\theta)|\widehat{B}-N_{\mathrm{h}}|^2 \\ &\leq (M_{\mathrm{N}}-m_{\mathrm{N}})^2 \end{aligned} \tag{49}$$

because we have $|\overline{B}-N_{\mathrm{h}}| \leq |M_{\mathrm{N}}-m_{\mathrm{N}}|$ and $|\widehat{B}-N_{\mathrm{h}}| \leq |M_{\mathrm{N}}-m_{\mathrm{N}}|$. Consequently, if we choose $\alpha_{\mathrm{C}}$ such that :

$$\alpha_{\mathrm{C}} \geq 3(M_{\mathrm{N}}-m_{\mathrm{N}})^2$$

which is condition (41), and thanks to (48)-(49), we observe that the function $l_{\mathrm{x}}^{\mathrm{h}}$ is convex. So in fact, $E_1$ is convex over $\overline{\mathcal{E}}(\Omega)$ and then (47) is proved. This concludes the proof. ■

This theorem is important since it permits to consider the minimization problem over all $BV(\Omega)^{\mathrm{T}+1}$ without any constraint. On a numerical point of view, this remark will be also important since we will not have to handle with Lagrange multipliers. We can also remark that the condition (41) is in fact natural : it means that the background must be sufficiently taken into account.

# 5 The Minimization Algorithm

In the preceding section, we saw that there was a unique solution in $BV(\Omega)^{\mathrm{T}+1}$ of the minimization problem (40). The aim of this section is to propose a suitable algorithm to approximate numerically this solution.

Before begining, we would like to insist on the fact that working numerically with $BV(\Omega)$ is something hard. Firstly, we can not write Euler-Lagrange equations. Anzellotti [7] proposes an extension of Euler-Lagrange equation but they are variational inequalities. In an image restoration background, Vese [63] gives a caracterisation of the solution using a dual formulation. However, both of them can not be used, at the moment, numerically.
Secondly, discretizing directly functions in $BV(\Omega)$ is still an opened question. For theses reasons, we propose an algorithm with two steps :

- Section 5.1 : we define a functional $E_\epsilon$ on a more regular space. We show that the associated minimization problem admits a unique solution in $W^{1,2}(\Omega)^{\mathrm{T}+1}$ (noted $(B_\epsilon, C_{1\,\epsilon}, .., C_{\mathrm{T}\,\epsilon})$), and that the functional $E_\epsilon$ $\Gamma-$converges to $E$ for the $L^2$-strong topology. Consequently, $(B_\epsilon, C_{1\,\epsilon}, .., C_{\mathrm{T}\,\epsilon})$ will converge for the $L^2$-strong topology to the unique solution of the initial problem.
- Section 5.2 : For a fixed $\epsilon$, we are going to construct a sequence $(B^{\mathrm{n}}, C_1^{\mathrm{n}}, .., C_{\mathrm{T}}^{\mathrm{n}})$ converging to $(B_\epsilon, C_{1\,\epsilon}, .., C_{\mathrm{T}\,\epsilon})$ for the $L^2-$strong topology. It will be found as a minimizing sequence of an extended functional. This part usually referenced as the half quadratic minimization.

Consequently, we are able to construct a sequence $(B^{\mathrm{n}}, C_1^{\mathrm{n}}, .., C_{\mathrm{T}}^{\mathrm{n}})$ converging to the unique minimum of the functional $E$ for the $L^2-$strong topology. We will end this section by presenting in section 5.3 the precise discretized algorithm. Its stability will be proved using the fixed point theorem.

## 5.1 A Quadratic Approximation

We first extend an idea developed in [19]. For a function $\varphi$ having hypotheses (23)-(24), we define the odd function $\varphi_\epsilon$ by :

$$\varphi_\epsilon(t) = \begin{cases} \dfrac{\varphi'(\epsilon)}{2\epsilon}t^2 + \varphi(\epsilon) - \dfrac{\epsilon\varphi'(\epsilon)}{2} & \text{if } 0 \le t \le \epsilon \\[2ex] \varphi(t) & \text{if } \epsilon \le t \le 1/\epsilon \\[2ex] \dfrac{\epsilon\varphi'(1/\epsilon)}{2}t^2 + \varphi(1/\epsilon) - \dfrac{\varphi'(1/\epsilon)}{2\epsilon} & \text{if } t \ge 1/\epsilon \end{cases} \tag{50}$$

We observe that for $\epsilon > 0$, $\varphi_\epsilon \ge \varphi$ and for all $t$, we have : $\lim_{\epsilon\to 0} \varphi_\epsilon(t) = \varphi(t)$. Using this definition, let us denote by $\phi_{1,\epsilon}$ and $\phi_{2,\epsilon}$ the two functions associated to $\phi_1$ and $\phi_2$. We then define the function $E_\epsilon$ by :

$$E_\epsilon : BV(\Omega)^{\mathrm{T}+1} \to R \tag{51}$$

$$E_\epsilon(B, C_1, .., C_\mathrm{T}) =$$

$$\begin{cases} \sum_{h=1}^{T} \int_\Omega C_\mathrm{h}^2 (B - N_\mathrm{h})^2 dx + \alpha_\mathrm{c} \sum_{h=1}^{T} \int_\Omega (C_\mathrm{h} - 1)^2 dx \\ +\alpha_\mathrm{b}^\mathrm{r} \int_\Omega \phi_{1,\epsilon}(|\nabla B|) dx + \alpha_\mathrm{c}^\mathrm{r} \sum_{h=1}^{T} \int_\Omega \phi_{2,\epsilon}(|\nabla C_\mathrm{h}|) dx \\ \qquad\qquad \text{if } (B, C_1, .., C_\mathrm{T}) \in W^{1,2}(\Omega)^{\mathrm{T}+1} \\ +\infty \quad \text{otherwise} \end{cases}$$

Then we have the following results :

**Theorem 3** *Under hypotheses (23)-(24)(for $N_\mathrm{h}$) and (25)-(26)(for $\phi_j$), the minimization problem :*

$$\inf_{(B, C_1, .., C_\mathrm{T}) \in W^{1,2}(\Omega)^{\mathrm{T}+1}} E_\epsilon(B, C_1, .., C_\mathrm{T}) \tag{52}$$

*admits a solution in $W^{1,2}(\Omega)^{\mathrm{T}+1}$. If moreover :*

$$\alpha_\mathrm{C} \geq 3(M_\mathrm{N} - m_\mathrm{N})^2 \tag{53}$$

*where the constants $m_\mathrm{N}, M_\mathrm{N}$ are defined by (26), then the solution is unique. We will denote it by $(B_\epsilon, C_{1\epsilon}, .., C_{\mathrm{T}\epsilon})$.*

**Preuve** The demonstration of that theorem is based on the same arguments as in the preceding section. In particular, we can show that the function $(B, C_1, .., C_\mathrm{T})$ are bounded. ■

**Proposition 1** *The sequence of functionals $E_\epsilon$ $\Gamma-$converges to the functional $E$ for the $L^{2^{\mathrm{T}+1}}-$strong topology as $\epsilon$ goes to zero. The sequence of the unique minima of $E_\epsilon$ (noted $(B_\epsilon, C_{1\epsilon}, .., C_{\mathrm{T}\epsilon})$) converges in $L^{2^{\mathrm{T}+1}}-$strong to the unique minimum of $E$.*

**Preuve** By construction, the sequence $E_\epsilon$ is a decreasing sequence converging pointwisely to the functional $\widetilde{E}$ defined by :

$$\widetilde{E} : \mathbf{BV}(\Omega) \to R$$

$$\widetilde{E}(B, C_1, .., C_\mathrm{T}) = \begin{cases} E(B, C_1, .., C_\mathrm{T}) & \text{if } (B, C_1, .., C_\mathrm{T}) \in \mathbf{W^{1,2}}(\Omega)^{\mathrm{T}+1} \\ +\infty & \text{otherwise} \end{cases}$$

Thanks to [42] (proposition 5.7), we can deduce that $E_\epsilon$ $\Gamma$-converges to the lower semi continuous envelope of $\widetilde{E}$ (for the $L^{2^{\mathrm{T}+1}}-$strong topology) noted $R(E)$. We then show that in fact $R(E) = E$ using some compacity results developed in for instance [23, 13]. ■

### 5.2 An extension using dual variables

Let $(B_\epsilon, C_{1\epsilon}, .., C_{\mathrm{T}\epsilon})$ be the unique minumum of the functional $E_\epsilon$ over $W^{1,2}(\Omega)^{\mathrm{T}+1}$. For a fixed $\epsilon$, our aim is to approximate it. To this end, we need the result recalled in the Appendix A and already used for the image restoration problem (see Sect. 2.2) : let us apply Theorem 4 to the functions $\phi_{1,\epsilon}$ and $\phi_{2,\epsilon}$ which fulfil desired hypotheses. We will denote by $\Psi_{1,\epsilon}$ and $\Psi_{2,\epsilon}$ the associated functions $\Psi$. We then define the functional $E^d_\epsilon$ defined by :

$$\begin{aligned} &E^d_\epsilon : (W^{1,2}(\Omega) \times L^2(\Omega)) \times W^{1,2}(\Omega)^{\mathrm{T}} \times L^2(\Omega)^{\mathrm{T}} \rightarrow R \\ &E^d_\epsilon(B, d_{\mathrm{B}}, C_1, .., C_{\mathrm{T}}, d_{\mathrm{C}_1}, .., d_{\mathrm{C}_{\mathrm{T}}}) = \\ &\qquad \sum_{h=1}^{T} \int_\Omega \left[C_{\mathrm{h}}^2 (B - N_{\mathrm{h}})^2 + \alpha_{\mathrm{c}} (C_{\mathrm{h}} - 1)^2\right] dx \\ &\qquad + \alpha_{\mathrm{b}}^{\mathrm{r}} \int_\Omega \left[d_{\mathrm{B}} |\nabla B|^2 + \Psi_{1,\epsilon}(d_{\mathrm{B}})\right] dx \\ &\qquad + \alpha_{\mathrm{c}}^{\mathrm{r}} \sum_{h=1}^{T} \int_\Omega \left[d_{\mathrm{C_h}} |\nabla C_{\mathrm{h}}|^2 + \Psi_{2,\epsilon}(d_{\mathrm{C_h}})\right] dx \end{aligned} \tag{54}$$

where we have introduced the variable $d_{\mathrm{B}}, d_{\mathrm{C}_1}, .., d_{\mathrm{C}_{\mathrm{T}}}$ associated to $B, C_1, .., C_{\mathrm{T}}$ respectively. To minimize the functional $E^d_\epsilon$, the idea is to minimize successively with respect to each variable : given the initial conditions $(B^0, d^0_{\mathrm{B}}, C^0_{\mathrm{h}}, d^0_{\mathrm{C_h}})$, we iteratively solve the following system :

$$B^{\mathrm{n}+1} = \underset{B \in W^{1,2}(\Omega)}{\mathrm{argmin}} \quad E^d_\epsilon(B, d^{\mathrm{n}}_{\mathrm{B}}, C^{\mathrm{n}}_{\mathrm{h}}, d^{\mathrm{n}}_{\mathrm{C_h}}) \tag{55}$$

$$d^{\mathrm{n}+1}_{\mathrm{B}} = \underset{d_{\mathrm{B}} \in L^2(\Omega)}{\mathrm{argmin}} \quad E^d_\epsilon(B^{\mathrm{n}+1}, d_{\mathrm{B}}, C^{\mathrm{n}}_{\mathrm{h}}, d^{\mathrm{n}}_{\mathrm{C_h}}) \tag{56}$$

$$C^{\mathrm{n}+1}_{\mathrm{h}} = \underset{C_{\mathrm{h}} \in W^{1,2}(\Omega)}{\mathrm{argmin}} \quad E^d_\epsilon(B^{\mathrm{n}+1}, d^{\mathrm{n}+1}_{\mathrm{B}}, C_{\mathrm{h}}, d^{\mathrm{n}}_{\mathrm{C_h}}) \tag{57}$$

$$d^{\mathrm{n}+1}_{\mathrm{C_h}} = \underset{d_{\mathrm{C_h}} \in L^2(\Omega)}{\mathrm{argmin}} \quad E^d_\epsilon(B^{\mathrm{n}+1}, d^{\mathrm{n}+1}_{\mathrm{B}}, C^{\mathrm{n}+1}_{\mathrm{h}}, d_{\mathrm{C_h}}) \tag{58}$$

Equalities (57)-(58) are written for $h = 1..T$. Notice that the order of the minimization procedure is not important for all the results presented below. The way to obtain each variable like described in (55) to (58) consists in solving the associated Euler-Lagrange equations. As we will see in section 5.3, the dual variables $d^{\mathrm{n}+1}_{\mathrm{B}}$ and $(d^{\mathrm{n}+1}_{\mathrm{C_h}})_{\mathrm{h}=1..\mathrm{T}}$ are given explicitly, while $B^{\mathrm{n}+1}$ and $(C^{\mathrm{n}+1}_{\mathrm{h}})_{\mathrm{h}=1..\mathrm{T}}$ are solutions of linear systems. Anyway, before going further, we need to know more about the convergence of this algorithm : does it converges and does $(B^{\mathrm{n}}, C^{\mathrm{n}}_1, .., C^{\mathrm{n}}_{\mathrm{T}})$ approximates $(B_\epsilon, C_{1\epsilon}, .., C_{\mathrm{T}\epsilon})$? This is the purpose of the following proposition :

**Proposition 2** *Let $(B^0, d^0_{\mathrm{B}}, C^0_{\mathrm{h}}, d^0_{\mathrm{C_h}}) \in W^{1,2}(\Omega)^{\mathrm{T}+1}$. Then the sequence defined by the system (55)-(56)-(57)-(58) is convergent in $L^2(\Omega)^{\mathrm{T}+1}$−strong. Moreover, the sequence $(B^{\mathrm{n}}, C^{\mathrm{n}}_1, .., C^{\mathrm{n}}_{\mathrm{T}})$*

*converges in* $L^2(\Omega)^{\mathrm{T}+1}$*−strong to the unique minimum of* $E_\epsilon$ *in* $W^{1,2}(\Omega)^{\mathrm{T}+1}$*, that is to say* $(B_\epsilon, C_{1\epsilon}, .., C_{\mathrm{T}\epsilon})$.

**Preuve** The basis of the proof is to write the variational optimality conditions associated to each step and to pass to the limit into them. To this end we needed some results about non-linear elliptic equations [43, 44] and we used the trick of Minty (see for instance [16, 19]). For more details, we refer to [19, 9, 36] where such kind of ideas have been developed. ∎

## 5.3 The discretized algorithm

Let us write explicitely the equations that the system (55)-(56)-(57)-(58) implies. Starting from an initial estimate $(B^0, d^0_{\mathrm{B}}, C^0_{\mathrm{h}}, d^0_{\mathrm{C_h}})$, the equations that will be solved are the following :

$$\sum_{h=1}^{T} C_{\mathrm{h}}^{\mathrm{n}2}(B^{\mathrm{n}+1} - N_{\mathrm{h}}) - \alpha^{\mathrm{r}}_{\mathrm{b}} div(d^{\mathrm{n}}_{\mathrm{B}} \nabla B^{\mathrm{n}+1}) = 0 \tag{59}$$

$$d^{\mathrm{n}+1}_{\mathrm{B}} = \frac{\phi'_{1,\epsilon}(|\nabla B^{\mathrm{n}+1}|)}{2|\nabla B^{\mathrm{n}+1}|} \tag{60}$$

$$C^{\mathrm{n}+1}_{\mathrm{h}}\left[\alpha_{\mathrm{c}} + (B^{\mathrm{n}+1} - N_{\mathrm{h}})^2\right] - \alpha_{\mathrm{c}} - \alpha^{\mathrm{r}}_{\mathrm{c}} div(d^{\mathrm{n}}_{\mathrm{C_h}} \nabla C^{\mathrm{n}+1}_{\mathrm{h}}) = 0 \tag{61}$$

$$d^{\mathrm{n}+1}_{\mathrm{C_h}} = \frac{\phi'_{2,\epsilon}(|\nabla C^{\mathrm{n}+1}_{\mathrm{h}}|)}{2|\nabla C^{\mathrm{n}+1}_{\mathrm{h}}|} \tag{62}$$

As we said in the previous section, (60) and (62) give explicitely the values of $d^{\mathrm{n}+1}_{\mathrm{B}}$ and $d^{\mathrm{n}+1}_{\mathrm{C_h}}$ while $B^{\mathrm{n}+1}$ and $C^{\mathrm{n}+1}_{\mathrm{h}}$ are solutions of a linear system. Once discretized using finite differences, the linear system can be solved by a Gauss-Seidel method for instance.

We next prove that the discretized algorithm described by (59) to (62) is unconditionally stable.

**Proposition 3** *Let* $\Omega^{\mathrm{d}}$ *correspond to the discretization of* $\Omega$*. Let* $\overline{\mathcal{E}}^{\mathrm{d}}(\Omega)$ *be the space of discrete functions* $(B, C_1, .., C_{\mathrm{T}})_{i,j}$ *such that :*

$$m_{\mathrm{N}} \leq B_{i,j} \leq M_{\mathrm{N}} \tag{63}$$

$$0 \leq C_{\mathrm{h}i,j} \leq 1 \qquad \textit{for } h = 1..T \tag{64}$$

$$0 < m_{\mathrm{c}} \leq \sum_{h=1}^{T} C_{\mathrm{h}} \leq T \tag{65}$$

$$\textit{where} \left\{ \begin{array}{l} m_{\mathrm{N}} = \inf_{\substack{(i,j)\\ h=1..T}} N_{\mathrm{h}i,j} \\ M_{\mathrm{N}} = \sup_{\substack{(i,j)\\ h=1..T}} N_{\mathrm{h}i,j} \end{array} \right. , \qquad m_{\mathrm{c}} = \frac{T\alpha^{\mathrm{r}}_{\mathrm{c}}}{\alpha_{\mathrm{c}} + (M_{\mathrm{N}} - m_{\mathrm{N}})^2 + 4} \tag{66}$$

*Then, for a given* $(B^{\mathrm{n}}, C_1^{\mathrm{n}}, \ldots, C_{\mathrm{T}}^{\mathrm{n}})$ *in* $\overline{\mathcal{E}}^{\mathrm{d}}(\Omega)$*, there exists a unique* $(B^{\mathrm{n}+1}, C_1^{\mathrm{n}+1}, \ldots, C_{\mathrm{T}}^{\mathrm{n}+1})$ *in* $\overline{\mathcal{E}}^{\mathrm{d}}(\Omega)$ *such that (59)-(62) are satisfied.*

Before proving this result, let us remark that the boundaries (63) and (64) can be justified if we consider the continuous case (see the proof of the Theorem 2). As for condition (65), it is also very natural if we admit the interpretations of the variables $C_{\mathrm{h}}$ : if this condition is false, this would mean that the background is never seen at some points which we refuse.
**Preuve** We first rewrite equations (55)-(57) taking into account (60)-(62) :

$$\sum_{h=1}^{T} C_{h\,i,j}^{n2}(B_{i,j}^{n+1} - N_{h\,i,j}) - \alpha_{\mathrm{B}}^{\mathrm{r}} div\left(\frac{\phi'_{1,\epsilon}(|\nabla B^n|)}{2|\nabla B^n|}\nabla B^{n+1}\right)_{i,j} = 0 \tag{67}$$

$$C_{h\,i,j}^{n+1}\left[\alpha_C + (B_{i,j}^{n+1} - N_{h\,i,j})^2\right] - \alpha_C - \alpha_C^{\mathrm{r}} div\left(\frac{\phi'_{2,\epsilon}(|\nabla C_h^n|)}{2|\nabla C_h^n|}\nabla C_h^{n+1}\right)_{i,j} = 0 \tag{68}$$

We now write the discretized equations in space. To this end, techniques developed in the Appendix B are used to discretize the divergence operator. Several possibilities have been considered and discussed. Anyway they can be written as follows :

$$div\left(\frac{\phi'(|\nabla A^n|)}{|\nabla A^n|}\nabla A^{n+1}\right)_{i,j} \approx \sum_{(k,l)\in D} p_{i+k,j+l}(A^n)A_{i,j}^{n+1} - \left(\sum_{(k,l)\in D} p_{i+k,j+l}(A^n)\right) A_{i,j} \tag{69}$$

where $D = \{(k,l) \neq (0,0) \in [-1,0,1]^2\}$ and $(p_{i+k,j+l})_{(k,l)\in D}$ verifying :

$$0 \leq p_{i+k,j+l} \leq 1 \qquad \text{and} \qquad \sum_{(k,l)\in D} p_{i+k,j+l} \leq 4 \tag{70}$$

Using this notation, and after some basic computation, we can re-write (67)-(68) in the following form :

$$B_{i,j}^{n+1} = \sum_{(k,l)\in D}\left(\frac{\alpha_{\mathrm{B}}^{\mathrm{r}} p_{i+k,j+l}(B^n)}{\sum_{h=1}^T C_h^{n2} + \alpha_{\mathrm{B}}^{\mathrm{r}}\sum_{(k,l)\in D} p_{i+k,j+l}(B^n)}\right) B_{i+k,j+l}^{n+1} + \sum_{h=1}^{T}\frac{N_h C_{h\ i,j}^{n2}}{\sum_{h=1}^T C_{h\ i,j}^{n2} + \alpha_{\mathrm{B}}^{\mathrm{r}}\sum_{(k,l)\in D} p_{i+k,j+l}(B^n)} \tag{71}$$

$$C_{h\,i,j}^{n+1} = \sum_{h=1}^{T}\left(\frac{\alpha_C^{\mathrm{r}} p_{i+k,j+l}(C_h^n)}{\alpha_C + (B^n - N_h)_{i,j}^2 + \alpha_C^{\mathrm{r}}\sum_{(k,l)\in D} p_{i+k,j+l}(C_h^n)}\right) C_{h\,i+k,j+l}^{n} + \frac{\alpha_C}{\alpha_C + (B^n - N_h)_{i,j}^2 + \alpha_C^{\mathrm{r}}\sum_{(k,l)\in D} p_{i+k,j+l}(C_h^n)} \tag{72}$$

To simplify notations, let us denote by $\mathbf{V}_{\mathrm{i,j}}^n \in R^{T+1}$, the vector defined by :

$$\mathbf{V}_{\mathrm{i,j}}^n = (B_{i,j}^n, {C_1}_{i,j}^n, .., {C_T}_{i,j}^n)^t$$

where the superscript denotes the transposition symbol. Let $(M_{i+k,j+l}(\mathbf{V}^n))_{(k,l)\in D}$ be diagonal matrices in $R^{T+1} \times R^{T+1}$ and $R(\mathbf{V}^n)_{i,j}$ be a $R^{T+1}$ valued vector such that equations (71)-(72) may be rewritten in the following form :

$$\mathbf{V}_{\mathrm{i,j}}^{n+1} = \sum_{(k,l)\in D} M_{i+k,j+l}(\mathbf{V}^n)\mathbf{V}_{i+k,j+l}^{n+1} + R(\mathbf{V}^n)_{i,j}$$

Now, for $\mathbf{W} \in \overline{\mathcal{E}}^{\mathrm{d}}(\Omega)$, we define the linear function $Q_{\mathbf{W}}(Z)$ by :

$$\begin{aligned} &Q_{\mathbf{W}} : \overline{\mathcal{E}}^{\mathrm{d}}(\Omega) \to R^{T+1} \\ &Q_{\mathbf{W}}(\mathbf{Z}) = \sum_{(k,l)\in D} M_{i+k,j+l}(\mathbf{W})\mathbf{Z}_{i+k,j+l} + R(\mathbf{W})_{i,j} \end{aligned} \tag{73}$$

Then we can show that :

$$Q_{\mathbf{W}}(\overline{\mathcal{E}}^{\mathrm{d}}(\Omega)) \subset \overline{\mathcal{E}}^{\mathrm{d}}(\Omega) \tag{74}$$

$$Q_{\mathbf{W}} \text{ is a contractive function on } \overline{\mathcal{E}}^{\mathrm{d}}(\Omega) \tag{75}$$

The first statement can be deduced directly from relations (71)-(72) and the definition of $\overline{\mathcal{E}}^{\mathrm{d}}(\Omega)$. Let us demonstrate (75). Let $\mathbf{Y}$ and $\mathbf{Z}$ be two elements in $\overline{\mathcal{E}}^{\mathrm{d}}(\Omega)$. Then we have :

$$\begin{aligned} |Q_{\mathbf{W}}(\mathbf{Z}) - Q_{\mathbf{W}}(\mathbf{Y})| &\leq \sum_{(k,l)\in D} \|M_{i+k,j+l}(\mathbf{W})\| \;\; |\mathbf{Z}_{i+k,j+l} - \mathbf{Y}_{i+k,j+l}| \\ &\leq \left( \sum_{(k,l)\in D} \|M_{i+k,j+l}(\mathbf{W})\| \right) |\mathbf{Z} - \mathbf{Y}|_\infty \\ &\leq K|\mathbf{Z} - \mathbf{Y}|_\infty \end{aligned}$$

where $\|\cdot\|$, $|\cdot|$ and $|\cdot|_\infty$ correspond to usual norms, and $K$ is a constant, independent of $\mathbf{W}$ and $i, j, k, l$ which is greater than all the coefficients of the diagonal matrices $M_{i+k,j+l}(\mathbf{W})$ but strictly inferior to 1. More precisely, we can establish that :

$$K = sup\left\{ \frac{4\alpha_{\mathrm{B}}^{\mathrm{r}}}{m_C + 4\alpha_{\mathrm{B}}^{\mathrm{r}}}, \frac{4\alpha_C^{\mathrm{r}}}{\alpha_C + 4\alpha_C^{\mathrm{r}}} \right\}$$

Consequently, thanks to the properties (74)-(75), we can apply the classical fixed point theorem to the function $Q_{\mathbf{W}}$. So, for $\mathbf{V}^n$ in $\overline{\mathcal{E}}^{\mathrm{d}}(\Omega)$, there exist a unique $\mathbf{V}^{n+1} \in \overline{\mathcal{E}}^{\mathrm{d}}(\Omega)$ such that :

$$\mathbf{V}^{n+1} = Q_{\mathbf{V}^n}(\mathbf{V}^{n+1})$$

```
/* Initializations (may be changed) */
B^0 ≡ 0
C_h^0 ≡ 1      ∀h
/* General loop (A stopping criterion could be used
instead) */
for(It=0;It≤ItNumber;It++) {
   /* *** Minimizing in B *** */
    - Compute coefficients (p_{i+k,j+l})_{(k,l)∈D} corresponding to
    the divergence discretization for B (see (69) with
    Appendix B)
    - Solve the linear system (71) by an iterative method
    (Gauss-Seidel) to find B^{n+1}
   /* *** Minimizing in C_h *** */
   for(h = 1; h ≤ T; h++) {
       - Compute coefficients (p_{i+k,j+l})_{(k,l)∈D} corresponding to
       the divergence discretization for C_h^n (see (69) with
       Appendix B)
       - Solve the linear system (72) by an iterative method
       (Gauss-Seidel) to find C_h^{n+1}
   } /* Loop on h */
} /* Loop on It */
```

Table 1: The detailled algorithm

that is to say $\mathbf{V}^{n+1}$ is the unique solution of (71)-(72). Moreover $\mathbf{V}^{n+1} \in \overline{\mathcal{E}}^{\mathrm{d}}(\Omega)$. This concludes the proof. ■

During this proof we needed to write explicitely the discretized equations to be solved. We give in Tab. 1 a sum-up of the precise algorithm. Notice that it is not necessary to compute explicitely the dual variables because they are directly replaced into the divergence operator. To conclude this section, we will notice that if $\alpha_{\mathrm{c}}^{\mathrm{r}} = 0$, the functions $(C_{\mathrm{h}}^{\mathrm{n+1}})_{\mathrm{h=1..T}}$ are in fact obtained explicitly by :

$$C_{\mathrm{h}}^{\mathrm{n+1}} = \frac{\alpha_{\mathrm{c}}}{\alpha_{\mathrm{c}} + (B^{\mathrm{n+1}} - N_{\mathrm{h}})^2} \tag{76}$$

As we can imagine, this case permits important reduction of the computational cost since $T$ linear systems are replaced by $T$ explicit expressions. We will discuss in Sect. 6 if it is worth regularizing or not the functions $C_{\mathrm{h}}$.

# 6 The Numerical Study

This section aims at showing quantitative and qualitative results about this method. Synthetic noisy sequences will be used to estimate rigorously the capabilities of our approach. The purpose of Sect. 6.1 is the quality of the restoration. The Sect. 6.2 is devoted to the motion detection and its sensibility with respect to noise. We will conclude in Sect. 6.3 by real sequences.

## 6.1 About the Restoration

To estimate the quality of the restoration, we used the noisy synthetic sequence presented in Fig. 1 **(a)(b)**. Figure 1 **(c)** is a representation of the noisy background without the moving objects. We mentioned the value of the Signal to Noise Ratio (SNR) usually used in image restoration to quantify the results quality. We refer to [38] for more details. We recall that the higher the SNR is, the best the quality is. Classically used to extract the foreground from the background, the median (see Fig. 1 **(d)**) appears to be inefficient. The average in time of the sequence (see Fig 1 **(e)**), although it permits a noise reduction, keeps the trace of the moving objects. The Fig. 1 **(f)** is the result that we obtained.

To conclude that section, let us mention that we also tried the case $\alpha_{\mathrm{c}}^{\mathrm{r}} = 0$, that is to say we did not regularized the functions $C_{\mathrm{h}}$. The resulting SNR was 14, to be compared with 14.4 ($\alpha_{\mathrm{c}}^{\mathrm{r}} \neq 0$). This leads to the conclusion that regularizing the functions $C_{\mathrm{h}}$ is not very important. However, this point has to be better investigated and more experimental results have to be considered before to conclude.

## 6.2 The Sensitivity of Motion Detection With Respect to Noise

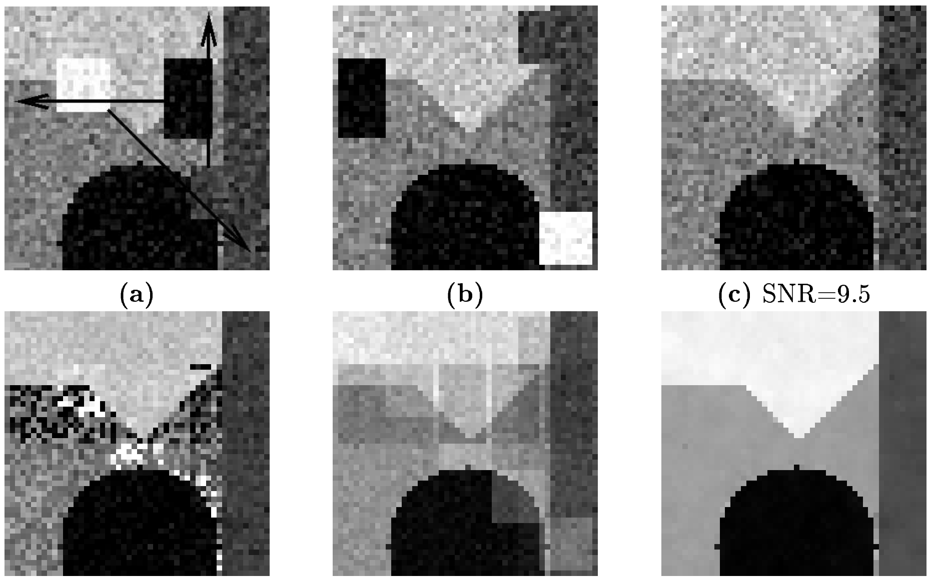
In this section, we aim at showing the robustness of our method with respect to noise. To this end, we choose a synthetic sequence (see Fig. 2) where a grey circle is translating from left to right in front of a textured background.

To estimate the sensitivity of the algorithm, we corrupted the sequence by gaussian noise of different variance (from 5 to 50). We give in Fig. 3 the value of the SNR of the corrupted sequences for each variance.

Results are reported in Fig. 4 and 5.

The first one presents five typical results obtained for different values of $\sigma$ ($\sigma$=5,15,25,35,45). The second one gives qualitative informations concerning the quality of the restoration and the motion detection. The criterion used to decide whether a pixel belongs to the background or not is : if $C_{\mathrm{h}}(i,j)>$threshold, then the pixel $(i,j)$ of the image number $h$ belongs to the background. Otherwise, it belongs to a moving object. The threshold has been fixed to 0.25 in all experiments.

We can observe that when the SNR of the data is more than 8 (corresponding to $\sigma \approx 25$), results are particularly precise : The SNR of the background is more than 20 and the error detections are less than 5 percent. When the SNR of the data is less than 8, the motion detection errors grow rapidly but the quality of the restored background still remains correct.



**(a)** **(b)** **(c)** SNR=9.5

**(d)** SNR=5.7 **(e)** SNR=9.8 **(f)** SNR=14.4

Figure 1: Results on a synthetic sequence (5 images) **(a)** Description of the sequence (first image) **(b)** Last image of the sequence **(c)** The noisy background without any objects **(d)** Mediane **(e)** Average **(f)** Restored background ($\alpha_{\mathrm{c}}^{\mathrm{r}} \neq 0$)

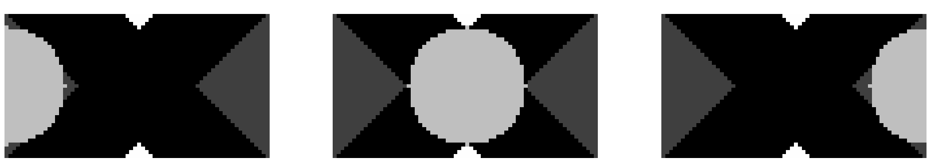


Figure 2: Three images of the initial synthetic sequence (35 images are available)

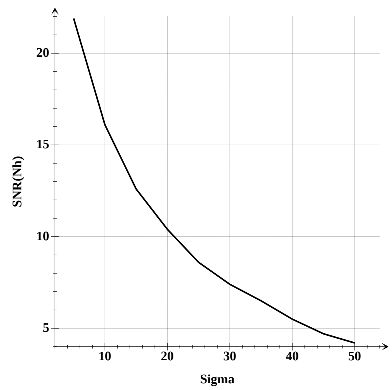



Figure 3: Signal to Noise Ratio of the data as a function of the variance.

See for instance the last row of Fig. 4 obtained for $\sigma = 45$ : the triangles on both sides are well recovered (observe the strong noise in the sequence).

Finally, notice that same parameters $(\alpha_{\mathrm{b}}^{\mathrm{r}}, \alpha_{\mathrm{c}}, \alpha_{\mathrm{c}}^{\mathrm{r}})$ have been used for all experiments. Generally speaking, we remarked that the algorithm performs well on a wide variety of sequences with the same set of parameters.

### 6.3 Some Real Sequences

The first real sequence is presented in Fig. 6 **(a)-(b)**. A small noise is introduced by the camera and certainly by the hard weather conditions. Notice the reflections on the ground which is frozen. We show in Fig. 6 **(c)** the average in time of the sequence. The restored background is shown in Fig. 6 **(d)**. As we can see, it has been very well found and enhanced. Figure 6 **(e)** is a representation of the function $C_{\mathrm{h}}$ (using a threshold of 0.5) and we show in Fig 6 **(f)** the associated dual variable $d_{C_{\mathrm{h}}}$.

The second sequence is more noisy than the first one. Its description is given in Fig. 7 **(a)**. To evaluate the quality of the restoration, we show a close-up of the same region for one original image (see Fig. 7 **(b)**), the average in time (see Fig. 7 **(c)**) and the restored background $B$ (see Fig. 7 **(d)**). The detection of moving regions is displayed in Fig. 7 **(e)**. Notice that some sparse motion have been detected at the right bottom and at the left side of the two persons. They correspond to the motion of a bush and the shadow of a tree due to the wind.

The last sequence is taken from an highway (see Fig. 8). We give two images (Fig. 8 **(a)** and **(b)**) and the corresponding motion detection below (Fig. 8 **(c)** and **(d)**). Finally, we show in Fig. 8 **(e)** the restored background. Notice that there is a black zone at the top of the road which comes from the fact that there are always cars in that region.

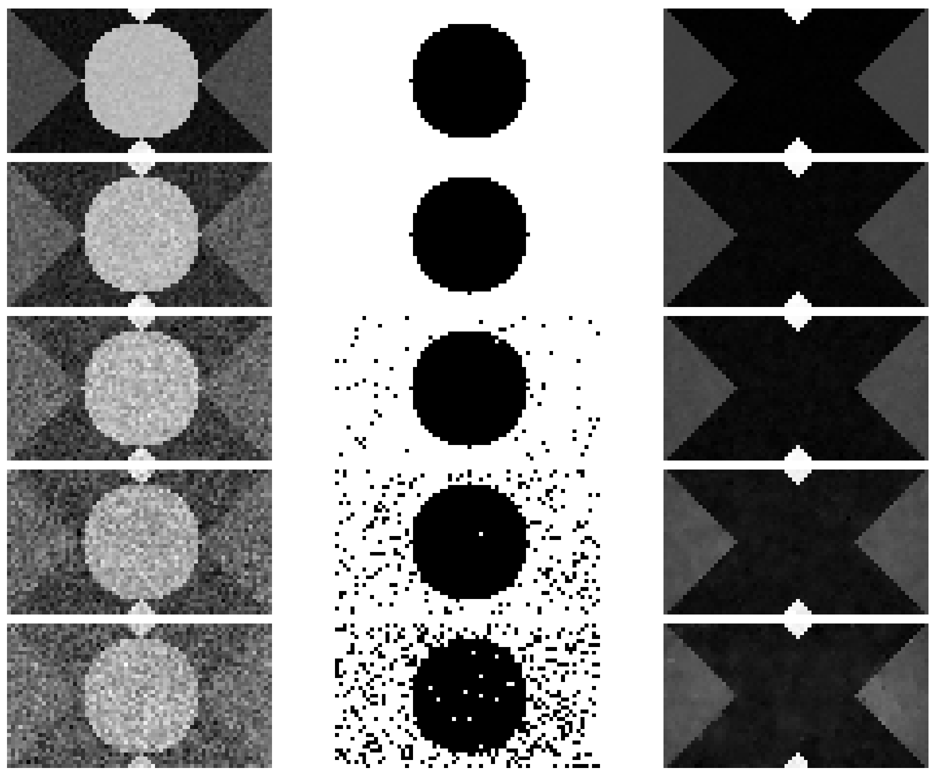


Figure 4: **Left :** One image of the noisy sequence. **Middle :** The motion detection based on variable $C_{\mathrm{h}}$ at the same time. **Right :** The restored background $B$. **From top to bottom :** Results for different variances of the gaussian noise (5,15,25,35,45).

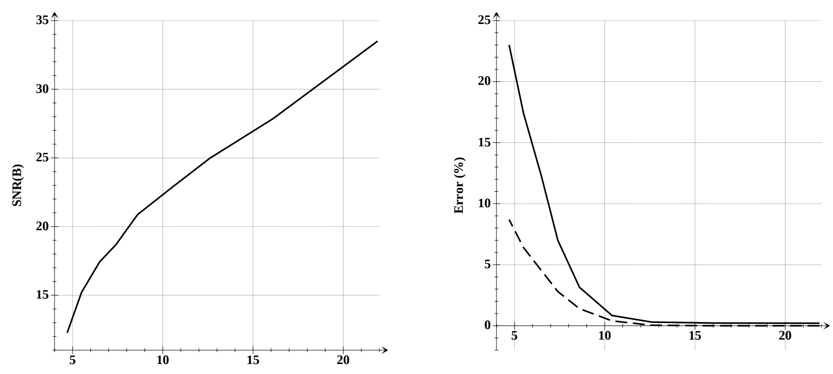



Figure 5: **Left :** SNR of the background as a function of the SNR of the data. **Right :** *dotted* (resp. *plain*) line : percentage of bad detections for the moving regions (resp. static background) as a function of the SNR of the data.

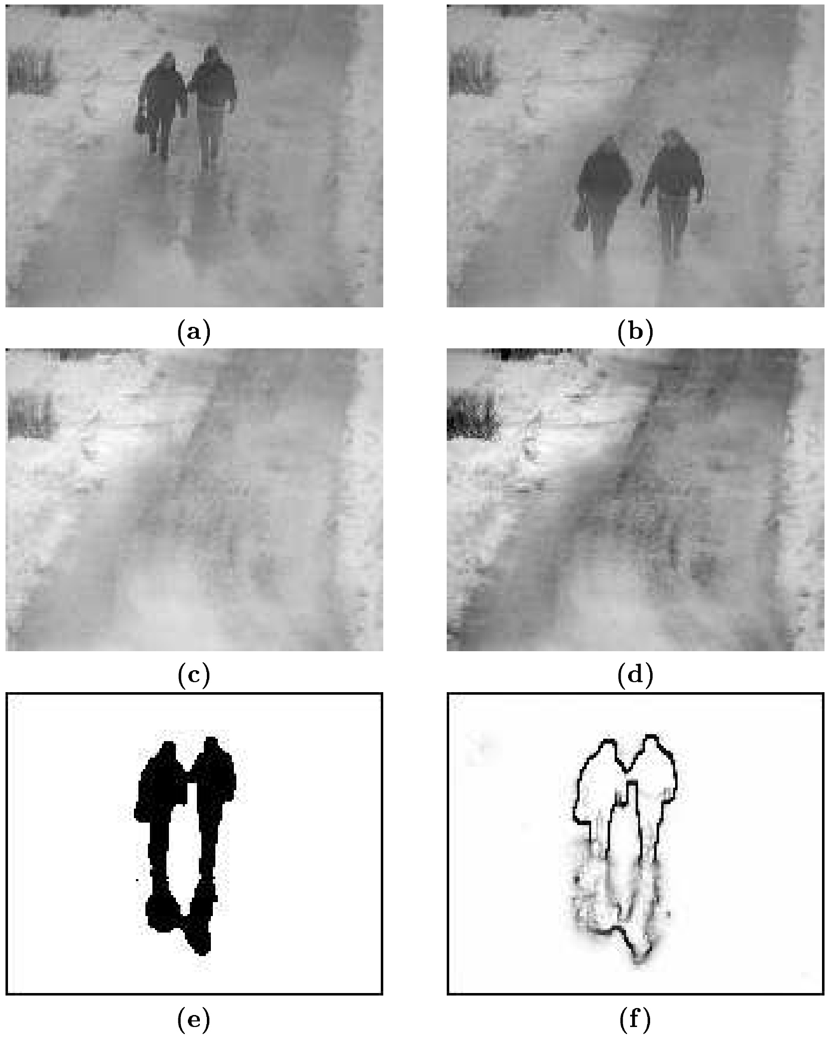



Figure 6: `Sweeden Sequence` : **(a)** and **(b)** Description of the sequence (55 images available). Two people are walking from top to bottom. This sequence is available from the web site `http://www.ien.it/is/is.html`. **(c)** The average over the time. **(d)** The restored background $B$. **(e)** Function $C_{\mathrm{h}}$ associated to the image **(a)** (a threshold of 0.5 has been used). **(f)** The dual variable $d_{C_{\mathrm{h}}}$ associated to the image **(a)**.

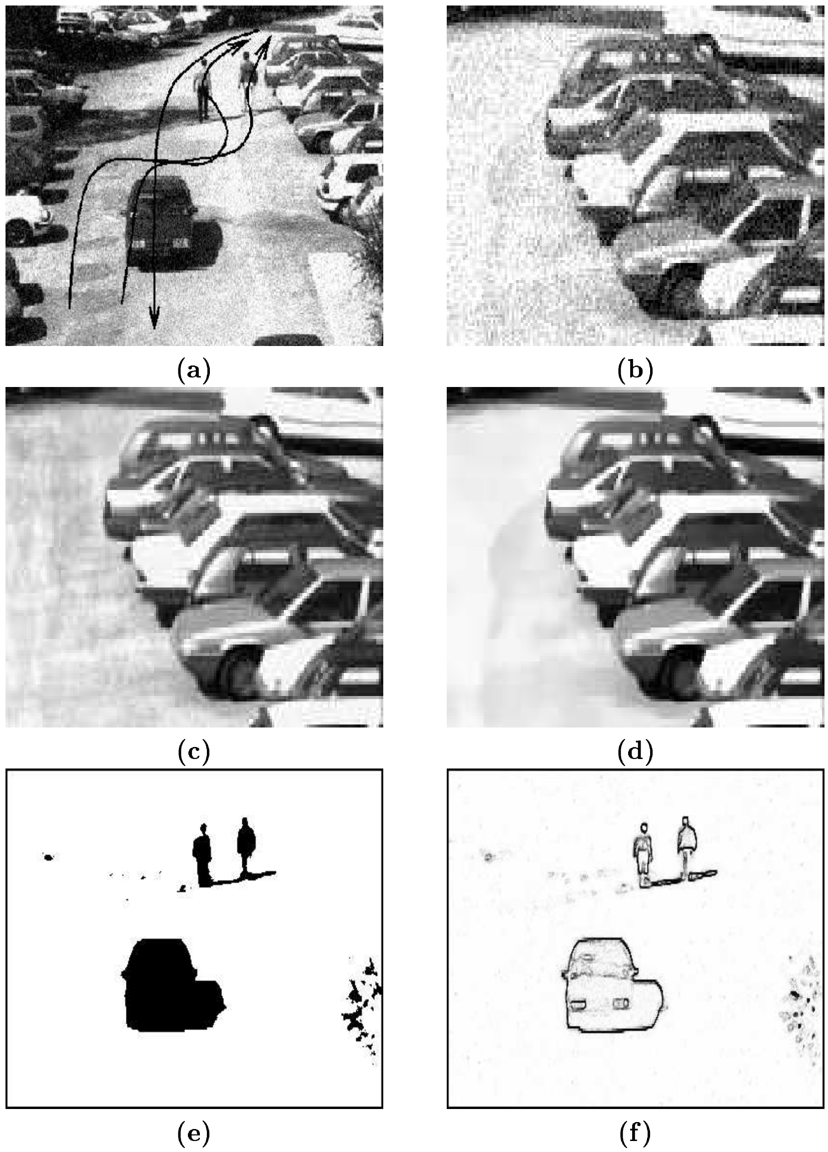


Figure 7: `INRIA Sequence :` **(a)** Description of the sequence (12 images available). **(b)** Zoom on a upper right part of the original sequence (without objects). **(c)** Zoom on the mean image. **(d)** Zoom on the restored background $B$. **(e)** The function $C_{\mathrm{h}}$ thresholded. **(f)** The dual variable $d_{C_{\mathrm{h}}}$.

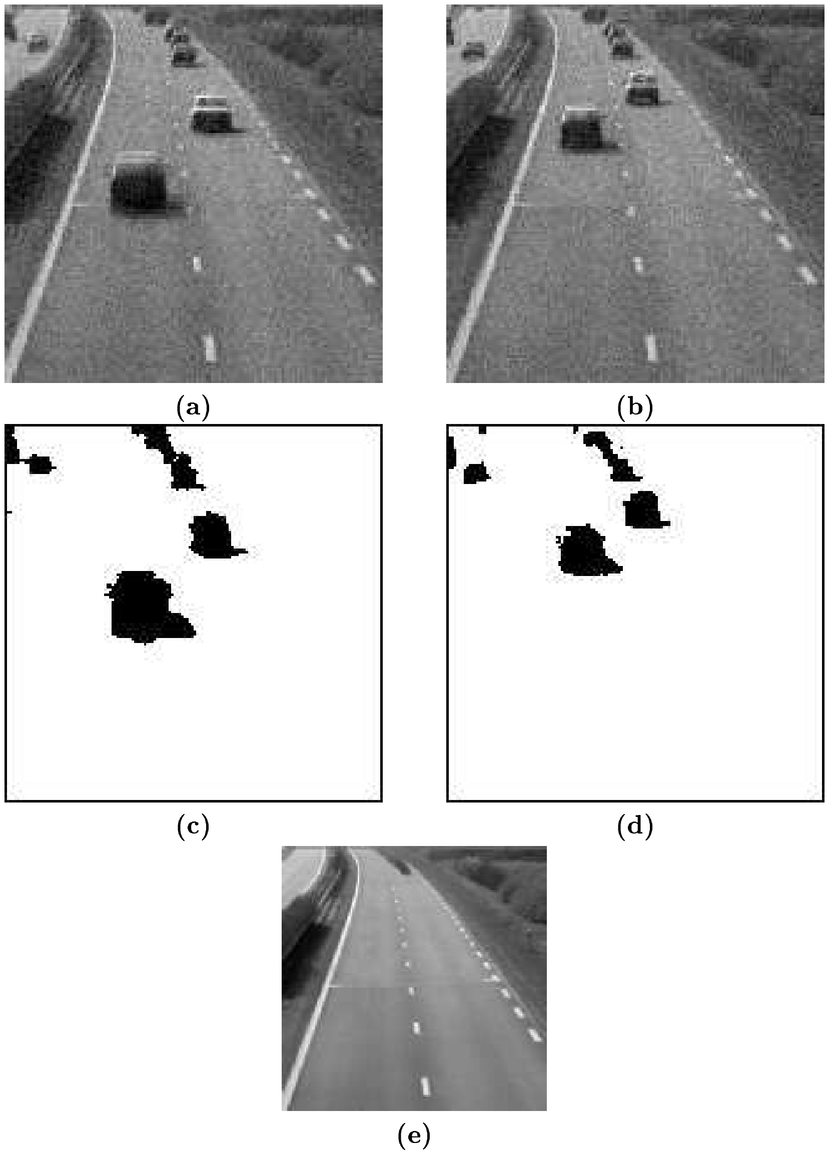



Figure 8: `Highway Sequence` : **(a)** and **(b)** Two images from the sequence (90 images available). **(c)** and **(d)** Corresponding $C_\mathrm{h}$ functions. **(e)** The restored background.

# 7 Conclusion

We have presented in this article an original coupled method for the problem of image sequence restoration and motion segmentation. A theoretical study in the space of bounded variations showed us that the problem was well-posed. We then proposed a convergent stable algorithm to approximate the unique solution of the initial minimization problem.

This original way to restore image sequence has been proved to give very promising result. A straightforward extension to color image sequences has recently been developed. To complete this work, several ideas are considered : use the motion segmentation part to restore also the moving regions, think about possible extensions for non-static cameras. This is the object of our current work.

# A The Half Quadratic Minimization Theorem

This theorem has been inspired by Geman and Reynolds [28] and proposed by Aubert [8].

**Theorem 4** *Let* $\varphi : [0, +\infty[\to [0, +\infty[$ *be such that:*

$$\varphi(\sqrt{t}) \quad \textit{is strictly concave on } ]0, +\infty[. \tag{77}$$

*Let* $L$ *and* $M$ *be defined as:* $L = \lim_{t \to +\infty} \frac{\phi'(t)}{2t}$ *and* $M = \lim_{t \to 0^+} \frac{\phi'(t)}{2t}$. *Then, there exists a strictly convex and decreasing function* $\psi : ]L, M] \to [\beta_1, \beta_2]$ *such that*

$$\varphi(t) = \inf_{L \le d \le M} (dt^2 + \psi(d)) \tag{78}$$

*where :* $\beta_2 = \lim_{t \to +\infty} \left( \phi(t) - t^2 \frac{\phi'(t)}{2t} \right)$ *and* $\beta_1 = \lim_{t \to 0_+} \phi(t)$ *Moreover, for every fixed* $t \geq 0$ *the value* $d_t$ *for which the minimum is reached is unique and given by:*

$$d_t = \frac{\phi'(t)}{2t} \tag{79}$$

In addition, we can give the expression of the function $\Psi$ with respect to $\phi$. If we note $\theta(t) = \phi(\sqrt{t})$,then :

$$\Psi(t) = \theta((\theta')^{-1}(t)) - t(\theta')^{-1}(t)$$

However, notice that this expression will never be used explicitely.

# B On Discretizing the Divergence Operator

Let $d$ and $A$ given at nodes $(i,j)$. The problem is to get an approximation of $div(d\nabla A)$ at the node $(i,j)$. We denote by $\delta^{x_1}$ and $\delta^{x_2}$ the finite difference operators defined by :

$$\begin{aligned}\delta^{x_1} A_{i,j} &= A_{i+\frac{1}{2},j} - A_{i-\frac{1}{2},j}\\ \delta^{x_2} A_{i,j} &= A_{i,j+\frac{1}{2}} - A_{i,j-\frac{1}{2}}\end{aligned}$$

Using that notation, Perona and Malik [52] proposed the following approximation :

$$\begin{aligned}div(d\nabla A)_{i,j} &= \frac{\partial}{\partial x_1}\left(d\frac{\partial A}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(d\frac{\partial A}{\partial x_2}\right) \approx \delta^{x_1}(d\delta^{x_1} A_{i,j}) + \delta^{x_2}(d\delta^{x_2} A_{i,j})\\ &\approx \begin{pmatrix} 0 & d_{i,j+\frac{1}{2}} & 0\\ d_{i-\frac{1}{2},j} & -S^{P} & d_{i+\frac{1}{2},j}\\ 0 & d_{i,j-\frac{1}{2}} & 0\end{pmatrix} \star A_{i,j}\end{aligned} \tag{80}$$

where the symbol $\star$ denotes the convolution and $S^{P}$ is the sum of the four weights in the principal directions. Notice that we need to estimate the function $d$ at intermediate nodes. Our aim is to extend this approximation so that we could take into account the values of $A$ at the diagonal nodes :

$$\begin{aligned}div(d\nabla A)_{i,j} &= \alpha_{P}\begin{pmatrix} 0 & d_{i,j+\frac{1}{2}} & 0\\ d_{i-\frac{1}{2},j} & -S^{P} & d_{i+\frac{1}{2},j}\\ 0 & d_{i,j-\frac{1}{2}} & 0\end{pmatrix} \star A_{i,j}\\ &+ \alpha_{D}\begin{pmatrix} d_{i-\frac{1}{2},j+\frac{1}{2}} & 0 & d_{i+\frac{1}{2},j+\frac{1}{2}}\\ 0 & -S^{D} & 0\\ d_{i-\frac{1}{2},j-\frac{1}{2}} & 0 & d_{i+\frac{1}{2},j-\frac{1}{2}}\end{pmatrix} \star A_{i,j}\end{aligned} \tag{81}$$

where $\alpha_{P}$ and $\alpha_{D}$ are two weights to be discussed, and $S^{D}$ is the sum of the four weights in the diagonal directions. Approximation (81) is consistent if and only if :

$$\alpha_{P} + 2\alpha_{D} = 1 \tag{82}$$

Now, there remains one degree of freedom. Two possibilities have been considered :

$$(\alpha_{P}, \alpha_{D}) = \text{ constant and for instance } = (\frac{1}{2}, \frac{1}{4}) \tag{83}$$

$$(\alpha_{P}, \alpha_{D}) = \text{ functions depending on } d \text{ (See Figure 9)} \tag{84}$$

To compare these different discretizations, we made numerical experiments with the image restoration problem where such kind of operator have to be discretized. We recall that for a given $d_{I}^{n}$, we need to find $I^{n+1}$ such that :

$$I^{n+1} - N - \alpha^{r} div(d_{I}^{n}\nabla I^{n+1}) = 0$$

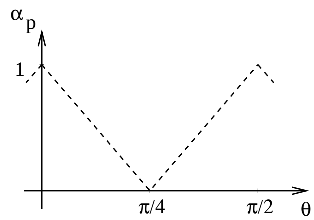



Figure 9: $\alpha_{\mathrm{P}} = \alpha_{\mathrm{P}}(\theta)$ is a $\pi/2$ periodic function where $\theta$ is the direction of the gradient of $d$. Notice that $\alpha_D$ can be deduced from the consistency condition is then computed thanks to the consistency condition.

We refer to section 2 for more details. The value of $d_{\mathrm{I}}^{\mathrm{n}} \left( = \frac{\phi'(|\nabla I^n|)}{2|\nabla I^n|} \right)$ at intermediate nodes is computed by interpolation (see [52]).

We tested these different discretizations on a noisy test image using quantitative measures. We checked that (81) permits to restore identically edges in principal or diagonal directions. Moreover, we observed that choosing $\alpha_{\mathrm{P}}$ adaptatively (84) gave more precise results than (83). We used this approximation (84) in our experiments.

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