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Abstract: We address the problem of asymptotic stabilization of certain configurations for the *general* N-trailer system (i.e. for which the hitching point of each trailer is not necessarily located on the rear axle of the preceding vehicle). In general, this system is not flat and, therefore, cannot be transformed into the classical chained form system. However, we show that it can be *approximated*, at some configurations, by the chained form system. This allows us to deduce simple time-varying continuous feedbacks which ensure local exponential stability of these particular configurations.

Key-words: Exponential stabilization, time-varying feedback, homogeneous feedback, general N-trailer system.

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Stabilisation exponentielle de certaines configurations du système à N remorques

Résumé : Nous considérons le problème de la stabilisation de certaines postures pour le système composé d'une voiture tractant N remorques dans le cas général (i.e. le point d'attache de chaque remorque n'est pas nécessairement localisé sur l'essieu arrière de la remorque précédente). Ce système n'est en général pas plat et, par conséquent, il est impossible de l'écrire sous forme chaînée. Nous montrons toutefois que, dans certaines configurations, le système peut être approximé par un système en forme chaînée. Ceci nous permet de déduire des retours d'état instationnaires continus qui stabilisent localement ces postures particulières, tout en assurant une convergence exponentielle.

Mots-clés : Stabilisation exponentielle, retour d'état instationnaire, retour d'état homogène, voiture avec N remorques.

1 Introduction

The N-trailer system (i.e. a cart-like or car-like mobile robot followed by N non-actuated trailers) has been much studied over the past few years. This kind of system is standard in mobile robotics, while its highly nonlinear characteristics lead to very challenging control problems. In most of the studies on this system, it has been assumed that each trailer is hitched at the middle of the rear axle of the preceding vehicle. A consequence of this assumption is that the kinematic equations of the system can be expressed —in some adequate coordinates— in the so-called *chained form*. This was proved in [13] for a cart with one trailer, and in [20] for an arbitrary number of trailers. The very simple structure of the chained form, and some specific properties associated with it —such as flatness [6] and homogeneity of the control vector fields with respect to a family of dilations [7]—, have been used to derive various solutions to both the stabilization problem (see e.g. [10, 11, 12, 14, 18, 19, 22] for a time-varying feedback approach, [1, 3] for a discontinuous approach, and [2, 21] for a hybrid approach), and the trajectory generation problem (see e.g. [6] and the numerous contributions on the subject in [8]).

While a car with a single trailer is still flat when the hitching point of the trailer is located at some distance of the car's rear axle [17], this is no longer the case when there are two —or more— successive trailers with "off-axle" hitch points. Then, it becomes impossible to transform the kinematic equations of the system into a chained form, and the results obtained for this latter class of systems cannot be applied. The fact that no simple canonical expression for the kinematic models of these systems is known may account for the limited research effort that they have raised up until now. For configurations around which the system is controllable —i.e. satisfies the Lie Algebra Rank Condition—, one may apply methods recently developed for general controllable driftless systems, e.g. [9] for the trajectory generation problem, and [11] for the asymptotic stabilization of a desired configuration of the system. However, the complexity of these methods makes their application to these systems extremely involved. There is a need for simpler approaches.

In this paper, we consider the stabilization problem of a configuration of the "general N-trailer system" —following [17], we use this expression to refer to the N-trailer system with possibly off-axle hitch points. We exhibit a set of configurations around which the system's equations can be approximated, in some sense, by the chained form system. This kind of approximation is sufficient to assert the local controllability of the system from that of the approximated system. It also implies, as we shall see, that time-varying feedback laws previously developed for chained-form systems can be used to achieve local exponential stability of these particular configurations. A practical interest of this approach is that it yields simple and easily tunable control laws.

The paper is organized as follows. We introduce in Section 2 the concept of a "control system with a chained form approximation" and point out some properties associated with such a system. We show in Section 3 that the general N-trailer system has a chained form approximation at certain configurations, and provide time-varying feedbacks for the local exponential stabilization of these configurations. Finally, simulation results are presented in Section 4.

2 Main Results

The results reported in the present paper rely on the properties of homogeneous systems, some of which are recalled hereafter. Given an n-tuple of positive reals $r = (r_1, \dots, r_n)$, the family of mappings $\delta_\lambda^r : \mathbb{R}^n \to \mathbb{R}^n$ indexed by a positive real λ and defined as

$$\delta_\lambda^r(x) = (\lambda^{r_1} x_1, \dots, \lambda^{r_n} x_n) \tag{1}$$

is called a *dilation of weight* r. Given a dilation δ_λ^r, a function $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ is *homogeneous of degree* τ *with respect to* δ_λ^r (in short δ_λ^r*-homogeneous of degree* τ) if

$$\forall x \in \mathbb{R}^n,\ \forall t \in \mathbb{R} :\ f(\delta_\lambda^r(x), t) = \lambda^\tau f(x, t). \tag{2}$$

A vector field $X(x,t) = \sum_{i=1}^n X_i(x,t) \frac{\partial}{\partial x_i} + \frac{\partial}{\partial t}$ is *homogeneous of degree* σ *with respect to* δ_λ^r if X_i is δ_λ^r-homogeneous of degree $\sigma + r_i$, for $i = 1, \dots, n$. A *homogeneous norm associated with a dilation* δ_λ^r is a continuous, positive definite mapping $\rho : \mathbb{R}^n \to \mathbb{R}$ which is δ_λ^r-homogeneous of degree 1. Finally, several subsequent statements make use of the following definition:

Definition 1 ([7]) *Given a dilation* δ_λ^r *and a* δ_λ^r*-homogeneous norm* ρ*, the origin* $x = 0$ *of* $\dot{x} = f(x,t)$ *is said to be* locally ρ-exponentially stable *if there exist an open neighborhood* U *of this point and two positive reals* K *and* γ*, such that for any* $(x_0, t_0) \in U \times \mathbb{R}^+$*, the solution* $x(t)$ *issued from* x_0 *at time* t_0 *satisfies* $\rho(x(t)) \leq K\rho(x(t_0))e^{-\gamma(t-t_0)}$.

For $n \geq 3$, a $(2,n)$ *single-chain system* (henceforth called, *a chained-form system*) is defined by

$$\begin{aligned} \dot{z} &= b_1(z)u_1 + b_2 u_2 \\ b_1(z) &= (1, z_3, \dots, z_n, 0)^T \\ b_2 &= (0, \dots, 0, 1)^T. \end{aligned} \tag{3}$$

Note that the vector fields b_1 and b_2 are homogeneous of degrees -1 and $-q$ respectively w.r.t. the dilation of weight $r(q) = (1, q+n-2, q+n-3, \dots, q)$. The possibility of setting q equal to any positive integer represents a degree of freedom which will be used later on. As mentioned earlier, non-flat systems cannot be directly transformed into the chained form, since a chained form system is flat. Nevertheless, it may happen that such systems still exhibit properties allowing to view them as if they were flat, in such a way that available stabilization techniques for chained-form systems can apply to them. To give a precise statement of one of these properties, we define a class of systems which behave locally in a "flat fashion". They are systems for which there exists a chained-form approximation that locally emulates their dynamics:

Definition 2 *A smooth control system*

$$\dot{x} = f(x,u), \quad x \in \mathbb{R}^n\ (n \geq 3),\ u \in \mathbb{R}^2$$

is said to have a chained-form approximation *at $\bar{x}$ if there exist a diffeomorphism $(x,u) \mapsto (z,v) \triangleq (\varphi(x), \psi(x,u))$ mapping $(\bar{x},u)$ into $(0,v)$ and a dilation $\delta_\lambda^{r(q)}$ with $r(q) = (1, q+n-2, q+n-3, \ldots, q)$ and $q \in \mathbb{N} - \{0\}$, such that in the new coordinates the system has the form*

$$\dot{z} = b_1(z)v_1 + b_2 v_2 + v_1 \varepsilon_1(z) + v_2 \varepsilon_2(z) \tag{4}$$

with b_1, b_2 defined by (3), and ε_1, ε_2 equal to a countable sum of δ_λ^r-homogeneous vector fields of degrees strictly larger than -1 and $-q$, respectively.

In what follows, we let $\mathcal{AC}_q^n(\bar{x})$ denote the set of smooth control systems on $\mathbb{R}^n$ that have a chained-form approximation at $\bar{x} \in \mathbb{R}^n$ for a given q. The next proposition asserts that any feedback control which ρ-exponentially stabilizes the origin of a chained-form system yields a corresponding feedback control which, when applied to any element of $\mathcal{AC}_q^n(\bar{x})$, asymptotically stabilizes the point $\bar{x}$ with exponential convergence.

Proposition 1 *Let $\Sigma \in \mathcal{AC}_q^n(\bar{x})$ and φ, ψ, be as in Definition 2. Assume that the time-varying feedback $v(z,t)$ with $v \in C^0(\mathbb{R}^n \times \mathbb{R}^+; \mathbb{R}^2)$ locally asymptotically stabilizes the origin of a chained form system, and that v_1 and v_2 are T-periodic in their second argument and δ_λ^r-homogeneous of degrees 1 and q respectively. Then, the feedback u defined by $u(x,t) = \psi_x^{-1}(v(\varphi(x),t))$[1], when applied to Σ, locally asymptotically stabilizes the point $x = \bar{x}$ with exponential convergence, i.e. there exist $\gamma, \eta > 0$ and a class $\mathcal{K}$ function h such that, if $\|x(0) - \bar{x}\| < \eta$, then $\|x(t) - \bar{x}\| \leq h(\|x(0) - \bar{x}\|)e^{-\gamma t}$ for all $t \geq 0$.*

Proof. The proof is based on [15, Prop. 4] (itself an extension of [16, Thm. 2]), according to which: *Given a dilation δ_λ^r, the origin of $\dot{z} = X^0(z,t) + \sum_{i=1}^\infty X^i(z,t)$ is locally ρ-exponentially stable* if *(i) The origin of $\dot{z} = X^0(z,t)$ is locally asymptotically stable, (ii) X^0 is δ_λ^r-homogeneous of degree 0, (iii) X^i is δ_λ^r-homogeneous of strictly positive degree for all* $i \geq 1$. Introducing the feedback $v(z,t)$ into (4), we obtain the closed-loop dynamics

$$\dot{z} = X^0(z,t) + v_1(z,t)\varepsilon_1(z) + v_2(z,t)\varepsilon_2(z) \tag{5}$$

with $X^0(z,t) \triangleq b_1(z)v_1(z,t) + b_2 v_2(z,t)$. By assumption, the origin of $\dot{z} = X^0(z,t)$ is locally asymptotically stable, and X^0 is δ_λ^r-homogeneous of degree 0, so conditions *(i)* and *(ii)* above are satisfied. Moreover, by Definition 2, ε_1 and ε_2 are sums of δ_λ^r-homogeneous v.f. of degrees strictly greater than -1 and $-q$ respectively, thus the last two terms in (5) are also sums of δ_λ^r-homogeneous v.f. of strictly positive degrees. Therefore, the third condition is satisfied as well, and the origin $z = 0$ of (5) is locally ρ-exponentially stable. This implies in turn that, given a δ_λ^r-homogeneous norm ρ, there exist constants $\gamma, \eta' > 0$, and a class $\mathcal{K}$ function h' such that if $\|z(0)\| \leq \eta'$, the closed-loop solution $z(t)$ satisfies

$$\|z(t)\| \leq h'(\|z(0)\|)e^{-\gamma t}, \tag{6}$$

[1] For any $x \in \mathbb{R}^n$, ψ_x^{-1} is the inverse of the map $u \mapsto \psi(x,u)$.

for all $t \geq 0$. This means that the solution will be confined to lie in the set $B = \{z \in \mathbb{R}^n : \|z\| \leq h'(\|z(0)\|)\}$, whose compactness follows from the continuity and properness of h'. From Definition 2 we have $x(t) = \varphi^{-1}(z(t))$ and $\bar{x} = \varphi^{-1}(0)$ and, since φ^{-1} is a diffeomorphism and B is compact, it follows that there exists a constant $M' \geq 0$ such that for all $t \geq 0$:

$$\begin{aligned} \|x(t) - \bar{x}\| &= \|\varphi^{-1}(z(t)) - \varphi^{-1}(0)\| \\ &\leq M'\|z(t)\|. \end{aligned} \tag{7}$$

The combination of (7) and (6) yields

$$\|x(t) - \bar{x}\| \leq M'h'(\|z(0)\|)e^{-\gamma t}. \tag{8}$$

Similarly, by virtue of the mean value theorem, there exists a constant M such that $\|z(0)\| \leq M\|x(0)-\bar{x}\|$ and thus, applying the non-decreasing function h' to both terms of this equality:

$$h'(\|z(0)\|) \leq h'(M\|x(0) - \bar{x}\|)$$

which, combined with (8) implies

$$\|x(t) - \bar{x}\| \leq h(\|x(0) - \bar{x}\|)e^{-\gamma t}, \tag{9}$$

with $h(\tau) = M'h'(M\tau)$ and $\tau \in \mathbb{R}$. One verifies immediately that h is a class $\mathcal{K}$ function and that $\eta = \inf_{\|z\|=\eta'} \|\varphi^{-1}(z)\| > 0$, hence any solution $x(t)$ that satisfies $\|x(0) - \bar{x}\| \leq \eta$ will also satisfy (9) for all $t > 0$. ∎

The set of driftless systems with two inputs is of special interest to us because it includes a number of kinematic models of mobile robots. The following proposition states sufficient conditions, for a system which belongs to this set, to have a chained form approximation.

Proposition 2 *Consider a control system*

$$\Sigma : \quad \dot{x} = f(x)u_1 + g(x)u_2 \tag{10}$$

with f, g analytic and such that

(i) $f(x) = \begin{bmatrix} f_1(x) \\ f_r(x_r) \end{bmatrix}$, $g(x) = \begin{bmatrix} g_1(x) \\ g_r(x_r) \end{bmatrix}$
$f_1(x), g_1(x) \in \mathbb{R}$, $x = (x_1, \dots, x_n)^T$ *and* $x_r = (x_2, \dots, x_n)^T$,

(ii) $f_1(\bar{x}) \neq 0$ *and* $f_r(\bar{x}_r) = 0$,

(iii) $(A_r, B_r) \triangleq \left(\frac{\partial f_r(\bar{x}_r)}{\partial x_r}, g_r(\bar{x}_r)\right)$ *is a controllable pair of matrices.*

Denote $a_i\,(i=0,\ldots,n-2)$ *the coefficients of the characteristic polynomial associated with the matrix* A_r *(i.e.,* $det(sI-A_r) = s^{n-1}+a_{n-2}s_{n-2}+\ldots+a_0$*). Then the coordinate transformation* $(x,u)\mapsto(z,v)\triangleq(\varphi(x),\psi(x,u))$ *with*

$$\begin{aligned}\varphi(x) &= \begin{bmatrix}(x_1-\bar{x}_1)\\ T_r^{-1}(x_r-\bar{x}_r)\end{bmatrix}\\ \psi(x,u) &= \begin{bmatrix}u_1\\ -u_1\alpha^T T_r^{-1}(x_r-\bar{x}_r)+u_2\end{bmatrix}\end{aligned} \tag{11}$$

$$\begin{aligned}\alpha &= (a_0,\ldots,a_{n-2})^T\\ T &= \begin{bmatrix}f_1(\bar{x}) & 0\\ 0 & T_r\end{bmatrix},\; T_r=(t_2,\ldots,t_n) \qquad (12)\\ t_{n-i} &= \begin{cases}B_r, & (i=0)\\ A_r t_{n-i+1}+a_{n-i-1}t_n, & \\ & (1\le i\le n-2)\end{cases}\end{aligned}$$

transforms (10) into (4), with ε_1 *and* ε_2 *as specified in Definition 2 when choosing* $q>n-2$*. Therefore,* $\Sigma\in\mathcal{AC}_q^n(\bar{x})$ *for* $q>n-2$*.*

Proof. Let $y=x-\bar{x}$, so that system (10) may also be written as $\dot{y}=f(y+\bar{x})u_1+g(y+\bar{x})u_2$. By a Taylor expansion of f and g at $y=0$, one obtains:

$$\begin{aligned}\dot{y} &= \begin{bmatrix}c & \gamma^T\\ 0 & A_r\end{bmatrix}\begin{bmatrix}y_1\\ y_r\end{bmatrix}u_1+\begin{bmatrix}f_1(\bar{x})\\ 0\end{bmatrix}u_1\\ &\quad+\begin{bmatrix}g_1(\bar{x})\\ B_r\end{bmatrix}u_2+u_1\eta_1^2(y_r)+u_2\eta_2^1(y_r)\end{aligned} \tag{13}$$

with $A_r=\partial f_r(\bar{x}_r)/\partial x_r$, $B_r=g_r(\bar{x}_r)$, $c=\partial f_1(\bar{x})/\partial x_1$, $\gamma^T=\partial f_1(\bar{x})/\partial x_r$, and η_1^2, η_2^1 countable sums of polynomial terms in $y_2,\ldots,y_n$ of degrees ≥ 2 and ≥ 1, respectively. It is well known from linear system theory (see e.g. [4]), that the matrix T_r given by (12) transforms the linear controllable system associated with the pair (A_r,B_r) into the control canonical form $(A_c,B_c):=(T_r^{-1}A_rT_r, T_r^{-1}B_r)$ with

$$A_c=\begin{bmatrix}0 & 1 & & 0\\ & \ddots & \ddots & \\ 0 & & 0 & 1\\ -a_0 & -a_1 & \cdots & -a_{n-2}\end{bmatrix}$$
$$B_c=(0,\ldots,0,1)^T.$$

Therefore, the linear diffeomorphism $y\mapsto z=T^{-1}y$ transforms system (13) into

$$\begin{aligned}\dot{z} &= \begin{bmatrix}1+cz_1+\frac{1}{f_1(\bar{x})}\gamma^T T_r z_r\\ A_c z_r\end{bmatrix}u_1+\begin{bmatrix}\frac{g_1(\bar{x})}{f_1(\bar{x})}\\ B_c\end{bmatrix}u_2\\ &\quad+T^{-1}\left(\eta_1^2(T_rz_r)u_1+\eta_2^1(T_rz_r)u_2\right).\end{aligned} \tag{14}$$

In view of the definition (11) of v, one readily verifies that system (14) is of the form (4) with

$$\begin{aligned}\varepsilon_1(z) &= \begin{bmatrix} cz_1 + \frac{1}{f_1(\bar{x})}(\gamma^T T_r + g_1(\bar{x})\alpha^T)z_r \\ 0 \end{bmatrix} \\ &\quad + T^{-1}\eta_1^2(T_r z_r) + \alpha^T z_r T^{-1}\eta_2^1(T_r z_r) \\ \varepsilon_2(z) &= \begin{bmatrix} \frac{g_1(\bar{x})}{f_1(\bar{x})} \\ 0 \end{bmatrix} + T^{-1}\eta_2^1(T_r z_r). \end{aligned} \tag{15}$$

It only remains to show that ε_1 and ε_2 satisfy the homogeneity conditions of Definition 2 for $q > n-2$. Let $M := \max\{r_i\}_{2\leq i\leq n} = q+n-2$. For ε_1, the only non-zero component of the first vector field (v.f.) is $cz_1 + (1/f_1(\bar{x}))(\gamma^T T_r + g_1(\bar{x})\alpha^T)z_r$. Since it is linear in z, and since $q = \min\{r_i\,(i=2,\ldots,n)\}$, it may be written as a finite sum of δ_λ^r-homogeneous v.f. of degrees not smaller than $\min\{1,q\} - r_1 = 0$. Simple calculations show that $T^{-1}\eta_1^2(T_r z_r)$ is a countable sum of quadratic and higher order terms in $z_2,\ldots,z_n$, and so is $\alpha^T z_r T^{-1}\eta_2^1(T_r z_r)$ due to the factor $\alpha^T z_r$. Therefore, each of these v.f. may be decomposed as a sum of δ_λ^r-homogeneous v.f. of degrees not smaller than $2q-M = q-n+2$. The homogeneity condition on ε_1 requires that the degrees of the v.f. arising in its decomposition be minored by -1. From what precedes, this is clearly satisfied for $q > n-2$.

Let us turn to ε_2 in (15). The first v.f. on the right-hand side is constant and defines a δ_λ^r-homogeneous v.f. of degree -1 $(= -r_1)$. For the second v.f (i.e. $T^{-1}\eta_2^1(T_r z_r)$), each of its components is a countable sum of polynomials in $z_2,\ldots,z_n$ of degree larger or equal to one. Therefore, we may decompose this vector field as a sum of δ_λ^r-homogeneous v.f. with degrees not smaller than $m - M = -n+2$ (since $n \geq 3$). From Definition 2, ε_2 must be a sum of vector fields homogeneous of degree larger than $-q$. This condition is satisfied if $-1 > -q$ and $-n+2 > -q$, that is if $q > \max\{1, n-2\} = n-2$, as announced in the proposition. ∎

3 Application

Let us consider the general N-trailer system with *off-axle hitching*, as shown on Fig. 1. In order to derive a kinematic model of the system, it is assumed that the vehicles' wheels roll on a plane without slipping. It is also assumed that the leading vehicle (the tractor) is a car-like vehicle equipped with a front steering wheel. When the leading vehicle is a unicycle, extension of the present results involves slightly more complicated equations, but is nonetheless straightforward.

The notation for various physical parameters and angles is detailed on Figure 1. The vehicles are numbered starting with the one farthest from the tractor. The relative orientations of the vehicles with respect to each other are given by $\{\alpha_i\}_{2\leq i\leq N+1}$, while α_{N+2} is the orientation of the car's front wheel. Besides these angles, it remains to determine the position and orientation of one of the vehicles, say vehicle M, in order to completely characterize the configuration of the system in the plane. To this purpose, one may consider a given curve $\mathcal{C}$

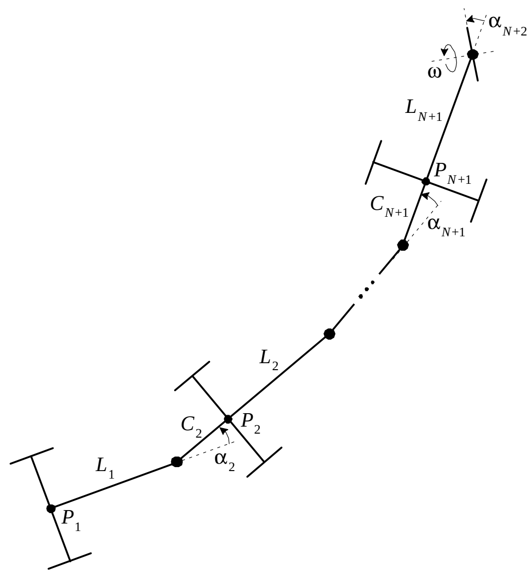

Figure 1: *General N-trailer system. The leading vehicle may be cart- or car-like.*

in the plane whose known curvature κ is a function of the curvilinear distance s measured along $\mathcal{C}$ from some point C_0 on the curve, and consider a set of Frénet coordinates (s, y, β) as shown on Figure 2. The coordinates s and y give the position of the point P_G, which is fixed to vehicle M, and β is the angle between vehicle M and the tangent to the curve $\mathcal{C}$ at the origin P_C of the moving Frénet frame. In the particular case when $\mathcal{C}$ is a straight line ($\kappa(s) \equiv 0$), s and y are nothing but the Cartesian coordinates of P_G w.r.t. a fixed frame. In former studies, see for instance [20] and [19], P_G was chosen as the mid-point P_1 on the rear axle of the last trailer because its Cartesian coordinates constitute a flat output for the system, thus facilitating the transformation of the model into the chained form. However, when the hitch offsets are all nonzero and $N \geq 2$, the system ceases to be flat, depriving P_1 of this special meaning. As shown further, a practical advantage of the proposed approach is to allow the selection of an arbitrary point P_G attached to *any* of the vehicles involved in the composition of the trailer system. We will thus subsequently assume that P_G is fixed to vehicle M for some $M \in \{1, \dots, N+1\}$.

The state vector x is defined by

$$\left\{ \begin{array}{lcl} x_1 & \triangleq & s \\ x_2 & \triangleq & y \\ x_3 & \triangleq & \beta \\ x_j & \triangleq & \alpha_{j-2} \quad (4 \leq j \leq n = N+4) \end{array} \right. \tag{16}$$

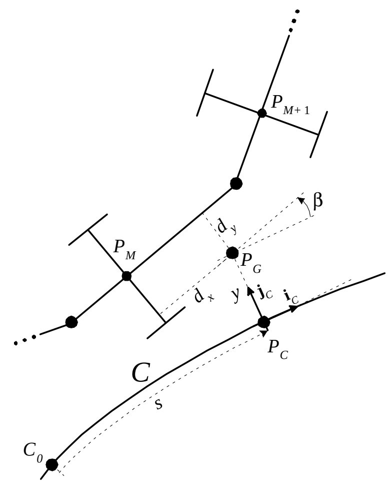

Figure 2: *Reference curve $\mathcal{C}$ and Frénet frame with origin P_C.*

The control inputs $u_1 = r\omega$, with r denoting the radius of the car's front wheel, and $u_2 = \dot{\alpha}_{N+2}$ are the *rolling* and *steering* velocities of the car's front wheel. With this notation, the model for the car with N trailers, obtained in the Appendix, is:

$$\dot{x} = f(x)u_1 + gu_2 \tag{17}$$

with

$$\begin{aligned}
f(x) &= \begin{bmatrix} \frac{W(x,\beta(x))}{1-\kappa(x_1)x_2} \\ W(x, \beta(x) - \frac{\pi}{2}) \\ \frac{G_M(x)}{L_M} - \frac{\kappa(x_1)W(x,\beta(x))}{1-\kappa(x_1)x_2} \\ \frac{G_2(x)}{L_2} - \frac{G_1(x)}{L_1} \\ \vdots \\ \frac{G_{N+1}(x)}{L_{N+1}} - \frac{G_N(x)}{L_N} \\ 0 \end{bmatrix} \\
g &= (0, \dots, 0, 1)^T \\
W(x, \cdot) &= \cos(\cdot)F_M(x) \\
&\quad -(\sin(\cdot)d_x + \cos(\cdot)d_y)\frac{G_M(x)}{L_M} \\
\beta(x) &= x_3,
\end{aligned}$$

and the recursive expressions

$$
\begin{array}{c}
F_{N+1}(x) = \cos(x_{N+4}) \\
G_{N+1}(x) = \sin(x_{N+4})
\end{array} \qquad (18)
$$

$$
\left.\begin{array}{rcl}
F_i(x) & = & \cos(x_{i+3})F_{i+1}(x) + \frac{C_{i+1}}{L_{i+1}}\sin(x_{i+3})G_{i+1}(x) \\
G_i(x) & = & \sin(x_{i+3})F_{i+1}(x) - \frac{C_{i+1}}{L_{i+1}}\cos(x_{i+3})G_{i+1}(x)
\end{array}\right\} (i = N, N-1, \ldots, 1).
$$

The posture stabilization problem addressed here consists in finding a feedback control which locally stabilizes some desired posture $\bar{x}$.

Henceforth, we assume that κ is constant. This means that $\mathcal{C}$ is either a straight line or a circle. Under this assumption, it is simple to verify that (17) has the form (10) of Proposition 2, with f and g analytic away from configurations such that $\kappa x_2 \neq 1$. Moreover, f and g are both independent of x_1, so that condition *(i)* in the proposition is met. As for condition *(ii)*, it may or may not be satisfied depending on the selection of $\overline{x}$. For instance, one readily verifies that any posture of the form $\bar{x} = (s_0, 0, \ldots, 0)^T$, when $\kappa = 0$, corresponding to the case where all vehicles are aligned, satisfies this condition. A natural question which then arises is whether there exist other postures which also satisfy this condition. The following result gives a positive answer to this question.

Lemma 1 *For any constant curvature κ belonging to some interval $(\kappa_{\min}, \kappa_{\sup})$ containing 0, there exist unique values $\bar{x}_3(\kappa), \ldots, \bar{x}_n(\kappa)$ in $(-\pi/2, +\pi/2)$ such that for*

$$\bar{x} \triangleq (s_0, 0, \bar{x}_3(\kappa), \ldots, \bar{x}_n(\kappa))^T,$$

with s_0 arbitrary, condition (ii) of Proposition 2 is satisfied for system (17).

(Proof in the Appendix).

The set of particular postures $\bar{x}$ pointed out in this lemma has in fact a simple geometrical interpretation. Indeed, the trailer system can move along the circle $\mathcal{C}$ only when all vehicles describe circular paths concentric to $\mathcal{C}$. The corresponding constant vehicle-to-vehicle angles $\bar{x}_i$ $(i = 3, \ldots, n)$ are therefore geometrically obtained by drawing these paths and the tangenting lines at the points P_i $(i = 1, \ldots, N+1)$.

In order to proceed, we must verify that condition *(iii)* of Proposition 2 is also satisfied, i.e. that the pair (A_r, B_r) is controllable. For $\kappa = 0$, the following proposition guarantees this property.

Proposition 3 *Given $\bar{x} = (s_0, 0, \ldots, 0)^T$ with $s_0 \in \mathbb{R}$, the pair $(A_r, B_r) \triangleq \left(\frac{\partial f_r(\bar{x})}{\partial x_r}, g_r\right)$, obtained from (17) with $\kappa = 0$, is controllable.*

(Proof in the Appendix).

Based on this result, one can invoke continuity arguments to deduce that if $|\kappa|$ is small enough and different from zero, then condition *(iii)* of Proposition 2 is still satisfied. The problem of determining *all* values of κ for which this condition is met is however more involved.

Having established that the general N-trailer system can be approximated, in the sense of Definition 2, by a chained-form system at the specific configurations pointed out in Lemma 1, we are now in the position of applying Proposition 1 in order to extend to this system previous feedback stabilization results obtained in the case of chained-form systems. We will illustrate this possibility by first recalling a result in [12].

Proposition 4 ([12, Prop. 3]) *Select $n-1$ real constants $a_2, \ldots, a_n$ such that the polynomial $P(s) = s^{n-1} + a_n s^{n-2} + \cdots + a_3 s + a_2$ is Hurwitz. Consider a set of weight vectors $r(q) = (1, q+n-2, q+n-3, \ldots, q)$ and associated homogeneous norms $\rho_{p,q}(z) = (\sum_{i=2}^{n} |z_i|^{\frac{p}{r_i(q)}})^{\frac{1}{p}}$. Then, for q large enough and $p > n-2+q$, the continuous time-periodic control*

$$\begin{aligned} v_1(z,t) = &\ -k_1 z_1(\sin^2 t + \mathrm{sign}(z_1)\sin t) \\ &\ -k_{n+1}\rho_{p,q}(z)\sin(t) \\ v_2(z,t) = &\ -v_1(z,t)\sum_{i=2}^{n} a_i z_i \left(\frac{\mathrm{sign}(v_1)}{\rho_{p,q}(z)}\right)^{n+1-i} \end{aligned} \tag{19}$$

with $k_1 > 0$, $k_{n+1} > 0$, globally ρ-exponentially stabilizes the origin of the chained-form system (3).

Consider now the general N-trailer system with $n = N+4$. Let $\bar{x}$ be any of the postures pointed out in Lemma 1 such that the pair $(A_r, B_r) \triangleq (\frac{\partial f_r(\bar{x})}{\partial x_r}, g_r)$ is controllable. By application of Proposition 1, the control $u(x,t)$, obtained from (19) by the change of coordinates specified in Proposition 2, locally exponentially stabilizes $\bar{x}$.

4 Simulation results

This control has been simulated in the case of a car pulling three trailers, so that $n = 7$. In the first simulation a "straight-in-line" posture, for which $\bar{x} = 0$ with $\kappa = 0$, is stabilized. In the second simulation, the stabilized posture is $\bar{x} = (0, 0, \bar{x}_3(\kappa), \ldots, \bar{x}_n(\kappa))$ with $\kappa = 1/15$. This corresponds to what may be called a "circular" posture. In both simulations, the vehicles' dimensions L_i $(i = 1, \ldots, 4)$ are set equal to 2 and the off-axle hitch distances C_i $(i = 2, \ldots, 4)$ are all taken equal to 1. The point P_G is fixed to the last trailer $(M = 1)$ with coordinates $(d_x, d_y) = (0, -1)$. The desired posture $\bar{x}$ and the initial state value x_0 are set as follows:

Simulation 1:
∘ $\bar{x} = 0$,
∘ $x_0 = (0, 5, 0°, 0°, 0°, 0°, 0°)$.

Simulation 2:
∘ $\bar{x} = (0, 0, 0°, 12.18°, 12.09°, 12.01°, 7.95°)$,

∘ $x_0 = (0, 5, 0°, 0°, 0°, 0°, 0°)$.

The gains $(a_2, \ldots, a_7)$ in the feedback law (19) are given by $(0.27, 1.4, 4.2, 6.4, 6.7, 3.9)$, whereas $k_1 = 2.5$ and $k_{n+1} = 6$. Finally, the value q involved in the weight vector $r(q)$ has been chosen equal to $n - 1 = 6$. Figures 3 and 4 show the initial and desired postures of the simulated system, as well as the path described by P_G.

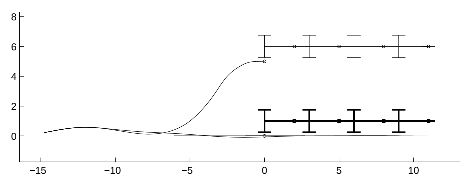

Figure 3: *Simulation 1: "Straight-in-line" posture stabilization (the stabilized desired posture is drawn with a thick trait).*

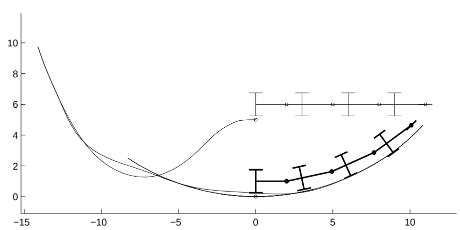

Figure 4: *Simulation 2: Stabilization of a "circular" posture.*

Appendix

Derivation of the kinematic model for the General N-Trailer System

In order to obtain the model, we will temporarily introduce an inertial reference frame $\Sigma_0 = (P_0, \{\mathbf{i}_0, \mathbf{j}_0\})$, (with arbitrary origin P_0 and orthonormal basis $\{\mathbf{i}_0, \mathbf{j}_0\}$) as well as the orientations θ_i, $(i = 1, \ldots, N+1)$, where θ_i represents the angle measured from $\mathbf{i}_0$ to the i-th vehicle's main axis. Similarly, we will use (x_i, y_i), $(i = 1, \ldots, N+1)$ to represent the position of P_i with respect to Σ_0, i.e. $P_i = x_i \mathbf{i}_0 + y_i \mathbf{j}_0$.

The model of the car (the tractor) is widely known and given by:

$$\begin{bmatrix} \dot{x}_{N+1} \\ \dot{y}_{N+1} \\ \dot{\theta}_{N+1} \\ \dot{\alpha}_{N+2} \end{bmatrix} = \begin{bmatrix} u_1 \cos(\theta_{N+1}) \cos(\alpha_{N+2}) \\ u_1 \sin(\theta_{N+1}) \cos(\alpha_{N+2}) \\ (u_1/L_{N+1}) \sin(\alpha_{N+2}) \\ u_2 \end{bmatrix}, \tag{20}$$

with $u_1 = \omega r$. In view of the cascaded structure of the system, it will be useful to derive expressions for $(\dot{x}_i, \dot{y}_i, \dot{\theta}_i)$ in terms of $(\dot{x}_{i+1}, \dot{y}_{i+1}, \dot{\theta}_{i+1})$. Let Q_i denote the position of the point where the i-th vehicle is hitched to the $(i+1)$-th one. Then, easy kinematic computations show that

$$\dot{Q}_i = \dot{P}_{i+1} + C_{i+1} \dot{\theta}_{i+1} \begin{bmatrix} \sin(\theta_{i+1}) \\ -\cos(\theta_{i+1}) \end{bmatrix}, \tag{21}$$

and also

$$\dot{Q}_i = \dot{P}_i + L_i \dot{\theta}_i \begin{bmatrix} -\sin(\theta_i) \\ \cos(\theta_i) \end{bmatrix}. \tag{22}$$

A third equation may be derived based on the rolling-without-slipping assumption, which translates into

$$-\dot{x}_i \sin(\theta_i) + \dot{y}_i \cos(\theta_i) = 0. \tag{23}$$

Equations (21)-(23) can be condensed into the equivalent one:

$$\begin{bmatrix} 1 & 0 & -L_i \sin(\theta_i) \\ 0 & 1 & L_i \cos(\theta_i) \\ -\sin(\theta_i) & \cos(\theta_i) & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_i \\ \dot{y}_i \\ \dot{\theta}_i \end{bmatrix} = \begin{bmatrix} \dot{x}_{i+1} + C_{i+1} \sin(\theta_{i+1}) \dot{\theta}_{i+1} \\ \dot{y}_{i+1} - C_{i+1} \cos(\theta_{i+1}) \dot{\theta}_{i+1} \\ 0 \end{bmatrix}$$

which in turn yields

$$\begin{bmatrix} \dot{x}_i \\ \dot{y}_i \\ \dot{\theta}_i \end{bmatrix} = \begin{bmatrix} \cos^2(\theta_i) & \cos(\theta_i) \sin(\theta_i) & C_{i+1} \cos(\theta_i) \sin(\alpha_i) \\ \cos(\theta_i) \sin(\theta_i) & \sin^2(\theta_i) & C_{i+1} \sin(\theta_i) \sin(\alpha_i) \\ -\frac{1}{L_i} \sin(\theta_i) & \frac{1}{L_i} \cos(\theta_i) & -\frac{C_{i+1}}{L_i} \cos(\alpha_i) \end{bmatrix} \begin{bmatrix} \dot{x}_{i+1} \\ \dot{y}_{i+1} \\ \dot{\theta}_{i+1} \end{bmatrix} \tag{24}$$

$$(i = N, N-1, \ldots, 1), \tag{25}$$

with $\alpha_i = \theta_{i+1} - \theta_i$, $(i = 1, \ldots, N)$. Together, this matrix equation and (20) allow us to develop the expressions $(\dot{x}_i, \dot{y}_i, \dot{\theta}_i)$ for $i = N+1, \ldots, 1$. After some simplifications, the resulting expressions are

$$\left.\begin{aligned} \dot{x}_i &= \cos(\theta_i) \Phi_i(\theta) u_1 \\ \dot{y}_i &= \sin(\theta_i) \Phi_i(\theta) u_1 \\ \dot{\theta}_i &= \frac{\Gamma_i(\theta)}{L_i} u_1 \end{aligned}\right\} \quad (i \; = \; N+1, N, \ldots, 1)$$

$$\dot{\alpha}_{N+2} \; = \; u_2$$

with $\theta_{N+2} \triangleq \alpha_{N+2}$, $\theta \triangleq (\theta_1, \ldots, \theta_{N+2})^T$ and:

$$\Phi_{N+1}(\theta) = \cos(\theta_{N+2})$$
$$\Gamma_{N+1}(\theta) = \sin(\theta_{N+2})$$
$$\left.\begin{array}{l} \Phi_i(\theta) = \cos(\alpha_i)\Phi_{i+1}(\theta) + \frac{C_{i+1}}{L_{i+1}}\sin(\alpha_i)\Gamma_{i+1}(\theta) \\ \Gamma_i(\theta) = \sin(\alpha_i)\Phi_{i+1}(\theta) - \frac{C_{i+1}}{L_{i+1}}\cos(\alpha_i)\Gamma_{i+1}(\theta) \end{array}\right\} \quad (i = N, N-1, \ldots, 1).$$

Assuming the point P_G fixed to the M-th vehicle for some $M \in \{1, \ldots, N+1\}$, the next step consists in expressing its velocity $\dot{P}_G$ with respect to Σ_0. This is readily accomplished by attaching a Cartesian frame $\Sigma_M(P_M, \{\mathbf{i}_M, \mathbf{j}_M\})$ fixed on the M-th vehicle and oriented so that the position of P_G with respect to Σ_M be given by $P_{G/M} = d_x\mathbf{i}_M + d_y\mathbf{j}_M$. Using the classical expression $\dot{P}_G = \dot{P}_M + \dot{P}_{G/M} + \omega_M \wedge P_{G/M}$, we get

$$\begin{bmatrix} \dot{x}_G \\ \dot{y}_G \end{bmatrix} = \begin{bmatrix} \dot{x}_M \\ \dot{y}_M \end{bmatrix} + \begin{bmatrix} -\sin(\theta_M)d_x - \cos(\theta_M)d_y \\ \cos(\theta_M)d_x - \sin(\theta_M)d_y \end{bmatrix} \dot{\theta}_M.$$

Let us now introduce the curve $\mathcal{C}$ and its associated Frénet frame and then find the expressions for $\dot{s}$, $\dot{y}$ and $\dot{\beta}$ (see Fig. 2). Calling θ_C the orientation of $\mathbf{i}_C$ with respect to $\mathbf{i}_0$ and using elementary kinematics, we get

$$\dot{s} = \frac{1}{1-\kappa(s)y}\left(\dot{x}_G\cos(\theta_C) + \dot{y}_G\sin(\theta_C)\right)$$
$$\dot{y} = -\dot{x}_G\sin(\theta_C) + \dot{y}_G\cos(\theta_C)$$
$$\dot{\theta} = \kappa(s)\dot{s}.$$

Finally, by posing $\beta \triangleq \theta_M - \theta_C$ and letting the state vector x be defined as in (16), the above expressions can be rearranged to give the complete model (17).

Proof of Lemma 1

Assuming a constant curvature $\kappa(x_1) \equiv \kappa$, our goal is to determine $\bar{x}_r(\kappa) = (0, \bar{x}_3(\kappa), \ldots, \bar{x}_n(\kappa))^T$ with $\bar{x}_i(\kappa) \in (-\frac{\pi}{2}, \frac{\pi}{2})$, such that

$$f_r(\bar{x}_r(\kappa)) = 0. \tag{26}$$

For conciseness in the notation, we will henceforth write $\bar{x}_j$, F_j and G_j instead of $\bar{x}_j(\kappa)$, $F_j(\bar{x}(\kappa))$ and $G_j(\bar{x}(\kappa))$, respectively. In view of (17), (26) is equivalent to the following three conditions:

$$(-d_x\cos(\bar{x}_3) + d_y\sin(\bar{x}_3))\frac{G_M}{L_M} = \sin(\bar{x}_3)F_M, \tag{27}$$
$$[1 + \kappa(d_x\sin(\bar{x}_3) + d_y\cos(\bar{x}_3))]\frac{G_M}{L_M} = \kappa\cos(\bar{x}_3)F_M, \tag{28}$$
$$\frac{G_1}{L_1} = \frac{G_2}{L_2} = \cdots = \frac{G_{N+1}}{L_{N+1}}. \tag{29}$$

We consider two cases depending on the value of κ. First, let us assume $\kappa = 0$. This assumption and (28) imply that $G_M = 0$. With (29), this implies in turn that $G_{N+1} = \sin(\bar{x}_{N+4}) = 0$ and therefore $\bar{x}_{N+4} = 0$. Using (29) again, as well as the definitions of F_i and G_i, we get $G_i = 0$ and $F_i = 1$ for $i = 1, \dots, N+1$. Therefore $\bar{x}_j = 0$ for $j = 4, \dots, N+4$. From this and (27), we find that $\bar{x}_3 = 0$ and consequently $\bar{x}_r(0) = (0, \dots, 0)^T \in \mathbb{R}^{n-1}$.

Now suppose $\kappa \neq 0$. In this case G_M does *not* vanish, otherwise one would have, by a similar reasoning as above, $G_i = 0$ and $F_i = 1$ for $i = 1, \dots, N+1$. Moreover, from (28), $\kappa \cos(\bar{x}_3) F_M = 0$ so that $\bar{x}_3 = \frac{\pi}{2} + \pi\mathbb{Z}$. Using any of these values in (27) it would come that $F_M = 0$, a contradiction. Combining (27) and (28) we find

$$(\kappa d_x + \sin(\bar{x}_3)) G_M = 0, \tag{30}$$

therefore

$$\bar{x}_3 = -\arcsin(\kappa d_x), \quad \text{provided} \quad |\kappa d_x| \leq 1, \tag{31}$$

since $G_M \neq 0$. In addition, from (18) and (28)

$$G_j = \sin(\bar{x}_{j+3}) F_{j+1} - C_{j+1} \cos(\bar{x}_{j+3}) \frac{G_j}{L_j},$$

which translates into

$$\frac{G_j}{L_j} = \frac{\sin(\bar{x}_{j+3})}{L_j + C_{j+1}\cos(\bar{x}_{j+3})} F_{j+1}, \quad (j = 1, \dots, N). \tag{32}$$

Using the definition of F_j, (29), and (32):

$$\frac{F_j}{F_{j+1}} = \frac{L_j \cos(\bar{x}_{j+3}) + C_{j+1}}{L_j + C_{j+1}\cos(\bar{x}_{j+3})}, \quad (j = 1, \dots, N). \tag{33}$$

Setting $C_{N+2} \triangleq 0$, and using (32) and (33),

$$\frac{G_j}{L_j} = \frac{\sin(\bar{x}_{j+3})}{L_j \cos(\bar{x}_{j+3}) + C_{j+1}} F_j, \quad (j = 1, \dots, N+1) \tag{34}$$

and, equivalently

$$\frac{G_{j+1}}{L_{j+1}} = \frac{\sin(\bar{x}_{j+4})}{L_{j+1}\cos(\bar{x}_{j+4}) + C_{j+2}} F_{j+1}, \quad (j = 0, \dots, N). \tag{35}$$

Using (29), (32) and (35), we obtain the following recursive expressions for the values of $\bar{x}_4, \dots, \bar{x}_{N+4}$:

$$\frac{\sin(\bar{x}_{j+3})}{L_j + C_{j+1}\cos(\bar{x}_{j+3})} = \frac{\sin(\bar{x}_{j+4})}{L_{j+1}\cos(\bar{x}_{j+4}) + C_{j+2}}, \quad (j = 1, \dots, N). \tag{36}$$

The next step consists in finding a relation between $\bar{x}_3$ and any $\bar{x}_i$, with $i \in \{4, \ldots, N+4\}$. Combining (28) and (34):

$$\frac{\sin(\bar{x}_{M+3})}{L_M \cos(\bar{x}_{M+3}) + C_{M+1}} = \frac{\kappa \cos(\bar{x}_3)}{1 + \kappa(\sin(\bar{x}_3) d_x + \cos(\bar{x}_3) d_y)} \triangleq A_\kappa \tag{37}$$

In order to find the values of $\bar{x}_{M+3}$ that solve this equation, we use the "half-angle" transformation $\tau \triangleq \tan(\bar{x}_{M+3}/2)$. Relation (37) then becomes

$$A_\kappa (C_{M+1} - L_M)\tau^2 - 2\tau + A_\kappa (L_M + C_{M+1}) = 0. \tag{38}$$

This quadratic (algebraic) equation in τ degenerates to a linear one when $A_\kappa (C_{M+1} - L_M) = 0$. However, under the assumptions $\kappa \neq 0$ and $|\bar{x}_3| < \frac{\pi}{2}$, A_κ cannot vanish, so this singular case occurs only when $C_{M+1} = L_M$. Its corresponding solution is $\tau = A_\kappa$, which tends to zero as κ does. On the other hand, when $A_\kappa (C_{M+1} - L_M) \neq 0$, the solutions to (38) are given by

$$\tau_{1,2} = \frac{1}{A_\kappa (C_{M+1} - L_M)} \pm \frac{\sqrt{L_M^2 - A_\kappa^2 (C_{M+1}^2 - L_M^2)}}{A_\kappa (C_{M+1} - L_M)}. \tag{39}$$

Since $\lim_{\kappa \to 0} A_\kappa = 0$, one verifies easily that

$$\lim_{\kappa \to 0} |\tau_1| = +\infty \tag{40}$$

$$\lim_{\kappa \to 0} \tau_2 = 0. \tag{41}$$

Since the absolute value of the solution $\bar{x}_{M+3} = 2\arctan(\tau_1)$ tends to π (i.e. $\bar{x}_{M+3}$ tends to leave the interval of interest $(-\pi/2, \pi/2)$) as κ tends to zero, we simply discard this solution. Therefore, with

$$\bar{x}_{M+3} = \begin{cases} A_\kappa, & \text{if } C_{M+1} = L_M \\ 2\arctan(\tau_2), & \text{otherwise,} \end{cases} \tag{42}$$

$|\bar{x}_{M+3}|$ can be made arbitrarily small by choosing a sufficiently small $|\kappa|$.

Notice that, once $\bar{x}_{M+3}$ has been determined, one can use a similar procedure to transform the relationships (36) into quadratic equations analogous to (38), giving $\bar{x}_{j+3}$ in terms of $\bar{x}_{j+4}$ and vice versa, for $j = 1, \ldots, N$. Again, by computing limits as above, one readily verifies that if $|\bar{x}_{M+3}|$ is small enough, then $\bar{x}_4, \ldots, \bar{x}_{M+2}$ as well as $\bar{x}_{M+4}, \ldots, \bar{x}_{N+4}$ exist and belong to $(-\frac{\pi}{2}, \frac{\pi}{2})$. As previously stated, this is accomplished for all $\kappa \in (\kappa_{\min}, \kappa_{\sup})$ with both $\kappa_{\min}$ and $\kappa_{\sup}$ sufficiently small. ∎

Proof of Proposition 3

The proof requires two auxiliary lemmas. The first one explicitly gives the first-order approximation of the recursive functions F_i, G_i at the origin. The second one details the structure of $\frac{\partial f(\bar{x})}{\partial x}$.

Lemma 2 *The expressions $F_i(x)$ and $G_i(x)$ ($i = 1, \ldots, N+1$) given by (18) can be written in the form*

$$\begin{aligned} F_i(x) &= 1 + o(x) && (43)\\ G_i(x) &= x_{i+3} - \sum_{j=i+4}^{N+4} (-1)^{i+j} \left(\prod_{k=i+1}^{j-3} \frac{C_k}{L_k} \right) x_j + o(x), && (44) \end{aligned}$$

where the terms $o(x)$ represent functions $h : \mathbb{R}^n \to \mathbb{R}$ such that $\lim_{x\to 0} \frac{|h(x)|}{\|x\|} = 0$.

Proof. We proceed by reverse induction on i (i.e. starting with $i = N+1$ and ending with $i = 1$) and we use Taylor series expansions at $x = 0$. For $i = N+1$, the result is clear:

$$\begin{aligned} F_{N+1}(x) &= \cos(x_{N+4}) = 1 + o(x) \\ G_{N+1}(x) &= \sin(x_{N+4}) = x_{N+4} + o(x). \end{aligned}$$

Now assume that (43) and (44) are true for $i = m+1$. Using (18) we readily obtain

$$\begin{aligned} F_m(x) &= (1 + o(x))(1 + o(x)) + \frac{C_{m+1}}{L_{m+1}} \sin(x_{m+3}) G_{m+1}(x) \\ &= 1 + o(x), \end{aligned}$$

and

$$\begin{aligned} G_m(x) &= (x_{m+3} + o(x))(1 + o(x)) \\ &\quad - \frac{C_{m+1}}{L_{m+1}}(1 + o(x)) \left[x_{m+4} - \sum_{j=m+5}^{N+4} (-1)^{m+j+1} \left(\prod_{k=m+2}^{j-3} \frac{C_k}{L_k} \right) x_j + o(x) \right] \\ &= x_{m+3} - \frac{C_{m+1}}{L_{m+1}} \left[x_{m+4} - \sum_{j=m+5}^{N+4} (-1)^{m+j+1} \left(\prod_{k=m+2}^{j-3} \frac{C_k}{L_k} \right) x_j \right] + o(x) \\ &= x_{m+3} - \frac{C_{m+1}}{L_{m+1}} x_{m+4} - \sum_{j=m+5}^{N+4} (-1)^{m+j} \frac{C_{m+1}}{L_{m+1}} \left(\prod_{k=m+2}^{j-3} \frac{C_k}{L_k} \right) x_j + o(x) \\ &= x_{m+3} - \frac{C_{m+1}}{L_{m+1}} x_{m+4} - \sum_{j=m+5}^{N+4} (-1)^{m+j} \left(\prod_{k=m+1}^{j-3} \frac{C_k}{L_k} \right) x_j + o(x) \\ &= x_{m+3} - \sum_{j=m+4}^{N+4} (-1)^{m+j} \left(\prod_{k=m+1}^{j-3} \frac{C_k}{L_k} \right) x_j + o(x), \end{aligned}$$

which is simply (44) with $i = m$. ∎

Lemma 3 *For $\kappa(x_1) \equiv 0$ and $\bar{x} = (s_0, 0, \dots, 0)^T$ with $s_0 \in \mathbb{R}$, the matrix $A \triangleq \frac{\partial f(\bar{x})}{\partial x}$ computed from (17) is*

$$\begin{bmatrix} 0 & 0 & 0 & A_{1,4} & A_{1,5} & \cdots & A_{1,N+3} & A_{1,N+4} \\ 0 & 0 & 1 & A_{2,4} & A_{2,5} & \cdots & A_{2,N+3} & A_{2,N+4} \\ 0 & 0 & 0 & A_{3,4} & A_{3,5} & \cdots & A_{3,N+3} & A_{3,N+4} \\ 0 & 0 & 0 & -\frac{1}{L_1} & \frac{L_1+C_2}{L_1L_2} & \cdots & \frac{(-1)^N C_3C_4\cdots C_N(L_1+C_2)}{L_1L_2\cdots L_N} & \frac{(-1)^{N-1} C_3C_4\cdots C_{N+1}(L_1+C_2)}{L_1L_2\cdots L_{N+1}} \\ 0 & 0 & 0 & 0 & -\frac{1}{L_2} & \cdots & \frac{(-1)^{N-1} C_4C_5\cdots C_N(L_2+C_3)}{L_2L_3\cdots L_N} & \frac{(-1)^{N} C_4C_5\cdots C_{N+1}(L_2+C_3)}{L_2L_3\cdots L_{N+1}} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & -\frac{1}{L_N} & \frac{L_N+C_{N+1}}{L_NL_{N+1}} \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \tag{45}$$

with

$$A_{1,j} = -d_y A_{3,j} \tag{46}$$

$$A_{2,j} = d_x A_{3,j} \tag{47}$$

$$A_{3,j} = \begin{cases} 0, & 4 \le j \le M+2 \\ \frac{(-1)^{M+j+1}}{L_M} \prod_{k=M+1}^{j-3} \frac{C_k}{L_k}, & M+3 \le j \le N+4. \end{cases} \tag{48}$$

Proof. By inspection of (17), one verifies that when $\kappa(x_1) \equiv 0$, x_1 does not appear explicitly in $f(x)$; it then follows that $\frac{\partial f(\bar{x})}{\partial x} = \frac{\partial f(0)}{\partial x}$. This latter matrix can be obtained by means of (43) and (44) by conserving only the linear terms in x. Since $A_{1,j}$ and $A_{2,j}$ are given in terms of $A_{3,j}$, let us first develop $f_3(x)$:

$$\begin{aligned} f_3(x) &= \frac{G_M(x)}{L_M} \\ &= \frac{1}{L_M}\left[x_{M+3} - \sum_{j=M+4}^{N+4} (-1)^{M+j} \left(\prod_{k=M+1}^{j-3} \frac{C_k}{L_k}\right) x_j\right] + o(x). \end{aligned}$$

Since $M \ge 1$, we see that the coefficients of $x_1, \dots, x_{M+2}$ are zero, whereas the remaining ones correspond to those in (48). The development for $f_1(x)$ yields

$$\begin{aligned} f_1(x) = W(x, x_3) &= \cos(x_3)F_M(x) - (\sin(x_3)d_x + \cos(x_3)d_y)\frac{G_M(x)}{L_M} \\ &= (1 + o(x)) - \frac{d_x(x_3 + o(x)) + d_y(1 + o(x))}{L_M} G_M(x) \\ &= -d_y \frac{G_M(x)}{L_M} + 1 + o(x), \end{aligned}$$

It follows directly that the $A_{1,j}$'s are in the form given by (46). Similarly, for $f_2(x)$:

$$\begin{aligned} f_2(x) = W(x, x_3 - \frac{\pi}{2}) &= \sin(x_3)F_M(x) + (\cos(x_3)d_x - \sin(x_3)d_y)\frac{G_M(x)}{L_M} \\ &= x_3 + d_x\frac{G_M(x)}{L_M} + o(x). \end{aligned}$$

The term x_3 accounts for the entry $A_{2,3} = 1$ in (45), whereas, by inspection of the next term, the $A_{2,j}$'s are given by (47).

Now, let us turn to $f_i(x) = \frac{G_{i-2}(x)}{L_{i-2}} - \frac{G_{i-3}(x)}{L_{i-3}}$, $(i = 4, \ldots, N+3)$. We start by deriving a relation between $G_{i-2}(x)$ and $G_{i-3}(x)$:

$$\begin{aligned} G_{i-3}(x) &= x_i - \sum_{j=i+1}^{N+4}(-1)^{i+j-3}\left(\prod_{k=i-2}^{j-3}\frac{C_k}{L_k}\right)x_j + o(x) \\ &= x_i - \left[\frac{C_{i-2}}{L_{i-2}}x_{i+1} - \frac{C_{i-2}}{L_{i-2}}\sum_{j=i+2}^{N+4}(-1)^{i+j-2}\left(\prod_{k=i-1}^{j-3}\frac{C_k}{L_k}\right)x_j\right] + o(x) \\ &= x_i - \frac{C_{i-2}}{L_{i-2}}G_{i-2}(x) + o(x). \end{aligned}$$

Using this relation, we readily determine $f_i(x)$ for $i = 4, \ldots, N+3$:

$$\begin{aligned} f_i(x) &= \frac{1}{L_{i-2}}G_{i-2}(x) - \frac{1}{L_{i-3}}\left(x_i - \frac{C_{i-2}}{L_{i-2}}G_{i-2}(x) + o(x)\right) \\ &= \left(\frac{1}{L_{i-2}} + \frac{C_{i-2}}{L_{i-2}L_{i-3}}\right)G_{i-2}(x) - \frac{1}{L_{i-3}}x_i + o(x) \\ &= -\frac{1}{L_{i-3}}x_i \\ &\quad + \frac{L_{i-3} + C_{i-2}}{L_{i-2}L_{i-3}}\left[x_{i+1} - \sum_{j=i+2}^{N+4}(-1)^{i+j}\left(\prod_{k=i-1}^{j-3}\frac{C_k}{L_k}\right)x_j\right] + o(x). \end{aligned} \tag{49}$$

The structure shown in (45) for the fourth row and up to the $N+3$-th one clearly follows from (49). The last row is obviously zero. ∎

Proof of Proposition 3. We use the fact (cf. [5, p. 140]) that a pair (A, B) is controllable iff every eigenvector v of A^T verifies $B^T v \neq 0$. The matrix A_r^T, directly obtained from Lemma 3, can be written as:

$$A_r^T = \begin{bmatrix} 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ 1 & 0 & 0 & & & & \vdots \\ A_{2,4} & A_{3,4} & A_{4,4} & 0 & & & \vdots \\ A_{2,5} & A_{3,5} & A_{4,5} & A_{5,5} & 0 & & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ A_{2,N+3} & A_{3,N+3} & A_{4,N+3} & A_{5,N+3} & \cdots & A_{N+3,N+3} & 0 \\ A_{2,N+4} & A_{3,N+4} & A_{4,N+4} & A_{5,N+4} & \cdots & A_{N+3,N+4} & 0 \end{bmatrix} \tag{50}$$

with entries $A_{i,j}$ given by:

- a) For $i = 2, 3$:

$$A_{2,j} = \begin{cases} 0, & 4 \leq j \leq M+2 \\ \frac{(-1)^{M+j+1} d_x}{L_M} \prod_{m=M+1}^{j-3} \frac{C_m}{L_m}, & M+3 \leq j \leq N+4 \end{cases}$$

$$A_{3,j} = \begin{cases} 0, & 4 \leq j \leq M+2 \\ \frac{(-1)^{M+j+1}}{L_M} \prod_{m=M+1}^{j-3} \frac{C_m}{L_m}, & M+3 \leq j \leq N+4. \end{cases}$$

- b) For $i = 4, \ldots, N+3$:

$$A_{i,j} = \begin{cases} 0, & 2 \leq j < i \\ -\frac{1}{L_{i-3}}, & j = i \\ (-1)^{i+j+1} \left(\frac{L_{i-3} + C_{i-2}}{L_{i-3} L_{i-2}} \right) \prod_{m=i-1}^{j-3} \frac{C_m}{L_m}, & i < j \leq N+4. \end{cases}$$

- c) For $i = N+4$:

$$A_{N+4,j} = 0, \quad (2 \leq j \leq N+4).$$

Since $B_r^T = (0, \ldots, 0, 1)$, one only has to show that if v is an eigenvector for A_r^T, then its last component v_{N+3} is nonzero. Since A_r^T is lower triangular, its eigenvalues are the diagonal entries:

$$\sigma(A_r^T) = \left(0, 0, -\frac{1}{L_1}, \ldots, -\frac{1}{L_N}, 0 \right).$$

Let v denote an eigenvector associated with an eigenvalue λ. For the remaining of the proof we will repeatedly use the matrix equation $\lambda v = A_r^T v$ which, after development, is equivalent to:

$$\lambda v_1 = 0 \tag{51}$$
$$\lambda v_2 = v_1 \tag{52}$$
$$\lambda v_i = \sum_{j=1}^{N+3} A_{j+1,i+1} v_j, \quad (i = 3, \ldots, N+3). \tag{53}$$

We consider two cases:

- **Case 1:** $\lambda = 0$. In this case, (52) implies that $v_1 = 0$. Let us distinguish three sub-cases:

  (i) $(i = 3, \ldots, M+1)$**.** We claim that $v_i = 0$ for $i = 3, \ldots, M+1$, and prove it by induction. Notice that when $M = 1$ there is nothing left to prove. We therefore assume that $M \in \{2, \ldots, N+1\}$. Recalling that $A_{2,j+1} = A_{3,j+1} = 0$ for $j = 3, \ldots, M+1$, then (53), with $i = 3$, gives:

  $$\begin{aligned} 0 &= A_{3,4} v_2 + A_{4,4} v_3 + \sum_{j=4}^{N+3} A_{j+1,4} v_j \\ &= -\frac{1}{L_1} v_3, \end{aligned}$$

  so that $v_3 = 0$. Now suppose that $v_i = 0$ for $i = 3, \ldots, k-1$, and develop (53) for $i = k$:

  $$\begin{aligned} 0 &= A_{3,k+1} v_2 + \sum_{j=3}^{k-1} A_{j+1,k+1} v_j + A_{k+1,k+1} v_k + \sum_{j=k+1}^{N+3} A_{j+1,k+1} v_j \\ &= -\frac{1}{L_{k-2}} v_k, \end{aligned}$$

  so that $v_k = 0$, as claimed.

  (ii) $(i = M+2, \ldots, N+2)$**.** We contend that $v_i = v_2$ for $i = M+2, \ldots, N+2$, and prove it by induction. For $i = M+2$, (53) becomes:

  $$\begin{aligned} 0 &= A_{3,M+3} v_2 + \sum_{j=3}^{M+1} A_{j+1,M+3} v_j + A_{M+3,M+3} v_{M+2} \\ &= \frac{(-1)^{2M+4}}{L_M} \left(\prod_{m=M+1}^{M} \frac{C_m}{L_m} \right) v_2 - \frac{1}{L_M} v_{M+2} \\ &= \frac{1}{L_M} (v_2 - v_{M+2}), \end{aligned}$$

so that $v_2 = v_{M+2}$. Now assume that $v_i = v_2$ for $i = M+2, \dots, k-1$, and develop (53) for $i = k$:

$$\begin{aligned} 0 &= A_{3,k+1} v_2 + \sum_{j=3}^{M+1} A_{j+1,k+1} v_j + \sum_{j=M+2}^{k-1} A_{j+1,k+1} v_j \\ &\quad + A_{k+1,k+1} v_k + \sum_{j=k+1}^{N+3} A_{j+1,k+1} v_j. \end{aligned} \tag{54}$$

The first sum on the right hand side vanishes by virtue of sub-case **(i)**. Using the induction hypothesis we can write (54) as

$$0 = \left(A_{3,k+1} + \sum_{j=M+2}^{k-1} A_{j+1,k+1} \right) v_2 - \frac{1}{L_{k-2}} v_k. \tag{55}$$

We have

$$A_{3,k+1} = \frac{(-1)^{M+k+2}}{L_M} \prod_{m=M+1}^{k-2} \frac{C_m}{L_m} \tag{56}$$

and

$$\begin{aligned} \sum_{j=M+2}^{k-1} A_{j+1,k+1} &= \sum_{j=M+2}^{k-1} (-1)^{j+k+3} \frac{L_{j-2} + C_{j-1}}{L_{j-2} L_{j-1}} \prod_{m=j}^{k-2} \frac{C_m}{L_m} \\ &= - \sum_{j=M+2}^{k-1} (-1)^{j+k} \frac{1}{L_{j-1}} \prod_{m=j}^{k-2} \frac{C_m}{L_m} \\ &\quad - \sum_{j=M+2}^{k-1} (-1)^{j+k} \frac{C_{j-1}}{L_{j-2} L_{j-1}} \prod_{m=j}^{k-2} \frac{C_m}{L_m}. \end{aligned} \tag{57}$$

From (56) and (57) we get

$$
\begin{aligned}
A_{3,k+1} + \sum_{j=M+2}^{k-1} A_{j+1,k+1} &= -\sum_{j=M+1}^{k-1} (-1)^{j+k} \frac{1}{L_{j-1}} \prod_{m=j}^{k-2} \frac{C_m}{L_m} \\
&\quad - \sum_{j=M+2}^{k-1} (-1)^{j+k} \frac{1}{L_{j-2}} \prod_{m=j-1}^{k-2} \frac{C_m}{L_m} \\
&= -\sum_{j=M+1}^{k-1} (-1)^{j+k} \frac{1}{L_{j-1}} \prod_{m=j}^{k-2} \frac{C_m}{L_m} \\
&\quad + \sum_{j=M+1}^{k-2} (-1)^{j+k} \frac{1}{L_{j-1}} \prod_{m=j}^{k-2} \frac{C_m}{L_m} \\
&= -(-1)^{2k-1} \frac{1}{L_{k-2}} \prod_{m=k-1}^{k-2} \frac{C_m}{L_m}.
\end{aligned}
$$

Therefore, (55) becomes

$$
\begin{aligned}
0 &= \left(-(-1)^{2k-1} \frac{1}{L_{k-2}} \prod_{m=k-1}^{k-2} \frac{C_m}{L_m} \right) v_2 - \frac{1}{L_{k-2}} v_k \\
&= \frac{1}{L_{k-2}} (v_2 - v_k),
\end{aligned}
$$

implying that $v_k = v_2$, as required.

(iii) $(i = N+3)$**.** Here we use the results of **(i)** and **(ii)** to show that $v_2 = 0$. Developing (53) for $i = N+3$, we get

$$
\begin{aligned}
0 &= A_{3,N+4} v_2 + \sum_{j=3}^{M+1} A_{j+1,N+4} v_j + \sum_{j=M+2}^{N+2} A_{j+1,N+4} v_j + A_{N+4,N+4} v_{N+3} \\
&= A_{3,N+4} v_2 + \sum_{j=M+2}^{N+2} A_{j+1,N+4} v_2 .
\end{aligned}
$$

After similar calculations as above, this equation reduces to

$$
0 = \frac{1}{L_{N+1}} v_2 ,
$$

so that $v_2 = 0$.

Summarizing, we have shown that $v_1 = \cdots = v_{N+2} = 0$, therefore, the eigenvectors associated with $\lambda = 0$ are of the form $v = (0, \ldots, 0, v_{N+3})^T$. We conclude that $v_{N+3} \neq 0$, otherwise v would vanish, in contradiction with its definition.

- **Case 2:** $\lambda = -\frac{1}{L_r}$ for any $r \in \{1, \ldots, N\}$. Let $\lambda = -\frac{1}{L_r}$ in (51) and (52), we get

$$v_1 = 0, \quad v_2 = 0.$$

We proceed by contradiction: assume that $v_{N+3} = 0$, we shall prove by (reverse) induction that $v_i = 0$ for $i = N+2, \ldots, 3$. Multiplying (53), with $i = N+2$, by $\frac{C_{N+1}}{L_{N+1}}$, we get:

$$\begin{aligned}
-\frac{C_{N+1}}{L_{N+1}} \frac{v_{N+2}}{L_r} &= \frac{C_{N+1}}{L_{N+1}} \left(\sum_{j=3}^{N+1} A_{j+1,N+3} v_j + A_{N+3,N+3} v_{N+2} \right) \\
&= \frac{C_{N+1}}{L_{N+1}} \left[-\sum_{j=3}^{N+1} (-1)^{j+N} \frac{L_{j-2} + C_{j-1}}{L_{j-2} L_{j-1}} \left(\prod_{m=j}^{N} \frac{C_m}{L_m} \right) v_j - \frac{1}{L_N} v_{N+2} \right] \\
&= -\sum_{j=3}^{N+1} (-1)^{j+N} \frac{L_{j-2} + C_{j-1}}{L_{j-2} L_{j-1}} \left(\prod_{m=j}^{N+1} \frac{C_m}{L_m} \right) v_j \\
&\quad -\frac{C_{N+1}}{L_{N+1}} \frac{1}{L_N} v_{N+2} \qquad (58)
\end{aligned}$$

Similarly, setting $i = N+3$ in (53):

$$\begin{aligned}
0 = -\frac{1}{L_r} v_{N+3} &= \sum_{j=3}^{N+2} A_{j+1,N+4} v_j \\
&= \sum_{j=3}^{N+2} (-1)^{j+N} \frac{L_{j-2} + C_{j-1}}{L_{j-2} L_{j-1}} \left(\prod_{m=j}^{N+1} \frac{C_m}{L_m} \right) v_j \qquad (59)
\end{aligned}$$

By addition of (58) and (59),

$$\begin{aligned}
-\frac{C_{N+1}}{L_{N+1}} \frac{1}{L_r} v_{N+2} &= (-1)^{2N+2} \frac{L_N + C_{N+1}}{L_N L_{N+1}} \left(\prod_{m=N+2}^{N+1} \frac{C_m}{L_m} \right) v_{N+2} - \frac{C_{N+1}}{L_{N+1}} \frac{1}{L_N} v_{N+2} \\
&= \frac{1}{L_{N+1}} v_{N+2},
\end{aligned}$$

which yields:

$$0 = \left(\frac{1}{L_{N+1}} + \frac{C_{N+1}}{L_{N+1} L_r} \right) v_{N+2}.$$

Therefore $v_{N+2} = 0$ since the term in parenthesis is always nonzero. We now suppose that $v_i = 0$ for $i = N+2, \ldots, k+1$ ($k \geq 3$), and proceed as above, i.e. we let $i = k$

in (53), multiply it by $\frac{C_{k-1}}{L_{k-1}}$, and add the resulting equation to the one obtained from (53) with $i = k+1$. The calculations are analogous to those above and give

$$0 = \left(\frac{1}{L_{k-1}} + \frac{C_{k-1}}{L_{k-1} L_r} \right) v_k ,$$

so that $v_k = 0$.

From the above two cases we deduce that the last component of any eigenvector v of A_r^T is different from zero, so that $B_r^T v \neq 0$. Therefore the pair (A_r, B_r) is controllable. ∎

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