

# A Primitive Recursive Algorithm for the General Petri Net Reachability Problem

Zakaria Bouziane

► **To cite this version:**

| Zakaria Bouziane. A Primitive Recursive Algorithm for the General Petri Net Reachability Problem.  
| [Research Report] RR-3404, INRIA. 1998. inria-00073286

**HAL Id: inria-00073286**

**<https://hal.inria.fr/inria-00073286>**

Submitted on 24 May 2006

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

***A Primitive Recursive Algorithm for the  
General Petri Net Reachability Problem***

Zakariae Bouziane

**N° 3404**

Avril 1998

————— THÈME 1 —————



***R**apport  
de recherche*





# A Primitive Recursive Algorithm for the General Petri Net Reachability Problem

Zakariae Bouziane

Thème 1 — Réseaux et systèmes  
Projet PAMPA

Rapport de recherche n3404 — Avril 1998 — 14 pages

**Abstract:** The general Petri net reachability problem is shown to be decidable in double exponential space. The previous algorithm is non-primitive recursive.

**Key-words:** Petri Net, Reachability, Decidability, Polynomial

*(Résumé : tsvp)*

# Un algorithme primitif récursif pour le problème de l'accessibilité dans les réseaux de Petri

**Résumé :** On montre que le problème d'accessibilité dans les réseaux de Petri est décidable en espace doublement exponentiel. L'algorithme existant avant est non primitif récursif.

**Mots-clé :** Réseau de Petri, Accessibilité, Décidabilité, Polynôme

## 1 Introduction.

Many aspects of the fundamental nature of computation are often studied via formal models, such as Turing machines, finite-state machines, and push-down automata. One formalism that has been used to model parallel computations is the Petri net [7, 8]. As a means of gaining a better understanding of the Petri net model, the decidability and computational complexity of typical automata theoretic problems concerning Petri nets have been examined. Examples of such problems include deadlock freedom, and liveness of the system. Solutions to these example problems proclaim, in a sense, the absence of “difficulties” for all states that are reachable in the system.

Another question is whether some arbitrary state can be reached from a fixed initial state. The later, the so-called general reachability problem, is of basic importance for many others. It is recursively equivalent to the liveness problem [2]. Moreover, a number of other problems in the representation of parallel and concurrent systems, in language generating systems, in algebra and in number theory can be shown to be effectively reducible or equivalent to the reachability problem.

Cardoza, Lipton, Mayr and Meyer [1, 6] have shown exponential space lower bound for the reachability problem, but the only known algorithm is non-primitive recursive [5, 4]. Even the decidability of this problem was an open question for many years.

## 2 Petri nets and polynomial equations

As usual,  $\mathbf{IN}$  denotes the set of nonnegative integers and  $\mathbf{IN}^m$  denotes the  $m$ -dimensional column vectors of natural numbers. For any integers  $a, b \in \mathbf{IN}$  such that  $a \leq b$ , the interval  $[a, b]$  denotes the set  $\{a, a + 1, \dots, b\}$ . For any vectors  $V_1, V_2 \in \mathbf{IN}^m$  we write:

- $V_1 = V_2$  if and only if  $V_1(i) = V_2(i)$ , for every  $i \in \{1, 2, \dots, m\}$ ,

- $V_1 + V_2$  to denote the vector of  $\mathbf{IN}^m$  whose  $i^{th}$  component is  $V_1(i) + V_2(i)$ ,
- $\sup(V_1, V_2)$  to denote the vector of  $\mathbf{IN}^m$  whose  $i^{th}$  component is  $\max(V_1(i), V_2(i))$  and
- $\|V_1\|_\infty = \max\{V_1(i) \mid i \in [1, m]\}$ .

A *Petri net* is a tuple  $N = (P, T, F, M_0)$  where,  $P = \{p_1, p_2, \dots, p_m\}$  is a finite set of *places*,  $T = \{t_1, t_2, \dots, t_n\}$  is a finite set of *transitions*,  $F : (P \times T) \cup (T \times P) \rightarrow \mathbf{IN}$  is a *flow function*<sup>1</sup>, and  $M_0 \in \mathbf{IN}^m$  is an *initial marking*. In all this paper  $m$  will designate the number of places and  $n$  the number of transitions of the Petri net.

Let  $N = (P, T, F, M_0)$  be a Petri net and  $M_f \in \mathbf{IN}^m$  be a marking.  $M_f$  is said to be *reachable* in  $N$ , and we note it  $M_f \in \mathcal{RS}(N)$ , if there exist a finite set of vectors  $H_1, H_2, \dots, H_k \in \mathbf{IN}^m$  and a sequence of transitions  $t_{i_1}, t_{i_2}, \dots, t_{i_k} \in T$  such that

$$\left\{ \begin{array}{l} M_0 = H_1 + F(., t_{i_1}) \\ H_j + F(t_{i_j}, .) = H_{j+1} + F(., t_{i_{j+1}}) \quad \text{for every } j \in [1, k-1] \\ H_k + F(t_{i_k}, .) = M_f \end{array} \right. \quad (1)$$

Note that system (1) has a solution if and only if it has a solution such that, for every  $j, j' \in [1, k]$ ,  $j \neq j'$ , we have  $M_0 \neq H_j + F(., t_{i_j}) \neq H_{j'} + F(., t_{i_{j'}})$ .

The **general Petri net reachability problem** is the problem of deciding for a given a Petri net  $N = (P, T, F, M_0)$  and a marking  $M_f \in \mathbf{IN}^m$  whether  $M_f \in \mathcal{RS}(N)$ .

Our computation of an upper bound for the reachability problem is based on a reduction of this later to the problem of solving a polynomial equation. For that we need to define the polynomials associated to a Petri net. We first

---

<sup>1</sup>We write  $F(., t)$  (resp.  $F(t, .)$ ) to denote the  $m$ -dimensional column vector whose  $i^{th}$  component is  $F(p_i, t)$  (resp.  $F(t, p_i)$ ).

code the set of vectors into integers by the help of an *injective function*  $\varphi$  defined by  $\varphi(V) = 2^{V(1)}.3^{V(2)}.5^{V(3)} \dots q^{V(m)}$  where  $q$  is the  $m^{\text{th}}$  prime number. Then, in order to simplify the notations, we put  $\alpha_0 = \varphi(M_f)$ ,  $\beta_0 = \varphi(M_0)$  and,  $\alpha_i = \varphi(F(., t_i))$  and  $\beta_i = \varphi(F(t_i, .))$ , for  $i \in [1, n]$ . Finally, let  $\mathbf{I}[X]$  be the set of polynomials over one variable  $X$  and whose coefficients are in  $\{0, 1\}$ . All the polynomials we will deal with in this paper are in  $\mathbf{I}[X]$ .

The reduction from the reachability problem to the problem of solving a polynomial system is given by the following lemma.

**Lemma 2.1**  *$M_f \in \mathcal{RS}(N)$  if and only if there exists a finite sequence of integers  $h_{i,j} \in \mathbf{N}$  such that the following polynomial equation is verified*

$$X^{\alpha_0} + \sum_{i=1}^n \sum_{j=1}^{k_i} X^{h_{i,j} \cdot \alpha_i} = X^{\beta_0} + \sum_{i=1}^n \sum_{j=1}^{k_i} X^{h_{i,j} \cdot \beta_i} \quad \in \mathbf{I}[X] \quad (2)$$

**Proof.** For every vectors  $V_1, V_2, V_3 \in \mathbf{N}^m$ , we have

$$[V_1 = V_2 + V_3] \iff [\varphi(V_1) = \varphi(V_2) \cdot \varphi(V_3)]$$

Thus, for every vectors  $V_1, V_2, V_3 \in \mathbf{N}^m$ , we have

$$[V_1 = V_2 + V_3] \iff [X^{\varphi(V_1)} = X^{\varphi(V_2) \cdot \varphi(V_3)}]$$

If we put  $l_i = \varphi(H_i)$ , for  $i \in [1, k]$ , then, from the reachability definition, we deduce that  $M_f \in \mathcal{RS}(N)$  if and only if we have

$$\begin{cases} \alpha_0 & = & l_1 \cdot \beta_{i_1} \\ l_j \cdot \alpha_{i_j} & = & l_{j+1} \cdot \beta_{i_{j+1}} \quad \text{for } j \in [1, k-1] \\ l_k \cdot \alpha_{i_k} & = & \beta_0 \end{cases}$$



Hence,  $M_f \in \mathcal{RS}(N)$  if and only if we have

$$\begin{cases} X^{\alpha_0} &= X^{l_1 \cdot \beta_{i_1}} \\ X^{l_j \cdot \alpha_{i_j}} &= X^{l_{j+1} \cdot \beta_{i_{j+1}}} \quad \text{for } j \in [1, k-1] \\ X^{l_k \cdot \alpha_{i_k}} &= X^{\beta_0} \end{cases}$$

We can choose  $\alpha_0 \neq l_j \cdot \alpha_{i_j} \neq l_{j'} \cdot \alpha_{i_{j'}}$ , for every  $j, j' \in [1, k]$  (This is equivalent to  $M_0 \neq H_j + F(t_{i_j}, \cdot) \neq H_{j'} + F(t_{i_{j'}}, \cdot)$ ).

Thus,  $M_f \in \mathcal{RS}(N)$  if and only if we have

$$X^{\alpha_0} + \sum_{j=1}^k X^{l_j \cdot \alpha_{i_j}} = X^{\beta_0} + \sum_{j=1}^k X^{l_j \cdot \beta_{i_j}} \in \mathbf{I}[X]$$

■

### 3 Solution of the polynomial equation

From lemma 2.1 we deduce that the reachability problem is equivalent to the problem of deciding whether system (2) has a solution. In this section we show that this later is decidable. Our proof consists in reducing system (2) to an equivalent system made of two equations (lemma 3.1). Then we show that the set of solutions to the first one can be effectively constructed (lemma 3.2). Finally, we construct this set and extract from it the solutions that also verify the second equation (lemma 3.3 and lemma 3.4).

**Lemma 3.1** *Equation (2) has a solution if and only if there exist two finite sequences of integers  $f_{i,j}, g_{i,j} \in \mathbf{IN}$  such that the two following polynomial equations are verified*

$$X^{\alpha_0} + \sum_{i=1}^n \sum_{j=1}^{k_i} X^{f_{i,j} \cdot \alpha_i} = X^{\beta_0} + \sum_{i=1}^n \sum_{j=1}^{k'_i} X^{g_{i,j} \cdot \beta_i} \in \mathbf{I}[X] \quad (3)$$

$$\sum_{j=1}^{k_i} (X^{\beta_i})^{f_{i,j} \cdot \alpha_i} = \sum_{j=1}^{k'_i} (X^{\alpha_i})^{g_{i,j} \cdot \beta_i} \quad \text{for every } i \in [1, n] \quad (4)$$

**Proof.** Equation (4) is equivalent to the proposition

$$f_{i,j} = g_{i,j} \quad \text{for } i \in [1, n] \text{ and } j \in [1, k_i]$$

Thus, the intersection of equation (3) and equation (4) is equivalent to equation (2).  $\blacksquare$

Let  $E_0$  be the set of uplets  $(P_1, \dots, P_{2.n})$  such that, for every  $i \in [1, n]$ ,  $P_i \in \mathbf{I}[X^{\alpha_i}]$  and  $P_{n+i} \in \mathbf{I}[X^{\beta_i}]$ , and  $X^{\alpha_0} + \sum_{i=1}^n P_i(X) = X^{\beta_0} + \sum_{i=n+1}^{2.n} P_i(X) \in \mathbf{I}[X]^2$ .  $E_0$  is then the set of solutions to equation (3).

Let  $l_0 = lcm\{\alpha_i, \beta_i \mid i \in [1, n]\}$  (*lmc* is the least common multiple) and  $l_1 = \min\{\lambda \cdot l_0 \mid \lambda \in \mathbf{IN} \text{ and } \max(\alpha_0, \beta_0) < \lambda \cdot l_0\}$ .

Let  $E_1$  be the set of uplets  $(Q_1, \dots, Q_{2.n}) \in E_0$  such that, for every  $i \in [1, 2.n]$ , we have  $\deg(Q_i) < l_1$ .  $E_1$  is finite and we can write  $E_1 = \{(Q_1^1, \dots, Q_{2.n}^1), \dots, (Q_1^h, \dots, Q_{2.n}^h)\}$ .

Let  $E_2$  be the set of uplets  $(R_1, \dots, R_{2.n})$  such that for every  $i \in [1, n]$ , we have  $R_i \in \mathbf{I}[X^{\alpha_i}]$ ,  $R_{n+i} \in \mathbf{I}[X^{\beta_i}]$ ,  $\max\{\deg(R_i), \deg(R_{n+i})\} < l_0$  and

$$\sum_{i=1}^n R_i(X) = \sum_{i=n+1}^{2.n} R_i(X)$$

$E_2$  is finite and we can write  $E_2 = \{(R_1^1, \dots, R_{2.n}^1), \dots, (R_1^k, \dots, R_{2.n}^k)\}$ .

**Lemma 3.2**  $E_0$  is equal to the set of uplets  $(P_1, \dots, P_{2.n})$  such that

$$(P_1, \dots, P_{2.n}) = (Q_1, \dots, Q_{2.n}) + X^{l_1} \cdot \sum_{i=1}^k S_i(X) \cdot (R_1^i, \dots, R_{2.n}^i)$$

for some  $(Q_1, \dots, Q_{2.n}) \in E_1$  and  $S_1, \dots, S_k \in \mathbf{I}[X]$  such that  $\sum_{i=1}^k .S_i(X) = \sum_{i=0}^a X^{i.l_0}$ , for some  $a \in \mathbf{IN}$ .

**Proof.**

- For every  $(Q_1, \dots, Q_{2.n}) \in E_1$ , for every  $l \in [1, 2.n]$  and for every monomial  $X^c$  in  $Q_l(X)$  we have  $0 \leq c < l_1$ .  
For every  $i \in [1, k]$ , for every  $l \in [1, 2.n]$  and for every monomial  $X^c$  in  $R_i^l(X)$  we have  $0 \leq c < l_0$ .  
Hence, for every  $i \in [1, k]$ , for every  $l \in [1, 2.n]$  and for every monomial  $X^c$  in  $X^{l_1} . X^{b.l_0} . R_i^l(X)$  we have  $l_1 + b.l_0 \leq c < l_1 + (b+1).l_0$ .  
Consequently,  $Q_l(X) + X^{l_1} . \sum_{i=1}^k .S_i(X) . R_i^l(X) \in \mathbf{I}[X]$ . Thus the sum

$$(Q_1, \dots, Q_{2.n}) + X^{l_1} . \sum_{i=1}^k .S_i(X) . (R_1^i, \dots, R_{2.n}^i)$$

is in  $E_0$ .

- Let  $(P_1, \dots, P_{2.n}) \in E_0$  and let  $d = \max\{\deg(P_j) \mid j \in [1, 2.n]\}$ . For every  $j \in [1, 2.n]$ , let  $Q_j$  to be the largest sub-polynomial of  $P_j$  of degree less than  $l_1$ , and let  $R_j^l$  to be the largest sub-polynomial of  $P_j$  such that any monomial  $X^c$  of  $P_j$  verify  $l_1 + (l-1).l_0 \leq c < l_1 + l.l_0$  ( $l$  is such that  $l_1 + (l-1).l_0 \leq d$ ).

The uplet  $(Q_1, \dots, Q_{2.n})$  is then in  $E_1$ .

For  $j \leq n$  we have  $P_j \in \mathbf{I}[X^{\alpha_j}]$ . Hence every monomial  $X^c$  of  $R_j^l$  verify

$$l_1 + (l-1).l_0 \leq c = \lambda . \alpha_j < l_1 + l.l_0, \text{ for some } \lambda \in \mathbf{IN}$$

Moreover, from the definition of  $l_0$  and  $l_1$ , we deduce that  $l_0 = \lambda_0 . \alpha_j$  and  $l_1 = \lambda_1 . \alpha_j$ , for some  $\lambda_0, \lambda_1 \in \mathbf{IN}$ . Thus

$$(\lambda_1 + (l-1).\lambda_0) . \alpha_j \leq c = \lambda . \alpha_j < (\lambda_1 + l.\lambda_0) . \alpha_j$$

Which means that

$$X^c = X^{l_1} . X^{(l-1) \cdot l_0} . X^{b \cdot \alpha_j} \text{ for some } b \cdot \alpha_j < l_0$$

In the case where  $n < j \leq 2.n$ , we use the same proof by replacing  $\alpha_j$  by  $\beta_{j-n}$ .

Consequently,  $(R_1^l, \dots, R_{2.n}^l) \in E_1$ . Thus  $(P_1, \dots, P_{2.n})$  can be written in the form of the equation in lemma 3.2. ■

**Lemma 3.3** *Equation (2) has a solution if and only if there exist  $S_1, \dots, S_k \in \mathbf{I}[X]$  such that the two following systems are verified*

$$Q_l(2^{\beta_l}) + 2^{l_1 \cdot \beta_l} \cdot \sum_{i=1}^k S_i(2^{\beta_l}) \cdot R_l^i(2^{\beta_l}) = Q_{n+l}(2^{\alpha_l}) + 2^{l_1 \cdot \alpha_l} \cdot \sum_{i=1}^k S_i(2^{\alpha_l}) \cdot R_{n+l}^i(2^{\alpha_l}), \forall l \in [1, n] \quad (5)$$

and

$$\sum_{i=1}^k S_i(X) = \sum_{i=0}^a X^{i \cdot l_0} \quad (6)$$

for some  $a \in \mathbf{IN}$  and  $(Q_1, \dots, Q_{2.n}) \in E_1$ .

**Proof.** From lemma 3.1 and lemma 3.2, we deduce that equation (2) has a solution if and only if there exist  $S_1, \dots, S_k \in \mathbf{I}[X]$  and  $a \in \mathbf{IN}$  such that

$$\sum_{i=1}^k S_i(X) = \sum_{i=0}^a X^{i \cdot l_0}, \text{ and}$$

$$Q_l(X^{\beta_l}) + X^{l_1 \cdot \beta_l} \cdot \sum_{i=1}^k S_i(X^{\beta_l}) \cdot R_l^i(X^{\beta_l}) = Q_{n+l}(X^{\alpha_l}) + X^{l_1 \cdot \alpha_l} \cdot \sum_{i=1}^k S_i(X^{\alpha_l}) \cdot R_{n+l}^i(X^{\alpha_l}), \forall l \in [1, n]$$

However, for  $l \in [1, n]$ , the polynomials  $Q_l(X^{\beta_l}) + X^{l_1 \cdot \beta_l} \cdot \sum_{i=1}^k S_i(X^{\beta_l}) \cdot R_l^i(X^{\beta_l})$

and  $Q_{n+l}(X^{\alpha_l}) + X^{l_1 \cdot \alpha_l} \cdot \sum_{i=1}^k S_i(X^{\alpha_l}) \cdot R_{n+l}^i(X^{\alpha_l})$  are in  $\mathbf{I}[X]$ . Hence, these two polynomials are equal if and only if they have the same value on integer 2. This ends the proof. ■

**Lemma 3.4** *The system made of system (5) and system (6) has a solution if and only if it has a solution with  $a \leq 2^{2^{c \cdot n \cdot l_1}}$ , for some constant  $c \in \mathbf{IN}$ .*

**Proof.** For  $i \in [1, k]$ ,  $l \in [1, n]$ , let  $A_{i,l} = 2^{l \cdot \beta_l} \cdot R_l^i(2^{\beta_l})$ ,  $A_{i,n+l} = 2^{l \cdot \alpha_l} \cdot R_{n+l}^i(2^{\alpha_l})$ ,  $A_{0,l} = Q_l(2^{\beta_l})$  and  $A_{0,n+l} = Q_{n+l}(2^{\alpha_l})$ . Let  $\Sigma = \{Z \in \mathbf{IN}^k \mid \sum_{i=1}^k Z(i) = 1\}$ . Hence, equation (5) is equivalent to the linear diophantine system

$$A_{0,l} + \sum_{i=1}^k A_{i,l} \cdot Y_{i,l} = A_{0,n+l} + \sum_{i=1}^k A_{i,n+l} \cdot Y_{i,n+l}, \quad \forall l \in [1, n] \quad (7)$$

and equation (6) is equivalent to the system

$$\begin{cases} Y_{i,l} = Z_0(i) + 2^{l_0 \cdot \beta_l} \cdot Z_1(i) + \dots + 2^{a \cdot l_0 \cdot \beta_l} \cdot Z_a(i) & , \forall i \in [1, k], \forall l \in [1, n] \\ Y_{i,n+l} = Z_0(i) + 2^{l_0 \cdot \alpha_l} \cdot Z_1(i) + \dots + 2^{a \cdot l_0 \cdot \alpha_l} \cdot Z_a(i) & , \forall i \in [1, k], \forall l \in [1, n] \end{cases} \quad (8)$$

for some  $Z_0, \dots, Z_a \in \Sigma$ .

However, for every  $l \in [1, n]$ , the set of solution

$$\mathcal{S}_l = \{(Y_{1,l}, \dots, Y_{k,l}, Y_{1,n+l}, \dots, Y_{k,n+l}) \in \mathbf{IN}^{2 \cdot k} \mid A_{0,l} + \sum_{i=1}^k A_{i,l} \cdot Y_{i,l} = A_{0,n+l} + \sum_{i=1}^k A_{i,n+l} \cdot Y_{i,n+l}\}$$

is semi-linear, and can be written in the form

$$\mathcal{S}_l = \left\{ V_l + \sum_{j=1}^h \lambda_j \cdot W_{j,l} \mid V_l \in \mathcal{S}_{l,0}, W_{1,l}, \dots, W_{h,l} \in \mathcal{S}_{l,1}, \lambda_1, \dots, \lambda_h \in \mathbf{IN} \right\}$$

where  $\mathcal{S}_{l,0} \subset \mathbf{IN}^{2 \cdot k}$  is the set of *minimal solutions* of the  $l^{\text{th}}$  equation in system (7) and  $\mathcal{S}_{l,1} \subset \mathbf{IN}^{2 \cdot k}$  is the set of *minimal solutions* of its homogeneous part  $\sum_{i=1}^k A_{i,l} \cdot Y_{i,l} = \sum_{i=1}^k A_{i,n+l} \cdot Y_{i,n+l}$ .

We only want the solutions  $Y_l = (Y_{1,l}, \dots, Y_{k,l}, Y_{1,n+l}, \dots, Y_{k,n+l}) \in \mathcal{S}_l$  that verify system (8). Let  $Y_l^0 = V_{l,0} + \sum_{j=1}^h \lambda_j \cdot W_{j,l}$  be such a solution. It can be written in the form

$$Y_l^0 = V_{l,0} + \sum_{j=1}^h \lambda_{j,1} \cdot W_{j,l} + 2^{l_0 \cdot \gamma_l} \cdot \sum_{j=1}^h \lambda_{j,2} \cdot W_{j,l}$$

where,  $\gamma_l = \max\{\alpha_l, \beta_l\}$  and  $0 \leq \lambda_{j,1} < 2^{l_0 \cdot \gamma_l}$ .

Thus, for every  $i \in [1, k]$ , we have

$$\begin{cases} V_{l,0}(i) + \sum_{j=1}^h \lambda_{j,1} \cdot W_{j,l}(i) & = Z_0(i) + 2^{l_0 \cdot \beta_l} \cdot V'_{l,0}(i) \\ V_{l,0}(k+i) + \sum_{j=1}^h \lambda_{j,1} \cdot W_{j,l}(k+i) & = Z_0(i) + 2^{l_0 \cdot \alpha_l} \cdot V'_{l,0}(k+i) \end{cases}$$

for some  $V'_{l,0} \in \mathbf{IN}^{2 \cdot k}$ . Let  $\delta_l = \min\{\alpha_l, \beta_l\}$  and let  $V_{l,1} \in \mathbf{IN}^{2 \cdot k}$ , such that, for every  $i \in [1, k]$ , we have  $V_{l,1}(i) = \frac{2^{l_0 \cdot \beta_l}}{2^{l_0 \cdot \delta_l}} \cdot V'_{l,0}(i)$  and  $V_{l,1}(k+i) = \frac{2^{l_0 \cdot \alpha_l}}{2^{l_0 \cdot \delta_l}} \cdot V'_{l,0}(k+i)$ . Hence,  $Y_l^1 = V_{l,1} + \frac{2^{l_0 \cdot \gamma_l}}{2^{l_0 \cdot \delta_l}} \cdot \sum_{j=1}^h \lambda_{j,2} \cdot W_{j,l}$  verifies system (8).

By repeating the previous process we will obtain a sequence of vectors  $V_{l,0}, V_{l,1}, \dots, V_{l,a}$  in  $\mathbf{IN}^{2 \cdot k}$ . Moreover, we have

$$\|V_{l,j+1}\|_\infty \leq \frac{1}{2^{l_0 \cdot \delta_0}} \cdot [\|V_{l,j}\|_\infty + h \cdot 2^{l_0 \cdot \gamma_l} \cdot \max\{\|W_{j,l}\|_\infty \mid j \in [1, h]\}]$$

Hence,

$$\|V_{l,j+1}\|_\infty \leq \max(\|V_{l,0}\|_\infty, h \cdot 2^{l_0 \cdot \gamma_l} \cdot \max\{\|W_{j,l}\|_\infty \mid j \in [1, h]\}) \cdot \frac{2^{l_0 \cdot \delta_l}}{2^{l_0 \cdot \delta_l} - 1}$$

From [3], we deduce that, for every  $W_{j,l} \in \mathcal{S}_{l,1}$  and for every  $V_{l,0} \in \mathcal{S}_{l,0}$ , we have

$$\begin{cases} \|W_{j,l}\|_\infty \leq \max\{A_{i,l}, A_{i,n+l} \mid i \in [1, k]\} & \leq 2^{\gamma \cdot (l_0 + l_1)} \\ \|V_{l,0}\|_\infty \leq 2 \cdot \max\{A_{i,l}, A_{i,n+l} \mid i \in [0, k]\} & \leq 2^{\gamma \cdot (l_0 + l_1) + 1} \end{cases}$$

Hence, the number  $h$  of elements in  $\mathcal{S}_{l,1}$  verifies

$$h \leq 2^{2 \cdot k \cdot \gamma \cdot (l_0 + l_1)}$$

However, the number  $k$  of elements in  $E_2$  verifies

$$k \leq 2^{n \cdot l_0}$$

Consequently,

$$h \leq 2^{2 \cdot \gamma \cdot (l_0 + l_1) \cdot 2^{n \cdot l_0}} \leq 2^{2^{c_0 \cdot n \cdot l_1}}$$

for some constant  $c \in \mathbf{IN}$ .

Which means that

$$\|V_{l,j}\|_\infty \leq 2^{2^{c_1 \cdot n \cdot l_1}}$$

for some constant  $c$ .

Thus, the elements of  $\mathcal{S}_l$  that verify equation (8) can be generated by a union of  $|\mathcal{S}_{l,0}|$  automaton  $\mathcal{A}_l = \cup\{\mathcal{A}_l(V_{l,i}) \mid V_{l,i} \in \mathcal{S}_{l,0}\}$ . Each automata  $\mathcal{A}_l(V_{l,i})$  is over the alphabet  $\Sigma$  and has nodes in the set of vectors

$$\{V \in \mathbf{IN}^k \mid \|V\|_\infty \leq 2^{2^{c_1 \cdot n \cdot l_1}}\}$$

the initial marking is  $V_{l,i}$  the final marking is  $\mathbf{0} = (0, \dots, 0)$  and there is an arc from  $V_1$  to  $V_2$  labeled  $Z$  if and only if

$$\begin{cases} V_1(i) + \sum_{j=1}^h \lambda_j \cdot W_{j,l}(i) = Z(i) + 2^{l_0 \cdot \delta_i} \cdot V_2(i) \\ V_1(k+i) + \sum_{j=1}^h \lambda_j \cdot W_{j,l}(k+i) = Z(i) + 2^{l_0 \cdot \delta_i} \cdot V_2(k+i) \end{cases}$$

for  $i \in [1, k]$  and  $\lambda_1, \dots, \lambda_h < 2^{l_0 \cdot \gamma}$ .

Finally, the set of solutions of equation (7) that verify equation (8) is completely determined by the intersection

$$\mathcal{L} = \bigcap_{l=1}^n \mathcal{L}(\mathcal{A}_l)$$

where  $\mathcal{L}(\mathcal{A}_l)$  is the language recognized by  $\mathcal{A}_l$ .

The number of nodes of every automaton  $\mathcal{A}_l(V_{l,j})$  is less than  $2^{2^{c_2 \cdot n \cdot l_1}}$ , for some constant  $c_2$ . Hence,  $\mathcal{L} \neq \emptyset$  if and only if it contains a sequence of length no longer than  $2^{n \cdot 2^{c_2 \cdot n \cdot l_1}} \leq 2^{2^{c_3 \cdot n \cdot l_1}}$ , for some constant  $c_3 \in \mathbf{IN}$ . Thus, the system made of system (7) and system (8) has a solution if and only if it has a solution with  $a < 2^{2^{c \cdot n \cdot l_1}}$ , for some constant  $c \in \mathbf{IN}$ .  $\blacksquare$

## 4 The general Petri net reachability problem is primitive recursive

Now, we are ready to compute a primitive recursive algorithm for the general Petri net reachability problem.

**Theorem 4.1** *There exist a constant  $c$  such that, for every Petri net  $N = (P, T, F, M_0)$  and every marking  $M_f \in \mathbf{IN}^m$ , we have*

*$M_f \in \mathcal{RS}(N)$  if and only if there exist  $H_1, H_2, \dots, H_k \in \mathbf{IN}^m$  and there exist  $t_{i_1}, t_{i_2}, \dots, t_{i_k} \in T$  such that*

$$\left\{ \begin{array}{l} M_0 = H_1 + F(\cdot, t_{i_1}) \\ H_j + F(t_{i_j}, \cdot) = H_{j+1} + F(\cdot, t_{i_{j+1}}) \text{ for every } j \in [1, k-1] \\ M_f = H_k + F(t_{i_k}, \cdot) \end{array} \right.$$

*and for every  $j \in [1, k]$  we have*

$$\varphi(H_j + F(t_{i_j}, \cdot)) \leq 2^{2^{c \cdot n \cdot l}}$$

*where  $l = \min\{\lambda \cdot l_0 \mid \lambda \in \mathbf{IN}, \max\{\alpha_0, \beta_0\} < \lambda \cdot l_0\}$ ,  $\alpha_0 = \varphi(M_0)$ ,  $\beta_0 = \varphi(M_f)$ , and  $l_0 = \varphi(\sup\{F(t_i, \cdot), F(\cdot, t_i) \mid i \in [1, n]\})$ ,*

**Proof.** This theorem is a consequence of the previous section. ■

## 5 Open problems

- Reduction of the gap between our upper bound and Cardoza, Lipton, Mayr and Meyer's bound.
- Complexity of :
  - The *liveness* problem.
  - The *regularity* and the *context-freedom* problems.
  - The *semi-linearity* problem.



## References

- [1] **Cardoza E., Lipton R., Meyer A.** “*Exponential space complete problems for Petri nets and commutative semi-groups*”, Proc. of the 8<sup>th</sup> annual ACM Symposium on Theory of Computing, pp. 50-54, STOC’1976.
- [2] **Hack M.** “*The recursive equivalence of the liveness problem and the reachability problem for Petri nets and vector addition systems*”, Proc. of the 15<sup>th</sup> Annual Symposium on SWAT, New Orleans, LA, pp. 156-164, SWAT’1974.
- [3] **Huet G.** “*An algorithm to generate the basis of solutions to homogeneous linear diophantine equations*”, Inf. Proc. Let. 7 (3), pp144-147, 1978.
- [4] **Kosaraju R.** “*Decidability of reachability in Vector Addition Systems*”, Proc. of the 14<sup>th</sup> Annual ACM Symposium on Theory of Computing pp. 267-280, STOC’1982.
- [5] **Mayr E.** “*An algorithm for the general Petri net reachability problem*”, Proc. of the 13<sup>th</sup> Annual ACM Symposium on Theory of Computing pp. 238-246, STOC’1981.  
Also SIAM J. Compu. 13, 3, pp. 441-46-, 1984.
- [6] **Mayr E., Meyer R.** “*The complexity of the word problems for commutative semi-groups and polynomial ideals*”, Advances in Mathematics 46, pp. 305-329, 1982.
- [7] **Peterson J.** “*Petri net theory and the modeling of systems*”, Prentice-Hall, Inc.,1981.
- [8] **Reisig W.** “*Petri nets : an introduction*”, Springer, Heidelberg, 1985.



---

Unit ´e de recherche INRIA Lorraine, Technop ˆole de Nancy-Brabois, Campus scientifique,  
615 rue du Jardin Botanique, BP 101, 54600 VILLERS LÈS NANCY  
Unit ´e de recherche INRIA Rennes, Irisa, Campus universitaire de Beaulieu, 35042 RENNES Cedex  
Unit ´e de recherche INRIA Rh ˆone-Alpes, 655, avenue de l'Europe, 38330 MONTBONNOT ST MARTIN  
Unit ´e de recherche INRIA Rocquencourt, Domaine de Voluceau, Rocquencourt, BP 105, 78153 LE CHESNAY Cedex  
Unit ´e de recherche INRIA Sophia-Antipolis, 2004 route des Lucioles, BP 93, 06902 SOPHIA-ANTIPOLIS Cedex

---

´Editeur  
INRIA, Domaine de Voluceau, Rocquencourt, BP 105, 78153 LE CHESNAY Cedex (France)  
<http://www.inria.fr>  
ISSN 0249-6399