Singularity Analysis and Asymptotics of Bernoulli Sums

Philippe Flajolet

INSTITUT NATIONAL DE RECHERCHE EN INFORMATIQUE ET EN AUTOMATIQUE

Singularity analysis and asymptotics of Bernoulli sums

Philippe Flajolet

No 3401
Avril 1998

THÈME 2

Rapport de recherche

ISSN 0249-6399

Singularity analysis and asymptotics of Bernoulli sums

Philippe Flajolet

Thème 2 — Génie logiciel
et calcul symbolique
Projet Algo

Rapport de recherche— Avril 1998 — 12 pages

Abstract: The asymptotic analysis of a class of binomial sums that arise in information theory can be performed in a simple way by means of singularity analysis of generating functions. The method developed extends the range of applicability of singularity analysis techniques to combinatorial sums involving transcendental elements like logarithms or fractional powers.

Unité de recherche INRIA Rocquencourt

Domaine de Voluceau, Rocquencourt, BP 105, 78153 LE CHESNAY Cedex (France)

Téléphone : (33) 01 39 63 55 11 – Télécopie : (33) 01 39 63 53 30

Analyse de singularité et asymptotique des sommes de Bernoulli

Résumé : L'analyse asymptotique d'une classe de sommes qui interviennent en théorie de l'information peut être effectuée de manière simple par analyse de singularité de séries génératrices. La méthode développée dans ce rapport étend en fait l'applicabilité des techniques d'analyse de singularité à des sommes combinatoires comprenant des éléments transcendants tels des logarithmes ou des puissances fractionnaires.

SINGULARITY ANALYSIS AND ASYMPTOTICS OF BERNOULLI SUMS

PHILIPPE FLAJOLET

ABSTRACT. The asymptotic analysis of a class of binomial sums that arise in information theory can be performed in a simple way by means of singularity analysis of generating functions. The method developed extends the range of applicability of singularity analysis techniques to combinatorial sums involving transcendental elements like logarithms or fractional powers.

1. BERNOULLI TRANSFORMS

Let $\{f_k\}$ be a sequence of real numbers and $p \in (0,1)$ a real parameter that is fixed. The *Bernoulli transform* of f_k (with parameter p) is defined as

$$S_n[f] := \sum_{k=0}^{n} \binom{n}{k} f_k p^k q^{n-k}, \qquad q = 1-p. \tag{1}$$

Given the coefficients f_k, the problem is to estimate asymptotically $S_n[f]$.

In probabilistic terms, the Bernoulli transform $S_n[f]$ is the expectation of f under the binomial, or Bernoulli, probability distribution $\mathcal{B}(n,p)$ (that is, the distribution of the number of successes in n independent trials with individual success probability p). For a sequence f_k that is "smooth", the concentration of the binomial distribution around its mean pn leads us to expect, as regards dominant asymptotics at least, that

$$S_n[f] \sim f_{\lfloor pn \rfloor}, \tag{2}$$

It is possible, but somewhat unwieldy, to obtain detailed asymptotic estimates in this way. Such estimates are however needed in variance computations or in the analysis of redundancy of codes [9], where cancellations are inherent. Jacquet and Szpankowski have considered in [6] a number of cases precisely motivated by information theory problems. For instance, a question as natural as that of estimating the entropy of the binomial distribution, namely,

$$\mathrm{H}_{n,p} = -\sum_{k=0}^{n} \pi_{n,k} \log \pi_{n,k}, \qquad \pi_{n,k} = \binom{n}{k} p^k q^{n-k},$$

is equivalent to the analysis of the Bernoulli transform of $f_k = \log k!$. On a different register, the analysis of the mean and variance of the logarithm of a binomial random variable requires the transforms of $\log k$ and $\log^2 k$.

The approach of [6] relies on a chain that is based on exponential generating functions, Cauchy coefficient integrals, and saddle point estimates under the form of "analytic depoissonization" in the sense of [7]. In this note, I propose a surprisingly

Date: April 1, 1998.

simple approach that is based on ordinary (rather than exponential) generating functions and singularity analysis [4, 13].

We restrict attention to coefficients f_k that are of at most polynomial growth. Representative cases meant to illustrate the principles of the approach are

$$f_k^{(1)} = \frac{1}{4^k}\binom{2k}{k},\ f_k^{(2)} = H_k,\ f_k^{(3)} = \frac{1}{\sqrt{k}},\ f_k^{(4)} = \log k,\ f_k^{(5)} = \log(k!). \tag{3}$$

There $H_k = 1 + \frac{1}{2} + \cdots + \frac{1}{k}$ is the kth harmonic number and by convention $f_0^{(3)} = f_0^{(4)} = 0$. The corresponding Bernoulli transforms are denoted by $S_n^{(j)}$, $j = 1\,..\,5$ and their asymptotic evaluation is summarized by the following statement.

Proposition 1. *The Bernoulli transforms of the basic sequences of Eq.* (3) *satisfy:*

$$\begin{aligned}
S_n^{(1)} &= \frac{1}{\sqrt{\pi p n}}\left(1 - \frac{3p-2}{8pn} + \frac{25p^2-60p+36}{128p^2n^2} + O(\frac{1}{n^3})\right)\\
S_n^{(2)} &= p\log(pn) + \gamma + \frac{1}{2n} - \frac{1}{12n^2} + O(\frac{1}{n^3})\\
S_n^{(3)} &= \frac{1}{\sqrt{pn}}\left(1 - \frac{3(p-1)}{8\,pn} + \frac{5(p-1)(5p-13)}{128\,p^2n^2} + O(\frac{1}{n^3})\right)\\
S_n^{(4)} &= \log(pn) + \frac{p-1}{2\,pn} - \frac{p^2-6p+5}{12\,p^2n^2} + O(\frac{1}{n^3})\\
S_n^{(5)} &= \log\left((pn)^{pn}e^{-pn}\sqrt{2\pi pn}\right) - \frac{p-1}{2} - \frac{p^2-3p+1}{12\,pn} + O(\frac{1}{n^2}).
\end{aligned} \tag{4}$$

To some extent in this range of problems, methods are at least as interesting as results. Indeed, the method developed below constitutes a useful general-purpose addition to the toolbox of singularity analysis [4, 13]. As discussed briefly in the last section of this note, applicability extends to many other situations, for instance tree recurrences, as considered in [2, 8, 12].

2. Singularity analysis of Bernoulli sums

Given a sequence f_k and its transform $S_n \equiv S_n[f]$, the corresponding ordinary generating functions are

$$S(z) := \sum_{n\ge 0} S_n z^n, \qquad f(z) = \sum_{k\ge 0} f_k z^k. \tag{5}$$

Then, the binomial theorem implies the fundamental relation,

$$S(z) = \frac{1}{1-qz} f\left(\frac{pz}{1-qz}\right), \tag{6}$$

as can be checked by the chain of equalities,

$$\frac{1}{1-qz} f\left(\frac{pz}{1-qz}\right) = \sum_k f_k p^k z^k (1-qz)^{-k-1} = \sum_{n,k} f_k p^k z^k \binom{n}{k} q^{n-k} z^{n-k}.$$

Recall that singularity analysis [4, 13] ensures the validity of a variety of transfers from the local properties of a generating function near its dominant singularity ($z \to 1$) to the asymptotics of its coefficients (as $n \to \infty$), for instance,

$$h(z) \sim (1-z)^{-\beta}\left(\log(1-z)^{-1}\right)^r \qquad \Longrightarrow \qquad [z^n]h(z) \sim \frac{n^{\beta-1}}{\Gamma(\beta)}\left(\log n\right)^r.$$

Conditions are that singularities must be isolated and that the expansion of the function holds "beyond" the circle of convergence in an "indented crown" $|z| < R$ (for some $R > 1$) and $|\arg(z-1)| > \epsilon$ (for some $\epsilon > 0$).

Now, $pz/(1-qz)$ maps conformally the unit disc on the interior disc of diameter $[-p/(1+q), 1]$. Thus, the function $S(z)$ is *a priori* analytic in the unit disc. Similarly if $f(z)$ has an isolated singularity at $z = 1$ and is continuable beyond the unit disc (in an indented crown), the same property holds for $S(z)$. Then, the observations just made entail the following property:

> *If the conditions of singularity analysis are satisfied by the generating function $f(z)$ of a sequence, then they also hold true for the generating function $S(z)$ of its Bernoulli transform.*

Direct analysis. In simpler cases, these observations are enough to conclude on the asymptotics of S_n from the relation (6) and derive an expansion to an arbitrary order. For instance, the first case of (3) has

$$f^{(1)}(z) = \frac{1}{\sqrt{1-z}}, \qquad S^{(1)}(z) = \frac{1}{1-qz}\,\frac{1}{\sqrt{1-\frac{pz}{1-qz}}} = \frac{1}{\sqrt{(1-qz)(1-z)}},$$

where the form of $S^{(1)}(z)$ results from (6). The singular expansion of $S^{(1)}(z)$ (at the singularity $z = 1$) as well as the asymptotic form of $S_n^{(1)}$ (as $n \to \infty$) read off immediately:

$$\begin{cases} S^{(1)}(z) &=& \dfrac{1}{\sqrt{p(1-z)}}\left(1 - \dfrac{1}{2}\left(\dfrac{q}{p}\right)(1-z) + \dfrac{3}{8}\left(\dfrac{q}{p}\right)^2(1-z)^2 + O((1-z)^3)\right) \\ S_n^{(1)} &=& \dfrac{1}{\sqrt{\pi p n}}\left(1 - \dfrac{3p-2}{8pn} + \dfrac{25p^2-60p+36}{128p^2n^2} + O(\dfrac{1}{n^3})\right). \end{cases}$$

Similarly, the second case of (3) has

$$\begin{cases} f^{(2)}(z) &=& \dfrac{1}{1-z}\log\dfrac{1}{1-z}, \qquad S^{(2)}(z) \;=\; \dfrac{1}{1-z}\log\dfrac{1-qz}{1-z}, \\ S^{(2)}(z) &=& \dfrac{1}{1-z}\log\dfrac{1}{1-z} + \dfrac{\log p}{1-z} + \dfrac{p-1}{p} + \cdots \\ S_n^{(2)} &=& H_n + \log p + O(q^n) \sim \log(pn) + \gamma + \dfrac{1}{2n} - \dfrac{1}{12n^2} + \cdots. \end{cases}$$

Indirect analysis. The truly interesting cases are the last three sequences in (3). The analysis then requires the singular expansions of the functions

$$L(z) = \sum_{k\ge 1}\frac{z^k}{\sqrt{k}}, \qquad M(z) = \sum_{k\ge 1}(\log k)\, z^k, \tag{7}$$

that have a polylogarithmic flavour [10]. We have

$$f^{(3)}(z) = L(z), \quad f^{(4)}(z) = M(z), \quad f^{(5)}(z) = \frac{1}{1-z}M(z).$$

In the next section, we establish a general result to the effect that generating functions involving $k^{-\alpha}$ and $\log k$ satisfy the conditions of singularity analysis; see Theorem 1 below. Granted this, the singular expansions for the remaining three cases of Proposition 1 follow automatically:

$$\begin{aligned}
S^{(3)}(z) &= \frac{\sqrt{\pi}}{\sqrt{p(1-z)}} + \frac{1}{p}\zeta(\frac{1}{2}) + \frac{\sqrt{\pi}(2p-3)\sqrt{p(1-z)}}{4p^2} + O(1-z) \\
S^{(4)}(z) &= \frac{1}{1-z}\log\frac{1}{1-z} + \frac{\log p - \gamma}{1-z} + \frac{1}{2p}\log(1-z) \\
&\quad + \frac{1}{2p}(1-2p-\log p+\gamma+\log(2\pi)) + O((1-z)\log(1-z)) \\
S^{(5)}(z) &= \frac{p}{(1-z)^2}(\log\frac{1}{1-z} + \log p - \gamma) + \frac{1}{1-z}(p-\frac{1}{2})\log(1-z) \\
&\quad + \frac{1}{1-z}((\frac{1}{2}-p)(1-\gamma) + p\log p + \log\sqrt{2\pi p}) + O(\log(1-z)).
\end{aligned}$$

The translation by the rules of singularity analysis is then done "at sight" and the last three estimates of Eq. (4) in Proposition 1 result.

3. Singularity analysis of polylogarithms

In this section, we show that the generalized polylogarithm of "fractional order",

$$\mathrm{Li}_{\alpha,r}(z) := \sum_{n\geq 1}(\log n)^r \frac{z^n}{n^\alpha}, \tag{8}$$

(r an integer, α an arbitrary complex number) initially defined for $|z| < 1$ satisfies the conditions of singularity analysis. This means analytic continuation as well as validity of the singular expansion in a sector that "goes out" of the unit circle. Note that, with the earlier notations of (7), we have $L(z) = \mathrm{Li}_{1/2,0}(z)$ and $M(z) = \mathrm{Li}_{0,1}(z)$.

Theorem 1. *The function* $\mathrm{Li}_{\alpha,r}(z)$ *is analytic in the slit plane* $\mathbb{C}\setminus[1,+\infty[$. *For* $\alpha \neq 1, 2, \ldots$, *the function* $\mathrm{Li}_{\alpha,0}(z)$ *satisfies the singular expansion*

$$\begin{cases} \mathrm{Li}_{\alpha,0}(z) \sim \Gamma(1-\alpha)t^{\alpha-1} + \displaystyle\sum_{j\geq 0}\frac{(-1)^j}{j!}\zeta(\alpha-j)t^j, \\ t = -\log z = \displaystyle\sum_{\ell=1}^{\infty}\frac{(1-z)^\ell}{\ell}. \end{cases} \tag{9}$$

as $z \to 1$ *in the sector* (ϵ *an arbitrary positive real*)

$$-\pi + \epsilon < \arg(1-z) < \pi - \epsilon.$$

For $r > 0$, *the singular expansion of* $\mathrm{Li}_{\alpha,r}(z)$ *is obtained by*

$$\mathrm{Li}_{\alpha,r}(z) = (-1)^r \frac{\partial^r}{\partial\alpha^r}\mathrm{Li}_{\alpha,0}(z), \tag{10}$$

and corresponding termwise differentiation of (9) *with respect to* α.

The proof decomposes into three lemmas. Lemma 1 establishes analytic continuation. This fact is classical and it appears in Ford's monograph [5] but we recall the proof based on the classical integral representation (11) as it is needed in the sequel. Lemma 2 establishes the form of the singular expansion as z tends to 1 while staying near the real axis; the analysis is then a direct application of Mellin transform techniques [3] to the *series* representation that defines the polylogarithmic functions. Finally, Lemma 3 extends the range of the singular expansion to a suitable sector that goes beyond the disc of convergence; this is achieved by subjecting the *integral* representation of analytic continuation to Mellin transform asymptotics along the lines of [14].

Lemma 1 (W. Ford, 1936). *The polylogarithm* $\mathrm{Li}_{\alpha,r}(z)$ *is analytically continuable to the complex plane slit along the ray* $[1,+\infty[$.

Proof. Consider an arbitrary function $g(s)$ analytic in $\Re(s)>0$, of at most polynomial growth there. The case of interest here is $g(s)=s^{-\alpha}(\log s)^r$ that is of small growth. The associated series

$$G(z) := \sum_{n\geq 1} g(n) z^n$$

admits, by the residue theorem, the integral representation

$$G(-y) := -\frac{1}{2i\pi}\int_{1/2-i\infty}^{1/2+i\infty} g(s) y^s \frac{\pi}{\sin \pi s}\, ds; \tag{11}$$

see Lindelöf's monograph [11]. By the fast decrease of the reciprocal sine kernel along vertical lines, the integral (11) giving $G(-y)$ converges and is analytic in y provided that y has an argument that lies in an interval $[-\pi+\epsilon, \pi-\epsilon]$, where ϵ is an arbitrary positive quantity. Since $G(z)$ is obviously analytic in the unit disc, this proves, by uniqueness of analytic continuation, that $G(z)$ is in fact analytic in the complex plane slit along the ray $[1,+\infty)$. $\square$

The next two lemmas relie crucially on properties of the Mellin transform, as detailed in [3, 14]. The Mellin transform of a function $h(t)$ is classically defined as

$$h^\star(s) = \int_0^\infty h(t) t^{s-1}\, dt.$$

The following properties are essential.

— Harmonic Properties. The transforms of harmonic sums and harmonic integrals have a factored form: if

$$F(t) = \sum_k \lambda_k f(\mu_k t), \qquad G(t) = \int \lambda(k) f(kt)\, dk,$$

then

$$F^\star(s) = \left(\sum_k \lambda_k \mu_k^{-s}\right)\cdot f^\star(s), \qquad G^\star(s) = \left(\int \lambda(k) k^{-s}\, dk\right)\cdot f^\star(s).$$

These properties derive from the rescaling rule for Mellin transforms and are detailed in [3] (Theorem 1, p. 10, Lemma 2, p. 24) in the case of sums, as well as in [14] (Theorem 4, p. 152) in the case of convolution integrals.

— Mapping Properties. The Mellin transform maps terms in the asymptotic expansion of an original function at 0 or $+\infty$ to singularities of the transformed function. The correspondence fares both ways: from original to transformed functions it is called the "Direct Mapping Property" and its proof is based on integral splittings; from transformed to original functions, it is called the "Converse Mapping Property" and it is based on a residue evaluation of inverse Mellin integrals, so that it holds under conditions of smallness of the transform at $\pm i\infty$. These mappings are described in [1], as well as in [3] (Theorems 3 and 4, pp. 16–22) and [14] (Theorem 5, p. 153).

Lemma 2. *The functions* $\mathrm{Li}_{\alpha,r}(z)$ *satisfy the singular expansions* (9) *and* (10), *as* $z \to 1$ *in the domain* (ϵ *an arbitrary positive real*)

$$\{|z|<1\} \cup \left\{ z \;\middle|\; -\frac{\pi}{2}+\epsilon < \arg(1-z) < \frac{\pi}{2}-\epsilon \right\}.$$

Proof. First, the asymptotic expansion as z tends to 1 from the left results from standard Mellin transform techniques applied to the sum defining $\mathrm{Li}_{\alpha,r}(z)$, as exposed in [3]. In the case of $\mathrm{Li}_{\alpha,0}$, the Mellin transform of $\lambda(t) = Li_{\alpha,0}(e^{-t})$ is by the Harmonic Property for sums (see also Example 9 of [3])

$$\lambda^\star(s) = \zeta(s+\alpha)\Gamma(s).$$

The "singular series" of $\lambda^\star(s)$ (that collects dominant contributions at singularities) results by separate consideration of the singularities that are simple poles, at $s = 1-\alpha$ and at $s = -j$, for integer j:

$$\lambda^\star(s) \asymp \Gamma(1-\alpha)\,\frac{1}{s-1+\alpha} + \sum_{j\geq 0}(-1)^j\,\frac{\zeta(\alpha-j)}{j!}\,\frac{1}{s+j}. \tag{12}$$

In accordance with the Converse Mapping Property, the asymptotic expansion as $t \to 0$, with t real positive, follows by a residue evaluation of the inverse Mellin transform: the map is $1/(s+\omega) \mapsto t^\omega$, and the result is the first line of (9) with $z = e^{-t}$.

The asymptotic expansion (9) remains valid provided that t stays in a sector of the right half-plane $\Re(t) > 0$ originating at 0, with an opening angle that is strictly less than π. This holds because the inverse Mellin integral still converges there and the residue evaluation that supports the asymptotic expansion of the original function $\lambda(t)$ applies in such a sector [3]. Thus, the singular expansion of $\mathrm{Li}_{\alpha,0}(z)$ as $z \to 1$ continues to be valid in the stated region, that is inside the unit disc.

The same reasoning applies to $\mu(t) = \mathrm{Li}_{\alpha,r}(e^{-t})$ whose Mellin transform is

$$\mu^\star(s) = \Gamma(s)\left((-1)^r\,\frac{\partial^r}{\partial\alpha^r}\zeta(s+\alpha)\right),$$

and it is easy to check that differentiation with respect to α propagates throughout in the argument used for the case $r = 0$. □

As a last step, one has to show persistence of the singular expansion in an indented crown that goes "outside" of the disc of convergence $|z| < 1$. Here is roughly what goes on: when y approaches -1 from above or below, the integral giving $G(-y)$ goes from a regime where it converges exponentially fast to a regime of slow convergence or even divergence. For $\mathrm{Li}_{1/2,0}(z)$ and $\mathrm{Li}_{0,1}(z)$, the situation is then analytically similar to what happens with the Laplace integrals of small arguments ($w \to 0$),

$$\int_0^\infty \frac{e^{-wt}}{\sqrt{1+t}}\,dt, \qquad \int_0^\infty e^{-wt}\log(1+t)\,dt;$$

see Chapter 3 of Wong's superb book [14] for a general theory.

Lemma 3. *The functions* $\mathrm{Li}_{\alpha,r}(z)$*satisfy the singular expansions* (9) *and* (10) *as* $z \to 1$ *in the domain* (ϵ *an arbitrary positive real*)

$$\{z \mid \epsilon \leq \arg(z-1) \leq \pi-\epsilon\} \cup \{z \mid -\pi+\epsilon \leq \arg(z-1) \leq -\epsilon\}.$$

Proof. By symmetry, it is enough to consider z in the upper half-plane $\Im(z) > 0$. The proof starts from the integral representation (11), taken with $y = -z$ approaching -1 from the bottom. We develop it in the case of $\mathrm{Li}_{\alpha,0}$, the other case being similar.

First consider the situation where $y = -z$ lies on the unit circle and set, in the notations of Lemma 1,

$$y = e^{i(-\pi+t)},$$

with t real and positive. Then, with $s = \frac{1}{2} + iw$, one has

$$G(y) \equiv G(-e^{i(-\pi+t)}) = -\frac{1}{2\pi}\int_{-\infty}^{+\infty} g(\frac{1}{2}+iw)\frac{\pi e^{i(-\pi+t)(1/2+iw)}}{\cos(i\pi w)}\, dw. \tag{13}$$

As a function of w, the integrand in (13) decays exponentially when $w \to -\infty$, and this decay holds for any *fixed* $t > 0$, since

$$\left|\frac{e^{i(-\pi+t)(1/2+iw)}}{\cos(i\pi w)}\right| = \frac{2e^{(\pi-t)w}}{e^{\pi w}+e^{-\pi w}}. \tag{14}$$

Set $G(-y) = G^-(-y) + G^+(-y)$ where G^+ and G^- represent respectively the contributions in (13) arising from the positive and from the negative part of the integration line. Then, given the uniform exponential decay of the integrand at $w = -\infty$, the quantity $G^-(-y)$ is an analytic function of t, and as such it admits a standard series expansion in powers of t. On the other hand, $G^+(-y)$ experiences a phase transition when t approaches 0, as the exponential decay of the integrand at $w = +\infty$ ceases to hold; see (14). More precisely, with $\gamma(t) = G^+(-y)$, the form (13) yields

$$\gamma(t) = \int_0^{+\infty} e^{-tw} h(w)\, dw,$$

where $h(w)$ is analytic at 0 and admits a full asymptotic expansion at ∞. For instance $g(w) = w^{-1/2}$ corresponds to

(15)

$$h(w) \underset{w\to 0}{=} -\frac{1}{\sqrt{2}}(1 + (\pi - i)w + \cdots), \qquad h(w) \underset{w\to\infty}{=} -(1-i)\frac{1}{\sqrt{2w}}(1 + \frac{i}{4w} + \cdots).$$

A standard method for the analysis of such integrals, see [14], is once more Mellin transforms. (The method parallels the treatment of harmonic sums in [3].) By the Harmonic Property for sums, the Mellin transform of $\gamma(t)$ is

$$\gamma^\star(s) = \Gamma(s)\int_0^{\infty} h(w) w^{-s}\, dw. \tag{16}$$

Then, by the Direct Mapping Property, the existence of standard asymptotic expansions of $h(w)$ at 0 and $+\infty$ (as illustrated by the typical expansion (15)) entails that $\gamma^\star(s)$ is a meromorphic function in the whole of the complex plane. In addition, the fast decrease of $\Gamma(s)$ towards $\pm i\infty$ legitimates the use of the inverse Mellin integral and of companion residue evaluations, in accordance with the Converse Mapping Property. Thus $\gamma(t)$ that represents $G^+(-y)$ admits an asymptotic expansion as $t \to 0$ that is of the form

$$\gamma(t) \sim \sum_{(\delta,\ell)} c_{\delta,\ell} t^\delta (\log t)^\ell, \tag{17}$$

$f(X)$	*Expectation*	*Variance*
$\dfrac{1}{X}$	$\dfrac{1}{pn} - \dfrac{p-1}{p^2n^2} + \dfrac{(p-1)(p-2)}{p^3n^3} + \cdots$	$\dfrac{1-p}{p^3n^3} + \dfrac{(2p-2)(2p-3)}{p^4n^4} + \cdots$
$\log X$	$\log(pn) + \dfrac{p-1}{2pn} - \dfrac{(p-1)(p-5)}{12\,p^2n^2} + \cdots$	$\dfrac{1-p}{pn} + \dfrac{(p-1)(p-3)}{2\,p^2n^2} + \cdots$
$\sqrt{X}$	$\sqrt{pn}\left(1 + \dfrac{p-1}{8pn} + \dfrac{(p-1)(p+7)}{128\,p^2n^2} + \cdots\right)$	$\dfrac{1-p}{4} + \dfrac{(p-1)(p+3)}{32\,pn} + \cdots$

TABLE 1. Mean and variance of functions $f(X)$ of a binomial random variable $X \in \mathcal{B}(n,p)$. (By convention, $f(0) = 0$.)

where the exponents δ form an unbounded nondecreasing sequence. Given the already noted analyticity of $G^-(-y)$, an expansion of type (17) also holds for $G(z)$ in terms of $t = \log(1/z)$, when $z \to 1$.

Finally, because of the fast decay of the Gamma function at ∞ in the transform (16), the asymptotic expansion in (17) remains valid when t tends to 0 inside any cone of the complex upper half-plane originating at 0 with an opening angle strictly less than π.

At this stage, one has thus established the *existence* of an asymptotic expansion of type (17) for $G(z)$ when $z \to 1$ from above (or from below, by symmetrical arguments). Such an expansion —we do not need to determine it explicitly— holds in a region that overlaps with the interior of the unit disc. Thus, this asymptotic expansion of $G(z)$ as $z \to 1$ *must* coincide with that of (9) and (10). The validity of the singular expansion (9), (10) is thus now ensured outside of the unit. □

For completeness, we mention that similar arguments yield the expansion of $\mathrm{Li}_{m,0}$ when $m \in \{1, 2, \ldots\}$, with $z = e^{-t}$ (there is a double pole at $s = 1 - m$),

$$\mathrm{Li}_{m,0}(z) = \frac{(-1)^m}{(m-1)!} t^{m-1}(\log t - H_{m-1}) + \sum_{j \geq 0,\ j \neq m-1} \frac{(-1)^j}{j!} \zeta(m-j) t^j,$$

an exact representation due to Zagier and Cohen [10, p. 387]. More generally, one has

$$\mathrm{Li}_{m,r}(z) = \mathrm{Res}\left((-1)^r \zeta^{(r)}(s+m)\Gamma(s)t^{-s}\right)_{s=1-m} + \sum_{j \geq 0,\ j \neq m-1} \frac{(-1)^j}{j!} \zeta(m-j) t^j,$$

where Res(.) denotes a residue. Thus, in such cases, a "special term" comes in that is provided by a residue at a multiple pole.

4. Conclusion

Generalized polylogarithms constitute a useful addition to the singularity analysis toolbox. The approach developed in this note is also well-suited to computer algebra and barely twenty instructions in the Maple system suffice to perform automatically all the formal computations underlying Proposition 1. Table 1 illustrates this further by providing detailed asymptotics estimates obtained in this way of mean and variance of the inverse, logarithm, and square-root of a binomial random variable.

Clearly the method applies to functions whose coefficients admit an analytic interpolation that is of polynomial growth in a half plane and has smooth asymptotic properties at infinity, typical examples being

$$\sum_{n\geq 0} \frac{z^n}{\sqrt{n^2+1}}, \qquad \sum_{n\geq 0} \frac{(\log n+\beta)^r}{(n+\gamma)^\alpha} z^n.$$

This makes it possible to analyse a whole range of combinatorial sums that involve "transcendental" elements. Direct consequences (see Case 5 of Proposition 1) are the estimation of the entropy of the binomial distribution,

$$\mathrm{H}_{n,p} = \frac{1}{2}\log n + \frac{1}{2} + \log\sqrt{2\pi p(1-p)} + O(\frac{1}{n}),$$

as determined to arbitrary order by Jacquet and Szpankowski in [6], or an alternative derivation of some of Krichevskiy's estimates regarding the redundancy of the "add-β rule" in universal coding (see [6, 9]).

Another instance of such transcendental elements is the tree recurrence of Knuth and Pittel [8] that arises in union-find algorithms

$$x_n = c_n + \sum_{k=0}^{n} p_{nk} x_k, \qquad p_{nk} = \frac{1}{n-1}\binom{n}{k}\left(\frac{k}{n}\right)^{k-1}\left(\frac{n-k}{n}\right)^{n-k-1},$$

and cases like $c_n = n^\beta$ are of interest. Similarly, the tree-shape parameter considered by Fill, Meir, and Moon [2, 12] leads to the recurrence

$$x_n = \log n + \frac{2}{n}\sum_{k=0}^{n-1} x_k.$$

Such tree-recurrences become amenable to singularity analysis thanks to Theorem 1 employed in conjunction with closure under Hadamard products. This will be explored in a future note.

Acknowledgements. This work was supported by the Long Term Research Project *Alcom-IT* (# 20244) of the European Union. Special thanks are due to Philippe Jacquet and Wojtek Szpankowski for kindly sharing ideas and preprints.

References

[1] G. Doetsch, *Handbuch der Laplace Transformation,* vol. 1–3, Birkhäuser Verlag, Basel, 1955.

[2] James Allen Fill, *On the distribution of binary search trees under the random permutation model*, Random Structures and Algorithms **8** (1996), no. 1, 1–25.

[3] Philippe Flajolet, Xavier Gourdon, and Philippe Dumas, *Mellin transforms and asymptotics: Harmonic sums*, Theoretical Computer Science **144** (1995), no. 1–2, 3–58.

[4] Philippe Flajolet and Andrew M. Odlyzko, *Singularity analysis of generating functions*, SIAM Journal on Discrete Mathematics **3** (1990), no. 2, 216–240.

[5] W. B. Ford, *Studies on divergent series and summability and the asymptotic developments of functions defined by maclaurin series*, 3rd ed., Chelsea Publishing Company, New York, 1960.

[6] Philippe Jacquet and Wojciech Szpankowski, *Entropy computations via analytic depoissonization*, Tech. Report CSD-TR-96-085, Purdue University, 1996, Revised February 1998, 22 pages.

[7] ———, *Analytical depoissonization and its applications*, Theoretical Computer Science (1998), In print, 67 pages.

[8] Donald E. Knuth and Boris Pittel, *A recurrence related to trees*, Proceedings of the American Mathematical Society **105** (1989), no. 2, 335–349.

[9] R. E. Krichevskiy, *Laplace's law of succession and universal encoding*, IEEE Transactions on Information Theory **44** (1998), no. 1, 296–303.

[10] L. Lewin (ed.), *Structural properties of polylogarithms*, American Mathematical Society, 1991.

[11] Ernst Lindelöf, *Le calcul des résidus et ses applications à la théorie des fonctions*, Collection de monographies sur la théorie des fonctions, publiée sous la direction de M. Émile Borel, Gauthier-Villars, Paris, 1905, Reprinted by Gabay, Paris, 1989.

[12] A. Meir and J. W. Moon, *On the log-product of subtree-sizes in random trees*, Random Structures and Algorithms **12** (1998), no. 2, 197–212.

[13] A. M. Odlyzko, *Asymptotic enumeration methods*, Handbook of Combinatorics (R. Graham, M. Grötschel, and L. Lovász, eds.), vol. II, Elsevier, Amsterdam, 1995, pp. 1063–1229.

[14] Roderick Wong, *Asymptotic approximations of integrals*, Academic Press, 1989.

INRIA Rocquencourt, F-78153 Le Chesnay (France), `Philippe.Flajolet@inria.fr`

Unité de recherche INRIA Lorraine, Technopôle de Nancy-Brabois, Campus scientifique,
615 rue du Jardin Botanique, BP 101, 54600 VILLERS LÈS NANCY
Unité de recherche INRIA Rennes, Irisa, Campus universitaire de Beaulieu, 35042 RENNES Cedex
Unité de recherche INRIA Rhône-Alpes, 655, avenue de l'Europe, 38330 MONTBONNOT ST MARTIN
Unité de recherche INRIA Rocquencourt, Domaine de Voluceau, Rocquencourt, BP 105,
78153 LE CHESNAY Cedex
Unité de recherche INRIA Sophia-Antipolis, 2004 route des Lucioles, BP 93, 06902 SOPHIA-ANTIPOLIS
Cedex

Éditeur
INRIA, Domaine de Voluceau, Rocquencourt, BP 105, 78153 LE CHESNAY Cedex
(France)
http://www.inria.fr
ISSN 0249-6399