# Solution of a Minimax Problem with Additive Final Cost

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# Solution of a minimax problem with additive final cost *

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**Abstract**

In this paper we study a minimax optimal control problem with finite horizon and additive final cost. We introduce a generalized problem where the original one can be embedded. For this generalized problem, we establish a dynamical programming principle and we present the associated Hamilton-Jacobi-Bellman (HJB) equation. This equation is defined in terms of a discontinuous Hamiltonian and we prove that the optimal cost of the generalized problem is the unique viscosity solution of this HJB equation.

**Key words:** *minimax problem, dynamical programming principle, Hamilton-Jacobi-Bellman equation, viscosity solution*

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# Solution d'un problème d'optimisation minimax avec coût final additif

**Résumé:** On analyse ici un problème de contrôle optimal de type minimax avec coût final additif. On introduit un problème auxiliaire généralisant le problème original et permettant d'utiliser les techniques de la programmation dynamique. Pour le problème généralisé, on établit un principe de la programmation dynamique et on présente aussi l'équation de Hamilton-Jacobi-Bellman (HJB) associée. On démontre que le coût optimal du problème généralisé est l'unique solution de viscosité de cette équation de HJB.

**Motclefs:** *problèmes minimax, principe de la programmation dynamique, équation de Hamilton-Jacobi-Bellman, solution de viscosité*

**AMS Classification:** 49S35, 49L05, 49L20, 49L25

# Contents

# 1 Introduction

The optimization of dynamic systems where the criterion is the maximum value of a function is a frequent problem in technology, economics and industry. This problem appears for example, when we attempt to minimize the maximum deviation of controlled trajectories with respect to a given "model" trajectory. Minimax problems differs from those usually considered in the optimal control literature where a cumulative cost is minimized. Since in some cases, minimax problems describe more appropriately decision problems arisen in controlled systems whose performance is evaluated with a unique scalar parameter, the minimax optimization has received considerable attention in recent publications (see, for example, [3, 4, 5, 6, 7, 8, 9, 10, 19]). Furthermore, the relation of minimax problems with the design of robust controllers can be seen in [11].

In addition, from the academic point of view, the minimax optimal control problem is of interest in the area of game theory because minimax problems can be seen as a game – see [14] – where a player applies ordinary controls and the other one – using complete and privileged information – chooses a stopping time. Problems of this type lead to the treatment of nonlinear partial differential inequalities akin to those appearing in the *obstacle problem* (with obstacle given in explicit or implicit form, see [7]). To find solutions of these systems we must consider generalized solutions – even discontinuous solutions – since commonly there does not exist classical solutions of such systems (see [1, 2]). The treatment of the infinite horizon problem also presents great analytical difficulties, because the optimal cost is neither necessarily lower semicontinuous nor upper semicontinuous. Moreover, the optimal cost cannot be approximated with a sequence of finite horizon problems. Studies concerning these issues can be seen in [15, 18]. Besides, it is also important to develop numerical methods to compute these solutions in approximate way because in general it is not possible to find exact analytical solutions. Numerical methods to obtain approximated open loop optimal controls are analyzed in [20] and for closed loop, see [4, 14].

Here, we analyze a minimax optimal control problem where the functional to be optimized, not only depends on the maximum of a function along the complete trajectory of the system but also it takes into account (in an additive fashion) another function of the final state of the system.

For a problem with this structure, a dynamical programming principle cannot be formulated merely in terms of the initial time and the initial state. To obtain a dynamical programming principle, we introduce an auxiliary parameter which "remembers" the past maximum values (the use of a similar procedure can be seen in [17, 18]). Using this parameter, we present an auxiliary problem which gives the solution of the original problem using a particular value of the auxiliary parameter. For this second optimal control problem, we establish the associated dynamical programing principle and a Hamilton-Jacobi-Bellman (HJB) equation. Finally, we prove that the optimal cost of the auxiliary problem is the unique solution of the associated HJB equation.

The paper is organized as follows. In §2, we present the optimization problem. In §3, we describe the auxiliary problem and its relation with the original one. We also establish there the dynamical programming equation. In §4, we give the HJB equation associated to the problem and we prove the uniqueness of solution in the viscosity sense.

# 2 The optimization problem

## 2.1 Presentation of the problem

We consider a minimax optimal control problem with finite horizon and final cost. More specifically, the problem consists in minimizing the functional $J$

$$\begin{aligned} J : [0,T] \times \mathbb{R}^m \times \mathcal{A} &\mapsto \mathbb{R} \\ (t,x,\alpha(\cdot)) &\mapsto J(t,x,\alpha(\cdot)) = \operatorname*{ess\,sup}_{s\in[t,T)} f(s,y(s),\alpha(s)) + \Psi(y(T)), \end{aligned}$$

where $y(\cdot)$ represents the state of a dynamic system which evolves accordingly to the following differential equation,

$$\begin{cases} y'(s) = g(s,y(s),\alpha(s)) & \forall\, t \le s < T \\ y(t) = x, & x \in \mathbb{R}^m . \end{cases}$$

$\mathcal{A} = L^\infty([0,T],A)$ is the set of admissible control policies and $A$ is the control set.

The optimal cost function is given by

$$u(t,x) = \inf_{\alpha(\cdot)\in\mathcal{A}} J(t,x,\alpha(\cdot)). \tag{1}$$

This problem is an extension of another one analyzed by Barron–Ishii in [7] and by Di Marco–González in [12, 13]. We are interested in analyzing the new problem from the continuous viewpoint.

After introducing an auxiliary problem, which generalized the problem presented above, we establish the dynamical programming principle corresponding to the new problem, and we present a Hamilton-Jacobi-Bellman (HJB) equation associated to the optimal cost. Finally, we prove that the optimal cost is the unique solution of this HJB equation.

## 2.2 General assumptions

Let $BUC([0,T] \times \mathbb{R}^m \times A)$ be the set of bounded and uniformly continuous functions in $[0,T] \times \mathbb{R}^m \times A$. We assume the following hypotheses hold:

($H_1$) $g : [0,T] \times \mathbb{R}^m \times A \mapsto \mathbb{R}^m$, $g \in BUC([0,T] \times \mathbb{R}^m \times A)$,

$$\|g(t,x,a)\| \le M_g,$$

$$\|g(t,x,a) - g(\hat{t},\hat{x},a)\| \le L_g(|t-\hat{t}| + \|x-\hat{x}\|), \quad \forall\, t,\hat{t} \in [0,T],\ \forall\, x,\hat{x} \in \mathbb{R}^m,\ \forall\, a \in A.$$

($H_2$) $f : [0,T] \times \mathbb{R}^m \times A \mapsto \mathbb{R}$, $f \in BUC([0,T] \times \mathbb{R}^m \times A)$,

$$m_f \le f(t,x,a) \le M_f,$$

$$|f(t,x,a) - f(\hat{t},\hat{x},a)| \le L_f(|t-\hat{t}| + \|x-\hat{x}\|), \quad \forall\, t,\hat{t} \in [0,T],\ \forall\, x,\hat{x} \in \mathbb{R}^m,\ \forall\, a \in A.$$

($H_3$) $\Psi : \mathbb{R}^m \mapsto \mathbb{R}$, $\Psi \in BUC(\mathbb{R}^m)$ and

$$|\Psi(x) - \Psi(\hat{x})| \le L_\Psi \|x-\hat{x}\|, \quad \forall\, x,\hat{x} \in \mathbb{R}^m.$$

($H_4$) $A$ is compact.

# 3 Auxiliary problem

## 3.1 Auxiliary variable and problem reformulation

In the problem presented above it is not possible to establish a dynamical programming principle only in terms of the variables $(t, x)$. In order to develop the analytical and numerical treatment of the problem we extend the state of the system introducing an auxiliary variable $y_{m+1}(\cdot)$. The initial value of this variable is $y_{m+1}(t) = \rho$ and the evolution of $y_{m+1}$ is given by the following expression:

$$y_{m+1}(\tau) = \max\left\{ y_{m+1}(\sigma), \operatorname*{ess\,sup}_{s\in[\sigma,\tau)} f(s,y(s),\alpha(s)) \right\}, \quad \sigma \in [t, T). \tag{2}$$

**Note 3.1** *The interpretation of the additional variable $y_{m+1}$ is the following one: $y_{m+1}$ remembers the maximum values of the function $f$ from the initial time $t$ to the current time $\tau$. To make it clear, let us consider $m_f(t,x) = \min_{a\in A} f(t,x,a)$ and $\rho \le m_f(x)$. Then, from (2), it follows that*

$$y_{m+1}(T) = \operatorname*{ess\,sup}_{s\in[t,T)} f(s,y(s),\alpha(s)) \,.$$

This variable $y_{m-1}$ is a solution of the differential inclusion given by

$$\frac{dy_{m-1}}{ds}(s) \in \overline{G}(f(s,y(s),\alpha(s)) - y_{m+1}(s)), \tag{3}$$

where

$$\overline{G}(w) = \left| \begin{array}{ll} 0 & \text{if } w < 0, \\ [0,\infty] & \text{if } w = 0, \\ \infty & \text{if } w > 0. \end{array} \right.$$

In this way, the original problem can be studied as an ordinary optimal control problem, i.e. a problem with pure final cost because $J(t,x,\alpha(\cdot)) = y_{m+1}(T) + \Psi(y(T))$. In this case, the optimal cost $v$ is:

$$v(t,x,\rho) = \inf\{y_{m+1}(T) + \Psi(y(T)) : \alpha(\cdot) \in \mathcal{A}\}, \tag{4}$$

where $t \in [0,T]$, $x \in \mathbb{R}^m$ and $\rho \in \mathbb{R}$, being $\rho$ the initial value for equation (3), i.e. $y_{m+1}(t) = \rho$.

**Remark 3.1** *From (2), it is clear that $\{y_{m+1}(s) : s \in (t,T]\} \subseteq [\nu, M_f]$, where $\nu = \max\{\rho, m_f\}$. Moreover, if $\rho \ge M_f$, then $y_{m+1}(s) = \rho$, $\forall\, s \in [t,T]$.*

## 3.2 Properties of the optimal cost $v$

The following properties brings some relations between the optimal costs $u$ and $v$ of the original and auxiliary problems. They are almost evident and can be proved without difficulties using the definitions of the original and auxiliary problems described above. Here we omit the complete proofs for the sake of brevity and we only sketch the lines of argument.

- *If $\rho \le \min_{a\in A} f(t,x,a)$, then*

$$v(t,x,\rho) = u(t,x). \tag{5}$$

  By replacing in (4) and comparing with (1), it follows that $v(t,x,\rho) = u(t,x)$,

- *If $\rho \geq \max_{a\in A} f(t,x,a)$, then $v(t,x,\rho) = \hat{u}(t,x) + \rho$, where $\hat{u}$ is the optimal cost of the control problem (1) when $f \equiv 0$, i.e. where the functional to be minimized is $\Psi(y(T))$.*
  In this case $y_{m+1}(s) = \rho$, $\forall\, s \in [t,T]$. Then, by replacing in (4), we have that
$$v(t,x,\rho) = \rho + \inf_{\alpha(\cdot)\in\mathcal{A}} \Psi(y(T)). \tag{6}$$

- *If $\rho_1 > \rho_2$, then*
$$v(t,x,\rho_1) \geq v(t,x,\rho_2)\,. \tag{7}$$
  Let $\rho_1 > \rho_2$ and $y_{m+1}^{\rho_i}(\cdot)$ the solution of the differential inclusion (3) when the initial condition is $\rho_i$. Then,
$$\begin{aligned} y_{m+1}^{\rho_1}(T) &= \max\{\rho_1, \operatorname*{ess\,sup}_{s\in[t,T)} f(s,y(s),\alpha(s))\} \\ &\geq \max\{\rho_2, \operatorname*{ess\,sup}_{s\in[t,T)} f(s,y(s),\alpha(s))\} = y_{m+1}^{\rho_2}(T)\,. \end{aligned} \tag{8}$$
  By replacing in (4) it results $v(t,x,\rho_1) \geq v(t,x,\rho_2)$.

- *$v$ is bounded $\forall\, \rho \in [-K,K]$.*
  This property follows from the definition of $v$ and the boundedness of $f$ and $\Psi$.

- *$v$ is Lipschitz continuous with respect to the three variables $t,x,\rho$.*
  This property follows from the hypotheses verified by $f,g$ and $\Psi$.

- *$v$ may be not semiconcave.*
  In effect, if $\Psi \equiv 0$, $u$ can be not semiconcave (see [12]), then $v$ has the same property (see examples in [14]).

### 3.3 Dynamical programming equation

It is clear that in this new problem with the state augmented by the variable $\rho$, the dynamical programming equation is given by
$$v(t,x,\rho) = \inf_{\alpha(\cdot)\in L^\infty([t,s],A)} v(s,y(s),y_{m+1}(s)) \tag{9}$$
We also have the final condition
$$v(T,x,\rho) = \max\left\{\rho, \min_{a\in A} f(T,x,a)\right\} + \Psi(x). \tag{10}$$

## 4 Hamilton-Jacobi-Bellman equation

Let us define
$$\begin{aligned} H_A(t,x,\rho,v) &= \inf\{v(t,x,f(t,x,a)) - v(t,x,\rho) \;: \\ &\qquad\qquad f(t,x,a) > \rho,\;\; v(t,x,f(t,x,a)) \leq v(t,x,\rho)\}, \\ H(t,x,\rho,\nabla v) &= \inf\left\{\frac{\partial v}{\partial t}(t,x,\rho) + \frac{\partial v}{\partial x}(t,x,\rho)g(t,x,a) \;:\; f(t,x,a) < \rho\right\}, \\ H_*(t,x,\rho,\nabla v) &= \inf\left\{\frac{\partial v}{\partial t}(t,x,\rho) + \frac{\partial v}{\partial x}(t,x,\rho)g(t,x,a) : f(t,x,a) \leq \rho\right\}. \end{aligned}$$

Taking into account the dynamical programming equation we obtain the following differential formulation. Strictly, it takes the following form

$$\left| \begin{array}{l} \min\{H_A(t,x,\rho,v),\ H(t,x,\rho,\nabla v)\} \geq 0\,, \\ \\ \min\{H_A(t,x,\rho,v),\ H_*(t,x,\rho,\nabla v)\} \leq 0\,, \end{array} \right. \tag{11}$$

with final condition

$$v(T,x,\rho) = \max\left\{\rho,\ \min_{a\in A} f(T,x,a)\right\} + \Psi(x)\,. \tag{12}$$

The solution in the viscosity sense of this system (11) is defined as follows.

- The function $w$ is a *subsolution* in the viscosity sense if,
  - $w$ is upper-semicontinuous on $(0,T)\times \mathbb{R}^m \times \mathbb{R}$,
  - $w$ is upper-bounded on $(0,T)\times \mathbb{R}^m \times [-K,K]$, $\forall K \in \mathbb{R}^+$,
  - $w$ is a non-decreasing function of the variable $\rho$,
  - given $(t,x,\rho)\in(0,T)\times\mathbb{R}^m\times\mathbb{R}$ and $\Phi \in C^1((0,T)\times\mathbb{R}^m\times\mathbb{R})$ such that $\Phi - w$ has a minimum in $(t,x,\rho)$ in a neighborhood $\mathcal{N}(t,x,\rho)$, it results

$$H(t,x,\rho,\nabla\Phi) \geq 0\,. \tag{13}$$

- The function $z$ is a *supersolution* in the viscosity sense if,
  - $z$ is lower-semicontinuous on $(0,T)\times\mathbb{R}^m\times\mathbb{R}$,
  - $z$ is lower-bounded on $(0,T)\times\mathbb{R}^m\times[-K,K]$, $\forall K\in\mathbb{R}^-$,
  - given $(t,x,\rho)\in(0,T)\times\mathbb{R}^m\times\mathbb{R}$ and $\Phi\in C^1((0,T)\times\mathbb{R}^m\times\mathbb{R})$ such that $\Phi - z$ has a maximum in $(t,x,\rho)$ in a neighborhood $\mathcal{N}(t,x,\rho)$, it results

$$\min\{H_A(t,x,\rho,z),\ H_*(t,x,\rho,\nabla\Phi)\} \leq 0\,. \tag{14}$$

By virtue of the especial definition of viscosity solution given above, we can use the following equivalent form of the HJB equation:

$$\left| \begin{array}{l} H(t,x,\rho,\nabla v) \geq 0\,, \\ \\ \min\{H_A(t,x,\rho,v),\ H_*(t,x,\rho,\nabla v)\} \leq 0\,. \end{array} \right. \tag{15}$$

## 4.1 The optimal cost as a viscosity solution

**Theorem 4.1** *The optimal cost $v$ is a solution in the viscosity sense of the system* (11).

**Proof.** The final condition is trivially verified from (9) and it was proved that $v$ is continuous, bounded and non-decreasing with respect to $\rho$. So it remains to prove the last property of the subsolution's definition and the last one of the supersolution's definition.

First, we will prove that $v$ is a *subsolution* of (11).

Let $(t,x,\rho)\in(0,T)\times\mathbb{R}^m\times\mathbb{R}$ and $\Phi\in C^1((0,T)\times\mathbb{R}^m\times\mathbb{R})$ such that $\Phi - v$ has a minimum in $(t,x,\rho)$ in a neighborhood $\mathcal{N}(t,x,\rho)$.

Let us prove that $H(t,x,\rho,\nabla\Phi)\geq 0$. Let $a\in A$ such that $f(t,x,a) < \rho$ and $\{\alpha_n\}$ be a control sequence such that $\alpha_n(s) = a$, for all $s\in[t,t+n^{-1}]$.

From equation (9), we have that

$$v(t,x,\rho) \leq v(t+n^{-1}, y_n(t+n^{-1}), \max\{\rho, \operatorname*{ess\,sup}_{s\in[t,t+n^{-1}]} f(s,y_n(s),\alpha_n(s))\}) \tag{16}$$

Since $\lim_{n\to\infty} \operatorname*{ess\,sup}_{s\in[t,t+n^{-1}]} f(s,y_n(s),\alpha_n(s)) = f(t,x,a)$, for all $n \geq n_0$, we get

$$\max\left\{\rho, \operatorname*{ess\,sup}_{s\in[t,t+n^{-1}]} f(s,y_n(s),\alpha_n(s))\right\} = \rho$$

and then, from (16), we obtain

$$v(t,x,\rho) \leq v(t+n^{-1}, y_n(t+n^{-1}), \rho). \tag{17}$$

By the minimality of $\Phi - v$ at $(t,x,\rho)$, it follows that

$$\Phi(t+n^{-1}, y_n(t+n^{-1}), \rho) - \Phi(t,x,\rho) \geq v(t+n^{-1}, y_n(t+n^{-1}), \rho) - v(t,x,\rho) \geq 0. \tag{18}$$

Since

$$\lim_{n\to\infty} \frac{y_n(t+n^{-1}) - x}{n^{-1}} = \lim_{n\to\infty} n \int_t^{t-n^{-1}} g(s,y(s),\alpha_n(s))ds = g(t,x,a),$$

dividing by $n^{-1}$ in (18) and taking limit when $n \to \infty$, we have that

$$\frac{\partial\Phi}{\partial t}(t,x,\rho) + \frac{\partial\Phi}{\partial x}(t,x,\rho)\, g(t,x,a) \geq 0.$$

Then, since $a$ is arbitrary, it results that

$$H(t,x,\rho,\nabla\Phi) \geq 0. \tag{19}$$

So, (13) and (19) imply that $v$ is subsolution.

Let us see that $v$ is a *supersolution* of (11).

Let $(t,x,\rho) \in (0,T) \times \mathbb{R}^m \times \mathbb{R}$ and $\Phi \in C^1((0,T) \times \mathbb{R}^m \times \mathbb{R})$ such that $\Phi - v$ has a maximum in $(t,x,\rho)$ in a neighborhood $\mathcal{N}(t,x,\rho)$.

We assume that $\exists \eta > 0$ such that

$$\min\{H_A(t,x,\rho,v),\ H_*(t,x,\rho,\nabla\Phi)\} \geq \eta > 0. \tag{20}$$

Let $\{\alpha_n\}$ be a minimizing control sequence such that

$$v(t,x,\rho) + \frac{1}{n^2} \quad \geq \quad v(t+n^{-1}, y_n(t+n^{-1}), \max\{\rho, \operatorname*{ess\,sup}_{s\in[t,t-n^{-1}]} f(s,y_n(s),\alpha_n(s))\}). \tag{21}$$

By the compactness of $A$, there exists $a \in A$ and a subsequence of $\{\alpha_n\}$ such that

$$\lim_{n\to\infty} \operatorname*{ess\,sup}_{s\in[t,t+n^{-1}]} f(s,y_n(s),\alpha_n(s)) = f(t,x,a).$$

The case $f(t,x,a) > \rho$ can be ruled out.

In effect, the inequality (20) implies that $H_A(t,x,\rho,v) \geq \eta$. But taking limit in (21) and by virtue of (20) and (7), it follows that $v(t,x,\rho) \geq v(t,x,\rho) + \eta$, which is absurd.

So, we have that $f(t,x,a) \leq \rho$. We will analyze the effect of this condition on the relation $H_*(t,x,\rho,\nabla\Phi) \geq \eta > 0$ (valid by virtue of (20)).

As $f(t,x,a) \le \rho$, then for all $\varepsilon > 0$, $f(t,x,a) < \rho + \varepsilon$. In consequence, there exists $n_\varepsilon$ such that $\forall n \ge n_\varepsilon$,

$$\operatorname*{ess\,sup}_{s\in[t,t+n^{-1}]} f(s, y_n(s), \alpha_n(s)) < \rho + \varepsilon. \tag{22}$$

Eventually, by redefining the controls $\alpha_n(\cdot)$ in zero measure sets, it is possible to affirm that $\forall n \ge n_\varepsilon, \forall s \in [t, t+n^{-1}]$,

$$f(s, y_n(s), \alpha_n(s)) < \rho + \varepsilon. \tag{23}$$

Since $f$ is Lipschitz–continuous, we have that $\forall n \ge n_\varepsilon$ , $\forall s \in [t, t+n^{-1}]$,

$$f(t, x, \alpha_n(s)) < \rho + \varepsilon + L_f(1 + M_g)n^{-1} < \rho + 2\varepsilon. \tag{24}$$

We consider the set of controls $Z_\varepsilon$ given by

$$Z_\varepsilon = \{a \in A : f(t,x,a) \le \rho + 2\varepsilon\}.$$

Then, for any $n \ge n_\varepsilon$ and $s \in [t, t+n^{-1}]$, it results $\alpha_n(s) \in Z_\varepsilon$.

As $(t,x,\rho)$ is a maximum point for the function $\Phi - v$, from (21), we have

$$\Phi(t+n^{-1}, y_n(t+n^{-1}), \rho) - \Phi(t,x,\rho) \le v(t+n^{-1}, y_n(t+n^{-1}), \rho) - v(t,x,\rho) \le n^{-2}. \tag{25}$$

It is possible to prove that there exists $\overline{g} \in \overline{Co(g(t,x,Z_\varepsilon))}$, where $Co(E)$ is the convex hull of the set $E$, such that (eventually using a suitable subsequence)

$$\lim_{n\to\infty} n \int_t^{t-n^{-1}} g(s, y_n(s), \alpha_n(s))ds = \overline{g}$$

and so, by virtue of (25), we get

$$\lim_{n\to\infty} \frac{\Phi(t+n^{-1}, y_n(t+n^{-1}), \rho) - \Phi(t,x,\rho)}{n^{-1}} = \frac{\partial\Phi}{\partial t}(t,x,\rho) + \frac{\partial\Phi}{\partial x}(t,x,\rho)\,\overline{g} \le 0. \tag{26}$$

On the other hand, as $Z_\varepsilon$ is closed, we have that $\overline{Co(g(t,x,Z_\varepsilon))} = Co(g(t,x,Z_\varepsilon))$.

As $\overline{g} \in Co(g(t,x,Z_\varepsilon))$ $\forall \varepsilon > 0$, we get that $\overline{g} \in Co(g(t,x,Z_0))$. Then, there exists a probability measure $\mu(\cdot)$ with support in $Z_0$ such that

$$\overline{g} = \int_{Z_0} g(t,x,a)d\mu(a). \tag{27}$$

From (20), if $a \in Z_0$, then $\dfrac{\partial\Phi}{\partial t}(t,x,\rho) + \dfrac{\partial\Phi}{\partial x}(t,x,\rho)g(t,x,a) \ge \eta > 0$.

Using this inequality and the integral expression (27) in (26) we have that,

$$0 \ge \frac{\partial\Phi}{\partial t}(t,x,\rho) + \frac{\partial\Phi}{\partial x}(t,x,\rho)\,\overline{g} = \int_{Z_0} \left(\frac{\partial\Phi}{\partial t}(t,x,\rho) + \frac{\partial\Phi}{\partial x}(t,x,\rho)g(t,x,a)\right) d\mu(a) \ge \eta. \tag{28}$$

This inequality contradicts the initial assumption (20) and so, we obtain by *reductio ad absurdum* that $v$ is a supersolution.

□

## 4.2 Uniqueness of the viscosity solution

**Theorem 4.2** *There is a unique solution of the system* (11).

**Proof.** Let $w$ be a subsolution and $z$ a supersolution of (11).

We will prove by *reductio ad absurdum* that $w \leq z$.

Let us suppose that $\exists K > 0$ such that

$$r := \sup\{w(t,x,\rho) - z(t,x,\rho) : (0,T) \times \mathbb{R}^m \times [-K,K]\} > 0, \tag{29}$$

then given $0 < \delta < \frac{r}{2}$ there exists $(t_\delta, x_\delta, \rho_\delta) \in (0,T) \times \mathbb{R}^m \times [-K,K]$ such that

$$w(t_\delta, x_\delta, \rho_\delta) - z(t_\delta, x_\delta, \rho_\delta) > \sup_{(0,T)\times\mathbb{R}^m\times[-K,K]} (w(t,x,\rho) - z(t,x,\rho)) - \delta > \frac{r}{2}.$$

Let $M > 0$ such that

$$\left| \begin{array}{l} w(t,x,\rho) \leq M, \quad \text{in } [0,T] \times \mathbb{R}^m \times [-2K, 2K], \\ z(t,x,\rho) \geq -M, \quad \text{in } [0,T] \times \mathbb{R}^m \times [-2K, 2K]. \end{array} \right. \tag{30}$$

As $w$ is upper semicontinuous, $\exists \omega_w : \mathbb{R}^+ \mapsto \mathbb{R}^+$ such that

$$\sup\{w(t',x',\rho') : |t'-t| + \|x'-x\| + |\rho'-\rho| \leq \eta\} < w(t,x,\rho) + \omega_w(\eta). \tag{31}$$

For each $\epsilon$ we consider the hump function

$$\Phi(t,x,s,y) = -\frac{|t-s|^2}{\epsilon^2} - \frac{\|x-y\|^2}{\epsilon^2} - \sigma(\frac{T}{t} - 1) - \frac{r}{8}(1 - \frac{t}{T}) + (\delta + \omega_w(\sqrt{\epsilon}))\,\xi(\|x - x_\delta\|). \tag{32}$$

The elements of $\Phi$ have the following properties:

- $\xi(\cdot) \in C^1(\mathbb{R})$ verifies

$$\left| \begin{array}{ll} \xi(\tau) = 0 & \tau = [0,1], \\ \xi(\tau) = -1 & \tau \geq 3, \\ |\xi'(\tau)| \leq 1. & \end{array} \right.$$

- $\sigma < \min\left\{M, \dfrac{r\,t_\delta}{8(T - t_\delta)}\right\}$.

- The derivatives of $\Phi$ are

$$\left\{ \begin{array}{l} \dfrac{\partial \Phi}{\partial t}(t,x,s,y) = -2\dfrac{(t-s)}{\epsilon^2} + \sigma\dfrac{T}{t^2} + \dfrac{r}{8T}, \\[2ex] \dfrac{\partial \Phi}{\partial x}(t,x,s,y) = -2\dfrac{(x-y)}{\epsilon^2} + (\delta + \omega_w(\sqrt{\epsilon}))\,\xi'(\|x - x_\delta\|)\,\dfrac{(x - x_\delta)}{\|x - x_\delta\|}, \\[2ex] \dfrac{\partial \Phi}{\partial s}(t,x,s,y) = 2\dfrac{(t-s)}{\epsilon^2}, \\[2ex] \dfrac{\partial \Phi}{\partial y}(t,x,s,y) = 2\dfrac{(x-y)}{\epsilon^2}. \end{array} \right. \tag{33}$$

We define the function

$$\phi^{\sqrt{\epsilon}}(t,x,\rho,s,y) = z(t,x,\rho) - w(s,y,\rho+\sqrt{\epsilon}) - \Phi(t,x,s,y). \tag{34}$$

As $w$ is nondecreasing, from (29) and (31) we have

$$w(t,x,\rho+\sqrt{\epsilon}) - z(t,x,\rho) \le r + \omega_w(\sqrt{\epsilon}),$$

so the function $\phi^{\sqrt{\epsilon}}$ has a minimum in $(0,T) \times \mathbb{R}^m \times [-K,K] \times [0,T] \times \mathbb{R}^m$.

Let $(t_0, x_0, \rho_0, s_0, y_0)$ be one of these minimizing points. From (30) and (32), we get

$$\frac{|t_0 - s_0|^2}{\epsilon^2} + \frac{\|x_0 - y_0\|^2}{\epsilon^2} + \sigma\left(\frac{T-t_0}{t_0}\right) + \frac{r}{8}\left(1 - \frac{t_0}{T}\right) \le 2M + \sigma\left(\frac{T-t_\delta}{t_\delta}\right) + \frac{r}{8}\left(1 - \frac{t_\delta}{T}\right).$$

If we define $M_1 = 2M + \sigma\frac{T}{t_\delta} + \frac{r}{8}$, the last inequality and the definition of $\Phi$ imply that

$$\left|\begin{array}{l} \dfrac{\sigma T}{M_1} < t_0, \\ |t_0 - s_0| < \sqrt{M_1}\,\epsilon, \\ \|x_0 - y_0\| < \sqrt{M_1}\,\epsilon, \\ \|x_0 - x_\delta\| \le 3. \end{array}\right. \tag{35}$$

In particular, as $(t_0, x_0, \rho_0, s_0, y_0)$ is a minimizing point

$$\phi^{\sqrt{\epsilon}}(t_0, x_0, \rho_0, s_0, y_0) \le \phi^{\sqrt{\epsilon}}(t_0, x_0, \rho_0, t_0, x_0),$$

and so, we have

$$\frac{|t_0 - s_0|^2}{\epsilon^2} + \frac{\|x_0 - y_0\|^2}{\epsilon^2} \le w(s_0, y_0, \rho_0 + \sqrt{\epsilon}) - w(t_0, x_0, \rho_0 + \sqrt{\epsilon}) \le \omega_w(\sqrt{\epsilon}). \tag{36}$$

From (36), we get

$$\left|\begin{array}{l} |t_0 - s_0| < \epsilon\sqrt{\omega_w(\sqrt{\epsilon})}, \\ \|x_0 - y_0\| < \epsilon\sqrt{\omega_w(\sqrt{\epsilon})}. \end{array}\right. \tag{37}$$

Let us define the function $\phi_1 \in C^1((0,T) \times \mathbb{R}^m \times \mathbb{R})$ as follows:

$$\phi_1(t,x,\rho) = w(s_0, y_0, \rho + \sqrt{\epsilon}) + \Phi(t, x, s_0, y_0).$$

From (34), $\phi_1 - z$ has a maximum in $(t_0, x_0, \rho_0)$, then, as $z$ is supersolution, it follows that

$$\min(H_A(t_0, x_0, \rho_0, z), H_*(t_0, x_0, \rho_0, \nabla\phi_1)) \le 0. \tag{38}$$

We can suppose w.l.g. that $H_A(t_0, x_0, \rho_0, z) = +\infty$. (see the justification of this argument in the Note 4.1 given below).

Therefore, it must be $\min(H_A(t_0, x_0, \rho_0, z), H_*(t_0, x_0, \rho_0, \nabla\phi_1)) = H_*(t_0, x_0, \rho_0, \nabla\phi_1))$, so

$$H_*(t_0, x_0, \rho_0, \nabla\phi_1)) \le 0. \tag{39}$$

Then, it must exist $\overline{a}$ such that $f(t_0,x_0,\overline{a}) \le \rho_0$ and

$$\frac{\partial\phi_1}{\partial t}(t_0,x_0,\rho_0)+\frac{\partial\phi_1}{\partial x}(t_0,x_0,\rho_0)\, g(t_0,x_0,\overline{a}) \le 0. \tag{40}$$

On the other hand, by defining $\phi_2 \in C^1((0,T)\times \mathbb{R}^m \times \mathbb{R})$

$$\phi_2(s,y,\rho) = z(t_0,x_0,\rho-\sqrt{\epsilon}) - \Phi(t_0,x_0,s,y)\,,$$

from (34) we have that $\phi_2 - w$ has a minimum in $(s_0,y_0,\rho_0+\sqrt{\epsilon})$.

As $w$ is a subsolution we have

$$H(s_0,y_0,\rho_0+\sqrt{\epsilon},\nabla\phi_2) \ge 0\,. \tag{41}$$

Since $|s_0-t_0| \le \epsilon\,\sqrt{\omega_w(\sqrt{\epsilon})}$ and $\|x_0-y_0\| \le \epsilon\,\sqrt{\omega_w(\sqrt{\epsilon})}$ we can suppose that $\epsilon$ is small enough such that

$$f(s_0,y_0,\overline{a}) \le f(t_0,x_0,\overline{a}) + 2L_f\,\epsilon\,\sqrt{\omega_w(\sqrt{\epsilon})} < \rho_0+\sqrt{\epsilon}$$

and so, by definition of $H$ we have that

$$\frac{\partial\phi_2}{\partial s}(s_0,y_0,\rho_0+\sqrt{\epsilon})+\frac{\partial\phi_2}{\partial y}(s_0,y_0,\rho_0+\sqrt{\epsilon})\, g(s_0,y_0,\overline{a}) \ge 0\,. \tag{42}$$

Besides,

$$\left|\begin{array}{lllll}
\dfrac{\partial\phi_2}{\partial s}(s,y,\rho) & = & -\dfrac{\partial\Phi}{\partial s}(t_0,x_0,s,y) & = & -2\dfrac{(t_0-s)}{\epsilon^2} \\[2ex]
\dfrac{\partial\phi_2}{\partial y}(s,y,\rho) & = & -\dfrac{\partial\Phi}{\partial y}(t_0,x_0,s,y) & = & -2\dfrac{(x_0-y)}{\epsilon^2} \\[2ex]
\dfrac{\partial\phi_1}{\partial t}(t,x,\rho) & = & \dfrac{\partial\Phi}{\partial t}(t,x,s_0,y_0) & = & -2\dfrac{(t-s_0)}{\epsilon^2}+\sigma\dfrac{T}{t^2}+\dfrac{r}{8T} \\[2ex]
\dfrac{\partial\phi_1}{\partial x}(t,x,\rho) & = & \dfrac{\partial\Phi}{\partial x}(t,x,s_0,y_0) & = & -2\dfrac{(x-y_0)}{\epsilon^2} \\
 & & & & +(\delta+\omega_w(\sqrt{\epsilon}))\,\xi'(\|x-x_\delta\|)\dfrac{(x-x_\delta)}{\|x-x_\delta\|}
\end{array}\right.$$

Using these expressions in (40) and (42) we have

$$\begin{aligned}
0 \le \frac{r}{8T} &\le -\sigma\frac{T}{t_0^2}+2\frac{\langle x_0-y_0, g(t_0,x_0,\overline{a})-g(s_0,y_0,\overline{a})\rangle}{\epsilon^2} \\
&\quad -(\delta+\omega_w(\sqrt{\epsilon}))\,\xi'(\|x_0-x_\delta\|)\frac{\langle x_0-x_\delta, g(t_0,x_0,\overline{a})\rangle}{\|x_0-x_\delta\|} \\
&\le \frac{2L_g}{\epsilon^2}\,\|x_0-y_0\|\,(\|x_0-y_0\|+|t_0-s_0|)+(\delta+\omega_w(\sqrt{\epsilon}))\,\|g(t_0,x_0,\overline{a})\| \\
&\le 4L_g\,\omega_w(\epsilon)+(\delta+\omega_w(\sqrt{\epsilon}))\,M_g\,.
\end{aligned}$$

Letting successively $\epsilon$ and $\delta$ go to zero, we have that $r \le 0$ which contradicts the inequality (29). Then $z \ge w$. Finally, taking into account that any solution is at the same time a subsolution and a supersolution, we obtain the uniqueness of solution.

□

**Note 4.1** *If* $(t_0, x_0, \rho_0, s_0, y_0)$ *realizes the minimum of*

$$\phi^{\sqrt{\epsilon}}(t,x,\rho,s,y) = z(t,x,\rho) - w(s,y,\rho+\sqrt{\epsilon}) - \Phi(t,x,s,y)\,, \tag{43}$$

*then* $\exists\,(t_0, x_0, \overline{\rho}_0, s_0, y_0)$ *that also realizes the minimum of* $\phi^{\sqrt{\epsilon}}$ *and the point* $(t_0, x_0, \overline{\rho}_0)$ *has the property:* $H_A(t_0, x_0, \overline{\rho}_0, z) = +\infty$.

**Proof.** Let us suppose that

$$\min(H_A(t_0,x_0,\rho_0,z), H_*(t_0,x_0,\rho_0,\nabla\phi_1)) = H_A(t_0,x_0,\rho_0,z)\,,$$

then

$$H_A(t_0,x_0,\rho_0,z) \le 0\,. \tag{44}$$

In consequence, there exists $a' \in A$, such that

$$\left|\begin{array}{l} f(t_0,x_0,a') > \rho_0 \\ \\ \text{and} \\ \\ z(t_0,x_0,f(t_0,x_0,a')) \le z(t_0,x_0,\rho_0)\,. \end{array}\right. \tag{45}$$

From the continuity of $f$ and the lower-semicontinuity of $z$, it is obvious that there exists at least one $\overline{a} \in A$ such that

$$f(t_0,x_0,\overline{a}) = \max\{f(t_0,x_0,a') \;:\; a' \in A \text{ and } (45)\} \tag{46}$$

We define:

$$\overline{\rho}_0 = f(t_0,x_0,\overline{a})\,. \tag{47}$$

Since $w$ is a subsolution, it is a non-decreasing function of the variable "$\rho$" and as $\overline{\rho}_0 + \sqrt{\epsilon} > \rho_0 + \sqrt{\epsilon}$, we have

$$w(s_0,y_0,\overline{\rho}_0+\sqrt{\epsilon}) \ge w(s_0,y_0,\rho_0+\sqrt{\epsilon})\,. \tag{48}$$

From the inequalities (45) and (48) it follows that

$$w(s_0,y_0,\overline{\rho}_0+\sqrt{\epsilon}) - z(t_0,x_0,\overline{\rho}_0) \ge w(s_0,y_0,\rho_0+\sqrt{\epsilon}) - z(t_0,x_0,\rho_0)\,.$$

So, we have that

$$\begin{aligned} w(s_0,y_0,\overline{\rho}_0+\sqrt{\epsilon}) \;&-\; z(t_0,x_0,\overline{\rho}_0) + \Phi(t_0,x_0,s_0,y_0) \\ &\ge\; w(s_0,y_0,\rho_0+\sqrt{\epsilon}) - z(t_0,x_0,\rho_0) + \Phi(t_0,x_0,s_0,y_0)\,. \end{aligned}$$

This last inequality proves that $(t_0, x_0, \overline{\rho}_0, s_0, y_0)$ is also a point where $\phi^{\sqrt{\epsilon}}$ has a minimum.

Let us now show that

$$H_A(t_0,x_0,\overline{\rho}_0,z) = +\infty \tag{49}$$

If we suppose that $H_A(t_0,x_0,\overline{\rho}_0,z) < \infty$, then

$$H_A(t_0,x_0,\overline{\rho}_0,z) \le 0\,. \tag{50}$$

In consequence, it would exist $a'' \in A$, such that

$$\left| \begin{array}{l} f(t_0, x_0, a'') > \overline{\rho}_0 \\ \text{and} \\ z(t_0, x_0, f(t_0, x_0, a'')) \leq z(t_0, x_0, \overline{\rho}_0) \,. \end{array} \right. \tag{51}$$

From (45)-(47) and (51) we have

$$z(t_0, x_0, f(t_0, x_0, a'')) \leq z(t_0, x_0, \overline{\rho}_0) = z(t_0, x_0, f(t_0, x_0, \overline{a})) \leq z(t_0, x_0, \rho_0)$$

and

$$f(t_0, x_0, a'') > \overline{\rho}_0 = f(t_0, x_0, \overline{a}) > \rho_0 \,. \tag{52}$$

These inequalities contradict the maximality of $f(t_0, x_0, \overline{a})$ given by (46). So, by *reductio ad absurdum* we get (49).

□

# Conclusions

In the minimax optimal control problem analyzed in this paper, by extending the state of the system, we have embedded the original problem into another one where it is possible to establish a dynamical programming principle and a HJB equation with unique solution in the viscosity sense. The optimal cost of the auxiliary problem is that unique solution and this property enables us to characterize the optimal cost and to deal with the numerical analysis of the problem.

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