# Stochastic Evolution via Graph Grammars

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**Abstract:** This is the second part in the series of papers where we are looking for new connections between computer science, mathematics and physics. These connections go through the central notions of computer science - grammar and graph grammar. In section 2 main definitions concerning random graph grammars are given. Section 3 and 4 are devoted to the simplest models of graph dynamics: we study the large time behaviour of local and global characteristics of growing one-dimensional complexes. Main emphasis is on looking for correct problems and models, discussing their qualitative behaviour. We consider asymptotic growth of the number of connected components, the degree of local compactness, topological chaos, phases of different topology etc. In section 4 we construct infinite cluster dynamics. We discuss some aspects which distinguish thermodynamic limit for graph grammars from that for Gibbs fields on a lattice. One of the central emerging notions is the statistically homogeneous infinite complex.

**Key-words:** Graph Grammar, Random Graph, Random Topology, Branching Process

*(Résumé : tsvp)*

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# Evolutions stochastiques via grammaires sur des graphes

**Résumé :** L'évolution de complexes uni-dimensionels est dècrite à l'aide de grammaires aléatoires sur des graphes. On considère le comportement asymptotique de certains caractéristiques topologiques de ces complexes, ainsi que de nouveaux phenomènes en limite thermodynamique.

**Mots-clé :** Grammaire graphique, graphe aléatoire, topologie aléatoire, processus de branchement

# Contents

# 1 Introduction

From the point of view of computer science we consider large graph grammars and their nondeterministic trajectories.

From probability theory point of view we consider here a kind of homogeneous stochastic processes with local interaction but without the underlying space.

From the point of view of topology we consider infinite complexes (in this paper they are one-dimensional). But the limiting spaces under consideration are not quite standard for topology: complexes here are infinite, groups have infinite number of generators, invariants equal either 0 or $\infty$. The problems are also not standard for topology: it is not homeomorphism or homotopy of spaces as a whole, but rather some homogeneity structure (without symmetry group).

We do not speak here about physics at all but it is present behind the curtains in the methods and mere ideology of the paper. Spin graph corresponds to a field on a lattice in physics, its stochastic dynamics is called stochastic quantisation method. But here the lattice itself is changing dynamically in time together with fields. Already two-dimensional complexes have important physical interpretation, we consider them in the next paper.

The plan of this paper is the following. In section 2 the main object is introduced: a countable Markov chain describing evolution of spin graphs. Only transient chains are of interest when the graph grows. Section 3 is an introduction to section 4 where several models for such countable chains are considered. We consider large time behaviour of the number of vertices, links, cycles, connected components. It appears that local characteristics of the growing graph have well defined limits exhibiting chaos properties. Phase transitions in topological characteristics are related to those in random graph theory.

In section 5 the initial graph is infinite. The central problem is the existence of the dynamics itself. We prove its existence under some conditions. For $t \to \infty$ even more interesting problems arise and we start to discuss them here.

# 2 Random Graph Grammars

## 2.1 Definitions

### 2.1.1 Spin Graphs

Consider a (finite or countable) graph $G$ with the set of vertices $V = V(G)$ and set of links $L = L(G)$. *Subgraph* of $G$ is a subset $V_1 \subset V$ of vertices together with some links connecting pairs of vertices from $V_1$ and inherited from $L$. *Regular subgraph* $G(V_1)$ of $G$ is a subset $V_1 \subset V$ of vertices together with ALL links connecting pairs of vertices from $V_1$ and inherited from $L$.

Links are not directed, there can be any finite number of links between two vertices. A link $l$ connecting vertices $u$ and $v$ is called incident to both of them and vice versa. A link is incident with subset $V'$ if it is incident with some vertex $v \in V'$. We assume everywhere that each vertex has at most finite number of links incident to it.

Distance $\rho(v, v_1)$ between two vertices $v, v_1$ of the graph is the minimal length of a path connecting these two vertices, that is the minimal number of links in such a path. Vertices connected by a link are called neighbours. Neighbourhood $O_d(v), O_1(v) = O(v)$, of vertex $v$ in $G$ is the regular subgraph with the set of vertices consisting of $v$ itself and of all vertices on distance not greater than $d$ from $v$.

*Spin graph* (also colored graph, marked graph, spin system etc.) is a pair $\alpha = (G, s)$ where $s = s(.)$ is a function $s : V \to S$ where $S$ is the set of "spin values", the alphabet. Isomorphism of spin graphs is an isomorphism of graphs respecting the spins. The empty spin graph $\emptyset$ is the empty graph with no spin.

If we say that the set $V$ of vertices is given as a set this means that the graph is labelled, for example we can enumerate $V$ by positive integers. It will be useful in the sequel to extend the definition of labelling: assume we have some abstract set $\Psi$ which will be called the space. Labelling then is an arbitrary embedding of $V$ into $\Psi$. We shall often consider also equivalence classes of spin graphs (they are called sometimes nonlabelled graphs). Isomorphism of spin graphs is given by a one-to-one mappings between their sets of vertices respecting spins and links. It will always be clear from the context whether we consider labelled or nonlabelled graphs. One of the difficulties we shall encounter later will be the absence of natural labellings (or the absence of natural space) for the set of vertices of the graphs evolving in time. One can even say that the set of vertices looses its exact meaning.

### 2.1.2 Operations, substitutions, transformations

Spin graphs dynamics is a random sequence (where moments $... < t_k < t_{k+1} < ...$, are also random) of spin graphs

$$\alpha_0 = (G_0, s_0), \alpha_{t_1} = (G_{t_1}, s_{t_1}), ..., \alpha_{t_k} = (G_{t_k}, s_{t_k}), ...$$

to be defined below. Graph $\alpha_{t_k}$ is obtained from $\alpha_{t_{k-1}}$ by a simple transformation from some fixed class of transformations.

We start with defining some basic operations on spin graphs. Most useful will be substitutions which are similar to substitutions in grammars.

The operations we often use are the following:

1. deleting a link;
2. identifying two vertices. More exactly, for each ordered pair of vertices $(v, v')$ we can identify $v$ and $v'$. The spin in the obtained vertex say equal to $s(v')$;
3. deleting a vertex, all links incident to this vertex are also deleted;
4. adding a link means that either a pair $v, v'$ of vertices in $G$ are joined by a new link $l$ or a link is added to a vertex $v$ with some new vertex $v'$.
5. changing function $s$ in one vertex;

6. nonconnected union of two spin graphs $\alpha \cup \beta$. Appending an isolated vertex is a particular case.

We shall consider other operations as well which are the products of the defined operations. The most important operation has the following definition.

**Definition 1** *Substitution (production)* $Sub = (\Gamma; \varphi : V_0 \to V(\Gamma'))$ *is defined by two "small" spin graphs* $\Gamma$ *and* $\Gamma'$*, subset* $V_0 \subset V = V(\Gamma)$ *and mapping* $\varphi : V_0 \to V' = V(\Gamma')$*; either of* $\Gamma$ *and* $\Gamma'$ *can be empty.*

**Definition 2** *Transformations* $T = T(Sub)$ *(which eventually will also be call substitutions) of a spin graph* $G$*, corresponding to a given substitution* $Sub = (\Gamma; \varphi : V_0 \to V(\Gamma'))$*, is defined in the following way. Fix an isomorphism* $\psi : \Gamma \to \Gamma_1$ *onto a spin subgraph* $\Gamma_1$ *of* $G$*. Consider nonconnected union. n this nonconnected union of* $G$ *and* $\Gamma'$ *delete all links of* $\Gamma_1$*, delete all vertices of* $\psi(V) \setminus \psi(V_0)$ *together with all links incident to them, identify each* $\psi(v) \in \psi(V_0)$ *with* $v' = \varphi(v) \in \Gamma'$*. The function* $s$ *on* $V(G) \setminus V(\Gamma_1)$ *is inherited from* $G$*, and on* $V(\Gamma')$ *it is inherited from* $\Gamma'$*. The resulting graph is denoted by* $G(Sub, \psi)$ *. Regular transformations are defined similarly, with the condition that only isomorphisms* $\psi$ *to regular subgraphs* $\Gamma_1$ *of* $G$ *are allowed.*

This can be formulated differently: one should find first a subgraph of $G$ isomorphic to $\Gamma$, delete the subset $V(\Gamma) \setminus V_0$ of vertices together will all links incident to them, some vertices in $V_0$ are identified and some links between vertices of $V_0$ are added.

Note that these definitions are in the same spirit as attaching of one topological space $T_1$ to another $T_2$ with the continuous mapping $\varphi : T_1' \to T_2$ where $T_1'$ is a closed subspace of $T_1$.

It is easy to see that the elementary transformations are transformations according to the above definition. For example, to append a vertex we take $\Gamma'$ consisting of one vertex, $\Gamma$ empty (we assume that there exists only one empty subgraph for each graph).

### 2.1.3 Graph Grammars

**Definition 3** *Graph grammar is defined by (an initial) spin graph* $\alpha_0 = (G_0, s_0)$ *(finite or infinite) and by a finite set of substitutions* $Sub_i, i = 1, ..., r$*. We call a graph grammar local if* $\Gamma$ *corresponding to all* $Sub_i$ *are connected. Language* $L(\alpha_0, \{Sub_i\})$ *is the set of all spin graphs which can be obtained from* $\alpha_0$ *by applying transformations arbitrary number of times in arbitrary order. More exactly,* $\alpha_0 \in L(\alpha_0, \{Sub_i\})$*, and if* $\alpha \in L(\alpha_0, \{Sub_i\})$ *then* $T\alpha \in L(\alpha_0, \{Sub_i\})$ *for arbitrary* $T = T(Sub_i)$*.*

The notion of locality can be made more precise. We say that graph $G_1$ is obtained from $G$ by a *d-local* transformation if the following conditions hold. Some finite integer $d > 0$ is fixed (transformation radius) so that the diameters of all $\Gamma$'s do not exceed $d$. For example, vertices on distance greater than $d$ cannot be connected by a link. Local transformations can be obtained by successively applying elementary local transformations bounded number

of times. Example is : deleting a set of vertices $V_1$ (or the subgraph $G_1 = G(V_1)$) can be obtained by deleting these vertices one-by-one and all links incident to them. One can denote the resulting graph by $G \setminus G_1$.

As for the elementary transformations, only cases 4 and 6 can be nonlocal. From many points of view (both in physics and in computer science) local transformations seem more natural. The role of nonlocal transformations is enigmatic and should be discussed each time separately. For example, the laws of physics nowadays are supposed to be local. Programming in computer science also prefers local conditions and local cycles.

There were recent activity in graph grammars in computer science framework, see [4] and proceedings of four conferences in graph grammars, the first one is [5]. Note however that graph grammars (more exactly, operations on spin graphs) appeared under another name earlier (see [10]). One should realize there exist (and this is quite inevitable) a lot of different variations on the theme of graph grammars. All of them use different generalizations of spin graphs (see for example, [4] , [5]). Examples are bipartite graphs (used in Petri nets), hypergraphs with hyperedges instead of edges. Choice of a particular definition depends on local historical tradition, personal taste and favourite applications. Most general framework for all these notions seems to be the category theory, which was in earlier years called abstract nonsense see [14]. Following this tradition one could also call such contsructions an abstract nonsense.

We want to emphasize, however, that many statements about such generalized graph grammars can be reformulated using only simple spin graphs by enlarging the set of spins, redefining a graph with new vertices and new edges, etc. But such formal equivalence can kill intuition. For example, one could consider abstract simplicial complexes and cell complexes (CW-complexes) as graph grammars. Obviously the geometrical intuition will play an important role. However, in general theory it is convenient to choose the simplest formal definition without entering infinite number of evident generalizations. More exactly this means that many results can be automatically put into more general framework.

### 2.1.4 Markov Process

We introduce a general class of Markov processes which we call random graph grammars. The states are spin graphs, that is pairs $(G, s)$ where $G$ is a graph and $s = s(v)$ is a function on vertices of $G$ taking values in some "spin space" $S$. Contrary to the processes with local interactions, where the graph (normally a lattice) is fixed, in our definition the graph changes (locally) together with spin values. The process is specified completely by the initial state and by the rates $c(\alpha, \beta)$ of transformations from the state $\alpha = (G, s)$ to the state $\beta = (G_1, s_1)$.

Let us fix some graph grammar. The only positive rates are those which correspond to the transformations defined by the substitutions from the fixed graph grammar. We assume here the analog of translation invariance, which takes here a more general form: rates $c(\alpha, \beta)$ depend only on the substitution done but not on the place (that is not on the choice of $\psi$) where it is done. Thus $c(\alpha, \beta)$ take only finite number of different values $c_i = c(Sub_i)$.

Thus each allowed morphism $\psi$ waits the corresponding exponential time (independently of the others). When the first of them succeeds it performs a transition (transformation), and the procedure starts again, and so on. To give conditions when this process can be defined globally (for all time interval $[0, \infty]$) we discuss separately two cases. When the initial graph is finite then we get a countable Markov chain and we should only prove that it is not exploding. When the initial graph is countable then the Markov chain has a continuum number of states and the construction of dynamics is a more complicated problem with new and unexpected thermodynamic limit aspects.

### 2.1.5 Examples

It is worth notice that the introduced class of processes covers many famous processes.

- Branching processes. For the Galton-Watson process graphs consist of isolated vertices, no links. Vertices represent particles, and spins represent particle types.
- Processes with a local interaction on a lattice. It is a local grammar and the graph is not changed at all. Only spins undergo local transitions.
- Queue is a random string of symbols, where the symbols are the customer types. For example in LIFO and FIFO protocols the changes may be only at one or at both ends of the string. At the same time it is a particular case of a random grammar because one can put a special "nonterminal symbols" $L$ at the end of the queue and $F$ at the beginning of the queue.. The substitutions have the form $L \to xL$ (customer $x$ arrived) and $Fy \to F$ (customer $y$ is served). In this cases the grammar is context free: customers arrive and served independently of the context. Generalization to queueing networks is discussed in [7].
- Random Walks. For example, states of a random walk on $Z_+$ are strings with one symbol (the length of the string is the position of the random walker). In other words, random walk is a one-sided linear grammar with one terminal symbol and one nonterminal symbol (at the end of the word). Random walks on a free noncommutative group are strings with particular substitutions at one end.
- Embedded graph grammar is a graph grammar with additional structure: for given embedding of the graph $G$ into some topological space $X$ (normally $R^n$), then embeddings of all its transforms $G(Sub, \psi)$ are given as well.
- Random fractals: imbedded graph grammars with or even without spins, see [8].
- Stochastic geometry. In stochastic geometry only embedded cell complexes were considered, see [11].
- Random graph theory. This beautiful field with many deep results was always very popular in combinatorics. Evolution of random graphs was studied already by Erdos

Figure 1: Bad substitution

and Renyi in 1960, and continued later by many authors, see [13], [6], [12]. Unfortunately in fact the only model considered was the following: the number of vertices held fixed, on each step we append a new link, choosing it with equal probability among all absent links. Below in this paper we consider more complicated evolution and discuss arising new problems in random graph theory.

- Quantum gravity models in physics, which now are the most popular fundamental models of nature. We shall consider such models in detail in the next paper. This paper considers in fact a caricature model (one-dimensional complexes instead of two-dimensional) of stochastic quantisation approach to discrete quantum gravity.

## 2.2 Finite dynamics

Even for the case when the initial graph $G$ is finite, the existence of the Markov process for graph grammars (that is the absence of explosion) is a more subtle question than for random grammars, one-dimensional analog of graph grammars, see [8].

One peculiarity is that parallel discrete time dynamics, analogous to $L$-systems in one-dimensional case, imposes strong restrictions on possible substitutions. The reason is that the conflicts arising due to simultaneous transformations in all vertices may not to have local rules for solving these conflicts. That is why continuous time is necessary here.

Second peculiarity is that one can easily give examples of local but exploding dynamics. The typical example is the following. Assume there are no spins and the only substitution of rate 1 consists in appending one extra link for arbitrary chosen pair of links incident with the same vertex. If some vertex $v$ has $N$ links incident to it, the rate of the next change in $v$ will be $\frac{N^2}{2}$, which is exploding. However there are sufficiently general cases where this perplexing situation does not arise.

It could be that distingushing between exploding uncontrollable growth and normal growth can have some biological and physical counterparts but we shall not try to pursue this issue here.

To get nonexploding process one could follow two lines:

- scale the transition rates appropriately. When the scale depends, for example, on the number of vertices this can be equivalent to random time scale. Or even one can consider discrete time process doing one substitution at each step, choosing the substitution to do via some random rule. We shall see examples in the next section.
- give simple sufficient conditions to avoid explosion without scaling the rates. General trick is to find a function $f(\alpha) \geq 0$ such that for each transformation $T$ the increment $f(T\alpha) - f(\alpha), T\alpha$ is the transformed $\alpha$, is uniformly bounded from above. We shall call, by evident analogy, such function $f$ a Lyapounov function. For each model considered below such Lyapounov function can be easily given. Now we come to exact definitions.

**Definition 4** *Fix some graph grammar and denote $n(\alpha, \Gamma)$ the number of possible embeddings of $\Gamma$ into the spin graph $\alpha$, $n(\alpha)$ be the maximum of $n(\alpha, \Gamma)$ for all possible substitutions $Sub = (\Gamma; \varphi : V_0 \to V(\Gamma'))$ with $c(Sub) > 0$. A function $f(\alpha)$ is called a Lyapounov function if it is an upper bound for $n(\alpha)$, that is $n(\alpha) \leq f(\alpha)$, and moreover for any transformation $T$*

$$f(T\alpha) - f(\alpha) < const$$

**Lemma 1** *If there exists a Lyapounov function then the random graph grammar is non exploding.*

From the definitions it follows that $f(\alpha(t))$, and also $n(\alpha(t))$ grow not faster than linearly with the number of jumps. The absence of explosion is quite obvious because under this condition the sequence of random variables $n(\alpha(t))$ is dominated (we do not come into details here) by the number of particles in the pure birth process on $Z_+$, starting from $N = n(\alpha(0))$, with transition rates $p(i \to i + d) = Ci$, where $C$ is the maximum of the transition rates for our process. But such pure birth process with linear growth of transition rates is known to be nonexploding.

**Remark 1** *In this case one can construct the minimal process as follows. Each image $\psi(\Gamma)$ waits (independetly of the others) exponential time and when the first one of them succeeds the substitution is done at the same moment. Afterwards the concurrence starts from scratch for the new spin graph.*

**Proposition 1** *Let there exist $d > 0$ such that the local graph grammar has the following property: for any $\alpha \in L(\alpha_0, \{Sub_i\})$ a vertex cannot have more than $d$ links incident to it. Then the Markov chain is not exploding, i.e. for the time interval $[0, t]$ and any initial state $(G, s)$, one can construct a process with a.s. finite number of jumps.*

Take here $f(\alpha) = const \mid V(G) \mid, \alpha = (G, s)$. Note that the number of different $d$-neighbourhoods is bounded by some $C(d)$ and thus the number of possible morphisms $\psi$ is bounded by $const\,|V(G)|$. Then for all $\alpha = (G, s)$

$$\sum_{\beta : \beta \neq \alpha} c(\alpha, \beta) < f(\alpha)$$

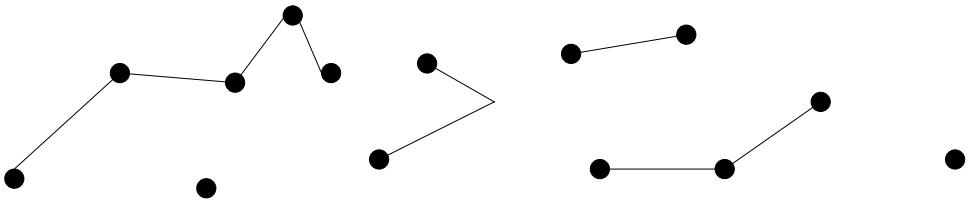

Figure 2: Random segments

Much less restrictive conditions are provided by the following proposition.

**Proposition 2** *Assume that the set $S$ of substitutions for a given graph grammar is subdivided into two parts: $S = S_1 \cup S_2, S_1 \cap S_2 = \emptyset$. Let $S_1$-transformations have linear Lyapounov function $f(\alpha) = C\,|V(\alpha)|$. Assume also that for all $\alpha, \alpha_1$ such that $\alpha_1$ can be obtained from $\alpha$ with one $S_2$-transformation we have $\sum_\beta c(\alpha,\beta) \geq \sum_\beta c(\alpha_1,\beta)$. Then we get a continuous time homogeneous countable Markov chain, which is non exploding.*

# 3 Two simple models

In this and in the next section we consider some models of dynamics for graphs without spins. In this section we discuss some degenerate models just to warm up. We use time scale to get rid of the exponential growth thus reducing the dynamics to more familiar random walk dynamics.

### 3.0.1 Random chains (model 1)

In the following model one can only append and delete links. Any existing link is deleted with rate $\mu$. Assume that we can append a new link only to vertices which are incident not more than with one link and that the second vertex of the appended link is a new vertex. Thus if we start with an isolated vertex, the state of the system is a finite number of linear chains (segments) and isolated vertices (chains of zero length). The rate of appending a new link at a vertex incident only with one link is $\lambda$ . Thus each connected segment gets a new link with rate $2\lambda$. For symmetry it is convenient to assume also that a link can be appended to an isolated vertex in two ways (i.e. that the rate is $2\lambda$).

Note that the number of vertices cannot decrease, thus this Markov chain is transient for all values of parameters.

**Modifications of the Markov chain** It will sometimes be convenient to consider a modified (with random time or measure change) process. One can use continuous or discrete time, parallel or simultaneous transformations of the graph etc. The choice mainly depends

on the personal taste of the author, on the techniques used, but also on some practical demands (for example in computer scince the choice between parallel and sequential evolution can be important). Qualitatively the choice of the model does not change much but formulas may differ. Moreover, it is important to understand what is really invariant with respect to the choice of particular scale and thus could have universal value. We give some examples:

- Perhaps the easiest way is to use the initial model with exponential growth. This we shall do in the next section. This is the typical branching process scale;
- In this section we use the embedded Markov chain. The time change is trajectorywise. In our process, as the time grows, events can become more and more condensed. We define time interval between any pair of consecutive events on each trajectory equal 1. Further on $t = 0, 1, 2, ...$ - the time for the embedded Markov chain. Note the analogy with a continuous time supercritical branching process where the asymptotical growth of the number of particles is $w \exp ct$ but for the corresponding embedded chain (random walk) it is only $ct$.
- Discrete time parallel evolution. In random grammars it corresponds to $L$-systems, see [8];
- Discrete time sequential evolution. This is similar to the embedded chain but the rates are simpler, we shall use this model once in this section.

**Random walk** Let $m(t)$ be the random number of connected components (segments) at time $t$, $m_k(t)$ - number of connected components with $k \geq 0$ links, $L(t) = \sum_k km_k(t)$ - the number of links at time $t$.

Equivalently the new process can be defined in the following way. On each step $t$ ($t$ is an integer) we choose what to do: either to append a new link to one of the ends of the segments or to delete a link. We make this choice with probabilities $p(t) = \frac{2\lambda m(t)}{2\lambda m(t)+\mu L(t)}$ and $q(t) = 1 - p(t)$ correspondingly. After we know what to do we choose randomly the place where to do it. For example, if we have chosen to delete a link then we choose the link to delete with probability $\frac{1}{L(t)}$. Taking into account the time change we get that the random variables $V(t), m(t)$ grow at most linearly. In fact, $L(t) \leq t$.

If we consider segments to be indistinguishable (non labelled) then, due to the symmetry, the state of the Markov process is the vector $(m_0, m_1, ...)$ and our process is a random walk in the infinite dimensional orthant $Z_+^\infty$ with points $(m_0, m_1, ...)$. It is clear that for all $\lambda$ and $\mu$ both $m(t)$ and $L(t)$ tend to infinity. We shall be interested in the asymptotic behaviour of $\frac{m(t)}{L(t)}$ and also of the proportions $\frac{m_k(t)}{m(t)}$ as $t \to \infty$. One could say that we shall study here the interplay between the number of connected components and the mean size of these components.

It is a luck here that the process restricted to the pairs $(m, L)$ is also Markov. It is a random walk $\xi(t)$ in $Z_+^2 = \{(m, L) : L \geq 0, m > 0\}$ with transition probabilities

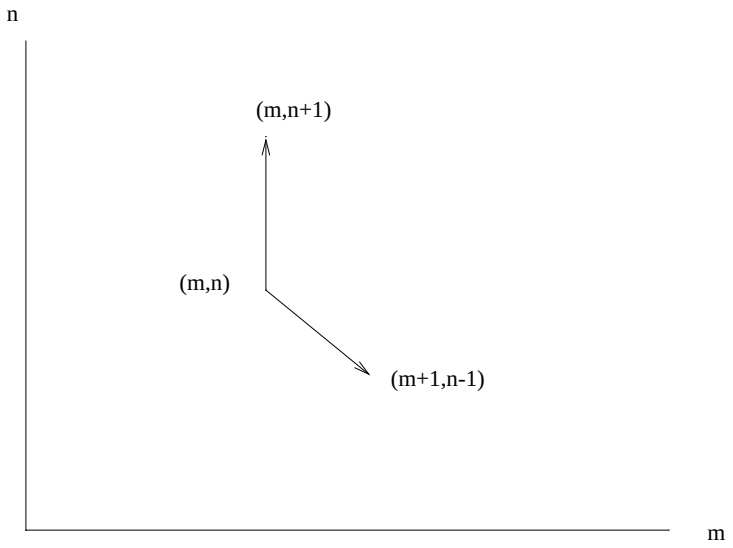


Figure 3: One step transitions

$$p((m,L) \to (m,L+1)) = \frac{2\lambda m}{2\lambda m + \mu L} = \frac{1}{1+\rho\alpha}, \alpha = \frac{L}{m}, \rho = \frac{\mu}{2\lambda},$$

$$p((m,L) \to (m+1,L-1)) = \frac{\mu L}{2\lambda m + \mu L} = \frac{\rho\alpha}{1+\rho\alpha}, L > 0$$

This random walk is always transient and goes to infinity a.s. Our main concern is the structure of its Poisson boundary.

**Theorem 1** *As* $t \to \infty$ *a.s.*

$$m(t) \sim \frac{\rho\alpha_{cr}}{1+\rho\alpha_{cr}} t, L(t) \sim \frac{1-\rho\alpha_{cr}}{1+\rho\alpha_{cr}} t, \alpha_{cr} = -\frac{1}{2} \pm \sqrt{\rho + \frac{1}{4}}$$

Proof. In polar coordinates let

$$M(r,\alpha) = (M_1(r,\alpha), M_2(r,\alpha)) = \frac{1}{1+\rho\alpha}(\rho\alpha, 1-\rho\alpha)$$

be the drift vector for the point $(r,\alpha)$. Then let us consider the set of points (a ray) where the drift points along this ray

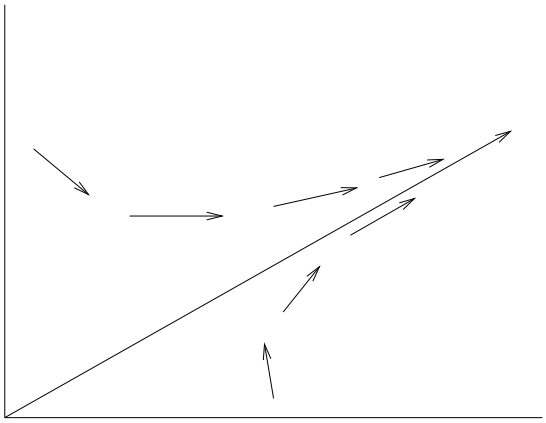

Figure 4: Vector field

$$\frac{M_2(r,\alpha)}{M_1(r,\alpha)} \equiv \frac{1-\rho\alpha}{\rho\alpha} = \alpha$$

Intuitively it is likely that this is the asymptotic direction along which the random walk goes to infinity. The equation is $\alpha^2 + \alpha - \rho^{-1} = 0$, and it has exactly one solution $\alpha_{cr}(\rho)$ in $[0,\infty)$. The vector field of mean drifts and the trajectories of the dynamical system $U(\tau)x = \lim_{N\to\infty} \frac{\xi(\tau N)}{N}, \xi(0) = [xN], x \in R_+^2, \xi = (m, L)$, are shown on the figure.

By standard techniques (for example, Lyapounov functions and exponential estimates) the theorem follows. This shows that the Poisson boundary consists of one point in all cases. We do not give detailed proof of this.

Now we consider the behaviour of $x_k(t) = \frac{m_k(t)}{m(t)}$.

**Theorem 2** *Deterministic limits* $\lim_{N\to\infty} x_k(t) = x_k > 0$ *exist in probability and* $\sum_{k=0}^{\infty} x_k = 1$.

Proof. From the results just obtained it follows that $p(t)$ and $q(t)$ tend to the limits

$$p = \frac{2\lambda}{2\lambda + \mu\frac{1-\rho\alpha_{cr}}{\rho\alpha_{cr}}}, q = 1 - p$$

This gives a hint to consider a simpler model with $p$ and $q$ constant and equal to these limits. Note that $m(t) \sim qt, L(t) \sim (p-q)t$ and then $x_k(t) \sim \frac{m_k(t)}{qt}$.

Let us consider in the space $Z_+^\infty = \{(m_0, m_1, ...)\}$ the vector field of mean drifts is

$$\Delta_0 \equiv E(m_0(t+1) - m_0(t) \mid (m_0(t), m_1(t), ...)) = -p\frac{m_0(t)}{m(t)}(t) + 2q\sum_{i=1}^{\infty}\frac{m_i(t)}{m(t)}$$

$$\Delta_k \equiv E(m_k(t+1) - m_k(t) \mid m(t)) =$$

$$= p\frac{m_{k-1}(t)}{m(t)} - (p+kq)\frac{m_k(t)}{m(t)} + 2q\sum_{i=k+1}^{\infty}\frac{m_i(t)}{m(t)}$$

Then

$$E(\frac{m_0(t+1)}{t+1} - \frac{m_0(t)}{t} \mid (m_0(t), m_1(t), ...)) =$$

$$= E(\frac{m_0(t+1) - m_0(t)}{t} \mid (m_0(t), m_1(t), ...)) - \frac{qx_0(t)}{t} + O(\frac{1}{t^2}) =$$

$$= \frac{1}{t}(-px_0(t) + 2q(1 - x_0(t)) - qx_0(t)) + O(\frac{1}{t^2})$$

Similarly

$$E(\frac{m_k(t+1)}{t+1} - \frac{m_k(t)}{t} \mid (m_0(t), m_1(t), ...)) =$$

$$= E(\frac{m_k(t+1) - m_k(t)}{t} \mid (m_0(t), m_1(t), ...)) - \frac{qx_k(t)}{t} + O(\frac{1}{t^2}) =$$

$$= \frac{1}{t}(-px_k(t) + 2q(1 - x_0(t) - ... - x_k(t)) - qx_k(t)) + O(\frac{1}{t^2})$$

This system is linear triangular and one can study convergence for each $k$ separately starting from $k = 0$. Note that the right hand part has a unique fixed point defined by

$$-px_0(t) + 2q(1 - x_0(t)) - qx_0(t) = 0$$

$$-x_k(t) + 2q(1 - x_0(t) - ... - x_k(t)) = 0$$

Moreover, the fixed point is stable as it is seen with the signs in the righthand side of the linear system. Using the constructed semimartingale we get convergence to this fixed point. Then we proceed by induction. We consider $x_j(t), j < k$, as constants and we get equations for $x_k(t)$ with the same signs. Calculation gives us positivity of all $x_k$.

This model does not have phase transitions, fixed points have smooth dependence on $p$.

### 3.0.2 Embedded Chains (model 1a)

One could consider also an embedded version of model 1. The segments are distinguishable and are embedded into the real axis as segments $[k, l]$, where $k \leq l$ are integers. Then the vertices are integer points, the links are unit segments $[k, k+1]$. Together with the problems considered above, for such embedded graph grammar one can study also the mean distances between neighbour components.

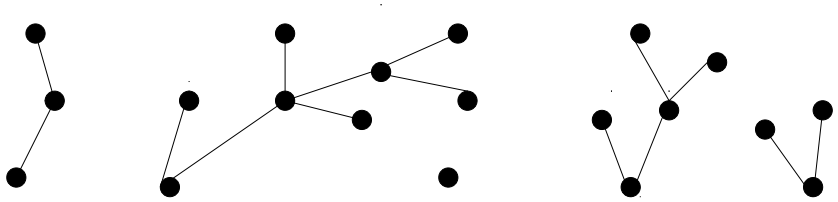

Figure 5: Trees

Reformulation of model 1 for this case gives the following rates: $\lambda$ - rate of appending a link at the empty interval $(k, k+1)$ with only one existing vertex (either $k$ or $k+1$), $2\lambda$ - to append a link if both vertices $k$ and $k+1$ exist, $\mu$ - rate of deleting a link. Once produced the vertices cannot dissappear in this model then finally each point of $Z$ will become a vertex. Then our process becomes the independent process as $t \to \infty$. Thus this model is not interesting.

More interesting is the model when isolated vertices automatically disappear. Then this model appears to be well known as the contact process describing infection process or Regge spin model in high energy physics. Here there exists critical value $\rho_{cr}$ of $\frac{\lambda}{\mu}$ such that below $\rho_{cr}$ there exists only one invariant measure which is the point measure on the configuration "no links, no vertices". Above $\rho_{cr}$ there are two invariant measures. See [3].

If in our model 1 we assume that isolated vertices also automatically disappear then the same effect holds. This shows, firstly, that the "monotonicity" effect giving two invariant measures (one of which is trivial) is not specific for embedded case. There will be two regions of parameters: one when the degeneration probability is equal to 1, and the second when the growth has some positive probability. In the latter case the condition that we escape to infinity gives the same linear growth of topology. Thus the degeneration is irrelevant to our purposes, that is why we do not consider it here.

### 3.0.3 Trees (model 2)

We change now model 1. Again we assume that there are only two substitutions: deleting a link with rate $\mu$ and appending for each vertex a new link with one new vertex with rate $\lambda$ (at time $t$ we call a link or a vertex new if they appear at time $t$ or later, ones appearing ealier than $t$ we call old). But, contrary to model 1, we allow appending a link to any vertex whatever the number of incident links it already has. As one of the vertices of the appended link is not incident to any other existing link then the only topology we can get is a disjoint union of connected trees.

This model is the simplest model which adds one more (local) characteristics to the already discussed number $m(t)$ of connected components and their volume (number $V$ of vertices and number $L$ of links, note that here we have $V - L = 1$ for one component). This characteristics is the degree $l_v$ (of local finiteness) of the vertex $v$, that is the number of links

incident to $v$. Interplay between local finiteness degree $El_v(N)$ and other characteristics can be studied for this model.

Note that for trees $V - L = m$. For this model we have also a random walk in $(V, L)$ coordinates. After the same time change the jump probabilities are the following

$$P((V,L) \to (V+1, L+1)) = \frac{\lambda V}{\lambda V + \mu L}$$

$$P((V,L) \to (V, L-1)) = \frac{\mu L}{\lambda V + \mu L}$$

Note that the drift vector is

$$(M_1, M_2) = \left( \frac{\lambda}{\lambda + \mu\alpha}, \frac{\lambda - \mu\alpha}{\lambda + \mu\alpha} \right), \alpha = \frac{L}{V}$$

From the equation

$$\frac{M_2}{M_1} = \alpha \Longleftrightarrow \frac{\lambda - \mu\alpha}{\lambda} = \alpha$$

as earlier, we get the direction along which we go straightly to infinity in $Z_+^2$: $\alpha_\infty = \frac{1}{1+\frac{\mu}{\lambda}}$. There is no phase transitions and we always go to infinity along the angle $0 < \alpha_\infty < 1$.

**Theorem 3** *As $t \to \infty$ we have all parameters $V, L, m$ growing asymptotically linearly in time, that is*

$$L \sim \frac{\lambda - \mu\alpha_\infty}{\lambda + \mu\alpha_\infty} t, m \sim \frac{\mu\alpha_\infty}{\lambda + \mu\alpha_\infty} t, V \sim \frac{\lambda}{\lambda + \mu\alpha_\infty} t$$

This could suggest that the degree of local finiteness equals the limit $\frac{L}{V} \to \frac{\lambda - \mu\alpha_\infty}{\lambda}$. However, this is wrong for $\mu = 0$. In this case $m = 1$ and in each appeared vertex the number of incident links tends to infinity. But sufficiently slowly (as $\log t$) with respect to the number of vertices. We shall not prove this here. Instead in the next section we shall consider this characteristics for more general case using another approach.

Models 1 and 2 are important to see what can occur in more complicated situations as, for example, model 3. In the next section we use however a different approach.

# 4 Large Time Behaviour of Finite Graph Dynamics

Here we present two basic models of graph dynamics without spins. Main attention is given to the topology of graphs. Note that the first topological characteristics of a graph are the numbers $V(t), L(t), C(t), m(t)$ of vertices, links, cycles and connected vertices correspondingly. First two are easy to calculate. In the model 3 we study an interplay between the last two. In model 4 $m(t) = 1$, thus $C(t)$ is also easy to calculate via Euler formula $V - L + C = m$. In model 4 we are looking for $C(t)$ at different scales.

## 4.1 Mean Field Dynamics (model 3)

**Definitions** Here the graph grammar consists of appending and deleting links. Each link is deleted with rate $\mu$. At each vertex $v$ we append with rate $\lambda_1$ a link incident to it and having a new vertex. Links appended in this way will be eventually called type 1 links. These two substitutions are the same for mean field and local models but the next one is different. For both models at each vertex $v$ with rate $\lambda_2$ we append a link (called type 2 link) connecting $v$ with another existing vertex $v' \neq v$. For the mean field model we choose $v'$ with equal probabilities $\frac{1}{V-1}$ among the existing $V-1$ vertices different from $v$. For the local model we choose $v'$ with equal probabilities among vertices in the $d$-neighbourhood of $v$ for some fixed $d$. Thus the graphs which can appear here may have more than one link between two vertices and may have cycles. Denote $L_i(t), i = 1, 2$, the set of type $i$ links at time $t, L_1(t) + L_2(t) = L(t)$.

Let at time zero we have one isolated vertex. For each $v \in V(G_t), l \in L(G_t)$ (independently of the others) jumps can occur with the following infinitesimal transition probabilities

$$P(G_{t+\Delta t} = G_t \cup (v, v_{new}) \mid G_t) = \lambda_1 \Delta t + o(\Delta t)$$

$$P(G_{t+1} = G_t \setminus l \mid G_t) = \mu \Delta t + o(\Delta t)$$

also for each $v, v' \in V(G_t)$

$$P(G_{t+1} = G_t \cup (v, v') \mid G_t) = \frac{1}{V(t)-1} \lambda_2 \Delta t + o(\Delta t)$$

for the mean field case, and

$$P(G_{t+1} = G_t \cup (v, v') \mid G_t) = \frac{1}{|O_d(v)|} \lambda_2 \Delta t + o(\Delta t)$$

for the local case. Thus for the local case loops (links $(v, v)$) can appear.

One can consider various topological (homotopy, homeomorphism etc.) and metrical characteristics of the one-dimensional complex: local and global. There are not many for one-dimensional complexes but hopefully interesting problems will be the same for other dimensions as well. Examples of local properties are local connectivity properties (how strongly a vertex is connected with others), example of global - the number of independent cycles $C(t)$ (number of generators in the first homology group or in the fundamental group). Remind that any connected one-dimensional complex is homotopic to a bundle $S \vee S \vee ... \vee S$ of circles $S$.

**Asymptotic Growth** $V(t)$ is defined by the following rates

$$P(V(t + \Delta t) = V(t) + 1 \mid V(t)) = \lambda_1 V(t) \Delta t + + o(\Delta t)$$

It is known in classical probability as the Yule process (see [18]).

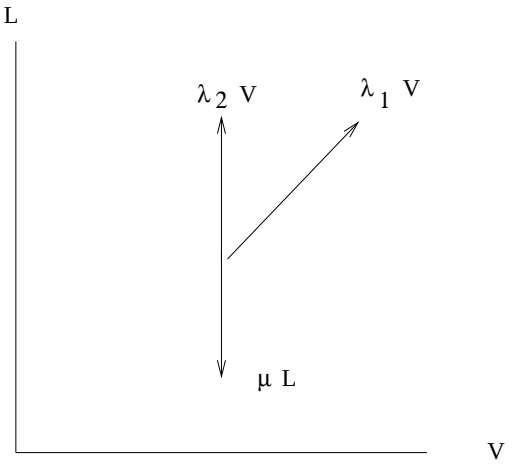


Figure 6: One step transitions

**Lemma 2** *We have*

$$V(t) \sim \xi \exp(\lambda_1 t)$$

where $E\xi = 1$ and $\xi$ is a positive random variable. Moreover

$$L_i(t) \sim \frac{\lambda_i}{\lambda_1 + \mu} \xi \exp(\lambda_1 t), i = 1, 2$$

Proof. This follows from well known techniques for supercritical branching processes.

The evolution of the pair $(V(t), L_1(t))$ does not depend on appending and deleting type 2 links and can be considered separately (in fact it is the model 2 considered earlier under some time change). Similarly evolution of $(V(t), L_2(t))$ does not depend on deletion of type 1 links. Thus, they can be considered separately. One can calculate $EL_i(t)$ from the following equations for the means:

$$EL_i(t + \Delta t) = EL_i(t)(1 - \mu \Delta t) + \lambda_i \Delta t \exp(\lambda_1 t) \Rightarrow \frac{dEL_i(t)}{dt} = -\mu EL_i(t) + \lambda_i \exp(\lambda_1 t)$$

It follows that

$$EL_i(t) \sim c_i \exp(\lambda_1 t), c_i = \frac{\lambda_i}{\lambda_1 + \mu}$$

**Remark 2** *Properties of the r.v. $\xi$ can be easily extracted from the generating function*

$$\begin{aligned} F(s,t) &= \sum s^k P(V(t) = k) = s \exp(-\lambda_1 t) \left[1 - (1 - \exp(-\lambda_1 t))s\right]^{-1} = \\ &= \exp(-\lambda_1 t) \sum_{j=1}^{\infty} s^j (1 - \exp(-\lambda_1 t))^{j-1} \end{aligned}$$

*(see [18], p. 109). In particular, putting $s = a\exp(\lambda_1 t)$ one can get that $\xi$ has exponential tail.*

### 4.1.1 Local properties - Ancient history

**Local Finiteness** Fix time $t$ and consider first the vertices which appeared long before time $t$. Note that all vertices can be ordered in a natural way: the vertex has number $i$ if it is the $i$th vertex appeared. Let $l_i(t)$ be the random number of links incident to the vertex $i$ at $t$; we put it zero if $i$ did not appear until time $t$.

**Lemma 3** *If $\mu \neq 0$ then for each increasing function $i = i(t) = o(\exp \lambda_1 t)$ there exists the limit of distributions of $l_i(t)$ as $t \to \infty$.*

$$\pi(k) = \lim_{t\to\infty} P(l_i(t) = k)$$

If we fix some increasing function $\varphi(t) = o(\exp \lambda_1 t)$ then this limit is uniform for all $i(t) \leq \varphi(t)$.

Proof. Write down the infinitesimal jump probabilities under the condition that the number of vertices is fixed

$$P(l_i(t+dt) = l-1 \mid l_i(t) = l) = \mu l dt$$

$$P(l_i(t+dt) = l+1 \mid l_i(t) = l) = (\lambda_1 + \lambda_2 + \lambda_2(V-1)\frac{1}{V-1})dt = (\lambda_1 + 2\lambda_2)dt$$

We see that these rates in fact do not depend on $V$ and thus $l_i(t)$ is a Markov chain. This chain is obviously ergodic, exponentially convergent and its transition rates do not depend on $i$. Denote this chain by $\mathcal{M}$, then $\pi(k)$ - its stationary probabilities.

**Topological Chaos**

**Theorem 4** *Denote $p_{ij}(k,l;t)$ the probability that at time $t$ vertices $i(t)$ and $j(t), i(t) < j(t) = o(\exp \lambda_1 t)$, have $k$ and $l$ incident links correspondingly. Then as $t \to \infty$*

$$p_{ij}(k,l;t) \to \pi_k \pi_l$$

Consider the pair $(l_i(t), l_j(t))$. This is not anymore a Markov chain. But let us write down again the conditional transition rates for this pair under the condition that the number of vertices is fixed:

$$P(l_i(t+dt) = l-1, l_j(t+dt) = k \mid l_i(t) = l, l_j(t) = k) = \mu l dt$$

$$P(l_i(t+dt)=l, l_j(t+dt)=k-1 \mid l_i(t)=l, l_j(t)=k) = \mu k dt$$

$$P(l_i(t+dt)=l+1, l_j(t+dt)=k \mid l_i(t)=l, l_j(t)=k) = (\lambda_1 + 2\lambda_2 - \frac{2}{V-1})dt$$

$$P(l_i(t+dt)=l+1, l_j(t+dt)=k+1 \mid l_i(t)=l, l_j(t)=k) = (2\lambda_2 \frac{1}{V-1})dt$$

etc. As $V(t) \to \infty$ a.s., the process $(k(t), l(t))$ tends to the Markov chain $\mathcal{M} \times \mathcal{M}$ which is the product of two Markov chains $\mathcal{M}$.

**Local triviality of homotopy** Let $P_i^t(O_d(i) = g)$ be the probability that at time $t$ the $d$-neighbourhood of the vertex $i$ is isomorphic to the graph $g$.

**Lemma 4** *As $t \to \infty$ the probabilities $P_i^t(O_d(i) = g), i = i(t) = o(\exp \lambda_1 t)$, tend to some $p(d; g)$. For mean field case they are positive only if $g$ is a tree.*

Proof. It repeats the proofs of previous assertions. Main remark is that it is very improbable to get type 2 link inside $O_d(i)$.

### 4.1.2 New history.

From the exponential asymptotics it is clear that the $\frac{V(t-s)}{V(t)}$ tends to zero if only $s \to \infty$ and $t \to \infty$ somehow. This means that most vertices are sufficiently young, most of them were born at a time $t - s$ for a constant $s$, $s$ does not depend on $t$.

Among such young vertices we shall specify an individual vertex differently. Fix $s$ and let $v(s; t, \omega)$ be the vertex (it is a.s. uniquely defined) which appearance time are the closest to $t - s$. Let $l(\tau)$ be the (Markov) process $\mathcal{M}$ of number of links in one vertex starting with $l(0) = 1$. Its transitions thus are $+1$ with rate $\lambda_1 + 2\lambda_2$ and $-1$ with rate $\mu l(s)$.

**Lemma 5** *Let $\eta(t, v)$ be the number of links at $v$ at time $t$, then as $t \to \infty$*

$$P(\eta(t, v(s; t, \omega)) = D) \to P(l(s) = D)$$

$$P(\eta(t, v(s_1; t, \omega)) = D_1, \eta(t, v(s_2; t, \omega) = D_2) \to P(l(s_1) = D)P(l(s_2) = D),$$

*$s_1 \neq s_2$. Also the probability $P(O_d(v(s; t, \omega)) = g)$ has a limit $P(s, g)$ which is nonzero only when $g$ is a connected tree.*

Proof. Take two vertices $1, 2$ which appeared say at the moments $t - s_1 < t - s_2$ correspondingly. The probability that the second one is a descendant of the first one tends to

zero as $t \to \infty$. Then the probability $P(l_1(t) = D_1, l_2(t) = D_2) \to P(l(s_1) = D_1)P(l(s_2) = D_2)$ as $t \to \infty$, which quite similar to the previous proofs.

For any $D$ denote $V(D,t)$ the number of vertices at time $t$ with $D$ incident links. The mean density of the number of vertices which appeared at $(t, t+dt)$ is $\rho(t) = \frac{d \exp \lambda_1 t}{dt} = \lambda_1 \exp \lambda_1 t$. Thus

$$\frac{EV(D,t)}{\exp \lambda_1 t} = \lambda_1 \int_0^t \exp(\lambda_1(s-t))P(l(t-s) = D)ds$$

We see that it tends to exponentially fast zero if $D \to \infty$ uniformly in $s$. Now we shall say it in a more precise way.

Let $V(> R, t)$ be the set of vertices at time $t$ such that $l_v(t) > R$.

**Lemma 6** *There exists $\varepsilon(R) > 0$ such that $\varepsilon(R) \to 0$ as $R \to \infty$ and*

$$\limsup_t P(|V(> R,t)| > \varepsilon(R)V(t)) \to_{R\to\infty} 0$$

**Phase diagram** Denote $V_D(t)(V_{>D}(t))$ the number of vertices belonging to a connected component of size $D$ (larger than $D$).

**Theorem 5** *If $\lambda_1 + 2\lambda_2 < \mu$ then there is a uniform in $t$ exponential bound $\frac{EV_{>D}(t)}{\exp \lambda_1 t} < a \exp(-bD)$. Moreover, there exists $\varepsilon(D) > 0$ such that $\varepsilon(D) \to 0$ as $D \to \infty$ and*

$$\limsup_t P(|V_{>D}(t)| > \varepsilon(D)V(t)) \to_{D\to\infty} 0$$

*Also $\frac{C(t)}{\exp \lambda_1 t} \to 0$.*

Proof. Denote $P_D(v,t)$ the probability that a given vertex belongs to a connected component with $D$ vertices.To estimate this probability we need the following lemma.

**Lemma 7** *If $\lambda_1 + 2\lambda_2 < \mu$ then each vertex $v$ has mean number of links $E(t) = El_v(t) \leq 1$. Moreover if $t_v$ is the time when the vertex $v$ appeared then for each $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that $El_v(t) \leq 1 - \delta$ for $t - t_v \geq \varepsilon$.*

Proof of the lemma. Denote $p_k(t)$ the probability that $l_v(t) = k$ and $E(t) = \sum kp_k(t)$. Then $E(t+\Delta t) = E(t) + (\lambda_1 + 2\lambda_2)\Delta t p_0(t) + (\lambda_1 + 2\lambda_2)\Delta t(1 - p_0(t)) - \sum \mu k \Delta t p_k(t)$. It follows that $\frac{dEl_v(t)}{dt} = \lambda_1 + 2\lambda_2 - \mu E(t)$. The fixed point of this dynamical system is $E_f = \frac{\lambda_1 + 2\lambda_2}{\mu}$ and if we start from some $a$ then we stay always inside $[0, \max(a, E_f)]$.

We know that for fixed $d$ the neighbourhood $O_d(v(s;t,\omega))$ of $v = v(s;t,\omega)$ is a random tree a.s. Moreover the distribution of the number of vertices on distance $d$ from $v$ coincides with the number of particles in some discrete time Galton-Watson process with one particle type. Exact statement is the following. For each $d$ as $t \to \infty$ it tends to Galton-Watson process with discrete time $t = 0, 1, ..., d$. The mean number of descendants in this G-W

process is bounded by $El_v(t)$ (the only difference is if there is only one particle it cannot die, we say in this case that there is reflection at zero). It can be proved as above. Note that the order of the limits is $\lim_{d\to\infty}\lim_{t\to\infty}$ and the first (time) limit exists for all values of parameters.

Then the mean maximal radius of the tree is the mean time for Galton-Watson process to degenerate. This process is subcritical under conditions of the lemma and thus the number of vertices on finite distance from $v$ is finite a.s. which is the connected component containing this vertex $v$.

For two different vertices the estimates are almost independent. Thus we can estimate the variance of $V_D(t)$ and thus prove the first assertion of the theorem. As far as we could prove that all $d$-neighbourhoods are trees with probability tending to 1 we get the result abour smallness of the number of cycles.

Now we shall consider another extreme case.

**Theorem 6** *If* $\lambda_2 > \frac{\mu}{2}$ *and* $\lambda_1 \ll \lambda_2$ *then the probability that there is a "giant" connected component of size* $O(\exp \lambda_1 t)$ *tends to* 1. *Moreover the number of cycles* $\liminf \frac{C(t)}{\exp \lambda_1 t} > 0$.

Proof. The simplest way to prove it is to reduce it to the "random graph" situation. Let $V_1(t)$ be the set of vertices appeared before $t-1$. Let us consider the set $L_1(t,\omega)$ of all type 2 links appeared in the time interval $[t-1,t]$. First note that these links are ordered correspondingly to the time when they appeared: $l_1, l_2, ....$ Let us use the following sieve process: we delete from this sequence each $l_i$ such that among $l_1, l_2, ..., l_{i-1}$ already there were a link between the same pair of vertices. Let $K(\omega)$ be the number of links left after the sieve. More exactly we consider only links both vertices of which belong to $V_1(t)$. The remaining set of links is also ordered and the conditional process on this set describing appending new links step by step has the following property: it coincides with equiprobable choice of a link among remaining links in $L(V_1(t))$. Consider the conditional distribution of these links under the condition that $L_1(t,\omega) = L_1, K(\omega) = K$: all such $C_{L_1}^K$ choices have equal probability.

Now we estimate most probable values of $K$ and $L_1$. Consider all vertices appeared before time $t-1$. From the equation $\frac{dEn_v(t)}{dt} = 2\lambda_2 - \mu E(t)$ we get (from the first condition of the theorem) that at time $t$ all such vertices will have $En_v(t) > 1$. From the second condition we get that most of the links join vertices from $V_1(t)$. More exactly: For any small $\varepsilon > 0$ there exist $\lambda_1(\varepsilon)$ such that for $\lambda_1 < \lambda_1(\varepsilon)$ we have

$$P((1-\varepsilon)En_v(t) < \frac{K(t)}{V_1(t)} < (1+\varepsilon)En_v(t)) \to_{t\to\infty} 1$$

To finish the proof one can simply apply the known results from the "random graph theory", see [12].

**Some improvements** I think that bounds for the phase diagram can be improved essentially and here we do some steps in this direction.

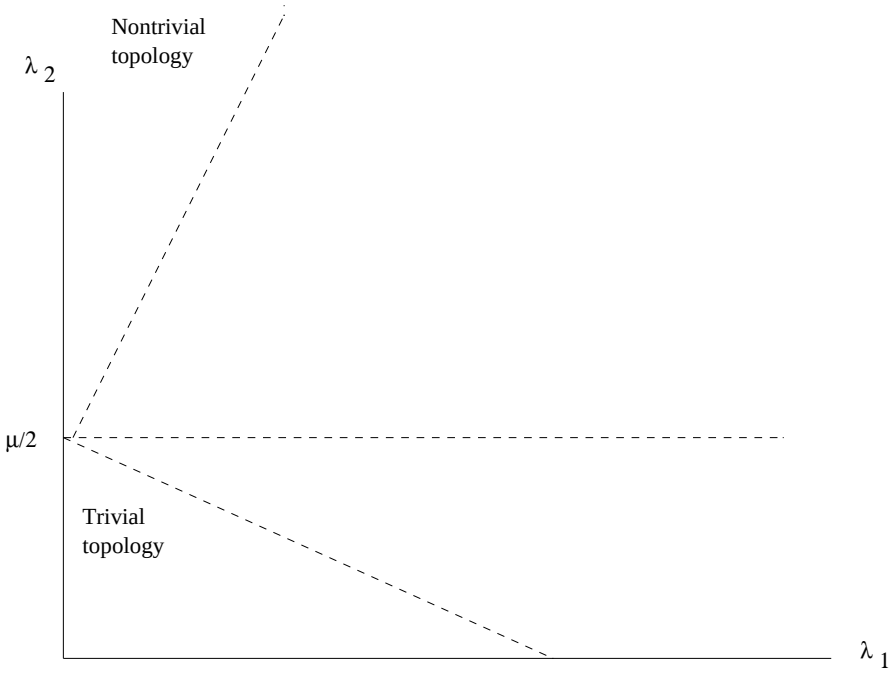


Figure 7: Phase diagram

First of all, in the latter theorem, we can estimate the number $r_v(t) = EL_1(t)$ more exactly.

In general an intuitive picture could be the following. We have links of type 1 and 2 mixed together. It is useful to look separately on them. Links of type 1 give many connected trees which stay finite as $t \to \infty$. This is confirmed by the following

**Lemma 8** *The probability that a vertex $v(s,\omega)$ in the graph $G_1(t)$ is contained in a connected component of size $D$ is bounded by $ca^{\log D}$ for some $c > 0, 0 < a < 1$.*

Proof. Assume that the earliest vertex $v_1$ in this component appeared at time $t-s$. And this means in particular that the process $l_v$ never reached zero state during time $s$. This gives the bound from above $a^s$ for some $a < 1$. Another source of estimates is when many vertices appeared during short time interval, for example $[t-1, t]$. The latter gives the worse bound and we take it.

These connected components can be considered as effective vertices of a new graph. That is the factor graph $F(t)$ of $G(t)$ which is obtained by identifying all vertices in each connected component of $G_1(t)$ and deleting all links which became loops because of the factorisation . That is we delete all type 1 links and links of type 2 which are loops. The number of vertices in the factor graph is $m_1(t)$

$$Em_1(t) \sim \frac{\mu}{\lambda_1 + \mu} \exp(\lambda_1 t)$$

by Euler formula $m = V - L_1$ and the asymptotic growth of $V(t)$ and $L_1(t)$, see above.

We can consider type 2 links as links of this factor graph. The number of type 2 links per each vertex then is

$$\frac{L_2(t)}{Em_1(t)} = \frac{2\lambda_2}{\mu}$$

If the type 2 links were uniformly distributed among vertices of $F(t)$ then by random graph theory we would get the result because $\frac{2\lambda_2}{\mu}$ is exactly the threshould suggested by random graph theory. In this case the separating line in the phase diagram could be $\lambda_2 = \frac{\mu}{2}$. We cannot however use the results of random graph theory because of absence of uniformity.

Consider the limiting (as $t \to \infty$) parameters assuming their existence

$$L = \lim \frac{EL(t)}{\exp \lambda_1 t}, V = \lim \frac{EV(t)}{\exp \lambda_1 t}, C = \lim \frac{EC(t)}{\exp \lambda_1 t}, m = \lim \frac{Em(t)}{\exp \lambda_1 t}$$

**Lemma 9** *If $\lambda_2 \geq \mu$ then $m > 0$ and $C > 0$.*

Proof. Note first that $\frac{L}{V} = \frac{\lambda_1+\lambda_2}{\lambda_1+\mu}$ motonically increases on each vertical line $\lambda_1 = const$. By the Euler formula

$$V(t) - L(t) + C(t) = m(t)$$

$m = C$ is the necessary and sufficient condition for $V(t) \sim L(t)$. It gives $\mu\alpha_{cr} \Longrightarrow \frac{\lambda_1+\lambda_2}{\lambda_1+\mu} = \frac{\lambda_2}{\mu} \Longrightarrow \lambda_2 = \mu$. Intuitively $m > 0$ follows as the extinction probability is always positive.

Note however that this does not prove the existence of a giant component $\lambda_2 \geq \mu$.

## 4.2 Local Dynamics (model 3a)

For the local model the asymptotic growth of $V(t), L_1(t), L_2(t)$ is the same as in the mean field case: in fact the derivation of these formulas does not use the way to choose the second vertex while appending type 2 links.

**Local properties** Only for $d = 1$ we have a simple processes $l_v(t)$ as for the mean field case. But the case $d = 1$ is degenerate. If we identify each time all links which connect the same pair of vertices then we get a tree. As $t \to \infty$ the probabilities $P_i^t(O_d(i) = g)$ tend to some $P_i(O_d(i) = g)$. For the local case they can be positive even when $g$ is not a tree. It follows that $\frac{\liminf C(t)}{\exp \lambda_1 t} > 0$ for all parameters. For $d = 1$ this is clear, we skip the proof for $d > 1$.

**Connected components** The number of connected components tends to infinity because if a component already appeared it will never be connected again with the rest of the world because of the locality restriction. Vertices, say $i$ and $j$, with positive probability finally will be in different components. In the local model there will not be giant component for all parameters. We conjecture that this holds for all $d < \infty$. Then this model should not have phase transitions.

## 4.3 Scales of topology (model 4)

We could try to study the structure of a giant component in the previous model but instead we shall define a new model 4. It is quite similar to model 3 but has a particular property that there is always only one connected component. In model 4 all transition are the same as in the model 3 with one exception: together with deleting type 1 link, we simultaneously identify the two vertices of the deleted link. We shall state some remarks and formulate new problems here.

It is clear that $m(t) \equiv 1$ and that $V(t) = L_1(t) + 1$ becomes a birth-death process.

**Lemma 10** *The Markov chain is ergodic if $\lambda_1 < \mu$. It is transient if $\lambda_1 > \mu$.*

If we consider only type 1 links and vertices then it is a birth and death process and the classification well known. Links of type 2 do not influence ergodicity because the killing rate of type 2 links at each vertex insreases with the number of links.

So we consider only transient case $\lambda_1 > \mu$ where for the means we get:

$$\begin{aligned} EL_1(t+\Delta t) &= EL_1(t)(1-\mu\Delta t+\lambda_1\Delta t) \Rightarrow \frac{dEL_1(t)}{dt} = (\lambda_1-\mu)EL_1(t) \Rightarrow \\ &\Rightarrow EL_1(t) \sim \exp(\lambda_1-\mu)t \end{aligned}$$

Similarly to model 3 then

$$EL_2(t) \sim c_2 \exp(\lambda_1 - \mu)t, c_2 = \frac{\lambda_2}{\lambda_1}$$

Then the vertices and type 1 links always give us a tree and each appended type 2 link augments the number of cycles by one. Thus $C(t) = L_2(t) \sim c_2 \exp(\lambda_1 - \mu)t$ for both local and mean field models.

Simplicity of behaviour for our main variables suggests to discuss two new and more complicated characteristics: scales of topology and the genus of the graph. The first one is related to the intuitive question: on what scale we see a nontrivial topology ?

But what scales are possible ? In fact this is the question about metric properties of the graph: more exactly, about the growth of the mean diameter $E(D(t))$ of the graph. In model 4 the diameter of the graph is the diameter of the tree with only type 1 links. We conjecture that it is linear that is asymptotically equal to $ct, c > 0$.

Consider the sets $\mathbf{C}(k,t), k = 1,2,3,...$, of all cycles of length $k$ which are not linear combinations of cycles from the sets $\mathbf{C}(1,t),...,\mathbf{C}(k-1,t)$. Let $c(k,t)$ be the rank of the group (with coefficients in $Z_2$), generated by $\mathbf{C}(k)$, that is the number of elements in its basis. Let $F(k,t) = \frac{1}{C(t)} \sum_{i=1}^{k} c(i,t)$. This is a distribution function and we would like to plot it with respect to scales.

To be more precise we shall specify the scales $ct^\alpha, 0 \leq \alpha \leq 1$. The case $\alpha = 0$ corresponds to the minimal scale. Now give exact definition of the topology density for this choice of scales:

$$T(\alpha) = \lim_{\varepsilon \to 0} \lim_{t \to \infty} \frac{1}{C(t)} \sum_{k=\varepsilon t^\alpha}^{\varepsilon^{-1} t^\alpha} c(k,t)$$

**Lemma 11** *For the local model $T(0) = 1$ and thus for all other $\alpha$ the density is zero.*

Proof. By definition each new type 2 link connects two vertices inside the $d$-neighbourhood. This means that the new cycle is inside this neighbourhood, it goes through a new link and some through already existing path between the two vertices.

For mean field model it is plausible that $T(1) = 1$. This means roughly that most independent cycles have the length linear in $t$. To get nontrivial topology on the scales $0 < \alpha < 1$ we could consider intermediate models: add type 2 links choosing them with equal probability in the $t^\alpha$-neighbourhood of a vertex (this resembles Van der Waals type models in statistical physics).

**Genus of the graph** Here we shall briefly discuss two more delicate properties of the graph which resemble some phenomena in physics. The first concerns embeddings of the graph in two-dimensional surfaces.

One can consider a graph $G$ as a topological space that is as an one-dimensional complex. Then an imbedding of the graph into a compact orientable surface $S$ is defined as its continuous mapping $\varphi$ into $S$, which is one-to-one on $\varphi(S)$. Imbedding is called cellular if

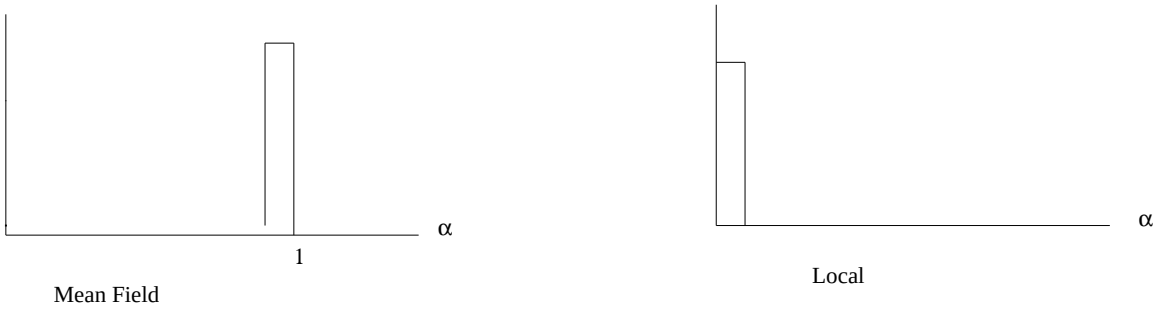


Figure 8: Scales of topology

$S \setminus \varphi(G)$ is topologically equivalent to a disconnected union of open disks. Genus $\rho(G)$ of the graph $G$ is the minimal genus of $S$ where the graph can be imbedded. Note that the genus is always finite as for each new line one could add a new handle for the surface. We shall mention here only the following result

**Lemma 12** *If* $\frac{\lambda_2}{\lambda_1} > 2$ *then with probability tending to one the genus* $\rho(G_t) > \frac{1}{6}(\frac{\lambda_2}{\lambda_1} - 2)\exp(\lambda_1 - \mu)t$.

Proof. The condition of Lemma means that asymptotically $L(t) > 3V(t)$. Then this follows from the lower bound for the genus in [22], p. 118.

The second problem concerns modelling the expansion of the Universe after the Big Bang. We would like to prove that two appearing vertices a.s. either collapse (becomes idetified) or fly away from each other. Unfortunately we do not get this in this model. Take some vertex $v$ and let a new vertex $v'$ appear as a new vertex while appending a type 1 link to vertex $v$. Thus at this moment the distance between $v$ and $v'$ is one. Once this link will be deleted a.s. and the vertices will be identified. In the same way one can show that any two fixed vertices will be identified a.s. The number of vertices increases however faster than vertices become identified. The correct formulation of the effect could be the following: for any two vertices there exist descendants (of a generation increasing with $t$) which fly away from each other with positive probability.

# 5 Dynamics of Infinite Spin Graphs

## 5.1 Infinite random graph grammars

First of all one needs to define probability measures on the set of all countable spin graphs. This is quite atndard for labelled graphs, but when lebelling (or space) is lost in the evolution we encounter difficulties here. The probability space $\Omega$ and $\sigma$-algebra are quite standard if we consider labelled graphs and the space $\Psi$ of labels is fixed (it is not supposed to be equipped with metrics or topology). Formally then $\Omega = S^{\Psi} \times (\Psi \times \Psi)^{Z_+}$, elements $\omega \in \Omega$

are pairs $(f, A)$ where $f \in S^{\Psi}$ is a function on the set of vertices (or labels) with values in $S$, and $A \in (\Psi \times \Psi)^{Z_+}$, the set of all symmetric functions on pairs of vertices to $Z_+$, giving the number of links for each pair of vertices.

Then the set of all spin graphs is a topological space with the product topology. Let $\Sigma(\Psi)$ be the standard $\sigma$-algebra in this space. Probability measures on this $\sigma$-algebra are defined via Kolmogorov theorem and generated by finite dimensional distributions $P(O_d(v) = \alpha)$ for different $d$ and $v$. We assume further on that $O_d(v)$ are all finite a.s.

This is approach will be sufficient for us in the next section. Afterwards we shall be forced to deviate from the standard Kolmogorov approach.

## 5.2 Existence

Here we construct dynamics for general initial countable graphs and prove that it is a (thermodynamic) limit of finite dynamics. We give below exact definitions and pose some constraints on the graph grammar. Moreover we shall show that, for small times, this dynamics has a remarkable property, what we called cluster dynamics in [8] . Some ideas of this sections are similar to [8] and we refer to it often.

We shall need some auxiliary notions now. Consider some graph $G$ and a subset $V' \subset V = V(G)$. Introduce the boundary of $V'$ of width $d$

$$\partial_d V' = \{v \in V \setminus V' : 0 < \rho(v, V') \leq d\}, \partial_1 V' = \partial V'$$

A subset $V_1$ of vertices is connected if the regular subgraph with this set of vertices is connected. For fixed random graph grammar $\mathcal{G}$ denote $\mathcal{G}^N(\beta)$ the corresponding Markov process starting with finite spin graph $\beta$.

Fix some initial countable spin graph $\alpha(0) = (G = G(0), s(0))$. Fix some vertex in $G(0)$ and denote it 0. Let $O_N = O_N(0)$ be the $N$-neighbourhood of 0.

Fix some random graph grammar and consider the family of Markov processes $\mathcal{G}^N = \mathcal{G}^N(O_N)$ on probability spaces $\Omega^N$ . We want to study the limit $N \to \infty$.

Fix now $N$ and $t$. We say that the vertex $v \in O_N$ was touched on the time interval $[0, t]$ if it belongs to one of the graphs $\psi(\Gamma)$ in the random sequence of transformations. Denote $Q(N, t; \omega)$ the set of all vertices in $G^N(0) = O_N$ which were touched by transformations of the process $\mathcal{G}^N$ during this time interval. Let $Q_v(N, t; \omega)$ be the connected component of $Q(N, t; \omega)$ containing the point $v$. It is empty if $v$ itself was not touched.

Let $R_v(N, t; \omega)$ be the connected component of $\overline{Q}(N, t; \omega) = Q(N, t; \omega) \cup \partial_d Q(N, t; \omega)$ for some $d > 0$, containing point $v$. For some $v$ they can coincide. Note that each $R_v(N, t; \omega)$ is the union of several $Q_w(N, t; \omega) \cup \partial_d Q_w(N, t; \omega)$ . Then we call the union of $Q_w(N, t; \omega)$ the core of $R_v(N, t; \omega)$

We shall do the following assumptions. Let $A(D)$ be the set of all graphs with the property that vertices cannot have more than some fixed number $D$ of incident links.

1. We assume the initial graph belongs to $A(D)$ and assume that all transformations of our graph grammar cannot withdraw us from this class of graphs. Under this

assumption with infinite initial graph we shall see later that there exists $t_0$ such that for $t < t_0$ $R_0(N,t;\omega)$ is finite a.s. and exponential estimates hold;

2. Radii of all $\Gamma$ entering the definition of substitutions do not exceed 1. This assumption is only to do the presentation shorter.

**Theorem 7** *The probabilities $P(Q_0(N,t;\omega) = B)$ tend as $N \to \infty$ to a limit which we denote $P(B)$. Moreover $\sum_B P(B) = 1$.*

To prove this put

$$P^N(A) = P(\overline{Q}(N,t;\omega) = A)$$

First, we shall get cluster representatiton (or polymer representation, see [9]) for

$$\frac{P^N(A)}{P^N(\emptyset)}$$

Note first that

$$P^N(\emptyset) = \prod P(\Gamma,\psi)$$

where the product is over all $\Gamma$ and all morphisms $\psi$ of $\Gamma$ into $G^N$, $P(\Gamma,\psi) = \exp(-\lambda(\Gamma)t)$ is the probability that the subgraph $\psi(\Gamma)$ does not undergo transition. Note that all $P(\Gamma,\psi)$ are close to 1 if $t$ is small enough. Then the cluster representation is the identity

$$\frac{P^N(A)}{P^N(\emptyset)} = \prod k_B^N, k_B^N = \frac{p(B;N)}{\prod^B P(\Gamma,\psi)}$$

where $B$ are all connected components of $A$ and $p(B;N)$ is the probability that in the finite chain starting with $B$ all points in the core of $B$ are touched and all vertices of the boundary are not touched. The product $\prod^B$ is over all pairs $(\Gamma,\psi)$ which touch $B$. Note that if $dist(B, G \setminus G^N)$ is larger than $d$ then $p(B;N)$ does not depend on $N$. In this case we denote it by $p(B)$.

This representation follows from the lemma just below.

**Lemma 1** *For any $A$ the following formula holds*

$$P^N(A) = \prod_B p(B,N) \prod P(\Gamma,\psi)$$

where the last product is over all $(\Gamma,\psi)$ such that $\psi(\Gamma)$ intersects with $\Lambda \setminus A$. Note that then it cannot intersect with the core of $A$.

Proof of this lemma is quite similar to the proof of the corresponding lemma in [8], this is lemma 2 on page 11.

We need also a cluster estimate.

**Lemma 13** *We have*

$$k_B^N < a^{|\Gamma|}$$

*uniformly in* $N$ *for some* $a = a(t) \to 0$ *if* $t \to 0$.

Now it is a standard cluster expansion techniques (see [9]) to get from the cluster representation a convergent series for the probability $P^N(B)$ that $B$ is the connected component of $A$ containing 0. Here the finiteness of $D$ is used.

The terms of this expansion do not depend on $N$ up to terms of order $c^N$ for some $c = c(D) < 1$. Exponential convergence of $P^N(B)$ to some $P(B)$, as $N \to \infty$, follows.

Denote the random field $\xi(v)$ on $V(G(0))$: $\xi(v) = 1$ if $v \in \overline{Q}(N, t; \omega)$ and $\xi(v) = 0$ otherwise. In the same way we can get cluster expansion for all correlation functions $\xi_E^N, E \subset V(G^N)$ and thus for all limiting correlation functions $\xi_E^N = \lim_{N\to\infty} \xi_E, E \subset V(G(0)$.

Limiting correlation functions define (by Kolmogorov theorem) the probability measure on $\{0,1\}^{V(G(0))}$ or on the set of all subsets of $V(G(0)$. For any initial graph $\alpha$ this defines a random point set $E(\omega, t, \alpha)$. These random sets satisfy the following properties:

- For small $t$ they consist a.s. of infinite number of finite connected components;
- For all $t_1 < t_2$ a.s.

$$E(\omega, t_2, \alpha) \subset E(\omega, t_1, \alpha)$$

We shall use these sets now for constructing infinite dynamics. One can say that they consists of the vertices which are touched by substitutions on the time interval $[0, t]$.

Note that for given $N$ the conditional distributions of trajectories on $B$ are independent for different connected components.

For any $t$ there exists cluster dynamics on the set of spin graphs. This means the following.

Let us consider some probability space $\Omega_1$ where the limiting random field $\xi(v)$ is defined. Denote $\mu_1$ the corresponding probability measure on $\Omega_1$. For any $\omega_1 \in \Omega_1$ consider some connected component $C_k(\omega_1)$ of $E(\omega, t, \alpha)$.

Inside a given connected component the process coincides with conditional finite dynamics under the condition that all vertices in the corresponding $Q_l$ are touched.

The global probability space is convenient to take as

$$\Omega_\infty = \Omega_\infty(\alpha(0)) = \prod_{v \in V(G(0))} (\{0,1\} \times \{(B, dyn(B))\})$$

where for each $v$ the pairs in $\{(B, dyn(B))\}$ are defined by the following conditions. $B$ is some connected set such that $v \in B \subset V(G(0))$, $dyn(B)$ is the set of all trajectories of the finite dynamics on the time interval $[0, t]$ starting with the regular subgraph on $B \cup \partial B$ and such that all vertices of $B$ are touched but all vertices of $\partial B$ are not touched. The probability measure $\mu$ on this space is defined as follows. The point of this space can be written as an array $(\xi_v, B_v, dyn(B_v); v \in V(G(0)))$. The projection of $\mu$ on $\{0,1\}^V = \{(\xi_v, v \in V(G(0)))\}$

is taken to be $\mu_1$. Thus the measure $\mu$ is defined by this projection and by the family of conditional distributions $\mu(\xi)$ on $\{(B_v, dyn(B_v); v \in V(G(0)))\}$. By definition $\mu(\xi)$ is the product of (thus independent) measures $\mu(B)$ correponding to the finite dynamics on $B \cup \partial B$.

Note that, contrary to the one-dimensional case we cannot construct cluster dynamics for all $t$ but only for small enough $t$. But the dynamics itself is contructed by semigroup property just glueing together dynamics for consecutive time intervals $[0, \varepsilon]$. The properties of the dynamics which hold uniformly in all initial conditions for $t < \varepsilon$ can be obtained for any finite $t$ as well.

## 5.3 Stochastic Local Homogeneity

We swallowed somehow the problem of labels for graphs at time $t$, now we come back to it. For small $t$ we assign labels in the following way. If the vertex was not touched then its label is kept the same as it was at time zero. If it was touched then this label refers to a finite graph which appeared instead of the corresponding connected component. For larger times the labelling can be done in principle recurrently on $n\varepsilon$ but it becomes more and more complicated with $n$. For larger times the control over labels seems to be finally lost.

We come now to a discussion of how to define probability measures on countable graphs.

**For labelled or embedded spin graphs** If the vertices are labelled with some fixed set $\Psi$ then there are no problems. Classical Kolmogorov approach is valid. We discussed it in the introduction to this section.

**Definition 5** *We say that random spin graph is locally stochastically homogeneous if for all $d$ the distribution on graphs $O_d(v)$ does not depend on $v$, that is for all $d$ and all spin graphs $\alpha$ the probabilities $p(d, \alpha) = P(O_d(v) = \alpha)$ do not depend on $v$.*

Note that the distance here is not related to any metrics on the set of labels but it is an internal graph characteristics. If there is a group $U$ acting transitively on the set of labels (it can be shifts as in grammars, see [8], or the permutation group on the set of labels) then one could define homogeneity with respect to the embedding space. For grammars, for example, this coincides with the above definition.

The space of spin graphs with labelled vertices however is too huge for us: labels do not carry useful information. Also we shall see below that labels do not survive during dynamics. For example, during time evolution vertices can appear and disappear and there is an amount of arbitrariness in the choice of labels for appearing vertices. That is why one would like to consider only unlabelled graphs.

**For unlabelled spin graphs** For this one could choose our probability space $\Omega$ as the set of all equivalence classes of countable spin graphs, which differ only by labelling. Very unsatisfying is that the notion of finite dimensional distribution (or correlation function is lost). The problem is that to have such notion one should be able to specify the vertices

where this correlation function is taken. There is only one general way to do: using Zermelo axiom of choice to define a function $f(A)$ on the set $\Delta$ of all equivalence classes of countable spin graphs $A$, where the value of $f(A)$ is a representative and some vertex in it. This is highly nonconstructive but there are several ways out of this desperate situation.

Another general way could be to restrict somehow the number of events. For example, $\sigma$-algebra could be defined as the minimal $\sigma$-algebra containing all sets $R(\Gamma)$ where $R(\Gamma)$ is the set of all equivalence classes containing (somewhere) the fixed finite spin graph $\Gamma$. The sets $R(\Gamma)$ are also defined to be the basis of open set of the topological space $\Omega$. Note that for grammars (linear graphs) $\Omega$ is the factor space of $S^Z$ - orbits of the translations in $S^Z$. But homogeneity means roughly that the probability of appearance of $\Gamma$ is the same anywhere. But then $P(R(\Gamma))$ is either 0 or 1.

Contructive ways amount to finding some constructive way to specify a point in the graph, a reference point. For example, if $\alpha = (G, s)$ with $G$ imbedded into some fixed space it is easy to construct $f(A)$: just choose $v_0 = f(A)$ closest to some fixed space point. In the general case it is very plausible that we have NO natural embedding: any attempt to fix a reference vertex will change homogeneous structure in the vicinity of the reference point. This resembles similar situation with a measurement process in quantum mechanics.

**Local Observer** If we want to define a random spin graph as a limit of some time evolution there is another way to choose a reference point. If we start with fixed or random initial spin graph (finite or infinite) one can introduce a local observer, that is some specified (random) vertex $v_0(t)$ at any time $t$. Local observer is defined by a function $f$ on the pairs $(v, Sub)$ taking values in $V(\Gamma') \cup O_1(\psi(\Gamma))$. Its interpretation is the following. At time 0 the local observer sits at some fixed vertex 0, after it is by induction. It continues to sit at the same vertex until it dissapears. If the vertex $v_0$ disappears due to the substitution $Sub$ (if it belongs to $\psi(\Gamma)$) then after the substitution $Sub$ the local observer jumps from $v$ to the vertex $f(v, Sub) \in V(\Gamma')$ if $V(\Gamma') \neq \emptyset$ and $O_1(\psi(\Gamma))$ otherwise.

Consider now the probabilities $p(t, g) = P(O_d(v_0(t)) = g)$. Normally they are different for all local observers (as we saw in [8]) but the asymptotic distributions far away from the local observer can be the same for different local observers. For example, take a deterministic limit of the random numbers (if it exists)

$$\lim_{D\to\infty} \frac{\natural\{g : \exists v, v \in V(g), dist(v, v_0) = D\}}{|V(g)|\,\natural\{v : dist(v, v_0) = D\}}$$

This limits define an unlabelled distribution (see the definition below). As everything depends on $t$ this also defines the dynamics of these distributions.

**Time as a reference frame** Time also can serve as a reference frame if we start with a finite spin graph. Assume first that the vertices cannot disappear. Then let $v_i$ be the $i$th appearing vertex (in time order). Consider the probabilities $P_t(O_d(v_i) = g)$ that at time $t$ its $d$-neighbourhood is isomorphic to some fixed graph $g$. Then we can consider different limits

$$\lim_{t\to\infty, i=i(t)\to\infty} P_t(O_d(v_i)=g), \lim_{i\to\infty}\lim_{t\to\infty} P_t(O_d(v_i)=g)$$

and it would be interesting to compare them.

If the vertices can disappear then more general approach is available. One can consider first the nonconnected union $C(\omega)=\cup_t C_t(\omega)$ of all complexes appearing during time evolution for fixed random trajectory $\omega$. The vertices of $C(\omega)$ are denoted by $(v,t), v\in C_t(\omega)$.

Let us construct a reasonable connected graph, identifying some vertices in $C$. Each appearing vertex can be characterized by its life time interval $[t_1,t_2]$, where $t_1$ is the time when it first appeared, $t_2$ - when it dissapeared. Thus, it was not touched to any substitution, that is it did not belong to $\psi(\Gamma)$, on the time interval $[t_1,t_2]$. Let us then identify all points $(v,t)$ in $C$ for fixed $v$ and $t\in[t_1,t_2]$.

Denote the resulted complex $C_{con}(\omega)$. Note that time can be deduced here from the $C_{con}(\omega)$ as the scaled distance from the initial vertex. .

**Unlabelled distributions** Now we shall try to construct the set of numbers which can be an analog of correlation functions for unlabelled case. Let us consider a partially ordered system (tree) $T$ of all finite unlabelled spin graphs with respect to inclusion $\Gamma\subset\Gamma'$.

Let for finite unlabelled spin graphs $\Gamma$ numbers $p(\Gamma)$ be given satisfying the following (compatibility) conditions:

$$0\le p(\Gamma)\le 1, \Gamma\subset\Gamma'\Rightarrow p(\Gamma)\le p(\Gamma'), \sum_{\Gamma'} p(\Gamma')=p(\Gamma)$$

for all $\Gamma$, where the sum is over all $\Gamma', \Gamma\subset\Gamma', \mid V(\Gamma')\setminus V(\Gamma)\mid=1$ such that different pairs $(\Gamma,\Gamma')$ are not isomorphic.

Such system is called a distribution on countable unlabelled spin graphs, or shortly, unlabelled distribution. It can be equally given using some subtree of $T$, for example, only $d$- neighbourhoods of a fixed vertex. What is then the natural definition of stochastically homogeneous unlabelled distribution ?

We should give now examples of how such systems could be obtained.

**Induced distributions** Take statistically homogeneous labelled spin graph. For some fixed vertex $v$ and unlabelled connected spin graph $\alpha$ take then the probability $P(\alpha)$ that there exists a connected spin graph containing vertex $v$ and isomorphic to $\alpha$. Can all statistically homogeneous unlabelled distributions obtained in this way ?

**Asymptotic correlation functions** Assume now that we have a probability measure $P$ on the set of countable labelled spin graphs with fixed vertex, a reference point. For example, assume there is a fixed "reference" vertex $v_0$ and $G(N)$ - the set of all graphs with radius $N$ and centre in $v_0$. Let $P_N$ be a distribution on $G(N)$. Consider then the limiting distribution as a weak limit point of $P_N$ (by compactness argument) as $N\to\infty$. This means

that for fixed $d$ and $g$ the probability $P_N(O_d(v_0) = g)$ tends to a limit $P(O_d(v_0) = g)$ as $N \to \infty$.

One can define in this case asymptotic stochastic homogeneity property of the limiting measure $P$: for fixed $d$ and $g$ there exists

$$p(d; g) = \lim_{r \to \infty} \frac{\sharp\{v : dist(v, v_0) = r, O_d(v) = g\}}{\sharp\{v : dist(v, v_0) = r\}}$$

as $dist(v, v_0) \to \infty$. Note that these numbers satisfy the above conditions for $p(\Gamma)$.

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