New Results on Painlevé Paradoxes

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Abstract: In this note, we deal with dynamics of mechanical system subject to unilateral constraints. In particular, we focus on the integration of such a system that is known to be closely related to so-called Linear Complementarity Problem (LCP). Except in very simple case like codimension 1, frictionless contraints, the problem of well-posedness (existence and uniqueness of solution) to such hybrid dynamic systems (smooth dynamics + LCP + shock dynamics) is a big challenge. We concentrate on the well-known Painlevé example [Painlevé, 1895], whose dynamics in a sliding regime may be singular depending on the friction coefficient. A new critical friction coefficient is exhibited below which contact forces remain bounded. Moreover a detailed analysis of the vector field nearby the singularity shows that the eventual divergence of the contact force does not call into question the model well-posedness.

Key-words: Painlevé example, hybrid dynamical system, LCP, generalized friction cone, singular ODE, Impact WithOut Collision.

(Résumé : tsvp)

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Nouveaux résultats sur les paradoxes de Painlevé

Résumé : Ce rapport traite de la dynamique des systèmes mécaniques avec contraintes unilatérales. L'accent est mis sur l'intégration d'un tel système qui est connue pour être étroitement liée à ce que l'on appelle les problèmes linéaires de complémentarité ("Linear Complementarity Problem", LCP). A l'exception de cas très simples de codimension 1, en l'absence de frottement, le problème du caractère bien-posé (existence et unicité de la solution) de ce type de systèmes dynamiques hybrides (dynamiques lisses + LCP + dynamiques des chocs) demeure un réel défi. Nous nous concentrerons sur un système communément appelé l'*exemple de Painlevé* [Painlevé, 1895], dont la dynamique dans un mode de glissement peut devenir singulière pour certaines valeurs du coefficient de frottement. Une nouvelle valeur critique du coefficient de frottement est exhibée en dessous de laquelle les forces de contact demeurent bornées. D'autre part une analyse détaillée du champ de vecteurs au voisinage des singularités montre que l'éventuelle divergence de la force de contact ne remet pas en question le caractère bien-posé du modèle.

Mots-clé : Exemple de Painlevé, système dynamique hybride, LCP, cône de frottement généralisé, EDO singulière, impact sans collision.

1 Introduction

This note aims at illustrating with a simple example the problems of wellposedness of the dynamics of mechanical systems with unilateral constraints and dry friction. The analysed example is the wellknown Painlevé system [Painlevé, 1895]. In particular we will focus on singularities of the dynamics in sliding regimes, i.e. configurations at which the contact force diverges to infinity. The problem of *inconsistencies*, that is configurations for which no continuous solution exists, will also be examined, as well as *indeterminacies*, i.e. configurations which lead to non-uniqueness of solutions. More precisely, there may be no bounded contact forces that permit to satisfy the unilateral constraints. Consequently the space within which solutions have to be defined and found must be augmented to discontinuous velocities and distributional interaction forces. For instance, some sort of *impact without collisions* (IW/OC) can be introduced when dry friction is present. This is a phenomenon such that velocity jumps can occur with zero initial normal velocity, primarily due to Amontons-Coulomb friction. It is related to *Kilmister's principle of constraints* : "a unilateral constraint must be verified with (bounded) forces each time it is possible, and with impulses if and only if it is not possible with bounded forces". Therefore this a priori stated principle tells us that if one is able to exhibit dynamical situations for which a bounded force cannot be found such that the contraints are satisfied, then one may use an impulsive force at the contact point. Concerning non-uniqueness of solutions, Painlevé also proposes his principle : "two rigid bodies, which under given conditions would not produce any pressure on one another, if they were ideally smooth, would likewise not act on one another if they were rough". The two above mentionned principles have not been given any experimental validation to the best of our knowledge. Note that inconsistencies due to Amontons-Coulomb have been known for a long time. Historically Jellet [Jellet, 1872] discovered the problems produced by Amontons-Coulomb's friction model and Painlevé [Painlevé, 1895, Painlevé, 1905] brought these problems to the attention of the scientific community. The possibility of applying velocity jumps to escape from inconsistent configurations was first proposed by Lecornu [Lecornu, 1905].

In modern language, Painlevé's problem can be recast into the framework of hybrid dynamical systems, i.e. systems which contain both continuous and discret-event state variables. Such systems are therefore discribed by differential equations corresponding to the *modes* of the systems, and by switching rules at the transitions between different modes [van der Schaft et al., 1996a, van der Schaft et al., 1996b]. In this work, we study the wellposedness of Painlevé's example following a hybrid dynamical system point of view. In other words, starting from a consistent mode, we analyze whether a smooth continuation of the solution in a another mode exists, and if it is unique : this is called the concept of initial conditions in [Heemels et al., 1997]. This kind of approach differs from the analysis in [Paoli et al., 1993, Marques, 1993] based on the use of penalized or discretized problems and the study of their limit(s). Our approach strongly relies on so-called *Linear Complementarity Problems* (LCP) which allow us to compute the contact forces.

This note is organized as follows. In section 2, we present the model and the regions of inconsistencies and indeterminacies. Section 3 is devoted to the study of the undetermined region, introducing analytical tools that will be useful in the sequel. In section 4, we will focus on the transitions from a consistent sliding mode when the contact force takes arbitrarily large values in the neighbourhood of a singular point. Conclusions are given in section 5.

2 Dynamics of Painlevé's example

Painlevé's example consists of a rigid homogeneous slender rod with mass m, length $2m$ and inertia $I = \dfrac{ml^2}{3}$. We will study the case where the rod is in contact with a rigid ground at one edge A, see Fig. 1.

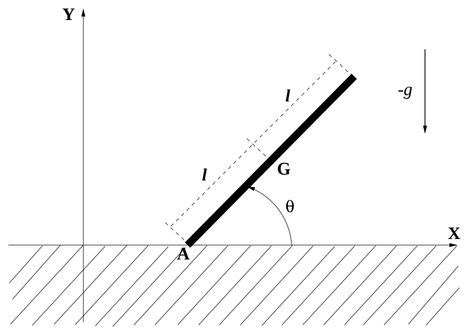

Figure 1: Painlevé's example

Let $G = (x, y)^T$ and $A = (x_A, y_A)^T$ be the Cartesian coordinates, with respect to a Galilean frame, of the center of mass and the contact point, respectively. The set of generalized coordinates is given by $q = (x, y, \theta)^T$, where θ is as in Fig. 1. The unilateral constraint corresponds to :

$$F(q) = y_A = y - l \sin\theta \geq 0 \tag{1}$$

Amontons-Coulomb dry friction model is supposed to hold at the contact point A with friction coefficient $\mu \geq 0$. The external forces acting on the rod are gravity, $(0, -mg)^T$, and the interaction force at A, $(F_N, F_T)^T$. Assuming that no torque acts at A, the dynamics of the system is given by :

$$\left\{ \begin{array}{lcl} m\ddot{x} & = & F_T \\ m\ddot{y} & = & -mg + F_N \\ I\ddot{\theta} & = & l\,(-\cos\theta F_N + \sin\theta F_T) \end{array} \right. \tag{2}$$

with

$$\left\{ \begin{array}{l} y_A F_N = 0,\;\; y_A \geq 0,\;\; F_N \geq 0 \\ \dot{x}_A = 0 \Rightarrow -\mu F_N \leq F_T \leq \mu F_N \\ \dot{x}_A > 0 \Rightarrow F_T = -\mu F_N \\ \dot{x}_A < 0 \Rightarrow F_T = \mu F_N \end{array} \right.$$

As pointed out by Van der Schaft and Schumacher [van der Schaft et al., 1996b], this mechanical system can be considered as a hybrid dynamical system with four modes.

Definition 1

Let X be the system state vector :

$$X = \begin{pmatrix} q \\ \dot{q} \end{pmatrix} = \begin{pmatrix} x \\ y \\ \theta \\ \dot{x} \\ \dot{y} \\ \dot{\theta} \end{pmatrix}$$

The four modes of the hybrid system are defined as :

- $M_I = \left\{X \in \mathbf{R}^6\right\}$ *with* $y_A(y, \theta) > 0$. *We have :*

$$F_N = F_T = 0$$

- $M_{II} = \left\{X \in \mathbf{R}^6\right\}$ *with* $y_A(y,\theta) = 0$ *and* $\dot{x}_A(\theta,\dot{x},\dot{\theta}) < 0$. *We have :*

$$F_N \geq 0, \;\; F_T = \mu F_N$$

- $M_{III} = \left\{X \in \mathbf{R}^6\right\}$ *with* $y_A(y,\theta) = 0$ *and* $\dot{x}_A(\theta,\dot{x},\dot{\theta}) > 0$. *We have :*

$$F_N \geq 0, \;\; F_T = -\mu F_N$$

- $M_{IV} = \left\{X \in \mathbf{R}^6\right\}$ *with* $y_A(y,\theta) = 0$ *and* $\dot{x}_A(\theta,\dot{x},\dot{\theta}) = 0$. *We have :*

$$F_N \geq 0, \;\; -\mu F_N \leq F_T \leq \mu F_N$$

Mode M_I corresponds to the flight phase, and no contact force acts on the rod. Modes M_{II} et M_{III} are identical through a redefinition of the Galilean frame. Finally mode M_{IV} does not create analytical difficulties. Indeed, as proved by Lötstedt [Lötstedt, 1981] and Baraff [Baraff, 1993, théorème 4], the calculation of the contact is always wellposed in the planar case with a single unilateral active sticking constraint, i.e. there is always a unique solution.

In the sequel, we will study in details the mode M_{II}.

Hypothesis 1

We shall suppose that the system has been initialized in mode M_{II}*, that is :*

$$\begin{cases} y_A(t_0) = 0 \\ \dot{y}_A(t_0) = 0 \\ \dot{x}_A(t_0) < 0 \end{cases}$$

Consequently, the only possible transitions are from M_{II} into modes M_I, M_{IV}, i.e. detachment of the rod or sticking at point A.

In mode M_{II}, the tangential component of the interaction force is given by $F_T = \mu F_N$. The dynamics of system (2) in this sliding regime can be rewritten as :

$$\begin{cases} m\ddot{x} &= \mu F_N \\ m\ddot{y} &= -mg + F_N \\ I\ddot{\theta} &= l(-\cos\theta + \mu \sin\theta) F_N \end{cases} \tag{3}$$

In order to take into account the unilateral feature of the contact at A, it is necessary to introduce a complementarity relationship between the normal component F_N and the normal acceleration $\ddot{y}_A$. Differentiating two times y_A in (1) and introducing the expressions of $\ddot{y}$ and $\ddot{\theta}$ from (3), one obtains :

$$\ddot{y}_A = -\mathcal{A} + \mathcal{B} F_N \geq 0, \;\; F_N \geq 0, \;\; F_N \ddot{y}_A = 0 \tag{4}$$

where

$$\mathcal{A}(\theta,\dot{\theta}) = g - l\dot{\theta}^2 \sin\theta \tag{5}$$

and

$$\mathcal{B}(\theta,\mu) = \frac{1}{m}(1 + 3\cos\theta(\cos\theta - \mu\sin\theta)) \tag{6}$$

Notice that the transition from LCP mode ($\ddot{y}_A = 0, \;\; F_N \geq 0$) towards LCP mode ($\ddot{y}_A > 0, \;\; F_N = 0$) in (4) corresponds to the transition of the hybrid system from mode M_{II} towards mode M_I.

mode	sgn($\mathcal{A}$)	sgn($\mathcal{B}$)	Solution(s) of the LCP
(1)	+	+	$F_N = \frac{\mathcal{A}}{\mathcal{B}}$
(2)	+	-	$\emptyset$
(3)	-	+	$F_N = 0$
(4)	-	-	$F_N = 0$ and $F_N = \frac{\mathcal{A}}{\mathcal{B}}$

Table 1: Solutions of the LCP

2.1 Inconsistencies and indeterminacies

According to the signs of $\mathcal{A}$ and $\mathcal{B}$, the tableau 1 follows [Lötstedt, 1981] :

Definition 2

We shall denote by

- $\mathcal{M}_i$ *the set of states of the system for which the rod is in mode* M_{II} *of the hybrid system and in mode (i) of the LCP,*
- $\mathcal{M}_i^+$ *and* $\mathcal{M}_i^-$ *the subsets of* $\mathcal{M}_i$ *with* $\dot{\theta} > 0$ *and* $\dot{\theta} < 0$ *respectively.*

Example

$$\mathcal{M}_1^+ = \left\{ X \in \mathbf{R}^6 \right\} \text{ with } X = \begin{pmatrix} x \\ y \\ \theta \\ \dot{x} \\ \dot{y} \\ \dot{\theta} \end{pmatrix} \text{ and } \begin{cases} y_A(y,\theta) = 0 \\ \dot{y}_A(\dot{y},\theta,\dot{\theta}) = 0 \\ \dot{x}_A(\dot{x},\theta,\dot{\theta}) < 0 \\ \mathcal{A}(\theta,\dot{\theta}) > 0 \\ \mathcal{B}(\theta,\mu) > 0 \\ \dot{\theta} > 0 \end{cases}$$

The inconstent states are those corresponding to $\mathcal{A} > 0$ and $\mathcal{B} < 0$, whereas the undetermined ones are characterized by $\mathcal{A} < 0$ and $\mathcal{B} < 0$. It therefore important to study sgn($\mathcal{B}$).
Easy computations show that :

- if $0 \leq \mu < \frac{4}{3}$, then $\forall \theta,\ \mathcal{B}(\theta,\mu) > 0$ and the system is always consistent [Pfeiffer, 1996].
- if $\mu \geq \frac{4}{3}$, then, setting :

$$\begin{cases} \theta_{c1}(\mu) & = & \arctan\left(\dfrac{3\mu - \sqrt{9\mu^2 - 16}}{2}\right) \\ \theta_{c2}(\mu) & = & \arctan\left(\dfrac{3\mu + \sqrt{9\mu^2 - 16}}{2}\right) \end{cases} \tag{7}$$

  it follows that

  – $\forall \theta \in [0, \theta_{c1}(\mu)) \bigcup (\theta_{c2}(\mu), +\infty),\ \mathcal{B}(\theta,\mu) > 0$ and the problem is consistent,
  – $\forall \theta \in (\theta_{C1}(\mu), \theta_{C2}(\mu)),\ \mathcal{B}(\theta,\mu) < 0$, the problem is either inconsistent or undetermined, depending on the sign of $\mathcal{A}(\theta,\dot{\theta})$,
  – $\mathcal{B}(\theta_{c1}(\mu),\mu) = \mathcal{B}(\theta_{c2}(\mu),\mu) = 0$.

Similarly for $\mathcal{A}(\theta,\dot\theta)$, one obtains that :

- if $\dot\theta > \sqrt{\dfrac{g}{l\sin\theta}}$, then $\mathcal{A}(\theta,\dot\theta) > 0$,
- if $\dot\theta < \sqrt{\dfrac{g}{l\sin\theta}}$, then $\mathcal{A}(\theta,\dot\theta) < 0$,
- $\mathcal{A}(\theta, \pm\sqrt{\dfrac{g}{l\sin\theta}}) = 0$.

Notice that the two critical values in (7) satisfy $0 < \theta_{c1} < \theta_{c2} < \dfrac{\pi}{2}$. In the following we will focus essentially on the case $\mu \geq \dfrac{4}{3}$.

Remark that if the solution $F_N = 0$ is "chosen" (in the case where the system is in the undetermined mode $\mathcal{M}_4$) or "imposed" (in the case where the system is in the consistent mode $\mathcal{M}_3$), then, from (3),

$$\ddot\theta = 0$$

Similarly, for the consistent mode $\mathcal{M}_1$ and for the undetermined one $\mathcal{M}_4$, substitution of the solution $F_N = \dfrac{\mathcal{A}(\theta,\dot\theta)}{\mathcal{B}(\theta,\mu)}$ into (3) yields the differential equation

$$\ddot\theta = \frac{3}{l}(-\cos\theta + \mu\sin\theta)\frac{g - l\dot\theta^2\sin\theta}{1 + 3\cos\theta(\cos\theta - \mu\sin\theta)} \tag{8}$$

which, setting $\mathcal{C}(\theta,\mu) = \dfrac{3}{ml}(-\cos\theta + \mu\sin\theta)$, can be rewritten as :

$$\ddot\theta = \mathcal{C}(\theta,\mu)\frac{\mathcal{A}(\theta,\dot\theta)}{\mathcal{B}(\theta,\mu)} \tag{9}$$

This strongly nonlinear differential equation will be at the core of our future analysis. Let

$$\left\{\begin{array}{lcl} \dot\theta_{c1}^{\pm} & = & \pm\sqrt{\dfrac{g}{l\sin\theta_{c1}}} \\ \dot\theta_{c2}^{\pm} & = & \pm\sqrt{\dfrac{g}{l\sin\theta_{c2}}} \end{array}\right.$$

Definition 3

The singular points of the LCP (4) are defined as

$$P_{c1}^{\pm} = \begin{pmatrix} \theta_{c1} \\ \dot\theta_{c1}^{\pm} \end{pmatrix} \text{ and } P_{c2}^{\pm} = \begin{pmatrix} \theta_{c2} \\ \dot\theta_{c2}^{\pm} \end{pmatrix}$$

The associated LCP modes are depicted in Fig. 2 for $\mu = 1.4$ together with the vector field directions in the various regions.

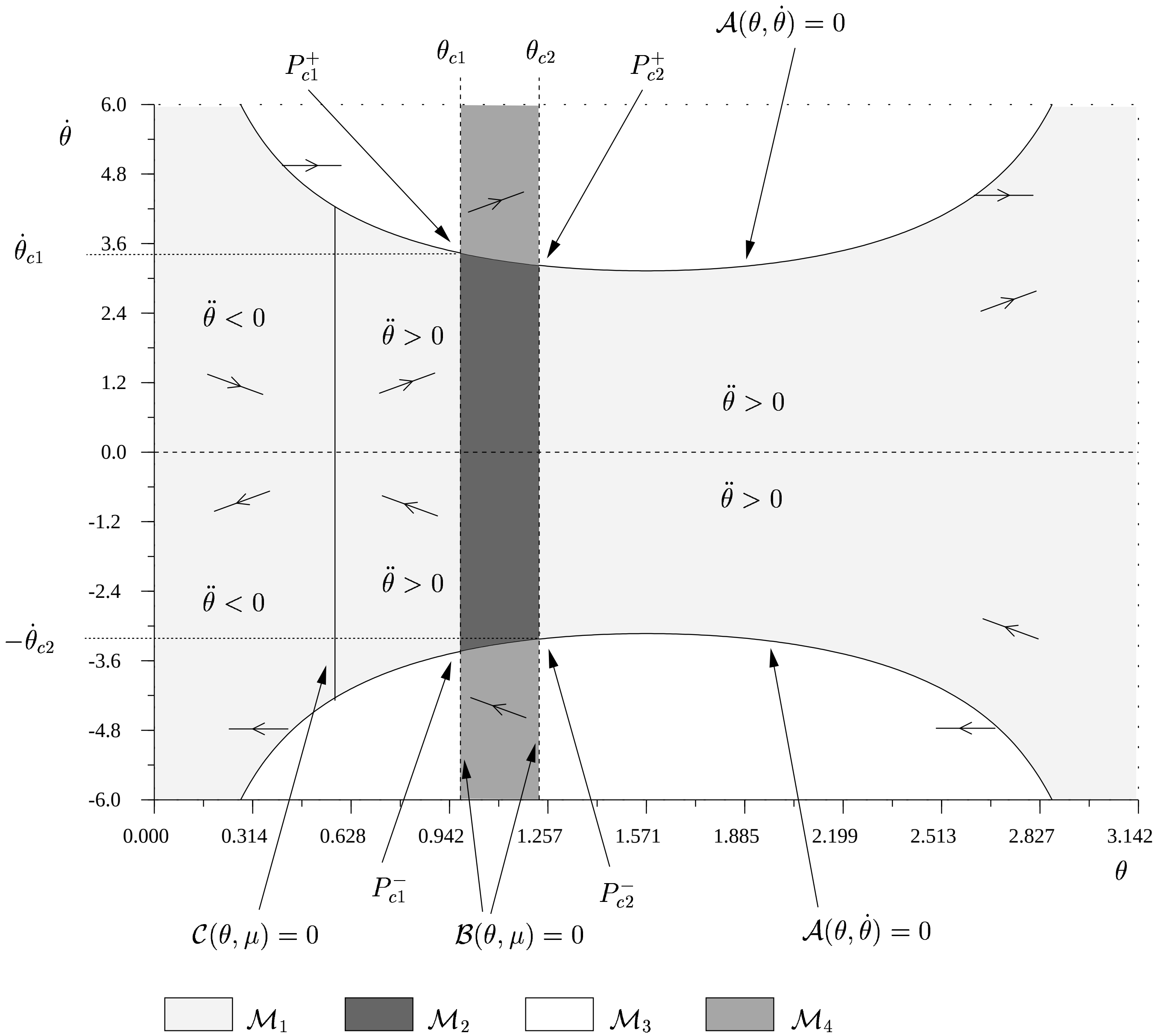

Figure 2: The different modes of the LCP for $\mu = 1.4$ ($m = 1$ kg, $g = 9.8$ m/s^2, $l = 1$ m)

2.2 The generalized friction cone

Let us rewrite the dynamics of motion (2) as :

$$\ddot{q} = F_N n_q + F_T t_q + F_e$$

subject to the constraint $0 \leq \mid F_T \mid \leq \mu F_N$, where $F_e = M^{-1} \begin{pmatrix} 0 \\ -g \\ 0 \end{pmatrix}$ is the vector of external generalized forces, $M = \begin{pmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & \frac{ml^2}{3} \end{pmatrix}$ is the inertia matrix of the mechanical system. It is not difficult to see that :

$$\left\{\begin{array}{lcl} n_q & = & M^{-1}\nabla F(q) = \dfrac{1}{m}\begin{pmatrix} 0 \\ 1 \\ -\dfrac{3}{l}\cos\theta \end{pmatrix} \\ t_q & = & \begin{pmatrix} 1 \\ 0 \\ \dfrac{3}{l}\sin\theta \end{pmatrix} \end{array}\right. \tag{10}$$

One deduces that the edges of the friction cone in the configuration space have direction :

$$e^{\pm} = n_q \pm \mu t_q = \frac{1}{m}\begin{pmatrix} \pm\mu \\ 1 \\ \dfrac{3}{l}(-\cos\theta \pm \mu\sin\theta) \end{pmatrix}$$

In other words, $e^{\pm}$ are the images of the $n_0 \pm \mu t_0$, with $n_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $t_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, under the mapping $M^{-1}J^T$, where J is the jacobian between $(\dot{x}_A, \dot{y}_A)$ and $\dot{q}$. Notice that the generalized cone in the (θ, y) plane is no longer symmetric around the kinetic normal vector n_q. It is noteworthy that the cone edge (q, e^+) can dip below the tangent plane. Indeed :

$$n_q^T M e^+ = \frac{1}{m}(1 + 3\cos\theta(\cos\theta - \mu\sin\theta)) = \mathcal{B}(\theta, \mu)$$

Most importantly in the sliding mode M_{II}, the interaction force is along e^+.

Lemma 1

> *The generalized interaction force dips below the tangent plane at q if and only if $\mu \geq \dfrac{4}{3}$ and $\theta_{c1}(\mu) < \theta < \theta_{c2}(\mu)$.*

Proof

This result is a direct consequence of the above analysis of the sign of $\mathcal{B}(\theta, \mu)$. ◇

Remarks

- Fig. 3 depicts the evolution of the generalized friction cone with θ for $\mu = 1.6$ ($l = 1$ m, $m = 1$ kg).
- Erdmann [Erdmann, 1994] and Moreau [Moreau, 1986] point out the relationship between inconsistencies and the generalized cone position relative to the tangent plane.

In the sequel, we will first prove that the solution $F_N = 0$, which is valid in modes $\mathcal{M}_3$ and $\mathcal{M}_4$, implies instantaneous detachment of the rod, that is transition into mode M_I of the hybrid system. Then we shall study the indeterminacy in mode $\mathcal{M}_4^+$ and we will prove that the solution $F_N(t) = \dfrac{\mathcal{A}(t)}{\mathcal{B}(t)}$ yields simultaneous sticking and detachment before the singularity $\theta = \theta_{c2}$ is reached. A similar study will allow us to prove that the system leaves the mode $\mathcal{M}_1^+$ or $\mathcal{M}_4^-$ before or when the orbit crosses the boundary curve $\mathcal{A}(\theta, \dot{\theta}) = 0$. Let $t_{\mathcal{A}}$ denote this instant. It will be prove that in mode $\mathcal{M}_1^+$, if

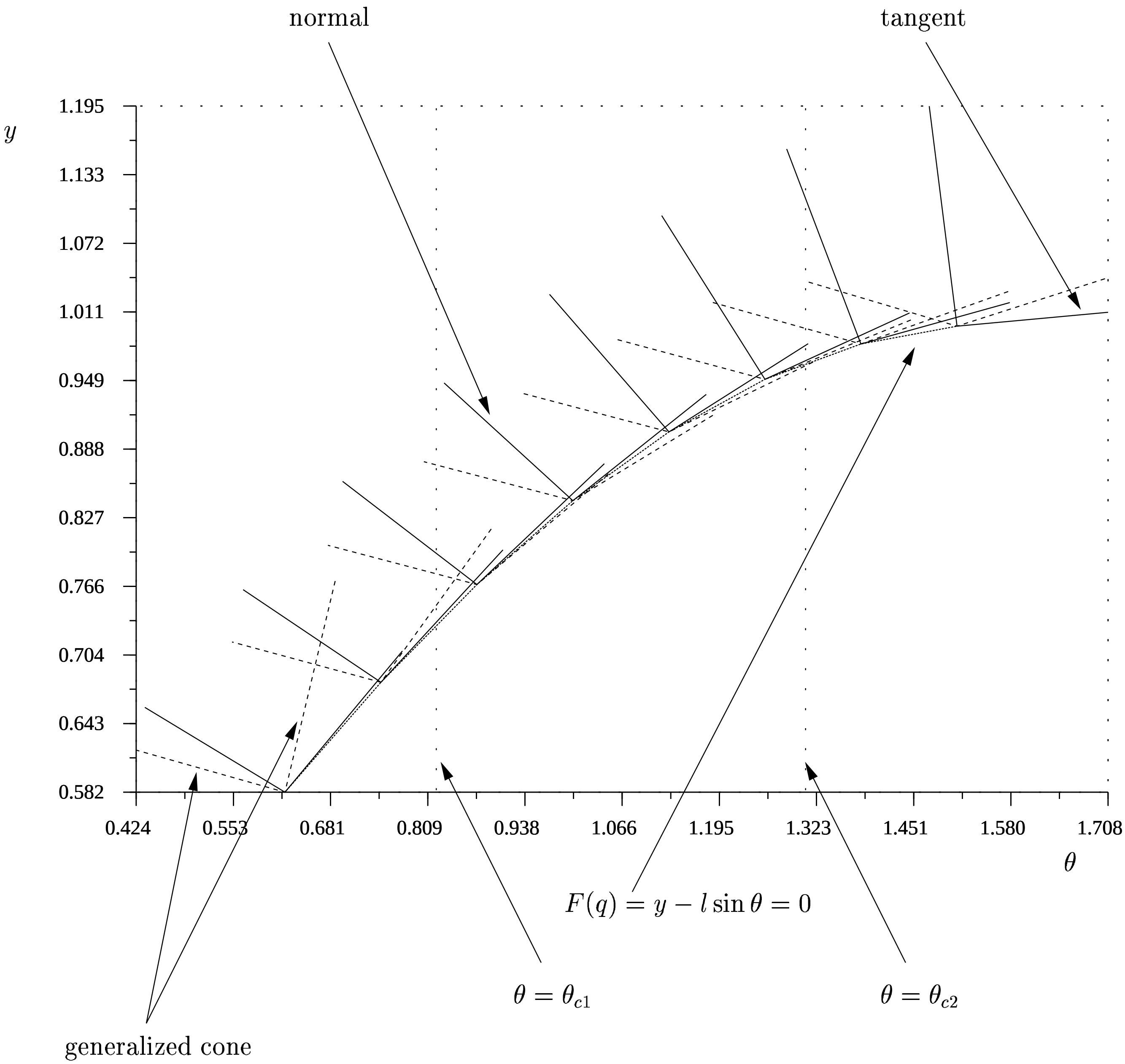

Figure 3: Evolution of the generalized friction cone with θ for $\mu = 1.6$ ($l = 1$ m, $m = 1$kg)

$\left(\theta(t_{\mathcal{A}}), \dot\theta(t_{\mathcal{A}})\right)^T \neq P_{c1}^+$, then detachment occurs at $t_{\mathcal{A}}$. Concerning mode $\mathcal{M}_4^-$, if $\left(\theta(t_{\mathcal{A}}), \dot\theta(t_{\mathcal{A}})\right)^T \neq P_{c1}^-$, the only solution to avoid entering the inconsistent mode $\mathcal{M}_2^-$ consists in an IW/OC. A more detailed study of the singular points P_{c1}^+ and P_{c1}^-, reachable from the modes $\mathcal{M}_1^+$ and $\mathcal{M}_4^-$, respectively, will be given to analyze the case for which $\left(\theta(t_{\mathcal{A}}), \dot\theta(t_{\mathcal{A}})\right)^T \in \left\{P_{c1}^+, P_{c1}^-\right\}$. In sight of orientation of the vector field in Fig. 2, it is easy to deduce that P_{c2}^+ and P_{c2}^- cannot be reached from any mode of the LCP. These analysis will allow us to determine the continuations from these two modes $\mathcal{M}_1^+$ and $\mathcal{M}_4^-$.

3 Study of mode $\mathcal{M}_4^+$

Let us first prove the following result :

Proposition 1

Let us assume that $\exists\, t_1 \mid X(t_1) \in \mathcal{M}_3 \bigcup \mathcal{M}_4$.
If $F_N(t_1) = 0$, *then* $\ddot{y}_A(t_1) > 0$.

Proof

One has $\mathcal{A}(t_1) < 0$, since $X(t_1) \in \mathcal{M}_3 \bigcup \mathcal{M}_4$. Hence from (4), $\ddot{y}_A(t_1) = -\mathcal{A}(t_1)$ and the result follows. $\diamond$

Hypothesis 2

We will assume that $\exists\, t_1 \mid X(t_1) \in \mathcal{M}_4^+$.

Lemma 2

If one chooses the solution $F_N(t) = \dfrac{\mathcal{A}(t)}{\mathcal{B}(t)}$ *for* $t \geq t_1$, *assuming at the same time that* $\dot{x}_A(t) < 0,\ \ \forall t \geq t_1$, *then the system escapes to infinite velocities in finite time.*

Proof

It is immediat that $\mathcal{C}(\theta, \mu) = \dfrac{3m}{l}(-\cos\theta + \mu \sin\theta) > 0$ for $\arctan\dfrac{1}{\mu} < \theta < \pi$, thus $\ddot{\theta} = \mathcal{C}(\theta, \mu)\dfrac{\mathcal{A}(\theta, \dot{\theta})}{\mathcal{B}(\theta, \mu)} > 0$ holds as long as the state remains in mode $\mathcal{M}_4^+$, since

$$\arctan\frac{1}{\mu} < \theta_{c1} < \theta(t) \leq \theta_{c2},\ \ \forall t \geq t_1$$

Consequently, $\dot{\theta}(t)$ is a strictly increasing function of time, under the same assumption. The same holds for $\theta(t)$ since $\dot{\theta}(t_1) > 0$. One finally deduces that the only solution to leave the mode $\mathcal{M}_4^+$ is to cross the verticale straight line $\theta = \theta_{c2}$. Let us assume now that $\dot{\theta}(t) \leq M < +\infty, \forall t \geq t_1$ with $\theta(t) < \theta_{c2}$, i.e. $\dot{\theta}(t)$ remains bounded. Thus there exists $t_{c2} > t_1$ such that $\theta(t_{c2}) = \theta_{c2}$. Our goal is to prove that $\lim_{t \to t_{c2}^-} \dot{\theta}(t) = +\infty$, hence a contradiction with the preceding assumption. Let $F(t) = 2\dot{\theta}(t)\ddot{\theta}(t)$ which is well defined as the product of to continuously differentiable functions for $t_1 \leq t < t_{c2}$. It follows that $G(t) = \int_{t_1}^{t} F(\tau)d\tau$ is also well defined on the same domain. More precisely :

$$G(t) = \dot{\theta}^2(t) - \dot{\theta}^2(t_1) \tag{11}$$

Let us now have a closer look at $\ddot{\theta}(t)$ given in (9) :

- $\mathcal{C}(\theta(t), \mu)$ is a continuous function and $\exists\, K_{\mathcal{C}} > 0 \mid \forall \theta \in [\theta(t_1), \theta_{c2}], \mathcal{C}(\theta, \mu) \geq K_{\mathcal{C}}$,
- $\mathcal{A}(\theta(t), \dot{\theta}(t)) < 0,\ \ \forall t \in [t_1, t_{c2})$. Note that $\dot{\mathcal{A}} = -l\dot{\theta}\left(\dot{\theta}^2\cos\theta + 2\dot{\theta}\ddot{\theta}\sin\theta\right) < 0$ since $0 < \theta_{c1} < \theta(t) \leq \theta_{c2} < \pi/2$, showing one more time that the orbit cannot cross the curve $\mathcal{A}(\theta, \dot{\theta}) = 0$. Thus, setting $K_{\mathcal{A}} = \mathcal{A}(\theta(t_1), \dot{\theta}(t_1)) < 0$, it follows that $\forall t \in [t_0, t_{c2}), \mathcal{A}(\theta(t), \dot{\theta}(t)) \leq K_{\mathcal{A}}$.

From both those observations (recall that $\mathcal{B}(\theta(t),\mu) < 0$) and setting $K = 2K_{\mathcal{C}}K_{\mathcal{A}} < 0$, one deduces that :

$$F(t) \geq K\frac{\dot{\theta}(t)}{\mathcal{B}(\theta,\mu)}, \quad \forall t \in [t_1, t_{c2})$$

Hence

$$G(t) \geq K\int_{t_1}^{t}\frac{\dot{\theta}(t)dt}{\mathcal{B}(\theta,\mu)} = K\int_{\theta(t_1)}^{\theta(t)}\frac{d\theta}{1+3\cos\theta(\cos\theta-\mu\sin\theta)}$$

Using a formal calculation tool (Maple), one calculate that a primitive function of

$$\frac{1}{1+3\cos\theta(\cos\theta-\mu\sin\theta)}$$

is given by :

$$L(\theta) = 2\frac{\operatorname{arctanh}\left(\dfrac{-2\tan\theta+3\mu}{\sqrt{9\mu^2-16}}\right)}{\sqrt{9\mu^2-16}}$$

and that $\lim_{\theta\to\theta_{c2}^-}\Re\left(L(\theta)\right) = -\infty$. The graph of the function $\Re\left(L(\theta)\right)$ for $\mu = 1.6$ is depicted in Fig. 4. From (11) and since $K < 0$, we deduce that $\lim_{\theta\to\theta_{c2}^-}\dot{\theta} = +\infty$, ending the proof. $\diamond$

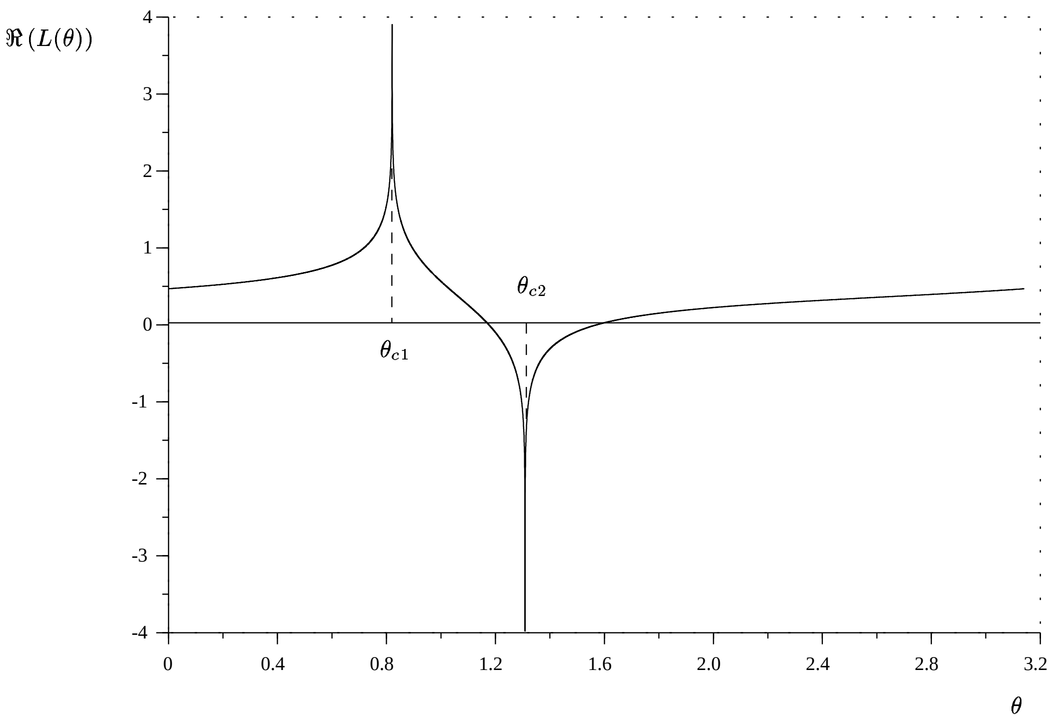

Figure 4: Graph of $\Re\left(L(\theta)\right)$, $\mu = 1.6$

Lemma 3

Suppose that one chooses $F_N(t) = \dfrac{\mathcal{A}(t)}{\mathcal{B}(t)}$ *for* $t \geq t_1$*, as the solution. Then the contact point* A *stops sliding at* t^* *with* $\theta(t^*) < \theta_{c2}$ *and the rod detaches at the same instant.*

Proof

Let us assume that this solution is choosen from instant t_1 and that the system remains in mode $\mathcal{M}_4^+$. From lemma 2, we know that the solution $F_N = \frac{\mathcal{A}}{\mathcal{B}}$ yields an energetical inconsistency. More precisely, let us denote the mechanical energy of the rod by $E_m(t)$. Since the only nonconservative force acting on the rod is $F_T = \mu F_N$, it follows that $\dot{E}_m(t) = \mu F_N(t)\dot{x}_A(t)$. $\dot{E}_m(t_1) < 0$, since $\dot{x}_A(t) < 0$ and $F_N(t) > 0$ (the system is dissipative). Hence from lemma 2, $\exists\, t_2 > t_1 \mid E_m(t_2) = E_m(t_1)$ and $X(t_2) \in \mathcal{M}_4^+$. By the use of the theorem of intermediate values, $\exists\, t^* \in]t_1, t_2] \mid \dot{E}(t^*) = 0$. One deduces that $\dot{x}_A(t^*) = 0$. Consequently, the rod enters mode M_{IV} of the hybrid system at $t = t^*$. Let us recall that Lötstedt [Lötstedt, 1981] has proved that there is always a unique solution for the contact force in this mode, as mentionned above. Here $F_N(t^*) = 0$ is a solution, since $\ddot{y}_A(t^*) = -A(t^*) > 0$, which ends the proof. $\diamond$

Proposition 2

- *If one chooses $F_N(t) = 0$ as the solution for $t \geq t_1$, then the rod detaches at time $t = t_1$.*
- *If one chooses $F_N(t) = \frac{\mathcal{A}(t)}{\mathcal{B}(t)}$ for $t \geq t_1$, then the contact point A sticks after a while and detaches instantaneously.*

Proof

Straightforward from proposition 1 and lemma 3. $\diamond$

Conclusions

- If one takes $F_N(t) = \frac{\mathcal{A}(t)}{\mathcal{B}(t)}$ as the solution for $t \geq t_1$, then the system leave mode M_{II} (and consequently mode $\mathcal{M}_4^+$) at time t^* and attains the boundary between modes M_I and M_{IV}, finally entering mode M_I.
- A proof similar to that of lemma 2 (which we will not detail here) allows us to prove that the verticale line $\theta = \theta_{c2}$ cannot be reached from mode $\mathcal{M}_1^-$.

4 Analysis of transitions from modes $\mathcal{M}_1^+$ and $\mathcal{M}_4^-$

In this first subsection we will first prove that the orbits of (9) cannot attain the vertical line $\theta = \theta_{c1}$, except at the singular points P_{c1}^+ or P_{c1}^-.

4.1 Behaviour of the orbits close to $\theta = \theta_{c1}$

Hypothesis 3

We will assume that $\exists\, t_1 \mid X(t_1) \in \mathcal{M}_1^+ \bigcup \mathcal{M}_4^-$.

Basically, in mode $\mathcal{M}_1^+$, $\dot{\theta} > 0$ and, if $\theta(t_1) < \theta_{c1}$, then $\mathcal{B}(\theta, \mu)$ is nonincreasing to 0. In mode $\mathcal{M}_4^-$, $\dot{\theta} < 0$ and $\mathcal{B}(\theta, \mu)$ is increasing to 0. At this stage, Pfeiffer et Glocker [Pfeiffer, 1996] claim that the contact force $F_N(t) = \dfrac{\mathcal{A}(t)}{\mathcal{B}(t)}$ take infinitely large values : one speaks of *Impact WithOut Collision* (IW/OC). These authors conclude that, since infinite forces do not exist in nature, such a phenomenon is due to an incompatibility between Amontons-Coulomb friction law and rigid body mechanics. Then they propose to give up the rigid body assumption, arguing that real bodies must deform under the action of infinite forces, see also [Neimark, 1995]. Mason and Wang [Mason et al., 1988] assert that an IW/OC must occur. Our goal is to prove that such a phenomenon does not occur systematically, i.e. the contact force remains bounded. It is worth noting first that, if $X(t_1) \in \mathcal{M}_1^+$ and $\theta(t_1) > \theta_{c2}$, due to the orientation of the vector field, the system cannot come back to the singularity $\theta = \theta_{c2}$. As a consequence, for the study of mode $\mathcal{M}_1^+$, we will confine ourselves to the case $\theta(t_1) < \theta_{c1}$.

Lemma 4

Let us assume that $\dot{x}_A(t) < 0,\ \forall t \geq t_1$, and that $\arctan\dfrac{1}{\mu} < \theta(t_1) < \theta_{c1}$ in the case where $X(t_1) \in \mathcal{M}_1^+$. Then :

$$\exists\, t_{\mathcal{A}} > t_1 \mid \mathcal{A}(\theta(t_{\mathcal{A}}), \dot{\theta}(t_{\mathcal{A}})) = 0 \textit{ et } \left\{ \begin{array}{lll} \theta(t_{\mathcal{A}}) \leq \theta_{c1} & \textit{si} & X(t_1) \in \mathcal{M}_1^+ \\ \theta(t_{\mathcal{A}}) \geq \theta_{c1} & \textit{si} & X(t_1) \in \mathcal{M}_4^- \end{array} \right.$$

Proof

The proof is based on a reductio ad absurdum similar to that of the proof of lemma 2. Remark that the condition $\theta(t_1) > \arctan\dfrac{1}{\mu}$ is always satisfied in mode $\mathcal{M}_4^-$. Consequently, $\mathcal{C}(\theta(t),\mu) > 0$ and thus $\ddot{\theta}(t) > 0, \forall t \geq t_1$, as long as the system remains in $\mathcal{M}_1^+ \bigcup \mathcal{M}_4^-$. Therefore, under the same assumption, $\dot{\theta}(t)$ is a increasing function of time. One deduces that, if the rod does not stop sliding, the only way to leave $\mathcal{M}_1^+ \bigcup \mathcal{M}_4^-$ is to cross one of the two curves $\mathcal{A}(\theta,\dot{\theta}) = 0$ or $\mathcal{B}(\theta,\mu) = 0$ (or both at the singular points). Assume that point A keeps sliding and that :

$$\exists\, K_{\mathcal{A}} \left\{ \begin{array}{lll} K_{\mathcal{A}} > 0 \text{ et } \forall t \geq t_1, \mathcal{A}(\theta(t),\dot{\theta}(t)) \geq K_{\mathcal{A}}, & \text{si} & X(t_1) \in \mathcal{M}_1^+ \\ K_{\mathcal{A}} < 0 \text{ et } \forall t \geq t_1, \mathcal{A}(\theta(t),\dot{\theta}(t)) \leq K_{\mathcal{A}}, & \text{si} & X(t_1) \in \mathcal{M}_4^- \end{array} \right.$$

One should remark that, in the case where $X(t_1) \in \mathcal{M}_1^+$, one has

$$K_{\mathcal{A}} = \mathcal{A}(\theta_{c1}, \dot{\theta}(t_{c1}))$$

since $\dot{A} = -l\dot{\theta}\left(\dot{\theta}^2\cos\theta + 2\dot{\theta}\ddot{\theta}\sin\theta\right) < 0$. The foregoing assumption is equivalent to assuming that the orbit does not cross the curve $A(\theta,\dot{\theta}) = 0$. It means that $\dot{\theta}(t)$ remains bounded as long as the system stays in the mode $\mathcal{M}_1^+ \bigcup \mathcal{M}_4^-$. Therefore there exists $t_{c1} > t_1$ such that $\theta(t_{c1}) = \theta_{c1}$. One knows that $\exists\, K_{\mathcal{C}} > 0 \mid \mathcal{C}(\theta,\mu) > K_{\mathcal{C}}, \forall\theta \in [\theta(t_1), \theta_{c1}]$. Using the final part of the proof of lemma 2, one finally ends up with a similar contradiction, hence the time $t_{\mathcal{A}}$ does exist. ◇

The problem is now to determine if the orbit crosses the curve $\mathcal{A}(\theta,\dot{\theta}) = 0$ before $\theta(t) = \theta_{c1}$.

Lemma 5

$$\textit{If } \exists\, t_{\mathcal{A}} \mid \mathcal{A}(t_{\mathcal{A}}) = 0 \textit{ and } \left\{ \begin{array}{lll} \theta(t_{\mathcal{A}}) \leq \theta_{c1} & \textit{si} & X(t_1) \in \mathcal{M}_1^+ \\ \theta(t_{\mathcal{A}}) \geq \theta_{c1} & \textit{si} & X(t_1) \in \mathcal{M}_4^- \end{array} \right. \textit{, then :}$$

- *if $X(t_1) \in \mathcal{M}_1^+$, the rod leaves the mode $\mathcal{M}_1^+$ by entering the mode $\mathcal{M}_3^+$ and detaches,*

- *if $X(t_1) \in \mathcal{M}_4^-$, then a IW/OC is required not to enter the inconsistent mode $\mathcal{M}_2^-$.*

Proof

At time $t_\mathcal{A}$, the system is on the boundary between modes $\mathcal{M}_1^+$ and $\mathcal{M}_3^+$ if $X(t_1) \in \mathcal{M}_1^+$, or on the boundary between modes $\mathcal{M}_4^-$ and $\mathcal{M}_2^-$, if $X(t_1) \in \mathcal{M}_4^-$. One has $F_N(t_\mathcal{A}) = \dfrac{\mathcal{A}(t_\mathcal{A})}{\mathcal{B}(t_\mathcal{A})} = 0$, since from the hypothesis $\mathcal{B}(t_\mathcal{A}) \neq 0$ and $\ddot{y}_A(t_\mathcal{A}) = -\mathcal{A}(t_\mathcal{A}) + \mathcal{B}(t_\mathcal{A})F_N(t_\mathcal{A}) = 0$. Thus we cannot conclude whether or not the rod detaches. However $\dot{\mathcal{A}}(t_\mathcal{A}) = -l\dot{\theta}^2(t_\mathcal{A})\cos\theta(t_\mathcal{A}) < 0$ hence $\mathcal{A}(t_\mathcal{A}^+) < 0$ and thus $X(t_\mathcal{A}^+) \in \mathcal{M}_3^+$, if $X(t_1) \in \mathcal{M}_1^+$, and $X(t_\mathcal{A}^+) \in \mathcal{M}_2^-$, if $X(t_1) \in \mathcal{M}_4^-$.
Proposition 2 allows us to conclude about the rod detachment in the case $X(t_1) \in \mathcal{M}_1^+$. $\diamond$

Remark

It is tempting to try the solution $F_N(t_\mathcal{A}^+) = 0$ on the boundary between modes $\mathcal{M}_4^-$ and $\mathcal{M}_2^-$, in order to pass the transition toward mode M_I. However in this case, we would have $\ddot{y}_\mathcal{A}(t_\mathcal{A}^+) = -\mathcal{A}(t_\mathcal{A}^+) < 0$ (the system entered mode $\mathcal{M}_2^-$), which is impossible.

Conclusion

One faces an ordinary differential equation $\dfrac{dx}{dt} = f(x,t)$ where $f(x,t)$ diverges in a closed subspace $\mathcal{D}$ of the state space. Nevertheless the foregoing study allows us to understand that our case cannot be recast into the framework of Filipov's recent results [Filipov, 1996] concerning this type of differential equations. Indeed, Filipov assumes that either the orbits rebound or cross the singular subspace $\mathcal{D}$. In our case, the vector field in a neighbourhood of $\mathcal{D}$ is tangent to $\mathcal{D}$, at least outside the singular points. The rest of the work concerns the analysis of the system behaviour in the neighbourhood of P_{c1}^+ or P_{c1}^-.

4.2 Study of the singular points

We shall confine ourselves to the study of P_{c1}^+ and P_{c1}^-, which are the only reachable singular points as long as the system remains in mode M_{II}.

Hypothesis 4

We will assume that $\exists\, t_{c1} \mid \left(\theta(t_{c1}), \dot{\theta}(t_{c1})\right)^T = P_{c1} \in \left\{P_{c1}^-, P_{c1}^+\right\}$ and that this point is reached by the solution $F_N(t) = \dfrac{\mathcal{A}(t)}{\mathcal{B}(t)}$.

In view of the vector field direction (Fig. 2), it is straightforward that :

- if $P_{c1} = P_{c1}^+$, then $X(t_{c1}^-) \in \mathcal{M}_1^+$,
- if $P_{c1} = P_{c1}^-$, then $X(t_{c1}^-) \in \mathcal{M}_4^-$.

The goal of the study is to examine if $\lim_{t\to t_{c1}^-} F_N(t) = \dfrac{\mathcal{A}(t)}{\mathcal{B}(t)}$ exists, is finite or is infinite. Since $\mathcal{C}(\theta(t),\mu)$ is a bounded positive function in a neighbourhood of P_{c1}, one sees from (9) that the existence of this limit is tantamount to that of $\lim_{t\to t_{c1}^-} \ddot{\theta}(t)$. Let $x_1(t) = \theta(t)$ and $x_2(t) = \dot{\theta}(t)$. The differential equation (9) can be rewritten as a first order system :

$$\left\{\begin{array}{lcll} \dfrac{dx_1}{dt} & = & x_2 & \text{(a)} \\ \dfrac{dx_2}{dt} & = & \mathcal{C}(x_1,\mu)\dfrac{\mathcal{A}(x_1,x_2)}{\mathcal{B}(x_1,\mu)} & \text{(b)} \end{array}\right. \quad (12)$$

Notice first that, in any neighbourhood of the singular point P_{c1} minus P_{c1}, equation (b) in (12) can be equivalently rewritten as :

$$\mathcal{B}(x_1,\mu)\frac{dx_2}{dt} = \mathcal{C}(x_1,\mu)\mathcal{A}(x_1,x_2)$$

4.2.1 Time scaling

Let us make the following time scaling :

$$\frac{dt}{ds} = \mathcal{B}(x_1,\mu) \tag{13}$$

This scale is valid if it is bijective, which is guaranteed by the fact that $\mathcal{B}(x_1,\mu)$ keeps a constant positive or negative sign, depending from which mode the singular point P_{c1} is reached. Then the system (12) is equivalent to :

$$\left\{\begin{array}{rcl} \dfrac{dx_1}{ds} & = & \mathcal{B}(x_1,\mu)x_2 \\ \dfrac{dx_2}{ds} & = & \mathcal{C}(x_1,\mu)\mathcal{A}(x_1,x_2) \\ \dfrac{dt}{ds} & = & \mathcal{B}(x_1,\mu) \end{array}\right. \tag{14}$$

4.2.2 Calculation of $\lim_{t\to t_{c1}^-} s(t)$

Let $t_1 < t_{c1} \mid X(t_1) \in \mathcal{M}_1^+$ if $P_{c1} = P_{c1}^+$ and $X(t_1) \in \mathcal{M}_4^+$ if $P_{c1} = P_{c1}^-$. Let $H(t) = \displaystyle\int_{t_1}^{t} \frac{\dot\theta(\tau)d\tau}{\mathcal{B}(\theta(\tau),\mu)}$. One has :

$$H(t) = \int_{\theta(t_1)}^{\theta(t)} \frac{d\theta}{1+3\cos\theta\,(\cos\theta-\mu\sin\theta)}$$

Using Maple, one sees that a primitive function of

$$\frac{1}{1+3\cos\theta\,(\cos\theta-\mu\sin\theta)}$$

is given by :

$$L(\theta) = 2\frac{\operatorname{arctanh}\left(\dfrac{-2\tan\theta+3\mu}{\sqrt{9\mu^2-16}}\right)}{\sqrt{9\mu^2-16}}$$

and that

$$\lim_{\theta\to\theta_{c1}^{\pm}} \Re\left(L(\theta)\right) = +\infty$$

Thus

$$\lim_{t\to t_{c1}^-} H(t) = +\infty$$

From the time scaling (13), it follows that

$$H(t) = \int_{t_1}^{t} \dot\theta(\tau)\dot s(\tau)d\tau$$

Since $\dot\theta(\tau)$ remains bounded and non zero on $[t_1, t_{c1}]$, ($\dot\theta(\tau) \geq \delta > 0$ in the neighbourhood of P_{c1}^+ and $\dot\theta(\tau) \leq \delta < 0$ in the neighbourhood of P_{c1}^-), one finally deduces that :

Lemma 6

- *if* $P_{c1} = P_{c1}^+$, *then* $\lim_{t\to t_{c1}-} s(t) = +\infty$.
- *if* $P_{c1} = P_{c1}^-$, *then* $\lim_{t\to t_{c1}-} s(t) = -\infty$.

4.2.3 Tangent linearization of system (14) around P_{c1}

Let us denote by $Y(s) = \begin{pmatrix} x_1(s) \\ x_2(s) \\ t(s) \end{pmatrix}$ the state vector of the transformed system (14) which can be rewritten in the more compact form :

$$\frac{dY}{ds} = F(Y) \tag{15}$$

The key to our analysis is that : $Y_{c1}^{\pm} = \begin{pmatrix} \theta_{c1} \\ \dot\theta_{c1}^{\pm} \\ t_{c1} \end{pmatrix}$ are *equilibrium points* of (15), i.e. $F(Y_{c1}^{\pm}) = 0$. The linearized model around $Y_{c1}^{\pm}$ is given by :

$$\frac{dY}{ds} = A^{\pm}(Y - Y_{c1}^{\pm}) \tag{16}$$

where

$$A^{\pm} = \frac{\partial F}{\partial Y}(Y_{c1}^{\pm}) = \begin{pmatrix} \alpha_1^{\pm} & 0 & 0 \\ \alpha_2^{\pm} & \alpha_3^{\pm} & 0 \\ \alpha_4^{\pm} & 0 & 0 \end{pmatrix} \tag{17}$$

with, after simplications,

$$\left\{ \begin{array}{lcl} \alpha_1^{\pm} & = & \dfrac{\partial \mathcal{B}}{\partial \theta}(\theta_{c1}, \mu)\dot\theta_{c1}^{\pm} = \dfrac{3\dot\theta_{c1}^{\pm}}{m}\left(\mu - 2\cos\theta_{c1}\left(\sin\theta_{c1} + \mu\cos\theta_{c1}\right)\right) \\ \alpha_2^{\pm} & = & \mathcal{C}(\theta_{c1}, \mu)\dfrac{\partial \mathcal{A}}{\partial \theta}(\theta_{c1}, \dot\theta_{c1}^{\pm}) = -\dfrac{\dot\theta_{c1}^{2}}{m} \\ \alpha_3^{\pm} & = & \mathcal{C}(\theta_{c1}, \mu)\dfrac{\partial \mathcal{A}}{\partial \dot\theta}(\theta_{c1}, \dot\theta_{c1}^{\pm}) = \dfrac{6}{m}\left(\cos\theta_{c1} - \mu\sin\theta\right)\dot\theta_{c1}^{\pm}\sin\theta_{c1} \\ \alpha_4^{\pm} & = & \dfrac{\partial \mathcal{B}}{\partial \theta}(\theta_{c1}, \mu) = \dfrac{3}{m}\left(\mu - 2\cos\theta_{c1}\left(\sin\theta_{c1} + \mu\cos\theta_{c1}\right)\right) \end{array} \right. \tag{18}$$

The evolutions of these various quantities as functions of the friction coefficient are depicted in figures 5 and 6 for $P_{c1} = P_{c1}^{+}$ and $P_{c1} = P_{c1}^{-}$ respectively.

4.2.4 Stability analysis around P_{c1}

It is not difficult to see that the eigenvalues of the jacobian matrix $A^{\pm}$ in (17) are $\left\{0, \alpha_1^{\pm}, \alpha_3^{\pm}\right\}$. One knows that :

- $\operatorname{sgn}(\alpha_1^{\pm}(\mu)) = \mp 1, \ \forall \mu > \frac{4}{3}$,
- $\alpha_1^{\pm}(\frac{4}{3}) = 0$,
- $\operatorname{sgn}(\alpha_3^{\pm}(\mu)) = \mp 1, \ \forall \mu \geq \frac{4}{3}$.

Since $A^{\pm}$ has 0 as an eigenvalue, one must resort to the center-manifold theorem to conclude about the nonlinear system stability around P_{c1}.

- Case $\mu > \frac{4}{3}$.
  In this case, the jacobian matrix $A^{\pm}$ has three distinct eigenvalues : $0, \alpha_1^{\pm}, \alpha_3^{\pm}$. First let us make the translation $\tilde Y = Y - Y_{c1}^{\pm}$. The nonlinear system (15) can be written as :

$$\dot{\tilde Y} = A^{\pm}\tilde Y + \left(F(\tilde Y + Y_{c1}^{\pm}) - A^{\pm}\tilde Y\right) = A^{\pm}\tilde Y + \tilde f(\tilde Y) \tag{19}$$

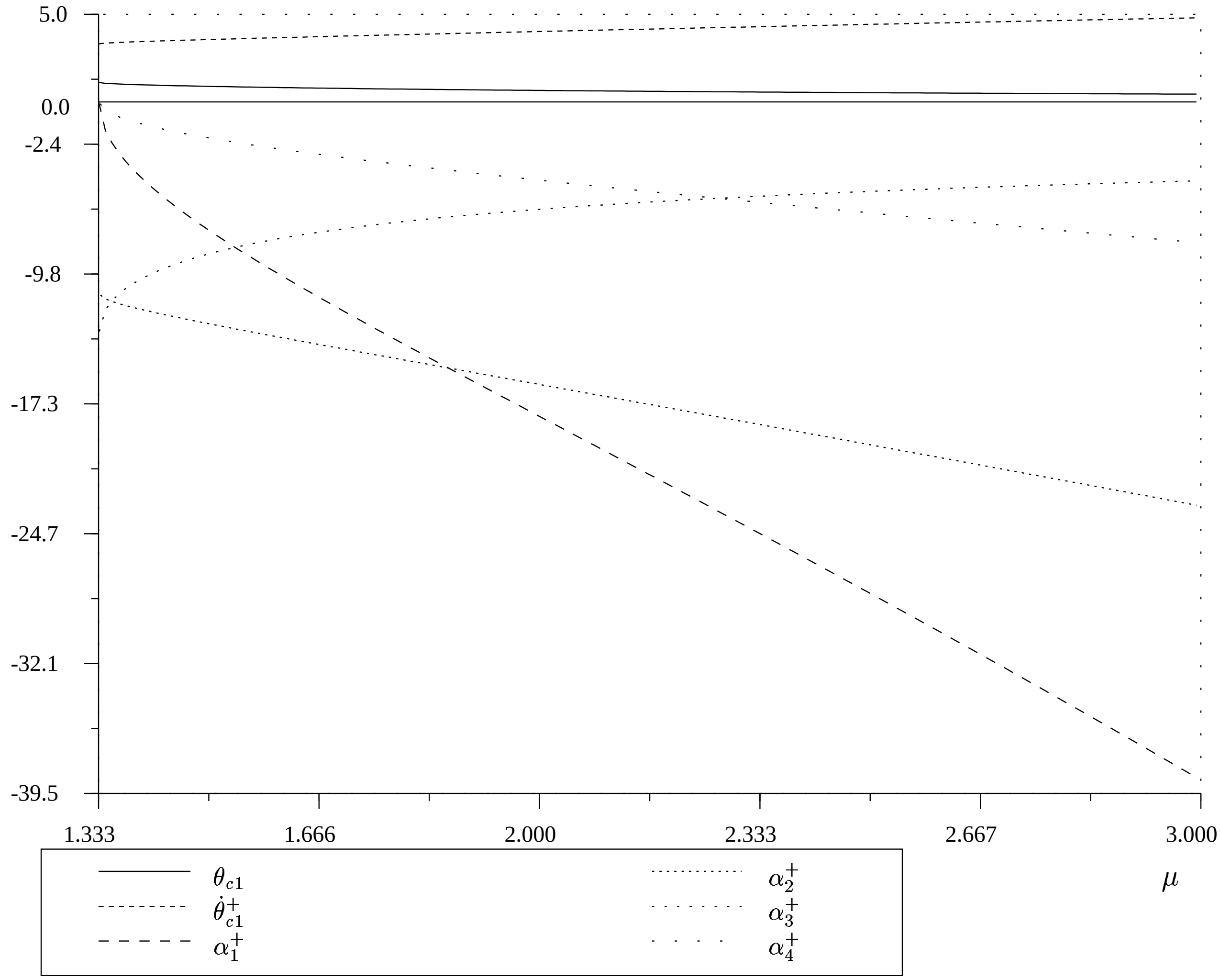

Figure 5: Characteristic curves for P_{c1}^- with varying friction coefficient ($m = 1$ kg, $g = 9.8$ m/s^2,$l = 1$m)

Let $J^\pm$ be the Jordan form of the matrix $A^\pm$ and $P^\pm$ denote the associated matrix such that : $P^{\pm T} A^\pm P^\pm = J^\pm$. We have :

$$J^\pm = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \alpha_1^\pm & 0 \\ 0 & 0 & \alpha_3^\pm \end{pmatrix}, \text{ and } P^\pm = \begin{pmatrix} 1 & 0 & -\dfrac{\alpha_1^\pm}{\alpha_4^\pm} \\ 1 & 0 & 0 \\ 1 & \dfrac{\alpha_3^\pm - \alpha_1^\pm}{\alpha_2^\pm} & 0 \end{pmatrix}$$

Let us do the variable change :

$$\begin{pmatrix} y \\ z \end{pmatrix} = P^\pm \tilde{Y}, \;\; y \in \mathbf{R}, \;\; z \in \mathbf{R}^2$$

We have :

$$\begin{cases} y &= \tilde{Y}_1 - \dfrac{\alpha_1^\pm}{\alpha_4^\pm}\tilde{Y}_3 \\ z_1 &= \tilde{Y}_1 \\ z_2 &= \tilde{Y}_1 - \dfrac{\alpha_1^\pm - \alpha_3^\pm}{\alpha_2^\pm}\tilde{Y}_2 \end{cases}$$

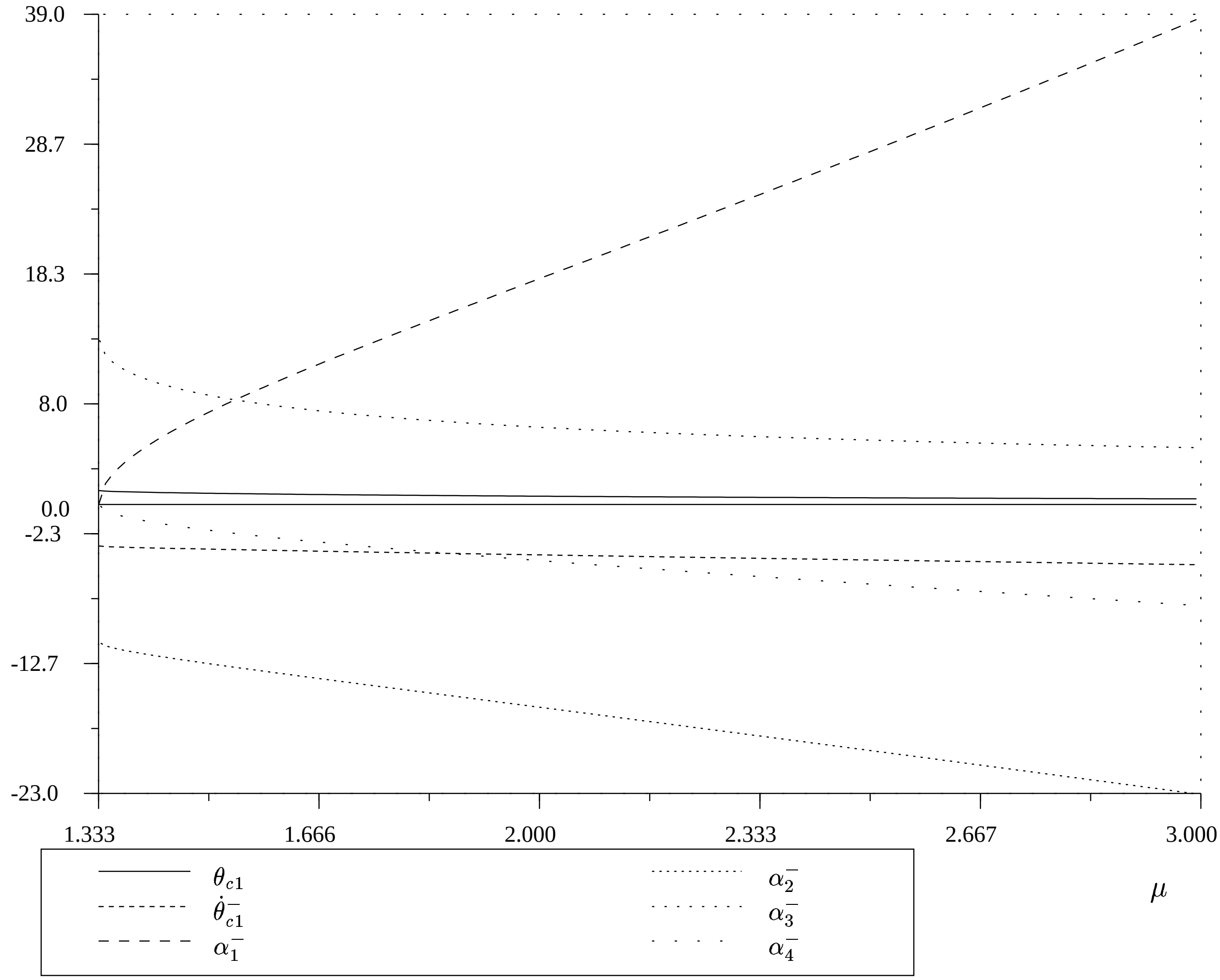

Figure 6: Characteristic curves for P_{c1}^{-} with varying friction coefficient (m = 1 kg, g = 9.8 m/s^2, l = 1m)

After manipulations, the system (19) can be rewritten as :

$$\left\{\begin{array}{lcl} \dot{y} & = & \dfrac{\alpha_2^{\pm}}{\alpha_1^{\pm}-\alpha_3^{\pm}}\mathcal{B}\left(\theta_{c1}+z_1,\mu\right)(z_1-z_2) \\ \dot{z}_1 & = & \alpha_1^{\pm} z_1+\mathcal{B}\left(\theta_{c1}+z_1,\mu\right)\left(\dot{\theta}_{c1}^{\pm}+\dfrac{\alpha_2^{\pm}}{\alpha_1^{\pm}-\alpha_3^{\pm}}(z_1-z_2)\right) \\ \dot{z}_2 & = & \alpha_3^{\pm} z_2+\mathcal{B}\left(\theta_{c1}+z_1,\mu\right)\left(\dot{\theta}_{c1}+\dfrac{\alpha_2^{\pm}}{\alpha_1^{\pm}-\alpha_3^{\pm}}(z_1-z_2)\right)- \\ & & \mathcal{A}\left(\theta_{c1}+z_1,\dot{\theta}_{c1}^{\pm}+\dfrac{\alpha_2^{\pm}}{\alpha_1^{\pm}-\alpha_3^{\pm}}(z_1-z_2)\right)\mathcal{C}\left(\theta_{c1}+z_1,\mu\right)\dfrac{\alpha_1^{\pm}-\alpha_3^{\pm}}{\alpha_2^{\pm}} \end{array}\right. \tag{20}$$

Consider the manifold $z = h(y) = 0$. It is straightforward to see that this subspace is invariant by the dynamics in (20) and that $h(0) = 0$; $\dfrac{\partial h}{\partial y} = 0$. Consequently, $z = 0$ is a *center-manifold* for (20). The evolution of the system along this center-manifold is defined by $\dot{y} = 0$. This subsystem is stable in the sense of Lyapunov. Hence, from the center-manifold theorem [Khalil, 1996, p. 171], one finally deduces the following result :

Lemma 7

If $\mu > \dfrac{4}{3}$, then le nonlinear system (14) is locally stable around P_{c1}.

- Case $\mu = \frac{4}{3}$.
In this case, $A^{\pm}$ has only one non-zero eigenvalue : $\alpha_3^{\pm}$. Let us do the same analysis as for the case $\mu > \frac{4}{3}$. The jacobian matrix given in (17) is :

$$A^{\pm} = \begin{pmatrix} 0 & 0 & 0 \\ \alpha_2^{\pm} & \alpha_3^{\pm} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Let us do the translation $\tilde{Y} = Y - Y_{c1}^{\pm}$. The nonlinear system (15) can be rewritten as :

$$\dot{\tilde{Y}} = A^{\pm}\tilde{Y} + \left(F(\tilde{Y} + Y_{c1}^{\pm}) - A^{\pm}\tilde{Y}\right) = A^{\pm}\tilde{Y} + \tilde{f}(\tilde{Y}) \tag{21}$$

Let $J^{\pm}$ be the Jordan form of the matrix $A^{\pm}$ and $P^{\pm}$ be the matrix defined by : $P^{\pm T}A^{\pm}P^{\pm} = J^{\pm}$. One has :

$$J^{\pm} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \alpha_3^{\pm} \end{pmatrix} \text{ and } P^{\pm} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & \dfrac{\alpha_3^{\pm}}{\alpha_2^{\pm}} & 0 \end{pmatrix}$$

Let us do the following transformation

$$\begin{pmatrix} y \\ z \end{pmatrix} = P^{\pm}\tilde{Y}, \;\; y \in \mathbf{R}^2, \;\; z \in \mathbf{R}$$

One has :

$$\left\{ \begin{array}{rcl} y_1 & = & \tilde{Y}_1 \\ y_2 & = & \tilde{Y}_3 \\ z & = & \tilde{Y}_1 + \dfrac{\alpha_3^{\pm}}{\alpha_2^{\pm}}\tilde{Y}_2 \end{array} \right.$$

System (21) is equivalent, after simplications, to :

$$\left\{ \begin{array}{rcl} \dot{y_1} & = & \mathcal{B}\left(\theta_{c1} + y_1, \mu\right)\left(\dot{\theta}_{c1}^{\pm} - \dfrac{\alpha_2^{\pm}}{\alpha_3^{\pm}}\left(y_1 - z\right)\right) \\ \dot{y_2} & = & \mathcal{B}\left(\theta_{c1} + y_1, \mu\right) \\ \dot{z} & = & \alpha_3^{\pm} z + \mathcal{B}\left(\theta_{c1} + y_1, \mu\right)\left(\dot{\theta}_{c1}^{\pm} - \dfrac{\alpha_2^{\pm}}{\alpha_3^{\pm}}\left(y_1 - z\right)\right) + \\ & & \mathcal{A}\left(\theta_{c1} + y_1, \dot{\theta}_{c1}^{\pm} - \dfrac{\alpha_2^{\pm}}{\alpha_3^{\pm}}\left(y_1 - z\right)\right)\mathcal{C}\left(\theta_{c1} + y_1, \mu\right)\dfrac{\alpha_3^{\pm}}{\alpha_2^{\pm}} \end{array} \right. \tag{22}$$

Contrary to the case $\mu > \dfrac{4}{3}$, the trivial manifold $z = h(y) = 0$ is no more invariant by the dynamics (22). Let

$$\left\{ \begin{array}{rcl} g_1(y, z) & = & \dot{y} \\ g_2(y, z) & = & \dot{z} - \alpha_3^{\pm} z \end{array} \right.$$

A center-manifold $z = h(y)$ must satisfy the partial differential equation [Khalil, 1996, p. 173] :

$$\frac{\partial h}{\partial y} g_1(y, h(y)) - \alpha_3^{\pm} h(y) - g_2(y, h(y)) = 0 \tag{23}$$

with the boundary conditions :

$$h(0) = 0; \ \ \frac{\partial h}{\partial y}(0) = 0 \tag{24}$$

(23) under the constraints (24) cannot be solved in closed form, but can be arbitrarily approximated by its Taylor-series. Notice that :

$$\mathcal{B}\left(\theta_{c1} + y_1, \mu\right) = \frac{\partial^2 \mathcal{B}}{\partial \theta^2}\left(\theta_{c1}, \mu\right) y_1^2 + O(|y_1|^3)$$

Let $z = h(y) = O(\|y\|^2)$. The system reduces to :

$$\dot{y} = \begin{pmatrix} \frac{\partial^2 \mathcal{B}}{\partial \theta^2}\left(\theta_{c1}, \mu\right) \dot{\theta}_{c1}^{\pm} y_1^2 \\ \frac{\partial^2 \mathcal{B}}{\partial \theta^2}\left(\theta_{c1}, \mu\right) y_1^2 \end{pmatrix} + O(\|y\|_2^3) \tag{25}$$

However $a = \frac{\partial^2 \mathcal{B}}{\partial \theta^2}\left(\theta_{c1}, \mu\right) = \frac{10}{m} \neq 0$, thus the origin of the reduced system is unstable. Hence, from the *reduction principle* [Khalil, 1996, p. 171], one deduces :

Lemma 8

If $\mu = \frac{4}{3}$, then P_{c1} is a unstable equilibrium point for the non-linear system (14).

Let us however notice that the interesting orbits are those for which $y_2(s) = \tilde{Y}_3(s) = t(s) - t_{c1} < 0$ and

$$\begin{cases} y_1(s) = \tilde{Y}_1(s) = \theta(s) - \theta_{c1} < 0, \text{ si } P_{c1} = P_{c1}^+ \\ y_1(s) = \tilde{Y}_1(s) = \theta(s) - \theta_{c1} > 0, \text{ si } P_{c1} = P_{c1}^- \end{cases}$$

Integration of (25) yields :

$$\begin{cases} y_1(s) &= \dfrac{1}{a\dot{\theta}_{c1}^{\pm}(s_0 - s) + \dfrac{1}{y_1(s_0)}} \\ y_2(s) &= y_2(s_0) - \dfrac{y_1(s_0)}{\dot{\theta}_{c1}^{\pm}} + \dfrac{1}{\dot{\theta}_{c1}^{\pm}\left(a\dot{\theta}_{c1}^{\pm}(s_0 - s) + \dfrac{1}{y_1(s_0)}\right)} \end{cases}$$

One finally obtains :

$$\begin{cases} \lim_{s \to \pm\infty} \theta(s) = \theta_{c1}^{\mp} \\ \lim_{s \to \pm\infty} t(s) = \left(t(s_0) - \dfrac{\theta(s_0) - \theta_{c1}}{\dot{\theta}_{c1}^{\pm}}\right)^{\mp} \end{cases}$$

Since this result is true on the center-manifold, one deduces that all the orbits pass through the point P_{c1}.

4.2.5 Calculation of $\lim_{t \to t_{c1}^-} \ddot{\theta}(t)$

In this paragraph, we will assume that $\mu > \frac{4}{3}$.

Let us now assume that the system enters at time $t_{\mathcal{V}}$ a neighbourhood $\mathcal{V}$ of $P_{c1}^{\pm}$ in which the above mentionned linearization is valid, i.e. the linear system (16) corresponds to a good approximation of the nonlinear system (14). This neighbourhood exists in view of lemma 7. A first step consists in the integration of the linearized model (16). Let $Y_{\mathcal{V}} = \begin{pmatrix} \theta(t_{\mathcal{V}}) \\ \dot{\theta}(t_{\mathcal{V}}) \\ t_{\mathcal{V}} \end{pmatrix}$ and $s(t_{\mathcal{V}}) = 0$. One obtains

$$\left\{ \begin{array}{rcl} x_1^{\pm}(s) & = & \theta_{c1} - (\theta_{c1} - \theta_v)\, e^{\alpha_1^{\pm} s} \\ t^{\pm}(s) & = & t_{\mathcal{V}} + \dfrac{\alpha_4^{\pm}}{\alpha_1^{\pm}} (\theta_{c1} - \theta(t_{\mathcal{V}})) \left(1 - e^{\alpha_1^{\pm} s}\right) \end{array} \right. \tag{26}$$

Moreover,

- if $\alpha_1^{\pm} \neq \alpha_3^{\pm}$:

$$x_2^{\pm}(s) = \dot{\theta}_{c1}^{\pm} - \left(\dot{\theta}_{c1}^{\pm} - \dot{\theta}(t_{\mathcal{V}})\right) e^{\alpha_3^{\pm} s} + \frac{\alpha_2^{\pm}}{\alpha_1^{\pm} - \alpha_3^{\pm}} (\theta_{c1} - \theta(t_{\mathcal{V}})) \left(e^{\alpha_3^{\pm} s} - e^{\alpha_1^{\pm} s}\right) \tag{27}$$

- if $\alpha_1^{\pm} = \alpha_3^{\pm}$

$$x_2^{\pm}(s) = \dot{\theta}_{c1}^{\pm} - \left(\dot{\theta}_{c1}^{\pm} - \dot{\theta}(t_{\mathcal{V}}) + \alpha_2^{\pm} (\theta_{c1} - \theta(t_{\mathcal{V}}))\, s\right) e^{\alpha_1^{\pm} s} \tag{28}$$

We focus on the calculation of $\lim_{t \to t_{c1}^-} \ddot{\theta}^{\pm}(t)$, that is, from lemma 6, $\lim_{s \to \pm\infty} \ddot{\theta}^{\pm}(s) = \dfrac{\frac{dx_2^{\pm}(s)}{ds}}{\frac{dt^{\pm}(s)}{ds}}$. From (26), one has :

$$\frac{dt^{\pm}}{ds}(s) = -\alpha_4^{\pm} (\theta_{c1} - \theta(t_{\mathcal{V}}))\, e^{\alpha_1^{\pm} s} \tag{29}$$

and from (27) and (28) :

- if $\alpha_1^{\pm} \neq \alpha_3^{\pm}$

$$\frac{dx_2^{\pm}}{ds}(s) = -\left(\dot{\theta}_{c1}^{\pm} - \dot{\theta}(t_{\mathcal{V}})\right) \alpha_3^{\pm} e^{\alpha_3^{\pm} s} + \frac{\alpha_2^{\pm}}{\alpha_1^{\pm} - \alpha_3^{\pm}} (\theta_{c1} - \theta(t_{\mathcal{V}})) \left(\alpha_3^{\pm} e^{\alpha_3^{\pm} s} - \alpha_1^{\pm} e^{\alpha_1^{\pm} s}\right) \tag{30}$$

- if $\alpha_1^{\pm} = \alpha_3^{\pm}$

$$\frac{dx_2^{\pm}}{ds}(s) = -\left(\alpha_1^{\pm} \left(\dot{\theta}_{c1}^{\pm} - \dot{\theta}(t_{\mathcal{V}})\right) + \alpha_2^{\pm} (\theta_{c1} - \theta(t_{\mathcal{V}})) \left(1 + \alpha_1^{\pm} s\right)\right) e^{\alpha_1^{\pm} s} \tag{31}$$

Finally, noting that $\theta(t_{\mathcal{V}}) \neq \theta_{c1}$ since $\theta(t)$ is strictly monotonous for $t \in [t_{\mathcal{V}}, t_{c1})$:

- if $\alpha_1^{\pm} \neq \alpha_3^{\pm}$:

$$\ddot{\theta}^{\pm}(s) = \frac{\alpha_1^{\pm} \alpha_2^{\pm}}{\alpha_4^{\pm} \left(\alpha_1^{\pm} - \alpha_3^{\pm}\right)} + \left(\frac{\dot{\theta}_{c1}^{\pm} - \dot{\theta}(t_{\mathcal{V}})}{\theta_{c1} - \theta(t_{\mathcal{V}})} - \frac{\alpha_2^{\pm}}{\alpha_1^{\pm} - \alpha_3^{\pm}}\right) \frac{\alpha_3^{\pm}}{\alpha_4^{\pm}} e^{(\alpha_3^{\pm} - \alpha_1^{\pm}) s} \tag{32}$$

- if $\alpha_1^{\pm} = \alpha_3^{\pm}$:

$$\ddot{\theta}^{\pm}(s) = \frac{\alpha_1^{\pm} \left(\dot{\theta}_{c1}^{\pm} - \dot{\theta}(t_{\mathcal{V}})\right)}{\alpha_4^{\pm} (\theta_{c1} - \theta(t_{\mathcal{V}}))} + \frac{\alpha_2^{\pm}}{\alpha_4^{\pm}} \left(1 + \alpha_1^{\pm} s\right) \tag{33}$$

As equations (33) and (32) have illustrated, the existence and the value of the limit of $\ddot{\theta}(t)$ when t tends to t_{c1}^- depends on the sign of $h^{\pm}(\mu) = \alpha_1^{\pm} - \alpha_3^{\pm}$. From (18), one has :

$$h^{\pm}(\mu) = \frac{3\dot{\theta}_{c1}^{\pm}}{m}\left(3\mu - 2\left(\mu+1\right)\cos\theta_{c1}\left(\cos\theta_{c1} + \mu\sin\theta_{c1}\right)\right)$$

Introducing the expression of θ_{c1} given in (7), one finally obtains :

- if $\frac{4}{3} \leq \mu < \mu_C$, then $\operatorname{sgn}\left(h^{\pm}(\mu)\right) = \pm 1$,
- if $\mu > \mu_C$, then $\operatorname{sgn}\left(h^{\pm}(\mu)\right) = \mp 1$.
- $h^{\pm}(\mu_C) = 0$

where

$$\mu_C = \frac{8}{3\sqrt{3}} \tag{34}$$

Passing to the limit $s \to \pm\infty$ in (32) and (33) allows us to state the following result :

Lemma 9

If $\exists\, t_{c1} \mid X(t_{c1}) = P_{c1}^{\pm}$, then $\exists\, t_{\mathcal{V}} \mid \forall t \in [t_{\mathcal{V}}, t_{c1})$, the nonlinear system (15) is equivalent to the linearized system (16) and one has :

- *if* $\frac{4}{3} < \mu < \mu_C$, *then* $\lim_{t \to t_{c1}^-} \ddot{\theta}^{\pm}(t) = \dfrac{\alpha_1^{\pm}\alpha_2^{\pm}}{\alpha_4^{\pm}\left(\alpha_1^{\pm} - \alpha_3^{\pm}\right)} < 0$,
- *if* $\mu > \mu_C$, *then*
  - *if* $\dfrac{\dot{\theta}_{c1}^{\pm} - \dot{\theta}(t_{\mathcal{V}})}{\theta_{c1} - \theta(t_{\mathcal{V}})} > \dfrac{\alpha_2^{\pm}}{\alpha_1^{\pm} - \alpha_3^{\pm}}$, *then* $\lim_{t \to t_{c1}^-} \ddot{\theta}^{\pm}(t) = \pm\infty$,
  - *if* $\dfrac{\dot{\theta}_{c1}^{\pm} - \dot{\theta}(t_{\mathcal{V}})}{\theta_{c1} - \theta(t_{\mathcal{V}})} < \dfrac{\alpha_2^{\pm}}{\alpha_1^{\pm} - \alpha_3^{\pm}}$, *then* $\lim_{t \to t_{c1}^-} \ddot{\theta}^{\pm}(t) = \mp\infty$,
  - *if* $\dfrac{\dot{\theta}_{c1}^{\pm} - \dot{\theta}(t_{\mathcal{V}})}{\theta_{c1} - \theta(t_{\mathcal{V}})} = \dfrac{\alpha_2^{\pm}}{\alpha_1^{\pm} - \alpha_3^{\pm}}$, *then* $\lim_{t \to t_{c1}^-} \ddot{\theta}^{\pm}(t) = \dfrac{\alpha_1^{\pm}\alpha_2^{\pm}}{\alpha_4^{\pm}\left(\alpha_1^{\pm} - \alpha_3^{\pm}\right)} > 0$.
- *if* $\mu = \mu_C$, *then* $\lim_{t \to t_{c1}^-} \ddot{\theta}^{\pm}(t) = \mp\infty$.

Figure 7 shows that all orbits have the same behaviour when $\mu < \mu_C$ ($\mu = 1.4$ in the case of the figure). Figure 8 shows that various behaviours for $\mu = 1.6 > \mu_C$.

Remark

We have proposed an alternative proof for the case $\frac{4}{3} \leq \mu < \mu_C$ in [Génot et al., 1997], which is recalled in appendix A for convenience.
The preceding developments show that the value of μ with respect to μ_C given in (34) plays a crucial role in the existence and the finiteness of $\lim_{t \to t_{c1}^-} \ddot{\theta}^{\pm}(t)$.

Definition 4

We will call the critical coefficient of the LCP (4) the coefficient μ_C given in (34).

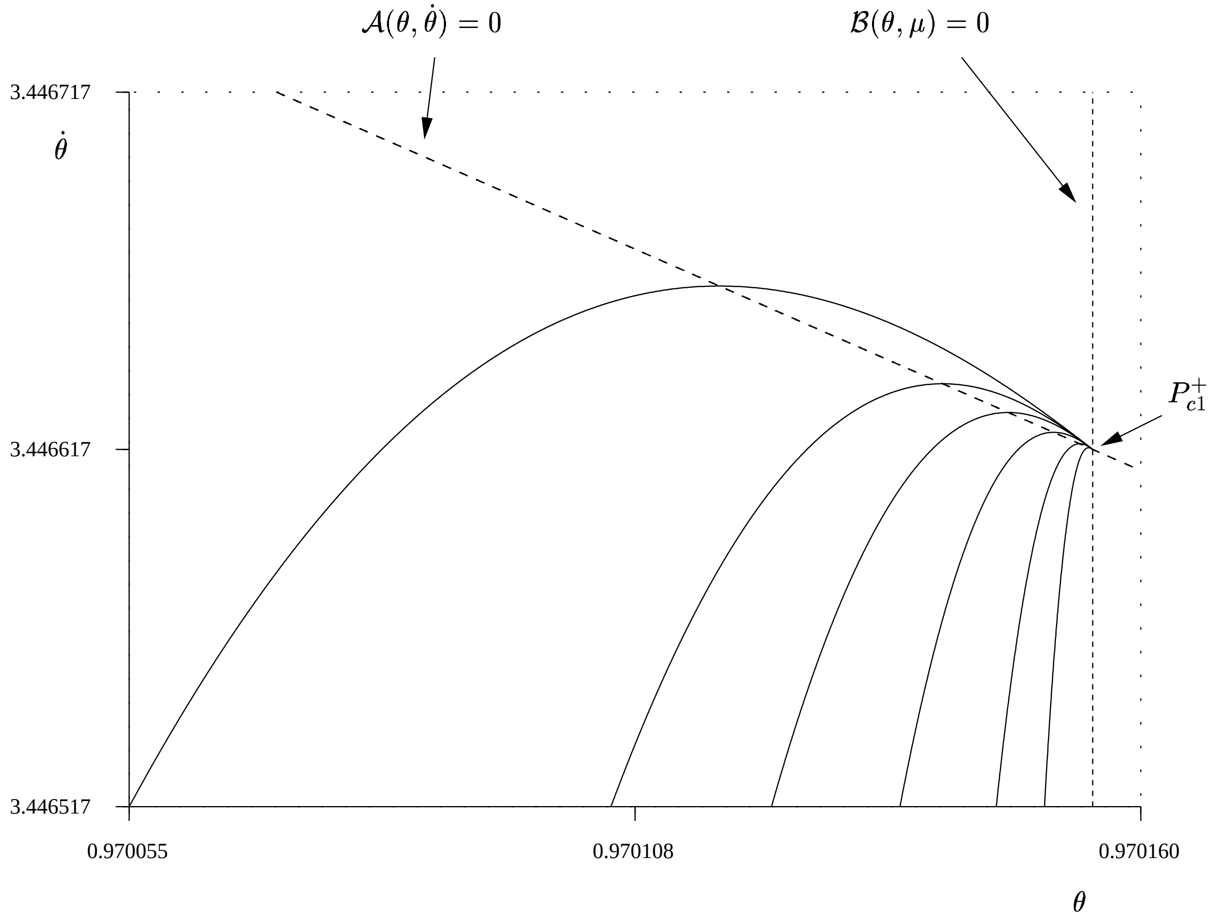

Figure 7: Behaviour of the orbits around P_{c1}^+ for $\mu = 1.4 < \mu_C$ ($m = 1$ kg, $g = 9.8$ m/s^2,$l = 1$m)

4.2.6 Extension to the case $\mu = \frac{4}{3}$

It is not difficult to see that the orbits of the nonlinear system (14) are continuous fonctions of μ (applying standard results about ordinary differential equations). Consequently, $\ddot\theta(s)$ possesses the same property. Passing to the limit $\mu \to \frac{4}{3}^+$ in (32), and noticing from (18) that $\frac{\alpha_1^\pm}{\alpha_4^\pm} = \dot\theta_{c1}^\pm$, one finally deduces the following :

Lemma 10

In the case where $\mu = \frac{4}{3}$, *if* $\exists\, t_{c1} \mid X(t_{c1}) = P_{c1}^\pm$, *then*

$$\lim_{t\to t_{c1}^-} \ddot\theta(t) = -\frac{\dot\theta_{c1}^\pm \alpha_2^\pm}{\alpha_3^\pm} < 0$$

4.2.7 Conclusions

From lemmas 9 and 10, it follows that in all cases, $\ddot\theta^\pm(t)$ always admits a finite or infinite limit when $t \to t_{c1}^-$.

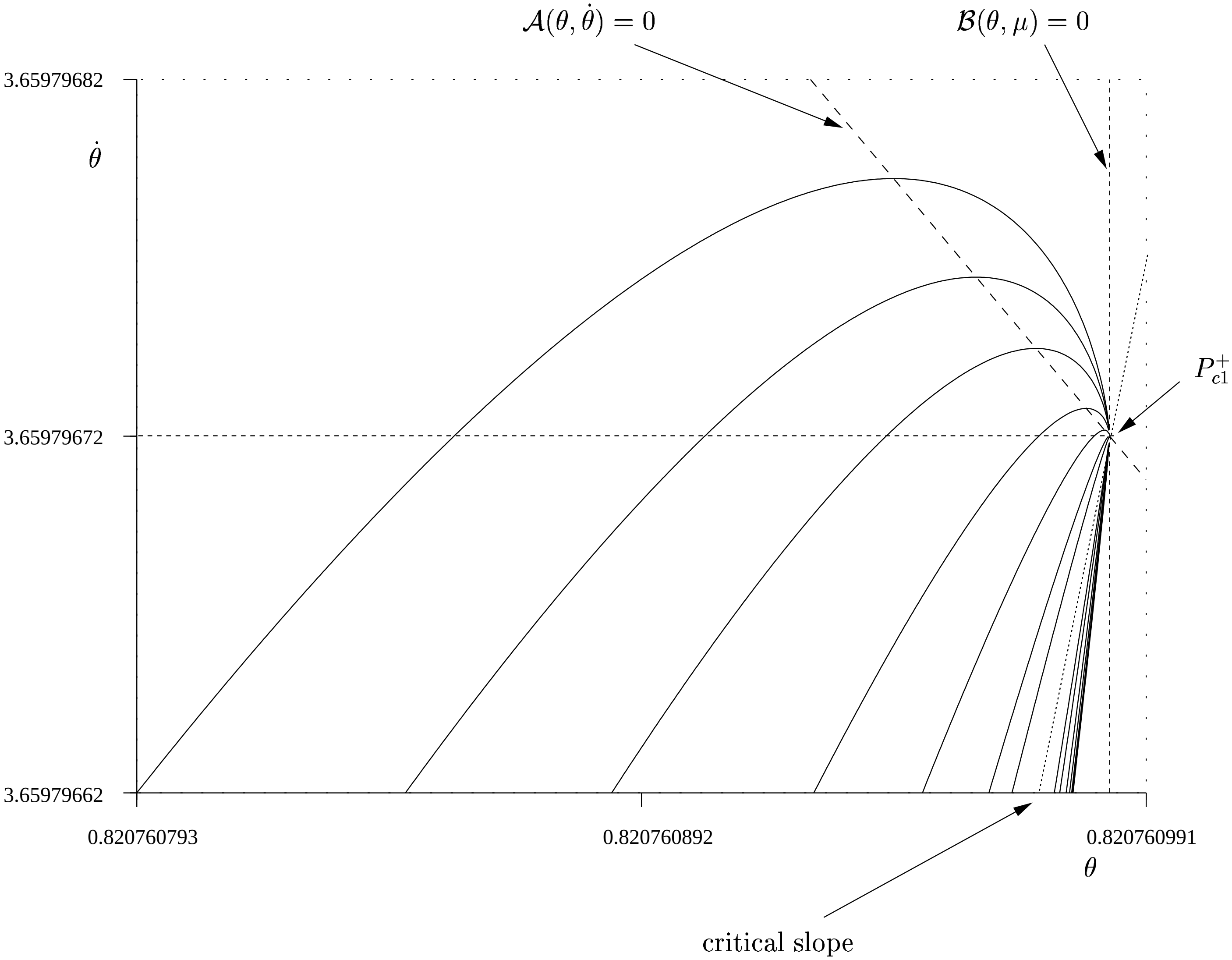

Figure 8: The various behaviour of the orbits around P^+_{c1} for $\mu = 1.6 > \mu_C$ ($m = 1$ kg, $g = 9.8$ m/s^2,$l =$ 1m)

Lemma 11

If from one of the lemmas 9 and 10, one should have $\lim_{t\to t^-_{c1}} \ddot\theta^{\pm}(t) = K \in [-\infty, 0)$, *then the point* $P^{\pm}_{c1}$ *is not reached and the system crosses the curve* $A(\theta,\dot\theta) = 0$ *before* $t = t_{c1}$.

Proof

If $X(t_{c1}) = P^+_{c1}$, then $X(t^-_{c1}) \in \mathcal{M}^+_1$ and if $X(t_{c1}) = P^-_{c1}$, then $X(t^-_{c1}) \in \mathcal{M}^-_4$. Therefore $\exists\, t_1 < t_{c1}$ such that :

- if $X(t_{c1}) = P^+_{c1}$, then $X(t_1) \in \mathcal{M}^+_1$ and $\theta(t_1) > \arctan\dfrac{1}{\mu}$,
- if $X(t_{c1}) = P^-_{c1}$, then $X(t_1) \in \mathcal{M}^-_4$.

One has $\ddot\theta^{\pm}(t_1) > 0$. However, from the hypothesis, $\lim_{t\to t^-_{c1}} \ddot\theta^{\pm}(t) < 0$. Consequently $\exists\, t_{\mathcal{A}},\ t_1 < t_{\mathcal{A}} < t_{c1} \mid \ddot\theta^{\pm}(t_{\mathcal{A}}) = 0$, corresponding to the searched time instant. $\diamond$

Remark

Lemma 4 allows us to conclude that if $X(t_1) \in \mathcal{M}_1^+$, then the rod detaches from the ground at $t_\mathcal{A}^+$ and if $X(t_1) \in \mathcal{M}_4^-$, then a IW/OC is necessary to prevent the system from entering the inconsistent mode $\mathcal{M}_2^-$. Some developments about IW/OC are given in appendix B.

Lemma 12

If from lemma 9, one should have $\lim_{t \to t_{c1}^-} \ddot{\theta}^{\pm}(t) = +\infty$, then, setting

$$p_N(\tau) = \int_{t_\mathcal{V}}^{\tau} F_N(t)dt$$

one has :

$$P_N = \lim_{\tau \to t_{c1}} p_N(\tau) < +\infty$$

and, setting $\dot{x}_A^{stick}(t_\mathcal{V}) = l\left(\dot{\theta}(t_\mathcal{V})\sin\theta(t_\mathcal{V}) - \dot{\theta}_{c1}^{\pm}\sin\theta_{c1}\right) - \frac{\mu}{m}P_N$, it follows that :

- *if $\dot{x}_A(t_\mathcal{V}) < \dot{x}_A^{stick}(t_\mathcal{V})$, then the rod keeps sliding, $P_{c1}^{\pm}$ is reached,*
- *if $\dot{x}_A(t_\mathcal{V}) > (=)\dot{x}_A^{stick}(t_\mathcal{V})$, then A sticks at $t^* < (=)t_{c1}$ and*
  - *if $X(t_1) \in \mathcal{M}_1^+$, then $X(t^{*+}) \in M_{IV}$,*
  - *if $X(t_1) \in \mathcal{M}_4^-$, then the rod detaches from the ground at the same instant : $X(t^*) \in M_I$.*

Proof

It is straightforward that :

$$p_N(\tau) = \int_{t_\mathcal{V}}^{\tau} F_N(t)dt = \int_0^{t^{-1}(\tau)} \mathcal{A}\left(\theta(s),\dot{\theta}(s)\right)ds = \int_0^{t^{-1}(\tau)} \left(g - l\dot{\theta}^2(s)\sin\theta(s)\right)ds$$

From the expressions of $\theta(s) = x_1^{\pm}(s)$ and $\dot{\theta}(s) = x_2^{\pm}(s)$ in (26) and (27) respectively, it follows that $P_N = \lim_{\tau \to t_{c1}^-} p_N(\tau) = \int_{t_\mathcal{V}}^{t_{c1}} F_N(t)dt$ is a convergent integral, i.e. $P_N < +\infty$.
Moreover notice that from (3) :

$$\ddot{x}_A = \ddot{x} + l\left(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta\right) = \frac{\mu}{m}F_N + l\left(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta\right) > 0$$

thus $\dot{x}_A$ is an increasing time function. Furthermore :

$$\dot{x}_A(\tau) = \dot{x}(\tau) + l\dot{\theta}(\tau)\sin\theta(\tau) = \dot{x}(t_\mathcal{V}) + \frac{\mu}{m}\int_{t_\mathcal{V}}^{\tau} F_N(t)dt + l\dot{\theta}(\tau)\sin\theta(\tau)$$

that is :

$$\dot{x}_A(\tau) = \dot{x}_A(t_\mathcal{V}) + \frac{\mu}{m}p_N(\tau) + l\left(\dot{\theta}(\tau)\sin\theta(\tau) - \dot{\theta}(t_\mathcal{V})\sin\theta(t_\mathcal{V})\right)$$

One deduces that, if $\dot{x}_A(t_\mathcal{V}) < \dot{x}_A^{stick}(t_\mathcal{V})$, then $\forall\tau \in [t_\mathcal{V}, t_{c1}], \dot{x}_A(\tau) < 0$, the system remains in mode M_{II} and the singular point $P_{c1}^{\pm}$ is attained. In the case where $\dot{x}_A(t_\mathcal{V}) > \dot{x}_A^{stick}(t_\mathcal{V})$, the proof of detachment from the ground if $X(t_1) \in \mathcal{M}_4^-$ is similar to that of lemma 3. However $F_N(t^*) = 0$ is not a solution if $X(t_1) \in \mathcal{M}_1^+$, since otherwise $\ddot{y}_A(t^*) = -A(t^*) < 0$. ◇

Remark

An energy based reasoning similar to that of proof of lemma 9 would have enabled us to conclude on sticking of A before the orbit attains the singular point P_{c1}. An important consequence of such a result would have been that the contact force F_N remains always bounded. However as proved in the previous lemma, there exist initial conditions such that P_{c1} is reached with infinitely large contact force. Nevertheless as long as the impulse P_N remains bounded, the system state is bounded as well.

Let us summarize the main ideas in the following :

Proposition 3

Assume that $\exists\, t_1 \mid X(t_1) \in \mathcal{M}_1^+ \bigcup \mathcal{M}_4^-$, *with* $\arctan \frac{1}{\mu} < \theta(t_1) < \theta_{c1}$. *Then*

- *if* $\frac{4}{3} \leq \mu < \mu_C$, *then detachment or sticking occurs and the contact force* $F_N(t) = \frac{\mathcal{A}(t)}{\mathcal{B}(t)}$ *remains always bounded,*
- *if* $\mu \geq \mu_C$, *then*
  - *either the orbit passes above the critical line passing through* $P_{c1}^{\pm}$ *and with slope* $\frac{\dot{\theta}_{c1}^{\pm} - \dot{\theta}(t_{\mathcal{V}})}{\theta_{c1} - \theta(t_{\mathcal{V}})} = \frac{\alpha_2^{\pm}}{\alpha_1^{\pm} - \alpha_3^{\pm}}$ *and the rod stops sliding or detaches,*
  - *or the orbit passes below the same critical line in the neighbourhood of* $P_{c1}^{\pm}$ *and*
    - *if* $\dot{x}_A(t_{\mathcal{V}}) < \dot{x}_A^{stick}(t_{\mathcal{V}})$, *then the rod keeps sliding and the orbit passes through* $P_{c1}^{\pm}$. *The contact force magnitude attains* $+\infty$ *but the impulse* P_N *remains bounded,*
    - *if* $\dot{x}_A(t_{\mathcal{V}}) \geq \dot{x}_A^{stick}(t_{\mathcal{V}})$, *then* A *sticks before* $P_{c1}^{\pm}$ *is reached. The contact force remains bounded,*
  - *or the orbit lies on the critical line and reaches* $P_{c1}^{\pm}$ *with*

$$\lim_{t \to t_{c1}^-} F_N(t) = \frac{\alpha_1^{\pm}\alpha_2^{\pm}}{\alpha_4^{\pm}\mathcal{C}(\theta_{c1},\mu)\left(\alpha_1^{\pm} - \alpha_3^{\pm}\right)} < +\infty$$

Proof

The proof is evident from lemmas 9, 10, 11 and 12.

5 Conclusions

The above analysis departs from previous studies on the Painlevé's example [Lötstedt, 1981, Mason et al., 1988, Pfeiffer, 1996] in the sense that it is not limited to only stating the existence of an inconsistent mode (mode $\mathcal{M}_2$ of the LCP) for all friction coefficient μ larger than $\frac{4}{3}$([1]). A detailed study of the dynamical behaviour when the system is initialized in a consistent mode $\mathcal{M}_1$ or $\mathcal{M}_3$ has been done (besides is it possible in pratice to initialize the system in the mode $\mathcal{M}_2$?).

[1] This value is not realistic in most applications. However as shown in [Moreau, 1988, Génot, 1998] a slight modification of the contact geometry allows one to recover more realistic critical frictional coefficient

5.1 Model consistency

It has been proved that if the system is initialized in the consistent modes $\mathcal{M}_1^+$ of the LCP, then the contact force can diverge to infinite values for a suitable choice of the initial conditions. The impulse however remains bounded. Consequently the system state does not diverge. This conclusion is essential concerning the well-posedness of this hybrid system. Indeed let us consider the generalized contact force as an input u and $\dot{q}$ as an output y. Then $u \in \mathcal{L}_p$, $p < +\infty$. The dissipativity property of a system relies strongly on its *supply rate* $< u, y >$ being locally integrable. This is the case for Painlevé's system, where the supply rate represents the power dissipated by dry friction. Although the theory of dissipative dynamical systems [Willems, 1972] is developed only for smooth systems, it is expected that it can be extended to such nonsmooth systems.
This analysis brought us to exhibit a new critical value of the friction coefficient

$$\mu_C = \frac{8}{3\sqrt{3}} > \frac{4}{3}$$

The value of the friction coefficient with respect to μ_C determines the transition from the sliding mode M_{II} in the vicinity of P_{c1}^+. If the system enters mode $\mathcal{M}_1^+$ at time t_1 with $\arctan \frac{1}{\mu} < \theta(t_1) < \theta_{c1}$. More precisely :

- if $\frac{4}{3} \leq \mu < \mu_C$, and if $\dot{x}_A(t_1) \ll 0$, the transition is done toward mode M_I (the rod detaches). The singular point P_{c1}^+ is not reached and the contact force attains 0 remaining always bounded,
- if $\mu \geq \mu_C$ and the system enters at $t_\mathcal{V}$ a vicinity of the singular point P_{c1}^+, then :
  - if the orbit is below a critical line (lemma 9), then depending on the sliding velocity at $t_\mathcal{V}$, either the system evolves toward mode M_{IV}, or the system enters mode M_{II}: then the orbit passes through P_{c1}^+ and the contact force reaches infinite values, the impulse remaining bounded,
  - if the orbit is above the critical line, then P_{c1}^- is not reached. The contact force remains bounded and depending again on the sliding velocity at $t_\mathcal{V}$, the transition is done towards mode M_I or mode M_{IV},
  - if the orbit lies on the critical line, then the singular point P_{c1}^+ is reached with a finite contact force.

5.2 Indeterminacy

A detailed study of the undetermined LCP mode $\mathcal{M}_4$ has been given. In this mode, two solutions are admissible : $F_N = 0$ and $F_N = \frac{\mathcal{A}}{\mathcal{B}}$. If one chooses $F_N(t_1) = 0$ as the solution, then the rod detaches instantaneously at time t_1 (proposition 1). In mode $\mathcal{M}_4^+$, if one chooses $F_N(t) = \frac{\mathcal{A}(t)}{\mathcal{B}(t)}$ as the solution for $t \geq t_1$, then the rod finally sticks and detaches at the same time instant (lemma 3). Therefore the transition is always made toward the hybrid system mode M_I. Nevertheless one should notice that the orbits corresponding to both cases are different. $F_N(t_1) = 0$ implies instantaneous detachment of the rod, whereas $F_N(t_1) = \frac{\mathcal{A}}{\mathcal{B}} > 0$ implies $\ddot{y}_A(t_1) = 0$. This has important consequencies for the numerical simulation. In order to solve this indeterminacy, Painlevé [Painlevé, 1905] proposes the following principle :

> Two rigid bodies which under given conditions would not produce any pressure on one another, if they were ideally smooth, would likewise not act on one another if they were rough.

In other words, if $X(t_1) \in \mathcal{M}_4$ (recall that $\mu \geq \frac{4}{3}$), one should choose $F_N(t_1) = 0$. Indeed, if one had $\mu = 0$ (frictionless case), the indetermined mode $\mathcal{M}_4$ (as well as the inconsistent mode $\mathcal{M}_2$) would not exist and one would have $X(t_1) \in \mathcal{M}_3$, with unique solution $F_N(t_1) = 0$. However one should notice that this rule is only an a priori principle. In particular Ivanov [Ivanov, 1986] argues that on one hand it has not been verified experimentally and on the other hand it does not assure continuity of the solution with respect to the initial data [Lötstedt, 1981].

5.3 Impact WithOut Collision

In mode $\mathcal{M}_4^-$, the solution $F_N = \dfrac{\mathcal{A}}{\mathcal{B}}$ brings the orbits toward the curve $\mathcal{A}(\theta, \dot{\theta}) = 0$, i.e. on the boundary with the inconsistent mode $\mathcal{M}_2^-$ and an IW/OC is required for the system not to enter $\mathcal{M}_2^-$. This raises several questions :

- How to calculate the associated velocity jump ?
- Is the jump unique ?
- What is the physical origin of this phenomenon ?

Some authors [Painlevé, 1905, Bolotov, 1906, Baraff, 1993] systematically associate sticking as the solution to inconsistency (or very large contact forces), as discussed in [Brogliato, 1996]. As illustrated in appendix B, the percussion leading to $\dot{x}_A(t_k^+) = 0$ and $\dot{y}_A(t_k^+) = 0$ belongs to the friction cone only if $X(t_k^-) \in \mathcal{M}_2 \bigcup \mathcal{M}_4$. As pointed out in the preceding paragraph, the IW/OCs are not a priori related to singularity of (9). They may only be necessary when the system enters an inconsitent mode.

5.4 Simulation aspects

As said above the contact force can become arbitrarily large close to the singular points. Then after a very short period :

- either the force F_N tends to zero and the bar detaches,
- or point A sticks without detaching, even if the system has been initialized with $\dot{x}_A(t_1) \ll 0$,
- or the force F_N tends toward a finite limit (this is however an isolated case, cf. proposition 3),
- or the force F_N reaches infinite values and the singular point is attained.

This creates problems for the choice of the integration step. Interestingly enough Moreau [Moreau, 1988] faces the same problem when the bar has a round edge. He exhibits a case where the tangential velocity of the contact point decreases rapidly. He calls such phenomenon a "frictional catastrophe" in the honor of Lecornu [Lecornu, 1905]. At the same instant the bar leaves the ground. His numerical algorithm provides various post-catastrophe velocities as a function of the integration step. However the case $\dot{x}_A(t_k^+) = 0$ seems to have the highest probability to occur.

5.5 Future works

In this work we have studied a system with single contact and gravity. Future work should concern the addition of conservative forces and the extension to more degrees of freedom and contact points. From the mathematical point of view it is expected that the above analysis in the neighbourhood of the singular point may serve to study singular vector fields whose numerator and denominator may simultaneously tend to zero.

A Another analysis of the case $\frac{4}{3} \leq \mu < \mu_C$

Hypothesis 5

Let us assume that $P_{c1} = P_{c1}^+$, i.e. the singular point is reached from $\mathcal{M}_1^+$.

Lemma 13

If $\ddot{\theta}$ admits a finite limit when $\theta \to \theta_{c1}^-$, then $\dot{\theta}$ also a finite limit at this point and :

$$\dot{\theta}_{c1}^- = \sqrt{\frac{g}{l \sin \theta_{c1}}} \tag{35}$$

$$\ddot{\theta}_{c1}^- = \frac{\dot{\theta}_{c1}^{-\,2} \cos \theta_{c1}(-\cos \theta_{c1} + \mu \sin \theta_{c1})}{4 \cos \theta_{c1}(\mu \cos \theta_{c1} + \sin \theta_{c1}) - 3\mu} \tag{36}$$

Proof

It is evident that the function $-\cos \theta + \mu \sin \theta$ is strictly positive for $\arctan \frac{1}{\mu} < \theta < \theta_{c1}$. Therefore if $\ddot{\theta}$ has a finite limit when $\theta \to \theta_{c1}^-$, then in view of equation (8), since $\mathcal{B} \to 0$, $\dot{\theta}_{c1}^-$ must satisfy equation (35). Developing in Taylor series with Young's rest both side of equation (8) and collecting the terms in $\ddot{\theta}_{c1}^-$ leads after trigonometric simplifications to equation (36). $\diamond$

Let us now focus on the expression of $\ddot{\theta}_{c1}^-$ given in equation (36). From equation (7) it can be shown that the numerator of $\ddot{\theta}_{c1}^-$ is positive. Reporting the expressions of θ_{c1}^- and $\dot{\theta}_{c1}^-$ from equations (7) and (35), tedious but straightforward computations show that the denominator is negative for $\mu < \mu_c = \frac{8}{3\sqrt{3}}$, and positive for $\mu > \mu_c$.

The main difficulty is now to prove that $\ddot{\theta}$ really admits a finite limit.

Lemma 14

When $\frac{4}{3} \leq \mu < \mu_C$, the trajectory $(\theta, \dot{\theta})$ crosses the curve $\mathcal{A}(\theta, \dot{\theta}) = 0$ before $\theta = \theta_{c1}$.

Proof

The proof consists in a *reductio ad absurdum.* The outline is as follows : if $\mathcal{A}$ remains strictly positive, we show that necessarily $\ddot{\theta}$ has a finite positive limit. But from equation (36), we know that this limit must be negative, hence a contradiction which shows that $\mathcal{A}$ must cross zero before $\theta = \theta_{c1}$.
Let us assume that $\forall t \in [0, t_{c1}], \mathcal{A} > 0$. In this case, $\ddot{\theta}$ is strictly positive. To simplify the notations, let us rewrite equation (8) as

$$\ddot{\theta} = f(\theta, \mu)(g - l\dot{\theta}^2 \sin \theta).$$

Differentiating once the previous equation, we obtain that

$$\theta^{(3)} = \dot{\theta}(H(\theta, \mu)\mathcal{A}(\theta, \dot{\theta}) - lf(\theta, \mu)\dot{\theta}^2 \cos \theta)$$

where

$$H(\theta, \mu) = \frac{\partial f}{\partial \theta} - 2lf^2 \sin \theta$$

Basic computations show that $H(\theta,\mu)$ always becomes strictly negative before θ reaches θ_c. So, since $f(\theta,\mu)$ is obviously positive, there exists a left vicinity of t_c for which $\theta^{(3)}(t)$ is strictly negative. Let us now come back to $\ddot{\theta}(t)$. It is a positive decreasing function in the above vicinity. So, $\ddot{\theta}$ admits the finite limit given in equation (36). But as we show in the proof on the precedent lemma, this limit is strictly negative. Therefore, since $\ddot{\theta}$ was strictly positive after θ passed $\arctan\frac{1}{\mu}$, there exists an instant $t_{\mathcal{A}} \in [0, t_{c1})$ where $\ddot{\theta}(t)$ crosses zero. Thus, at this instant, $\mathcal{A}(t_{\mathcal{A}}) = 0$, in contradiction with the hypothese. ◇

B Impact WithOut Collision

Hypothesis 6

Let us assume that $X(t_k^-) \in \mathcal{M}_2$.

In this case the LCP has no solution and an IW/OC is required to prevent penetration into the inconsistent region. Let us recall the following basic rules on tangential impacts :

Definition 5

If an IW/OC occurs at time t_k*, then the following conditions have to be fulfilled :*

1. $T(t_k^+) \leq T(t_k^-)$,
2. $\dot{y}_A(t_k^+) = 0$,
3. $P \in \mathcal{C}$, *where* $P \in \mathbf{R}^2$ *is the percussion vector and* $\mathcal{C}$ *the friction cone.*
4. *The LCP admits a solution at* t_k^+.

Conditions 1 et 2 impose that the post-impact velocities $\dot{x}_A(t_k^+)$ and $\dot{\theta}(t_k^+)$ lie in a compact subspace of $\mathbf{R}^2$. The impact dynamics for Painlevé's example are :

$$\begin{aligned} m\Delta\dot{x} &= \Lambda_t && (37)\\ m\Delta\dot{y} &= \Lambda_n && (38)\\ \frac{ml^2}{3}\Delta\dot{\theta} &= l\left(-\cos\theta\Lambda_n + \sin\theta\Lambda_t\right) && (39) \end{aligned}$$

From 2 and $\dot{y}_A(t_k^-) = 0$, it follows that :

$$\Lambda_n = ml\cos\theta\Delta\dot{\theta}, \ \ ,\Lambda_t = \frac{ml}{3\sin\theta}\left(1+3\cos^2\theta\right)\Delta\dot{\theta} \tag{40}$$

It is easy to see that $\Delta\dot{\theta} = 0$ is not possible because of condition 4. Now condition 3 says that :

$$\Lambda_n \geq 0, \ \mid \Lambda_t \mid \leq \Lambda_n \tag{41}$$

Thus $\Delta\dot{\theta} > 0,\ \Lambda_n > 0,\ \Lambda_t > 0$. One deduces from (41) that $1 + 3\cos\theta\left(\cos\theta - \mu\sin\theta\right) = \mathcal{B}(\theta,\mu) \leq 0$ which is satisfied from assumption 6. Moreover notice that no IW/OC is possible in mode $\mathcal{M}_1$. Setting $\mu(\theta) = \dfrac{\Lambda_t}{\Lambda_n} = \dfrac{1+3\cos^2\theta}{3\sin\theta\cos\theta} > 0$, one finds :

$$\Delta T = \Lambda_n\left(a\Lambda_n + b\right) \tag{42}$$

where :

$$a = \frac{2\,(1+3\cos^2\theta)}{9m\sin^2\theta\cos^2\theta} > 0,\;\; b = \mu(\theta)\dot{x}_A(t_k^-) < 0$$

Finally condition 1 means that :

$$0 \le \Lambda_n \le -\frac{b}{a} \tag{43}$$

Straightforward computations lead to :

$$\dot{x}_A(t_k^+) = \dot{x}_A(t_k^-) + \frac{4\Lambda_n}{3m\sin\theta\cos\theta} \tag{44}$$

Conditions grouping (41) and (43) are :

$$0 < \Lambda_n \le -\frac{b}{a} = -\frac{3}{2}m\sin\theta\cos\theta\dot{x}_A(t_k^-) \tag{45}$$

which in turn imply :

$$\dot{x}_A(t_k^-) < \dot{x}_A(t_k^+) \le -\dot{x}_A(t_k^-) \tag{46}$$

Now we investigate condition 4 :

- Case $\dot{x}_A(t_k^+) < 0$.
  From (44) :

  $$\Lambda_n < -\frac{3}{4}m\sin\theta\cos\theta\dot{x}_A(t_k^-) \tag{47}$$

  One sees that if $\dot{x}_A(t_k^+) < 0$, then $\dot{\theta}^2(t_k^+) \ge \frac{g}{l\sin\theta}$ is need to escape from the inconsistent zone $\mathcal{M}_2$ towards $\mathcal{M}_4$. But, from (40) :

  $$\dot{\theta}(t_k^+) = \dot{\theta}(t_k^-) + \Delta\dot{\theta} = \dot{\theta}(t_k^-) + \frac{\Lambda_n}{ml\cos\theta} > \dot{\theta}(t_k^-)$$

  hence $\Lambda_n \ge \left(\sqrt{\frac{g}{l\sin\theta}} - \dot{\theta}(t_k^-)\right) ml\cos\theta$. There may be cases where this last condition is not compatible with (47).

- Case $\dot{x}_A(t_k^+) > 0$.

  The system enters mode M_{III}. Thus $\mathcal{B}(\theta,\mu) = \frac{1}{m}\,(1+3\cos\theta\,(\cos\theta+\mu\sin\theta)) > 0$ and the LCP is well-posed.

- Case $\dot{x}_A(t_k^+) = 0$.
  The case of a static contact is known to be always well-posed. From (44) :

  $$\Lambda_n = -\frac{3}{4}m\sin\theta\cos\theta\dot{x}_A(t_k^-)$$

  which corresponds to the solution proposed in [Painlevé, 1905, Bolotov, 1906, Baraff, 1993]. This solution has the maximal dissipation property, as deduced from (42).

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