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The Optimal Time-Continuous Mass Transport Problem and its Augmented Lagrangian Numerical Resolution

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Abstract: This paper presents the mass transport problem in its time-continuous formulation and introduces an augmented Lagrangian numerical technique for its resolution.

(Résumé : tsvp)

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Le problème de transport the masse à coût optimal et sa résolution numérique par Lagrangien Augmenté

Résumé : Ce rapport présente le problème de transport de masse dans sa formulation en temps continu et introduit une méthode numérique de Lagrangien augmenté pour le résoudre.

his paper presents the mass transport problem in its time-continuous formulation and introduces an augmented Lagrangian numerical technique for its resolution.

1 Introduction

The mass transport problem was first considered by Monge in 1780 in his "mémoire sur les remblais et les déblais". A modern mathematical treatment of this problem has been initiated by Kantorovich in 1942. A recent comprehensive review of the mass transport problem can be found in Mc Cann and Gangbo [9].

The modern formulation of the problem is the following :

Two bounded, positive measurable functions ρ_0 and ρ_T with compact support in $\mathbb{R}^d$, called "densities", are given. We further require that they have same mass, normalized to 1 :

$$\int_{\mathbb{R}^d} \rho_0(x)dx = \int_{\mathbb{R}^d} \rho_T(x)dx = 1. \tag{1}$$

The problem is now to find an application M from $\mathbb{R}^d$ to $\mathbb{R}^d$ which realizes the transport from ρ_0 to ρ_T in the following sense : For all borel set A, M satisfies

$$\int_{M^{-1}(A)} \rho_0(x)dx = \int_A \rho_T(x)dx = 1, \tag{2}$$

and achieves the minimal cost

$$\int_{\mathbb{R}^d} c(x, M(x))\rho_0(x)dx. \tag{3}$$

The cost function $c(.,.)$ is fixed and can be taken for example as :

$$c(x,y) = |x-y|^r \tag{4}$$

for a given $r > 0$. Notice that condition (2) is equivalent to

$$\int f(M(x))\rho_0(x)dx = \int f(x)\rho_T(x)dx \tag{5}$$

for all continuous function f, and does not *a priori* require M to be one-to-one.

In this paper we concentrate on the case $r = 2$ (even though Monge treated $r = 1$) which exhibits remarquable properties (see [3], [7] and references in [9]).

The results presented in these papers, for the cost function $c(x,y) = |x-y|^2$ can be synthetized in the following theorem, where we use notation $\nabla = (\partial_1, ..., \partial_d)$ for the gradient operator :

Theorem 1.1 *There is a unique optimal application M defined on the support of ρ_0 satisfying (2). The application M is characterized as the unique application of this class which can be written as the gradient of a convex potential Φ :*

$$M(x) = \nabla\Phi(x). \tag{6}$$

If moreover ρ_0 and ρ_T are strictly positive and Hölder continuous on their supports, which we further assume to be strictly convex, then the potential Φ has Hölder continuous derivatives up to the second order and satisfies in the classical sense the Monge-Ampère equation :

$$det(\partial^2\Phi(x))\rho_T(\nabla\Phi(x)) = \rho_0(x), \tag{7}$$

where det denotes determinants for $d \times d$ matrices.

It is also possible to show that the optimal value of the cost is nothing but the square of the Wasserstein distance (also called Tanaka distance) between densities ρ_0 and ρ_T, usually defined by :

$$d_{Wa}(\rho_0, \rho_T)^2 = \inf \int |x-y|^2 d\mu(x,y), \tag{8}$$

where μ spans the space of probability measures $\mathbb{R}^d \times \mathbb{R}^d$ with marginals ρ_0 and ρ_T. Kantorovitch distance is defined likewise but with $r = 1$ instead of 2.

These mathematical objects can be used in a wide range of applications. Data assimilation of Lagrangian tracers in weather forecasting and interpolation in image processing are probably the most obvious possible applications. At a more theoretical level, let us mention the most recent works known to us, in the fields of functional analysis (Franck Barthe [16], on reverse Brascamp-Lieb inequalities) and non-linear pde's (Felix Otto [20], on Fokker-Planck and Hele-Shaw equations). References on less recent works can be found in [9].

In the present paper, we provide a reformulation of the mass transport problem using a continuous time variable $t \in [0,T]$, where the time interval $[0,T]$ is arbitrarily chosen and fixed. Then we introduce a numerical method of resolution based on the time-continuous formulation and using an augmented Lagrangian technique. The reformulation is based on the following observation :

Theorem 1.2 *The square of the Wasserstein distance is equal to :*

$$\inf_{(\rho,v)} T \int_{\mathbb{R}^d} \int_0^T \rho(t,x)|v(t,x)|^2 dxdt, \tag{9}$$

where the time horizon $T > 0$ is arbitrarily fixed and the infimum is performed over all pair $\rho(t,x) \geq 0$ $v(t,x) \in \mathbb{R}^d$ of density and velocity fields satisfying the conservation law

$$\partial_t\rho + \nabla.(\rho v) = 0 \tag{10}$$

for $0 < t < T$ and $x \in \mathbb{R}^d$, and subject to the initial and final conditions

$$\rho(0,.) = \rho_0, \ \ \rho(T,.) = \rho_T. \tag{11}$$

Moreover, the infimum is achieved by the unique pair (ρ, v) defined from Φ by :

$$\int f(t,x)\rho(t,x)dtdx = \int f(t, x + t\frac{\nabla\Phi(x) - x}{T})\rho_0(x)dtdx, \tag{12}$$

$$\int f(t,x)\rho(t,x)v(t,x)dtdx = \tag{13}$$
$$\int \frac{\nabla\Phi(x)-x}{T} f(t, x+t\frac{\nabla\Phi(x)-x}{T})\rho_0(x)dtdx,$$

for all continuous function f.

As a matter of fact, such a time-continuous formulation was implicitely contained in the original problem addressed by Monge : "le problème des remblais et des déblais", a Civil Ingeneering problem where parcels of materials have to be displaced with optimal cost. Eliminating the time variable was just a clever way of reducing the dimension of the problem. We give however in the following remarks sereval reasons to keep to the time-continuous formulation.

-Data interpolation

The time-dependent density $\rho(t,.)$ provides a natural interpolant of the data ρ_0 and ρ_T. The velocity field $v(t,x)$ which moves ρ_0 toward ρ_T is also a valuable additional information. Notice that formula (12) has been used, independently of the time-continuous framework, by Mc Cann [8] as a tool for his derivation of the Brünn-Minkovski inequality.

-Generalization to Riemannian manifold

The transport problem can be set on a compact Riemannian manifold. The cost function $c(x,y)$ is now the geodesic distance between points x and y of the manifold. (See [10] for an application to meteorological modelization.) In the time-continuous formulation, this extension is even simpler because (9), (10) can directly be transposed in the Riemannian framework, with trivial change of notations, and is well suited to numerical computations. As a matter of fact, our numerical calculations will be performed on the periodic box $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ (section 3).

-Relationship with Fluid Mechanics

The formal optimality condition for fields ρ and v (which can readily be deduced from (12) and (13)) is :

$$(\partial_t + v.\nabla)v = 0. \tag{14}$$

This equation models the evolution of a pressureless gas [14], a very crude model in Fluid Mechanics. We therefore obtain an interpretation of the transport problem which opens the way to more involved analogies with Fluid Mechanics.

-Variational formulation of Euler equations

The Euler equations for an ideal incompressible fluid

$$(\partial_t + v.\nabla)v = -\nabla p, \quad \nabla.v = 0, \tag{15}$$

in which $p(t, x)$ is the fluid pressure, obey a least action principle. This principle conveys fondamental geometrical properties used by Arnold in [1].

The Euler equations have been studied as a minimization problem by Shnirelman [5], [6] and Brenier [2], [18]. In this latter work, an optimal transport problem is used to characterize the limits of the minimizing sequences. In this generalized transport problem the density and velocity fields depend on an additional variable $a \in [0, 1]$. These quantities are now denoted $\rho_0(x, a)$, $\rho_T(x, a)$, for the data and $\rho(t, x, a)$, $v(t, x, a)$ for the unknowns. Constraints (11), (10) are enforced for each value of a. The problem is now to minimize

$$T \int_{\mathbb{R}^d} \int_0^T \int_0^1 \rho(t, x, a)|v(t, x, a)|^2 dxdtda. \tag{16}$$

The new feature of this problem lies in the additional constraint on the densites at each point (t, x) :

$$\int_0^1 \rho(t, x, a)da = 1. \tag{17}$$

This equation comes from the incompressibility constraint and pressure $p(t, x)$ is indeed the associated Lagrange multiplier.

When a is a discrete variable and da the counting measure, we recover the homegenized vortex sheet model discussed in [4].

The numerical method introduced in the present paper can in principle be directly generalized to such problems in which time cannot be eliminated.

-Interpolation of the L^2 and the Wasserstein distances

When two densities ρ_0 and ρ_T must be compared, it is very natural to use the Wasserstein distance. However, sometimes, the simpler L^2 distance :

$$d_{L^2}(\rho_0, \rho_T)^2 = \int |\rho_0(x) - \rho_T(x)|^2 dx \tag{18}$$

may be more appropriate. Both situations occur in the case of data assimilation for meteorological forecasting as pointed out by Cullen (see [11]). From a theoretical point of view, as explained in [19], there is also an interesting relationship between these two distances. (Fore example, the heat equation can be seen as the gradient flow of the Dirichlet integral with respect to the L^2 distance as well as the gradient flow of the entropy with respect to the Wasserstein distance.)

A weighted combination of these distances may therefore be desired for practical applications. A nice interpolation of the L^2 and Wasserstein distances between ρ_0 and ρ_T, for

$\theta \in [0,1]$, is naturally provided by the time continuous formulation, where (11), (10) are unchanged and the cost functional is replaced by :

$$\int_{\mathbb{R}^d} \int_0^1 [(1-\theta)\rho(t,x)|v(t,x)|^2 + \theta(\partial_t \rho(t,x))^2] dx dt \tag{19}$$

(here T is normalized to 1). Indeed, $\theta = 0$ and $\theta = 1$ respectively give back the Wasserstein and the L^2 distances. A remarkable feature of this problem is its formal optimality condition

$$(1-\theta)(\partial_t + v.\nabla)v + \theta \partial_{tt} \nabla \rho = 0, \tag{20}$$

which is nothing but the Boussinesq equation without gravity term ([17]).

2 Justification of the time-continuous formulation.

The proof of theorem 1.2 is straightforwardly obtained by using Lagrangian coordinates. Let us consider a density field ρ and a velocity field v satisfying (10), (11). We use Lagrangian coordinates and define $X(t,x)$ by :

$$X(0,x) = x, \quad \partial_t X(t,x) = v(t, X(t,x)), \tag{21}$$

so that, for all test functions f,

$$\int f(t,x)\rho(t,x) dx dt = \int f(t, X(t,x))\rho_0(x) dx dt, \tag{22}$$

$$\int f(t,x)\rho(t,x)v(t,x) dx dt = \int \partial_t X(t,x) f(t, X(t,x))\rho_0(x) dx dt. \tag{23}$$

Notice first that (11) and (22) imply that $M(x) = X(T,x)$ satisfies condition (5), just as the optimal map $\nabla\Phi(x)$ does. Next, we observe that

$$T \int_{\mathbb{R}^d} \int_0^T \rho(t,x)|v(t,x)|^2 dx dt = T \int_{\mathbb{R}^d} \int_0^T \rho_0(x)|v(t, X(t,x))|^2 dx dt$$

(by (22))

$$= T \int_{\mathbb{R}^d} \int_0^T \rho_0(x)|\partial_t X(t,x)|^2 dx dt$$

(by (21))

$$\geq \int_{\mathbb{R}^d} \rho_0(x)|X(T,x) - X(0,x)|^2 dx$$

(by Jensen's inequality)

$$= \int_{\mathbb{R}^d} \rho_0(x)|X(T,x) - x|^2 dx$$

(by (21) again)

$$= \int_{\mathbb{R}^d} \rho_0(x)|\nabla\Phi(x) - x|^2 dx$$

(because both $X(T,x)$ and $\nabla\Phi(x)$ satisfy condition (5), as already mentioned).

Thus, the optimal choice of (ρ, v) corresponds to

$$X(t,x) = x + \frac{t}{T}(\nabla\Phi(x) - x), \tag{24}$$

and, therefore, is given by (12), (13). This completes the proof of theorem 1.2.

3 The numerical method

Few papers have appeared on the numerical resolution of this class of problems, whether in the Monge-Ampère (theorem 1.1) or the Wasserstein minimization (theorem 1.2) formulation. In a series of papers on the numerical resolution of the semi-geostrophic equation [25] [23] [24] [22], the authors compute explicitely the solution Φ of equation (7). A similar method is used in [13] for the design of antennas. A domain decomposition method for problem (7) has been proposed in [15] and also used on a simplified Semi-geostrophic model in [26]. Finally a Lagarangian discretisation method for the computation of the Euler flow between initial and final prescribed density can be found in [12]. This last problem is closely related to problem (9).

In this section, we propose a new approach to solving the optimal mass transport problem based on an augmented Lagrangian technique. Such methods are commonly used in fluid mechanics and elasticity [21]. To the best of our knowledge, this is the first convergent numerical solver used for the computation of the discretization of problem (9).

3.1 The Augmented Lagrangian

The following compact notations are used throughout this section :

- $D = \mathbb{R}^d/\mathbb{Z}^d$ is the periodic unit cube, $[0,T]$ is a fixed time interval.
- ∇_x is the spatial gradient in $\mathbb{R}^d$.
- Δ_x is the spatial Laplacian in $\mathbb{R}^d$.
- ∂_t is the time derivative.
- . and $|.|$ denote the inner product and the Euclidean norm in $\mathbb{R}^d$.
- $\nabla_{t,x} = \{\partial_t, \nabla_x\}$ is the time-space gradient in $\mathbb{R} \times \mathbb{R}^d$.
- $\Delta_{t,x} = \partial_{t^2} + \Delta_x$ is the time-space Laplacian in $\mathbb{R} \times \mathbb{R}^d$.
- For two vectors in $\mathbb{R} \times \mathbb{R}^d$, a, b and a', b', $\{a,b\}.\{a',b'\} = aa' + b.b'$ denotes the inner product.

We use the Lagrangian formulation of the time-continuous mass transport problem of theorem (1.2) now set on the periodic domain D. The Lagrangian is given by :

$$L(\phi,\rho,m) = \int_0^T \int_D [\frac{|m|^2}{2\rho} - \partial_t\phi\rho - \nabla_x\phi.m] - \int_D [\phi(0,.)\rho_0 - \phi(T,.)\rho_T], \qquad (25)$$

where ϕ is the Lagrange multiplier of constraints (11), (10). Given initial and final densities ρ_0 and ρ_T, the unique solution of the mass transport problem is given by the resolution of the saddle point problem :

$$\inf_{\rho,m} \sup_{\phi} L(\phi,\rho,m). \qquad (26)$$

The (formal) optimality conditions for this problem are :

$$\left\{ \begin{array}{ll} \partial_t\phi + \dfrac{|m|^2}{2\rho} = 0 & \text{in }]0,T[\times D \\ \dfrac{m}{\rho} = \nabla_x\phi & \text{in }]0,T[\times D \\ \partial_t\rho + \nabla_x.m = 0 & \text{in }]0,T[\times D \\ \rho(0,.) = \rho_0, \;\; \rho(T,.) = \rho_T & \quad\text{in } D\ . \end{array} \right. \qquad (27)$$

We remark that m can be eliminated :

$$\left\{ \begin{array}{ll} \partial_t\phi + \dfrac{|\nabla_x\phi|^2}{2} = 0 & \text{in }]0,T[\times D \\ \partial_t\rho + \nabla_x.(\rho\nabla_x\phi) = 0 & \text{in }]0,T[\times D \\ \rho(0,.) = \rho_0, \;\; \rho(T,.) = \rho_T & \quad\text{in } D\ . \end{array} \right. \qquad (28)$$

System (28) is also known as the Eikonal and Transport equations in Geometrical Optics. In this framework, ϕ is called the phase and the curves $t \to X(t,x)$, defined by

$$\partial_t X(t,x) = \nabla_x\phi(t,X(t,x)), \;\; X(0,x) = x, \qquad (29)$$

are called characteristics (or rays). Here, the characteristics are just straight lines and ϕ is linked to the convex potential Φ (theorem 1.1) by

$$\nabla\phi(0,x) = \frac{\nabla\Phi(x) - x}{T}.$$

3.2 Reformulation.

In this section we reformulate the problem using the terminology of [21]. The following proposition is a straightforward consequence of the observation that, for positive ρ, we have, pointwise in time and space,

$$\frac{|m(t,x)|^2}{2\rho(t,x)} = \sup_{\{a,b\}\in K} [a(t,x)\rho(t,x) + b(t,x).m(t,x)] \qquad (30)$$

where

$$\begin{array}{l} K = \{\{a,b\} : \mathbb{R}\times\mathbb{R}^d \to \mathbb{R}\times\mathbb{R}^d, \;\; s.\ t. \\ a + \frac{|b|^2}{2} \leq 0 \;\; pointwise \;\; in\ ]0,T[\times D\}\ . \end{array} \qquad (31)$$

Lemma 3.1 *Using the following variables and notations :*

$$\left\{\begin{array}{l} \mu = \{\rho, m\} \\ q = \{a, b\} \\ F(q) = \left\{\begin{array}{l} 0 \ \mathit{if}\ q \in K \\ +\infty \ \mathit{else} \end{array}\right. \\ G(\phi) = \int_D [\phi(0,.)\rho_0 - \phi(T,.)\rho_T] \\ <\mu, q> = \int_0^T \int_D \mu.q \ , \end{array}\right. \tag{32}$$

we can write problem (26) as :

$$\sup_{\mu} \inf_{\phi, q} [F(q) + G(\phi) + <\mu, \nabla_{t,x}\phi - q>] \ . \tag{33}$$

Proof.

Using (30) in (25), we obtain for (26)

$$\inf_{\rho,m} \sup_{\phi} \int_0^T \int_D [\{ \sup_{\{a,b\} \in K} \mu.q \} - \mu.\nabla_{t,x}\phi] - G(\phi) \ . \tag{34}$$

Here $q = \{a, b\}$ is meant to be the dual variable of $\mu = \{\rho, m\}$. We now remark that :

$$\int_0^T \int_D [\sup_{\{a,b\} \in K} \mu.q] = \sup_q [\int_0^T \int_D \mu.q - F(q)] \ .$$

This last equation is used in (34) and after exchange of signs, we find (33).

Such Lagrangian formulations are used in [21] for solving problems of the form

$$\min_v \{F(Bv) + G(v)\} \ ,$$

where F, G are convex functionals and B is a linear operator. In order to fully comply with the hypothesis on F, G, and B used in [21], we lack coercivity on F. A simple way to fix this problem is to replace F by the perturbed function F_{ϵ_1} :

$$F_{\epsilon_1}(q) = F(q) + \epsilon_1 <q, q> \ .$$

We should mention that, in practice, we obtain fully satisfactory results just with $\epsilon_1 = 0$.

3.3 Augmented Lagrangian and the numerical method ALG2

In this section,we simply apply the augmented Lagrangian technique of [21] (chapter 3). First, we define the "augmented" Lagrangian :

$$\begin{array}{rl} L_r(\phi, q, \mu) = & F_{\epsilon_1}(q) + G(\phi) + <\mu, \nabla_{t,x}\phi - q> \\ & + \frac{r}{2} <\nabla_{t,x}\phi - q, \nabla_{t,x}\phi - q> \end{array} \tag{35}$$

where r is a positive parameter. Then, a simple algorithm, called ALG2, based on relaxations of the Uzawa algorithm is proposed to solve the problem :

$$\sup_{\mu} \inf_{\phi,q} L_r(\phi,q,\mu). \tag{36}$$

The introduction and the use of this technique is supported by the following remark (Theorem 2.1 [21]) : If (ϕ,q,μ) is a saddle point of (25), then it is also a saddle point of (35). The converse is also true. This means that problem (26) and (36) have same solutions. Note however that the existence of saddle points for infinite dimensional problems is not guaranteed. Sufficient conditions and theoretical references are given in [21]. We detail below the algorithm. It is a three step iterative method which constructs a sequence (ϕ^n, q^n, μ^n) converging to the saddle point.

ALG2:

- $(\phi^{n-1}, q^{n-1}, \mu^n)$ are given.
- Step A: Find ϕ^n such that :

$$L_r(\phi^n, q^{n-1}, \mu^n) \leq L_r(\phi, q^{n-1}, \mu^n), \ \ \forall \phi. \tag{37}$$

- Step B: Find q^n such that :

$$L_r(\phi^n, q^n, \mu^n) \leq L_r(\phi, q, \mu^n), \ \ \forall q. \tag{38}$$

- Step C : Do

$$\mu^{n+1} = \mu^n + r(\nabla_{t,x}\phi^n - q^n) \tag{39}$$

  (where $r > 0$ is the parameter of the Augmented Lagrangian).
- Go back to step A.

Step A and B are simply a relaxation method for the minimization part of the saddle point problem. Step C is a gradient step for the dual problem.

3.4 Interpretation of the method

We now interpret each of these steps in terms of our functions F and G.

3.4.1 Step A

We can differentiate L_r with respect to ϕ. Step A is simply the equation (in ϕ^n) :

$$\frac{\partial L_r(\phi^n, q^{n-1}, \mu^n)}{\partial \phi} = 0$$

which can be rewritten :

$$G(\phi) + r < \nabla_{t,x}\phi^n - q^{n-1}, \nabla_{t,x}\phi > + < \mu^n, \nabla_{t,x}\phi >= 0, \ \ \forall\phi.$$

After integrating by part in space and time, we see that this is the variational formulation of the following Laplace equation with periodic boundary conditions in space and Neumann boundary conditions in time :

$$\left\{\begin{array}{lr} \Delta_{t,x}\phi^n = \nabla_{t,x}.(\mu^n - rq^{n-1}) & \text{in }]0,T[\times D \\ -\partial_t\phi^n(0,.) = \rho^n(0,.) - ra^{n-1}(0,.) - \rho_0 & \text{in } D \\ \partial_t\phi^n(T,.) = \rho^n(T,.) - ra^{n-1}(T,.) - \rho_T & \text{in } D\ . \end{array}\right. \tag{40}$$

Recall that $\mu^n = \{\rho^n, m^n\}$ and $q^{n-1} = \{a^{n-1}, b^{n-1}\}$.

Note that this problem is well posed only if we have the classical compatibility conditions.

$$\int_0^T\int_D [\nabla_{t,x}.(\mu^n - rq^{n-1})] = \int_D [\rho^n(0,.) - ra^{n-1}(0,.) - \rho_0 + \rho^n(T,.) - ra^{n-1}(T,.) - \rho_T]$$

In practice, for programing simplicity, we use the perturbed Laplace equation :

$$\Delta_{t,x}\phi^n - \epsilon_2\phi^n = \nabla_{t,x}.(\mu^n - rq^{n-1})$$

where ϵ_2 is a small positive parameter.

3.4.2 Step B

We cannot differentiate L_r with respect to q. So q^n is obtained by solving

$$\inf_q [F_{\epsilon_1}(q) + \frac{r}{2} < \nabla_{t,x}\phi^n - q, \nabla_{t,x}\phi^n - q > + < \mu^n, \nabla_{t,x}\phi^n - q >],$$

which is equivalent to :

$$\inf_{q\in K} < q - \frac{r\nabla_{t,x}\phi^n + \mu^n}{r+\epsilon_1}, q - \frac{r\nabla_{t,x}\phi^n + \mu^n}{r+\epsilon_1} > .$$

It is important to notice that this minimization can be performed pointwise in space and time. Let us set :

$$p^n(t,x) = \{\alpha^n(t,x), \beta^n(t,x)\} = \frac{r\nabla_{t,x}\phi^n(t,x) + \mu^n(t,x)}{r+\epsilon_1}.$$

Then $q^n(t,x) = \{a^n(t,x), b^n(t,x)\}$ is obtained by solving :

$$\inf_{\{a,b\}\ \ s.t. a + \frac{|b|^2}{2} \leq 0} [(a - \alpha^n(t,x))^2 + |b - \beta^n(t,x)|^2].$$

This is a simple one dimensional projection problem which can be computed analytically or using a Newton method.

3.4.3 Step C

Step C is simply the pointwise update :

$$\mu^{n+1}(t,x) = \mu^n(t,x) + r(\nabla_{t,x}\phi^n(t,x) - q^n(t,x)).$$

3.5 Cost and convergence criterium

Amongst these three steps, only Step A is global. This means that the cost of Step B and C are of order O(N) where N is the number of points of the space time lattice. The Laplace equation (step A) can be solved in O(N Log N) operations. The cost of this methods is therefore of order $Niter \times NLogN$ where $Niter$ is the necessary number of iteration n to converge.

We do not have theoretical estimates on the speed of convergence of the method. To be able to produce numerical estimates and also for the practical purpose of stopping the computation we need to define a convergence criterium. The optimality conditions (28) are useful for that purpose. We can indeed use the residual of the "Eikonal equation" :

$$res^n = \partial_t \phi^n + \frac{|\nabla_x \phi^n|^2}{2}$$

which is a by-product of the algorithm. This quantity converges to 0 as we approach the solution of problem. The normalized convergence criterium used is

$$crit^n = \sqrt{\frac{\int_0^T \int_D \rho^n |res^n|}{\int_0^T \int_D \rho^n |\nabla_x \phi^n|^2}}\,. \tag{41}$$

4 Numerical Result

We present in this section numerical tests. The normalized space-time domain is discretized using a regular $32 \times 32 \times 33$ lattice. Initial and final densities are represented. They are all based on Gaussians densities, deformation of gaussians or rearrangements of the support of gaussians.

The parameters of the method are taken as $\epsilon_1 = 0$, $\epsilon_2 = 0.001$ and $r = 1$

Convergence history shows the convergence criterium (41) along the iterations of method ALG2. As usual for gradient method convergence rate quickly decays. Convergence history even shows small erratic behavior after the first thousands iterations.

Finally, we give for different time steps the level curves of ρ. The final value of ρ at time step 33 always match ρ_T. Small numerical errors can be seen on the first two tests, for which we used a higher number of contour lines.

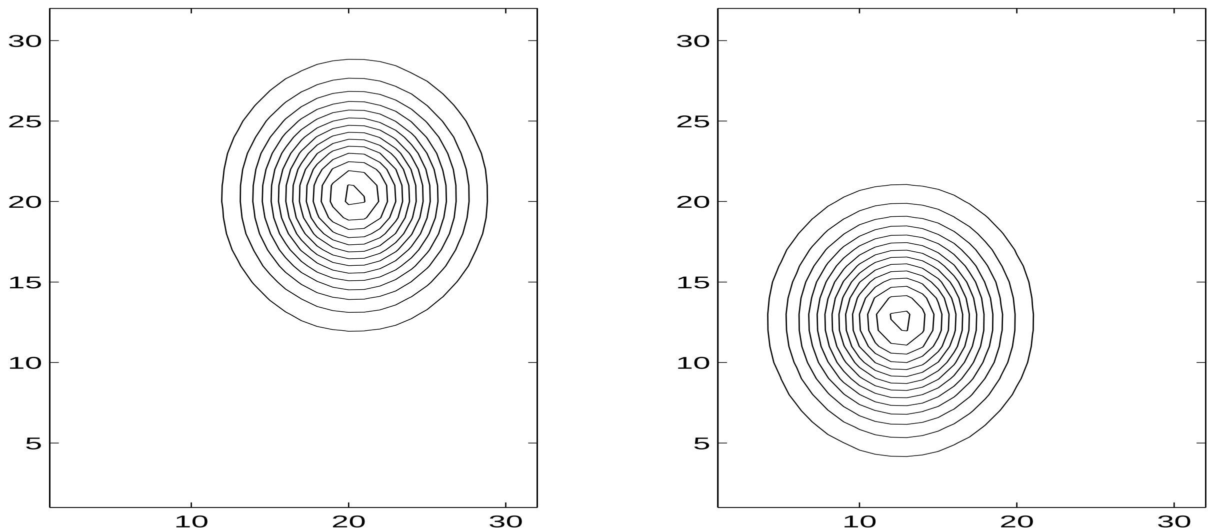

Figure 1: Initial and final densities (contour plots)

4.1 Test 1

The exact solution of this problem set in free space is the translation of the gaussian. As we use periodic boundary conditions in space a small amount of the mass is transferred across the periodic boundaries.

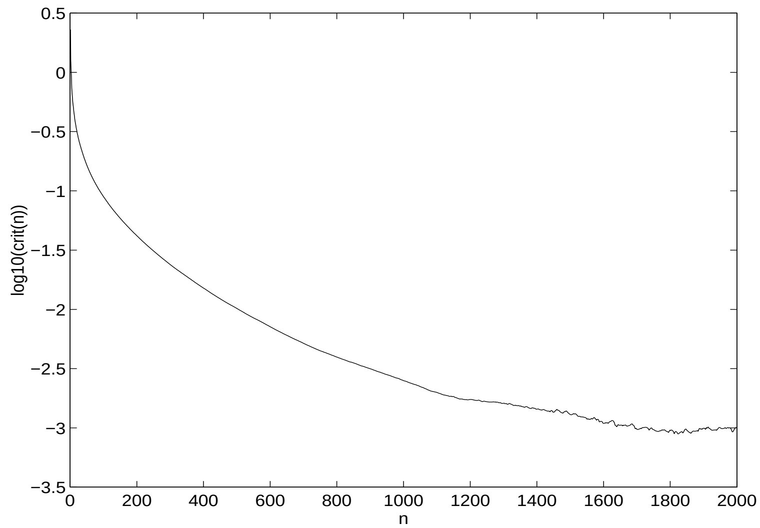

Figure 2: Convergence history

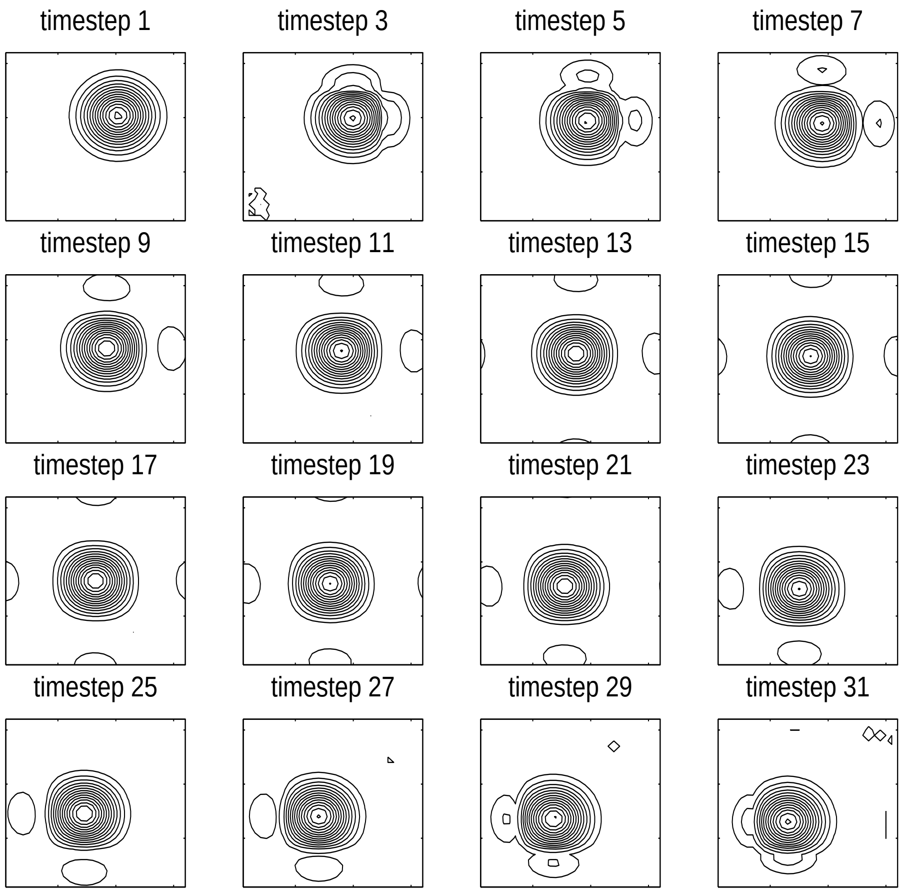

Figure 3: Contours plots of the density at successive time steps

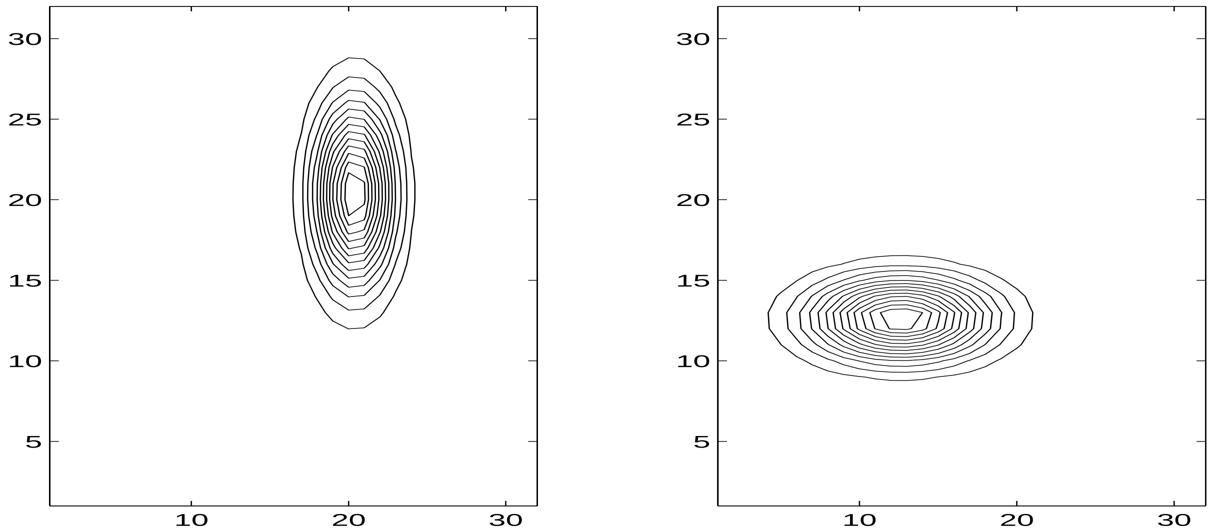

Figure 4: Final and initial densities (contour plots)

4.2 Test 2

Here we perform the translation and rotation of an elliptically deformed gaussian. The periodic boundary conditions in space are again responsible for the small amount of mass transferred across boundaries. Small errors produce scattered pebbles.

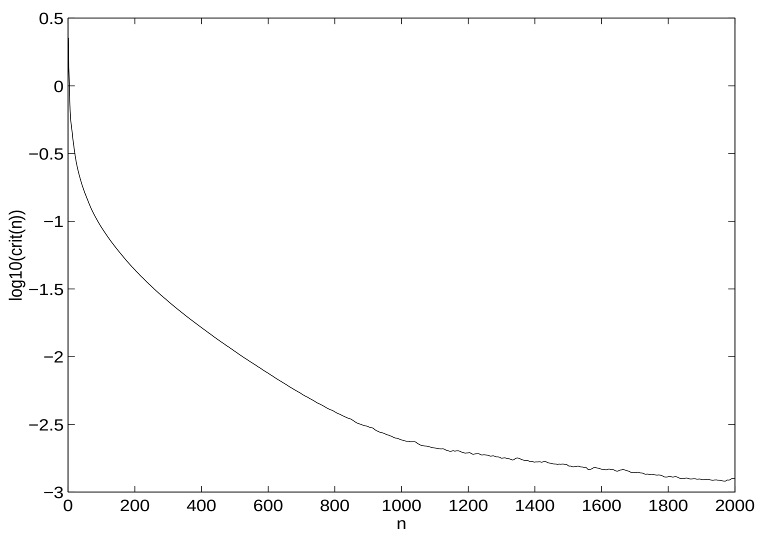

Figure 5: Convergence history

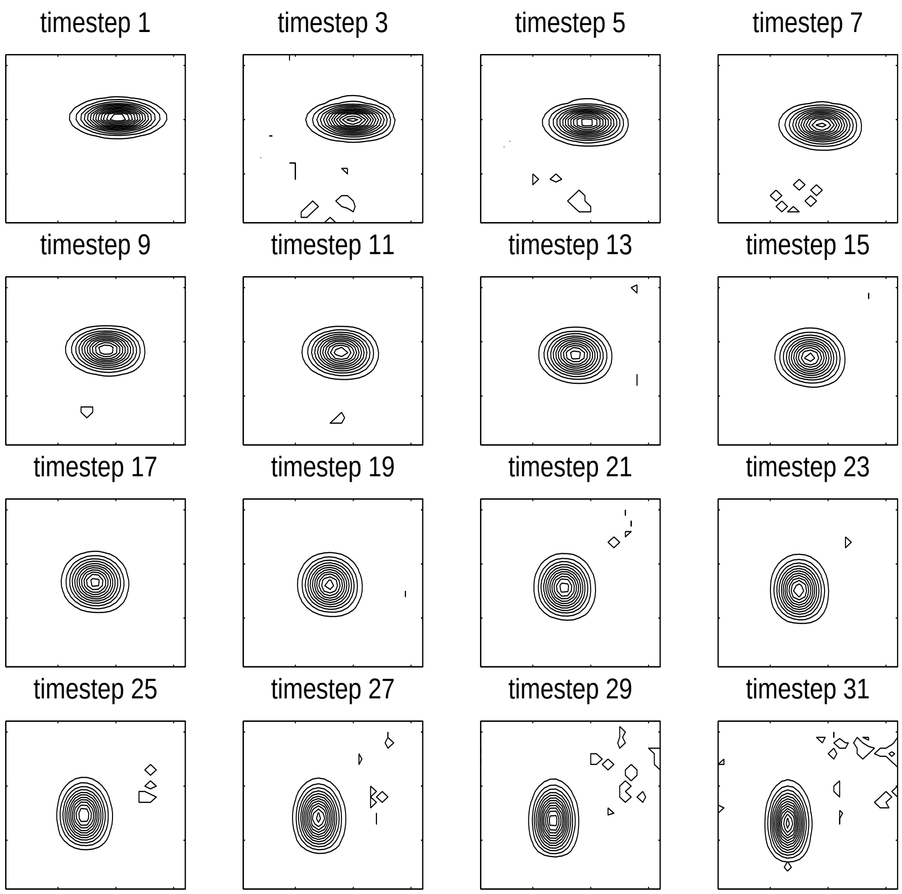

Figure 6: Contours plots of the density at successive time steps

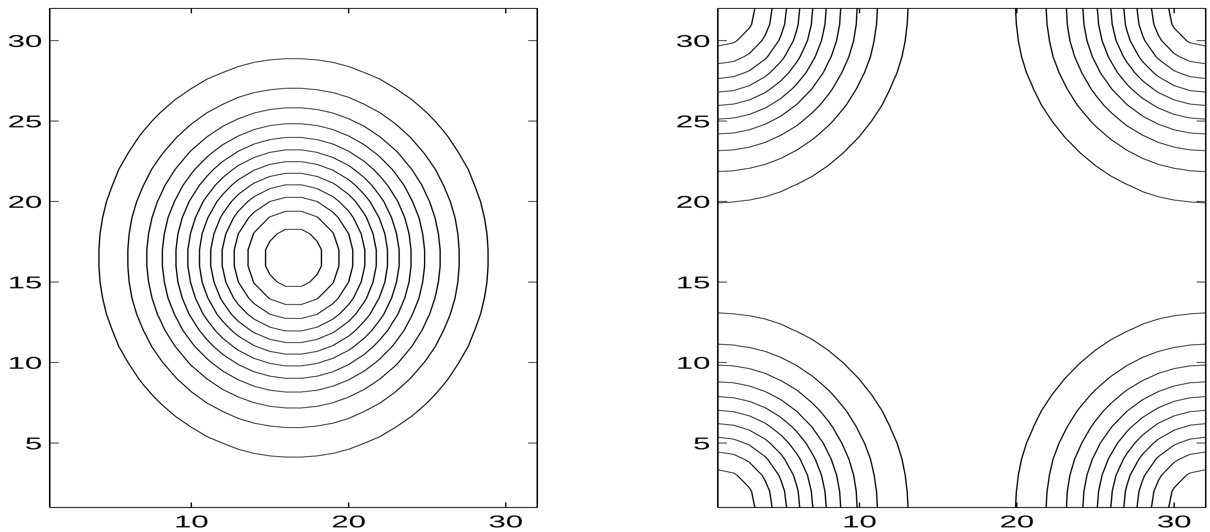

Figure 7: Initial and final densities (contour plots)

4.3 Test 3

Because of the periodic spatial domain, this experiment amounts to shift an array of gaussians. As we optimize the transportation cost (9), the solution splits each gaussian in four and send each part to the nearest corner rather that shifting the all array.

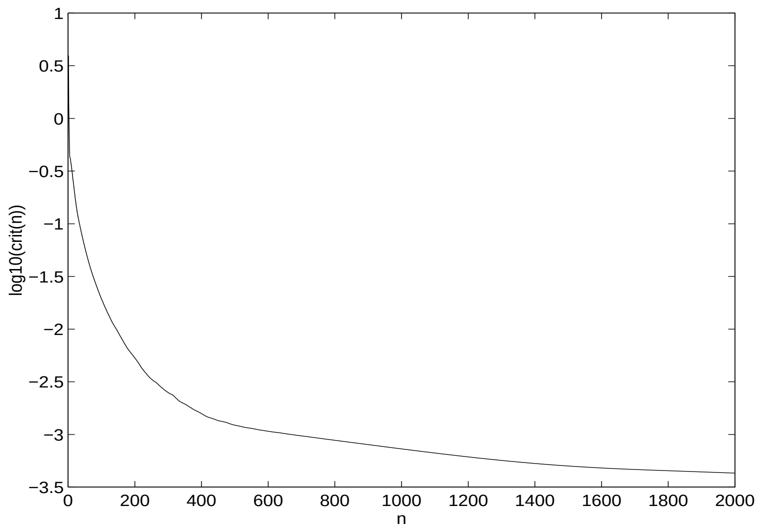

Figure 8: Convergence history

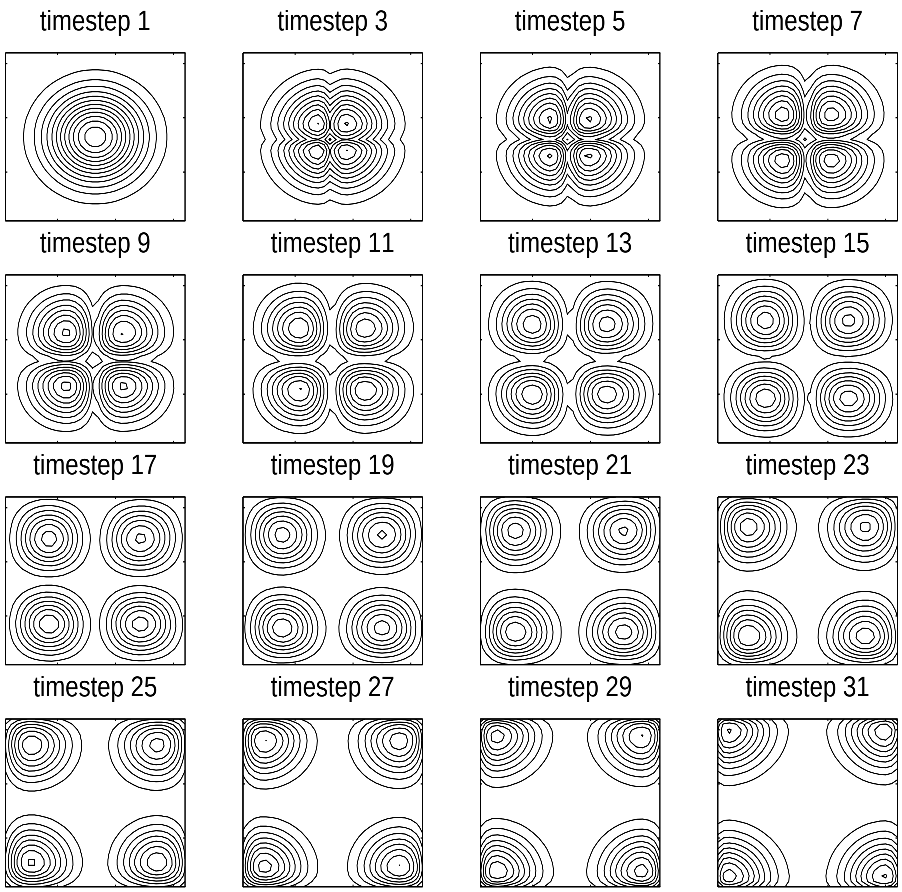

Figure 9: Contours plots of the density at successive time steps

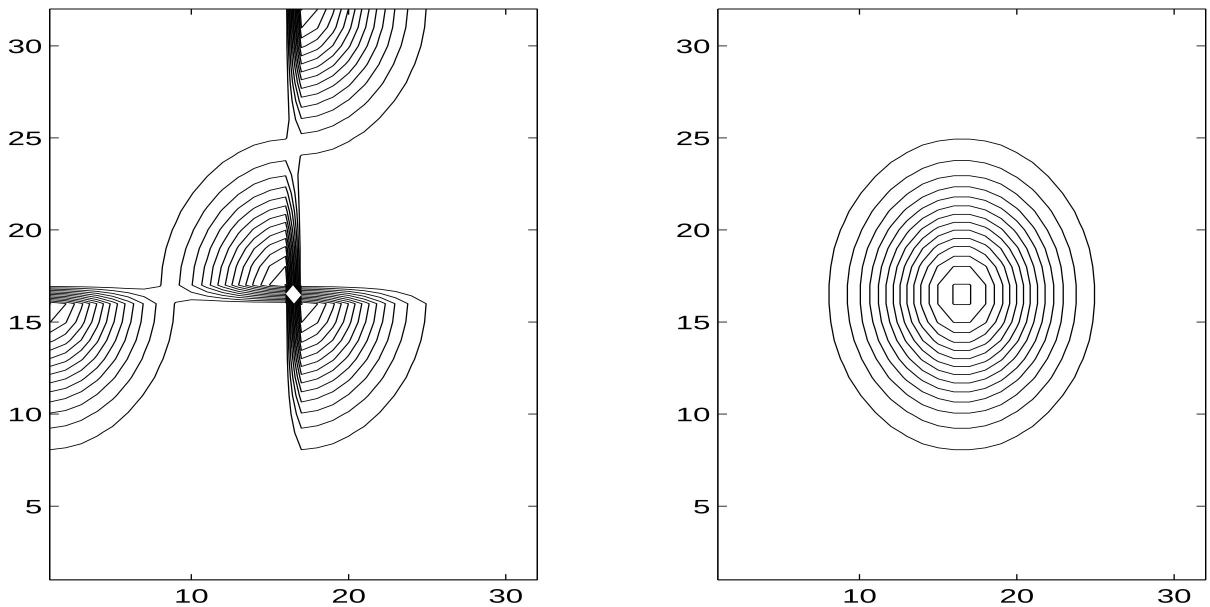

Figure 10: Initial and final density (contour plots)

4.4 Test 4

This test consist in shifting pieces of a gaussian. The initial density has sharp discontinuities.

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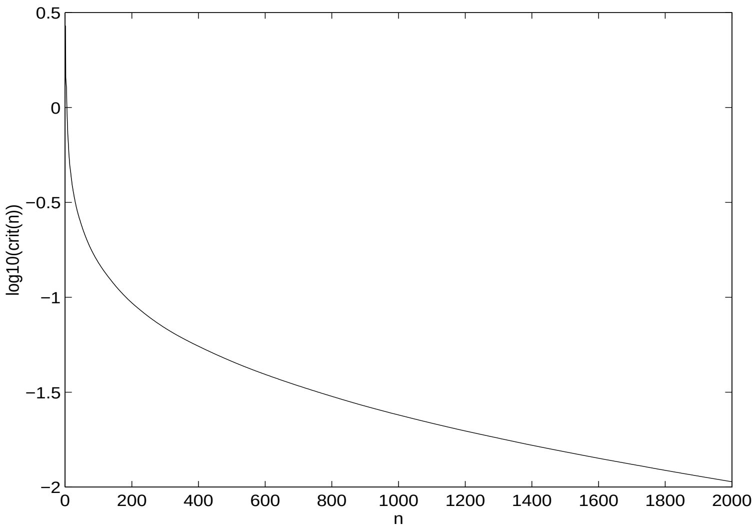

Figure 11: Convergence history

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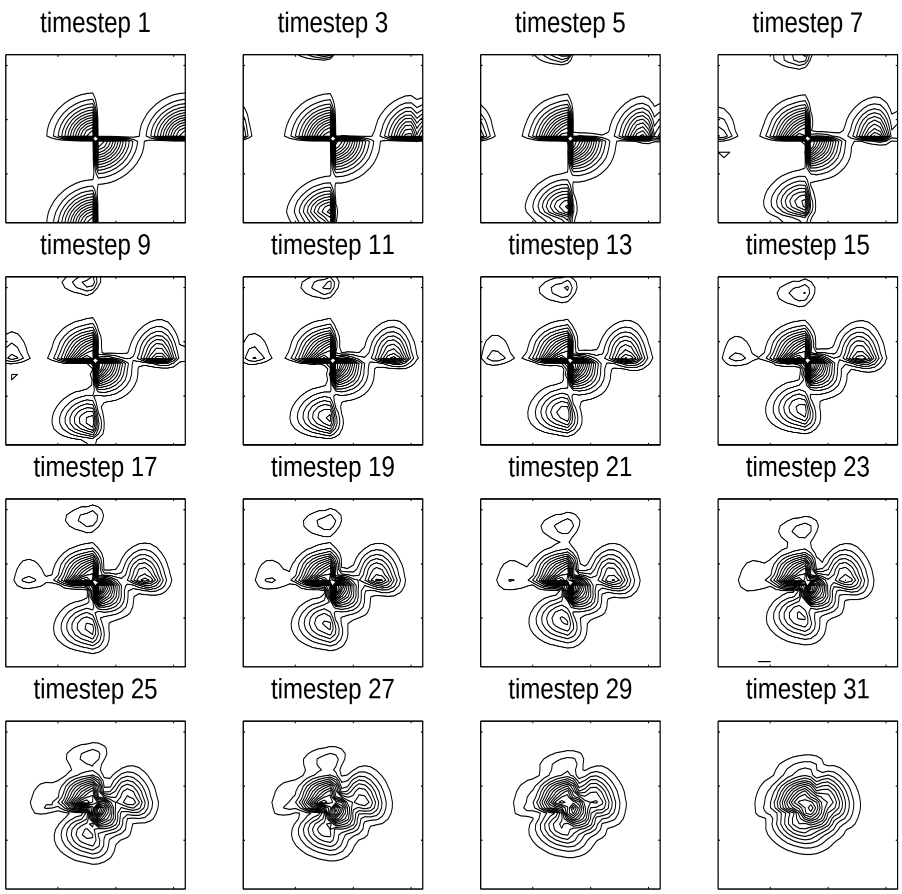

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