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N° 3328

Décembre 1997

THÈME 1

Rapport de recherche

ISSN 0249-6399

Dobrushin's Mean-Field Approximation for a Queue With Dynamic Routing*

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Thème 1 — Réseaux et systèmes
Projet MEVAL

Rapport de recherche n°3328 — Decembre 1997 — 39 pages

Abstract: A queueing system is considered, with a large number N of identical infinite-buffer FCFS single-servers. The system is fed by a Poisson flow of rate λN, with i.i.d. service times, under the non-overload condition. Each arriving task joins a server by selecting it from an independently (and 'completely randomly') chosen collection of $m \ll N$ servers, on the basis of some fixed dynamic routing policy. We discuss various properties of such a system as $N \to \infty$. In particular, a natural limiting random process describing statistical properties of the system turns out to be deterministic.

Key-words: queueing theory, single-server, dynamic routing, Markov process, generator, convergence, functional differential/difference equation, initial-boundary value problem, fixed points

(Résumé : tsvp)

* Work supported by the London Mathematical Society; St John's College, Cambridge; Dublin Institute for Advanced Studies; I.H.E.S., Bures-sur-Yvette; the Russian RFFI Grant 96-01-00150; the EC Grant, Contracts CHRX-CT 93-0411 and ERBMRXT-CT 960075A, and the INTAS Grant 93-820. The work was terminated during the visit of the first author at INRIA-Rocquencourt.

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L'approximation champ moyen de Dobrushin appliquée à un système de files d'attente avec routage dynamique

Résumé : On considère un système de N files d'attente identiques de type PAPS. Les arrivées forment un processus de Poisson d'intensité totale λN et les durées de service sont des variables i.i.d. arbitraires. Chaque nouveau client est dirigé vers une file sélectionnée au hasard, uniformément parmi $m << N$. On discute les propriétés de ce système lorsque $N \to \infty$, en montrant notamment que le processus donnant les principales caractéristiques statistiques tend à devenir déterministe.

Mots-clé : files d'attente, routage dynamique, processus de Markov, générateur, convergence, équation différentielle fonctionnelle, problème frontière, point fixe

1 Introduction

This paper is a continuation of the earlier work [30] conceived and executed with the active participation of Roland Dobrushin, one of the last papers completed before his untimely death. We should here like to say a few words about Roland's contribution. His influence on paper [30] (and on the queueing theory at the whole) was enormous, extending far beyond any conventional level of co-authorship: a tribute which could well be made of his influence on the numerous other areas of mathematics in which he was active. He was prolific of ideas and methods, and his enthusiasm and generosity in sharing them kept his co-workers in his orbit long after a particular collaborative work was complete. This does not mean that his work was considered above criticism: indeed in some cases his approach was proved wholly or partially wrong. Remarkably however even so the resulting byproduct of the research turned out be no less interesting than those initially conceived.

One of the central ideas in queueing theory conceived by Dobrushin in the early 70's and successively applied to various models of queueing networks (QNs) was the mean-field approximation. We refer the reader to the papers [4, 16, 18, 1, 14, 2, 25, 5] and [30] written with his participation or under his (direct or indirect) supervision; see also the reviews [3, 17, 15] and related papers [20] and [22, 23]. In the West, a similar approach was proposed independently (and in a different set-up) in [12, 13, 32, 21, 31] (see also the review [19]) and continued in [8, 9, 10]. Dobrushin was motivated by an urge of creating new methods of studying complex QNs where 'traditional' approaches inherited from the theory of queues to an isolated server failed. He quickly understood the merits of the mean-field approach, especially the fact that this approximation works the better the 'more complex' QNs are considered. Within a certain class of models the complexity of a QN can be measured in the number N of servers it contains; the mean-field approximation becomes exact in the limit $N \to \infty$.

The remaining part of this section discusses the results of [30]. The basic model considered there is a queueing system $\mathcal{S}_N$, with N identical infinite-buffer FCFS single-servers, fed by a Poisson arrival flow of rate $N\lambda$ and with i.i.d. exponential service times of mean μ^{-1}, where $0 < \lambda < \mu$. Upon its arrival each task chooses m servers at random (i.e., independently of the pre-history of the queueing system (QS) and with probability $1/(N^m)$) and then selects, among the chosen ones, the server with the lowest queue-size, i.e., the lowest number of tasks in the buffer (including the task in service). If there happen to be more than one server with lowest queue-size, the task selects one of them randomly. This model may be considered as a very simple example of dynamic routing (DR); see [19] and the references therein, and the recent paper [26].

One is interested in the 'typical' behaviour of a server in $\mathcal{S}_N$, as $N \to \infty$. Formally, it means that $\forall\ t \geq 0$ and $k = 0, 1, \ldots,$ we consider the fraction $q_k(t) = M_k(t)/N$ where $M_k(t)$ is the (random) number of servers with the queue-size k at time t. Clearly, $0 \leq q_k(t) \leq 1$, $\sum_k q_k(t) = 1$, and $Q(t) = \big(q_k(t)\big)$, $t \geq 0$, forms a Markov process (MP). Technically, it is more convenient to pass to the tail probabilities $r_k(t) = \sum_{j \geq k} q_k(t)$; the state space of the corresponding MP $\big\{U_N(t) = \big(r_k(t)\big)$, $t \geq 0\big\}$, is the set $\mathcal{U}_N$ of non-increasing non-negative sequences $\mathbf{u} = (u_k,\, k = 0, 1, \ldots)$ with $u_0 = 1$, $\sum_{k>1} u_k < \infty$ and with the u_k's multiple of $1/N$ (which implies that $u_k = 0$ for all k large enough). It is convenient to prolong the sequences $\mathbf{u} \in \mathcal{U}_N$ to the negative k's by the value 1; in future we use such a continuation without a reminder. The generator of $\{U_N(t)\}$ is an operator $\mathbf{A}$ acting on functions $f\colon \mathcal{U}_N \to \mathbf{C}^1$ and given by

$$\begin{aligned}\mathbf{A}_N f(\mathbf{u}) \;&=\; N \sum_{k>0} \big(u_k - u_{k+1}\big) \Big[f\left(\mathbf{u} - \frac{\mathbf{e}_k}{N}\right) - f(\mathbf{u}) \Big] \\ &\quad + \lambda N \sum_{k>0} \big((u_{k-1})^2 - (u_k)^2\big) \Big[f\left(\mathbf{u} + \frac{\mathbf{e}_k}{N}\right) - f(\mathbf{u}) \Big]. \qquad (1.1)\end{aligned}$$

Here, $\mathbf{e}_k$ stands for the sequence with the k–th entry 1 and all others 0, the addition of the sequences is componentwise.

Process $\{U_N(t)\}$ is positive-recurrent and thus possess a unique invariant distribution, π_N; given any initial distribution ϖ, the distribution of $U_N(t)$ approaches π_N as $t \to \infty$. The main result of [30] is that, as $N \to \infty$, the expected value $\mathsf{E}_{\pi_N}\, r_k(t)$ converges to the value $\{a_k\}$, where

$$a_k = \left(\frac{\lambda}{\mu}\right)^{(m^k - 1)/(m-1)}, \; k \geq 0. \qquad (1.2)$$

Pictorially speaking, it means that, as $N \to \infty$, an 'average' server in the QS will have $\geq k$ tasks in the buffer with probability a_k. It is interesting to compare $\mathcal{S}_N$ with another QS, $\mathcal{L}$, where the arriving task chooses the server completely randomly (i.e., independently of the pre-history and with probability $1/N$). Clearly, $\mathcal{L}$ is equivalent to an isolated $M/M/\infty$ queue with the arrival and service rates λ and μ, respectively (which justifies omitting subscript N in this notation). More precisely, the average server in $\mathcal{L}$ will have $\geq k$ tasks in the buffer with the geometrical probability

$$a_k^0 = (\lambda/\mu)^k, \;\; k \geq 1, \qquad (1.3)$$

(independently of N), which is much larger than a_k. See the comment below.

In fact, as was shown in [30], the whole process $\{U_N(t)\}$ is asymptotically deterministic as $N \to \infty$. More precisely, let $\mathcal{U}$ denote the set of the non-increasing non-negative sequences $\mathbf{u} = (u_k,\ k \in \mathbf{Z})$ with $u_k = 1$ for $k \leq 0$ and $\sum_{k \leq 0} u_k < \infty$. Then, if the distribution ϖ of initial state $U_N(0)$ approaches a Dirac delta-measure concentrated at a point $\mathbf{g} = \{g_k\} \in \mathcal{U}$, the distribution of $\{U_N(t)\}$ is concentrated in the limit at the 'trajectory' $\mathbf{u}(t) = \{u_k(t)\}$, $t \geq 0$, giving the solution to the following system of differential equations

$$\dot{u}_k(t) = \mu\big(u_{k+1}(t) - u_k(t)\big) + \lambda\big[\big(u_{k-1}(t)\big)^2 - \big(u_k(t)\big)^2\big],\ k \geq 1, \tag{1.4}$$

with the boundary condition $u_0(t) = 0$, $t \geq 0$, and the initial condition $u_k(0) = g_k$, $k \geq 1$. Point $\mathbf{a} = (a_k)$ (see (1.2)) is a (unique) fixed point for system (1.4) in $\mathcal{U}$.

These results illustrate the essence of the mean-field approximation for QS $\mathcal{S}_N$. Equations (1.4) describe a 'self-compatible' evolution of vector $\mathbf{u}(t)$, or, equivalently, of the probability distribution $\mathbf{q}(t) = \{q_k(t)\}$ defined by $q_k(t) = u_k(t) - u_{k+1}(t)$, $t \geq 0$, $k = 0, 1, \ldots$. [As before, $\mathbf{u}(t)$ is simply the sequence of the tail probabilities for $\mathbf{q}(t)$.] We will compare system (1.4) with the linear system

$$\dot{y}_k(t) = \mu\big(y_{k+1}(t) - y_k(t)\big) + \lambda\big(y_{k-1}(t) - y_k(t)\big),\ k \geq 1, \tag{1.5}$$

describing the evolution of the probability distribution $\mathbf{q}^{(0)}(t) = \big(q^0_k(t),\ q^0_k(t) = y_k(t) - y_{k+1}(t)\big)$ in a standard $M/M/1/\infty$ queue with the arrival and service rates λ and μ, respectively. The μ–terms in (1.4) and (1.5) are the same; they correspond with the departure of the tasks and 'push' the probability mass in $\mathbf{q}(t)$ and $\mathbf{q}^{(0)}(t)$ towards $k = 0$. On the other hand, the λ–terms (different in both SQ) correspond with the arrival of the tasks; these terms shift the probability mass to larger k's. The λ–term in (1.4) is smaller than the one in (1.5) when $u_k(t)$ is small; pictorially speaking, system (1.4) provides (for the same values of λ and μ) more 'protection', for large k, against the shift to the right (which may lead to an 'explosion', when the relation $\sum_{k>1} u_k(t) < \infty$ or $\sum_{k>1} y_k(t) < \infty$ may fail as $t \to \infty$). Because of this, the entries a_k of sequence $\mathbf{a}$ (see (1.2)) giving the fixed point of (1.4) decrease 'super-exponentially', in contrast with the exponential decay of the tail probabilities in the fixed point $\mathbf{a}^0 = (a^0_k)$ of (1.5).

2 Results

This paper aims to extend the above picture to a larger class of QS, where the service-time of a task may not be exponential, and the selection of a server, among a total N, is made on the basis of a general dynamic DR policy. Such a generalization arises naturally and in particular was, it seems, on Dobrushin's mind. For a general class of QSs under consideration we establish a similar result: the (Markov) process describing the evolution of a (suitably defined) state of the QS converges to a limiting deterministic process that follows the solution of an initial-boundary value problem for a system of equations (2.9) similar to (1.4). In the 'continuous' version, where $k \in \mathbf{Z}_+$ is replaced with $x \in \mathbf{R}_+$, (1.4) becomes a partial integro-differential equation; see (2.10). Using modern terminology, we call them functional differential equations (FDE). We analyse the fixed points of the limiting FDEs and establish their relation to the invariant distributions of the MPs under consideration. The proofs, as in [30], are based on methods from the theory of differential equations and convergence theorems for MPs.

In analogy with the 'standard' (or linear) queueing theory, one can interpret the results of [30] as a generalisation of the Erlang formula for the equilibrium distribution of the queue size in an $M/M/1$ queue, whereas the results of the present paper may be treated as a generalisation of the Pollacheck-Khinchin type formulas for the waiting-time distribution in an $M/GI/1$ queue.

An example of the QS that falls into our class is as follows. Suppose that the above QS $\mathcal{S}_N$ is modified so that the task selects, among m randomly chosen servers, the one that has the s–th shortest (or equivalently, the $(m-s)$–th longest) queue size, $1 \leq s \leq m$. In particular, $s = 1$ corresponds with the above selection of the shortest queue while $s = m$ with the selection of the longest queue. [Such a 'strange' DR policy can be justified when we want a significant portion of the servers in the QS to have a low queue size so that their buffers could be used for other purposes.] The analogue of system (1.4) for this model is

$$\dot{u}_k(t) = \lambda\Big[h_{m,s}\big(u_{k-1}(t)\big) - h_{m,s}\big(u_k(t)\big)\Big] + \mu\big(u_{k+1}(t) - u_k(t)\big),\ k = 1, 2, \dots, \quad (2.1)$$

where

$$h_{m,s}(x) = \sum_{l=0}^{s-1} \binom{m}{l} (1-x)^l x^{m-l},\ 0 \leq x \leq 1. \quad (2.2)$$

Set $\theta_0 \equiv \theta_0(m,s) = \sup\big[h_{m,s}(x)/x,\ 0 < x \leq 1\big]$. It turns out that for $\theta_0\lambda/\mu < 1$, system (2.1) has a unique fixed point $\mathbf{a} = \{a_k\} \in \mathcal{U}$, with

$$a_k \sim c_0 \left(\frac{\lambda}{\mu}\right)^{c_1(m-s+1)^k},\ m > s, \quad \text{or} \quad a_k \sim c_0 \left(\frac{\lambda}{\mu}\right)^{c_1 k},\ m = s, \quad k \gg 1,$$

where $c_0, c_1 > 0$ are constants (depending on m, s). In particular, $\theta_0(m, m) = m$.

The material of this paper is organised as follows: in the rest of this section we introduce the class of models under consideration (in fact, the class of MPs generalising the above $U(t)$) and state our main results. In Sections 3 and 4 we discuss properties of the limiting equations, for the discrete and continuous case, respectively, and in Section 5 establish the convergence of the MPs to their deterministic counterparts described by these equations. In Section 6 we discuss various examples and generalisations.

The notation for various spaces related to the MPs under consideration is as follows. The set $\overline{\mathcal{V}}$ is defined as the collection of non-increasing left-continuous functions $\mathbf{R} \to [0,1]$ taking value 1 for $x \leq 0$. $\mathcal{V}$ is the subset of $\overline{\mathcal{V}}$ consisting of functions $u(x)$ with $\int_0^\infty dx\, u(x) < \infty$. Furthermore, $\overline{\mathcal{U}}^{(L)}$ and $\mathcal{U}^{(L)}$, $L \geq 1$, are subsets of $\overline{\mathcal{V}}$ and $\mathcal{V}$, respectively, formed by the functions constant on each interval $\big(k/L, (k+1)/L\big]$, $k \geq 1$. Finally, $\mathcal{U}_N^{(L)}$, $N \geq 1$, denotes the subset of $\mathcal{U}^{(L)}$ consisting of functions with values multiple to $1/N$; functions from $\mathcal{U}_N^{(L)}$ vanish for $x > 0$ large enough. As before, we use the notation $\mathbf{u} = (u(x),\, x \geq 0)$, for an element of $\overline{\mathcal{V}}$ (or for one of its subsets just introduced). Space $\overline{\mathcal{V}}$ can be endowed with the Prokhorov metric. This metric metrizes the topology of pointwise convergence where $\mathbf{u}_n \to \mathbf{u}$ iff $u_n(x) \to u(x)$ at each point $x \in \mathbf{R}$ of continuity of $\mathbf{u}$; its restriction to $\overline{\mathcal{U}}$ (or $\overline{\mathcal{U}}^{(L)}$, if one prefers) is equivalent to, for example, the metric δ given by

$$\delta(\mathbf{u}, \mathbf{u}') = \sup_{k \geq 1} |u_k - u'_k|/k. \tag{2.3}$$

With the Prokhorov metric, $\overline{\mathcal{V}}$ is a complete separable compact space, and so is $\overline{\mathcal{U}}$. In what follows, the convergence in $\overline{\mathcal{V}}$ (or in its subspace) is understood in the above metric(s) (so, $\mathrm{C}(\overline{\mathcal{V}})$ is the space of functions $\overline{\mathcal{V}} \to \mathbf{C}^1$ continuous in the Prokhorov metric whereas $\mathrm{C}(\overline{\mathcal{U}})$ is the space of functions $\overline{\mathcal{U}} \to \mathbf{C}^1$ continuous in metric δ). Furthermore, the convergence of probability measures on $\overline{\mathcal{V}}$ (or on any of its subspaces) means the weak convergence with respect to this metric (although in some cases stronger forms of convergence can be established).

The spaces $\mathcal{U}_N^{(L)}$, $\mathcal{U}^{(L)}$ and $\overline{\mathcal{U}}^{(L)}$ are naturally isomorphic to the above sequence spaces $\mathcal{U}_N$, $\mathcal{U}$ and $\overline{\mathcal{U}}$. We will repeatedly use this fact and substitute one space with another, which will considerably simplify the notation.

Fix a monotone increasing C^2 function h: $[0,1] \to [0,\infty)$, with $h(0) = 0$. Set

$$\theta_0 = \sup\, \big[\ h(x)/x :\ 0 < x \leq 1\big].$$

$$\theta_1 = \sup\, \big[\ h'(x) :\ 0 \leq x < 1\big]. \tag{2.4}$$

$$\theta_2 = \sup \left[\, h''(x): \; 0 \le x < 1 \right].$$

Suppose that $B(x)$, $x \in \mathbf{R}$, is a distribution function on $\mathbf{R}_+$ (i.e., B is non-decreasing and left-continuous, $B(x) = 0$ for $x \le 0$ and $B(x) \to 1$ as $x \to \infty$). We assume $B(x)$ has a finite second moment:

$$\int_0^\infty (1 + x^2)\, dB(x) < \infty, \tag{2.5}$$

although some of the results do not require this. We also assume throughout the paper that the following *non-overload* condition holds:

$$\theta_0 \lambda \int_0^\infty x\, dB(x) < 1; \tag{2.6}$$

again we do not claim that this condition is necessary for all assertions below.

Set $\mathbf{N}^{(L)} = \{l/L\colon\ l \ge 1\}$ and $b^{(L)}(y) = B(y + 1/L) - B(y)$, $y \in \mathbf{N}^{(L)}$. Then, plainly, condition (2.6) implies that

$$\theta_0 \lambda \sum_{y \in \mathbf{N}^{(L)}} y b^{(L)}(y) < 1. \tag{2.7}$$

We consider continuous-time MPs on spaces $\mathcal{U}_N^{(L)}$, with the generators $\mathbf{A} \equiv \mathbf{A}_N^{(L)}$ of the form

$$\begin{aligned} &\mathbf{A} f(\mathbf{u}) \\ &= \; NL \sum_{x \in \mathbf{N}^{(L)}} \left[u(x) - u\left(x + 1/L\right) \right] \left[f\left(\mathbf{u} - \frac{\mathbf{e}_x^{(L)}}{N} \right) - f(\mathbf{u}) \right] \\ &\quad + \lambda N \sum_{x,y \in \mathbf{N}^{(L)}} b^{(L)}(y) \Big[h\big(u(x-y)\big) - h\big(u(x)\big) \Big] \left[f\left(\mathbf{u} + \frac{\mathbf{e}_x^{(L)}}{N} \right) - f(\mathbf{u}) \right]. \end{aligned} \tag{2.8}$$

Here, $\mathbf{e}_x^{(L)}$ stands for the function set of $\left[x - 1/(2L), x + 1/(2L)\right]$. As before, such a MP is denoted by $\{U_N^{(L)}(t),\, t \ge 0\}$. An example is the generator (1.1) (with $\mu = 1$): here $L = 1$, $B(l) = \mathbf{1}\{y > 1\}$ and $h(x) = x^m$, $x \in [0, 1]$. A more general example is with $L = 1$, $B(y) = \mathbf{1}\{y > 1\}$ and $h = h_{m,M}$ (see (2.2) above). In Section 6 we give further examples of QSs with DR policies which lead to processes with generators of the form (2.8) (we also consider some generalisations); here we only mention that such a QS consist, as before, of N servers with a common input. The arrival flow is again Poisson, of rate λN. However, tasks may arrive in 'batches', and $b^{(L)}(y)$,

$y \in \mathbf{N}^{(L)}$, is the probability that the size of a batch equals yL. In other words, an arriving task is composed of l 'mini-tasks' with probability $b^{(L)}(l/L)$; we assume that each mini-task needs an exponentially distributed time for its execution, of mean $1/L$, independently of the whole pre-history of the QS. We also observe that the corresponding DR policy is based on the number of mini-tasks in the buffers of the servers. [As before, this includes the mini-tasks in service.] The QS corresponding with a MP $\{U_N^{(L)}(t)\}$ under consideration is denoted by $\overline{\mathcal{S}}_N$.

Theorem 1 *Process $\{U_N^{(L)}(t)\}$ is positive recurrent and thus possesses a unique invariant distribution $\pi \equiv \pi_N^{(L)}$; for any initial distribution ϖ, the distribution of $U_N^{(L)}(t)$ converges, as $t \to \infty$, to π.*

As in [30], our main goal is to analyse the mean-field limit $N \to \infty$. In addition, we can let $L \to \infty$ which corresponds to passing from batches to generally-distributed service-times of tasks. More precisely, when talking of $N \to \infty$, L being fixed, we always assume that the distribution ϖ of the initial state $U_N^{(L)}(0)$ converges to the delta-measure concentrated at a point $\mathbf{g} = \{g_k,\ k \geq 1\} \in \overline{\mathcal{U}}^{(L)}$. This case is called discrete. On the other hand, when speaking of the double limit $N, L \to \infty$, we assume that ϖ converges to the delta-measure concentrated at a point $\mathbf{g} = \{g(x),\ x \geq 0\} \in \mathcal{V}$. Here, we speak of a continuous case. Before we state our results about convergence to these limits, we discuss some properties of the limiting objects. [First of all, from (2.8) it can immediately be seen that the limiting generators must be differential operators of the first order which means that the corresponding limiting MPs must be deterministic.]

Namely, A) in the discrete case, given $\mathbf{g} = (g_k) \in \overline{\mathcal{U}}$, consider, in $\overline{\mathcal{U}}$, the initial-boundary value problem for the system of ordinary FDEs:

$$\dot{u}_k(t) = L\big[u_{k+1}(t) - u_k(t)\big] + \lambda \sum_{l \geq 1} b^{(L)}(l/L)\big[h\big(u_{k-l}(t)\big) - h\big(u_k(t)\big)\Big],\ t \geq 0,\ k \geq 1,$$

$$u_k(0) = g_k,\ k \geq 1,\ \ u_k(t) = 1,\ t \geq 0,\ k \leq 0. \tag{2.9}$$

By using the isomorphism between $\overline{\mathcal{U}}$ and $\overline{\mathcal{U}}^{(L)}$, the solution to (2.9) induces a trajectory in the latter space.

Similarly, B) in the continuous case given $\mathbf{g} = \{g(x)\} \in \overline{\mathcal{V}}$, consider, in $\overline{\mathcal{V}}$, the initial-boundary value problem for the partial FDE

$$\dot{u}(x,t) = \frac{\partial}{\partial x} u(x,t) + \lambda \int_0^\infty dB(z)\Big[h\big(u(x-z,t)\big) - h\big(u(x,t)\big)\Big],\ t \geq 0,\ x > 0,$$

$$u(x,0) = g(x),\ x > 0, \tag{2.10}$$

$$u(x,t) = 1,\ t \geq 0,\ x \leq 0.$$

Solutions of these problems are denoted by $\mathbf{u}(\mathbf{g},t)$; by a weak solution we mean a solution to the corresponding integral equation. See Section 4. As one could expect, problem B) is formally obtained from A) by letting $L \to \infty$.

Invariant solutions $\mathbf{a}$ to (2.9) and (2.10) satisfy, respectively, the equations

$$L\,[a_k - a_{k+1}] = \lambda \sum_{l \geq 1} b^{(L)}(l/L)\big[h(a_{k-l}) - h(a_k)\big],\ k \geq 1,\ \ a_k = 1,\ k \leq 0, \tag{2.11}$$

and

$$\frac{\partial}{\partial x} a(x) = -\lambda \int_0^\infty dB(z)\big[h\big(a(x-z)\big) - h\big(a(x)\big)\big],\ x > 0,\ a(x) = 1,\ x \leq 0, \tag{2.12}$$

again by a weak solution to (2.12) we mean a solution to the corresponding integral equation.

Theorem 2 *(a) $\forall \mathbf{g} \in \overline{\mathcal{U}}$, there exists in $\overline{\mathcal{U}}$ a unique solution $\mathbf{u}(\mathbf{g},t)$, $t \geq 0$, to problem (2.9). If $\mathbf{g} \in \mathcal{U}$ then $\mathbf{u}(\mathbf{g},t) \in \mathcal{U}\ \forall\, t \geq 0$.*

(b) There exists in $\mathcal{U}$ a unique solution $\mathbf{a}$ to (2.11). Furthermore $\forall\ \mathbf{g} \in \mathcal{U}$, $\mathbf{u}(\mathbf{g},t) \to \mathbf{a}$ as $t \to \infty$.

Theorem 3 *(a) $\forall \mathbf{g} \in \overline{\mathcal{V}}$, there exists in $\overline{\mathcal{V}}$ a unique weak solution $\mathbf{u}(\mathbf{g},t)$, $t \geq 0$, to problem (2.10). If $\mathbf{g} \in \mathcal{V}$ then $\mathbf{u}(\mathbf{g},t) \in \mathcal{V}\ \forall\, t \geq 0$. If $\mathbf{g}$ is C^1 for $x > 0$ and B is absolutely continuous relative to the Lebesgue measure and has a continuous Radon – Nicodym derivative $dB(y)/dy$, $y > 0$, the above solution is strong.*

(b) There exists in $\mathcal{V}$ a unique weak solution $\mathbf{a}$ to (2.12). If B is absolutely continuous relative to the Lebesgue measure and has a continuous Radon – Nicodym derivative $dB(y)/dy$, $y > 0$, the solution is strong. Furthermore, $\forall\ \mathbf{g} \in \mathcal{V}$, $\mathbf{u}(\mathbf{g},t) \to \mathbf{a}$ as $t \to \infty$.

Denote by $\mathsf{E}_{\mathbf{g}}$ the expectation in the distribution of the MP $\{U_N^{(L)}(t)\}$ with the initial state $\mathbf{g} \in \mathcal{U}_N^{(L)}$ and define the operator semi-group of process $\{U_N^{(L)}(t)\}$ by

$$\mathbf{T}_N^{(L)}(t) f(\mathbf{g}) = \mathsf{E}_{\mathbf{g}}\, f(U(t)). \tag{2.13}$$

As before, denote by E_π the expectation in the invariant distribution π of process $\{U_N^{(L)}(t)\}$ (see Theorem 1). Given $x > 0$, the map $\mathbf{u} \in \mathcal{U}_N^{(L)} \mapsto u(x)$ defines a random variable on the probability space $\big(\mathcal{U}_N^{(L)}, \pi\big)$. Furthermore, the function $x \mapsto w_N^{(L)}(x)$, where $w_N^{(L)}(x) = \mathsf{E}_\pi\, u(x)$ determines an element $\mathbf{w}_N^{(L)} = \{w_N^{(L)}(x)\}$ of $\overline{\mathcal{U}}^{(L)}$ (or, equivalently, of sequence space $\overline{\mathcal{U}}$).

Theorem 4 *(a)* $\forall f \in \mathrm{C}(\overline{\mathcal{V}})$, *uniformly in t within a bounded interval in $\mathbf{R}_+$,*

$$\lim_{N\to\infty} \sup_{\mathbf{g}\in\mathcal{U}_N^{(L)}} \left|\mathbf{T}_N^{(L)}(t)f(\mathbf{g}) - \mathbf{u}(\mathbf{g},t)\right| \;=\; 0. \tag{2.14}$$

$$\lim_{N,L\to\infty} \sup_{\mathbf{g}\in\mathcal{U}_N^{(L)}} \left|\, \mathbf{T}_N^{(L)}(t)f(\mathbf{g}) - \mathbf{u}(\mathbf{g},t)\,\right| \;=\; 0. \tag{2.15}$$

(b) As $N \to \infty$ or $N, L \to \infty$,

$$\mathbf{w}_N^{(L)} \to \mathbf{a} \tag{2.16}$$

where $\mathbf{a}$ *is the solution to (2.11) or (2.12), respectively.*

As in [30], we can also prove a comparison theorem between QS $\overline{\mathcal{S}}_N^{(L)}$ and the isolated $M/GI/1/\infty$ queue $\overline{\mathcal{L}}^{(L)}$ described below. The distribution of the arrival batch-size in both models is the same (and equals B). On the other hand, the rate of arrival in QS $\overline{\mathcal{L}}^{(L)}$ is set to be $\lambda\theta_0$ (while that in $\overline{\mathcal{S}}_N^{(L)}$ is $N\lambda$). Finally, process $U_N^{(L)}(t)$ describing QS $\overline{\mathcal{S}}_N^{(L)}$ starts at time 0 from an initial state $\mathbf{g} = \{u(x)\} \in \mathcal{U}_N^{(L)}$, and the same vector $\mathbf{g}$ is taken as the probability distribution of the number of mini-tasks in the buffer in system $\overline{\mathcal{L}}^{(L)}$ at time 0. It is useful to write down the linear problem corresponding with QS $\overline{\mathcal{L}}^{(L)}$

$$\dot{y}_k(t) = L\big[y_{k+1}(t) - y_k(t)\big] + \lambda\theta_0 \sum_{l\geq 1} b^{(L)}(l/L)\big[y_{k-l}(t) - y_k(t)\big],\ t \geq 0,\ k \geq 1, \tag{2.17}$$

$$y_k(0) = g(k/L),\ k \geq 1,\ y_k(t) = 1,\ t \geq 0, k \leq 0, \tag{2.18}$$

where $\mathbf{g} \in \mathcal{V}$.

Denote by $\psi^{(L)}(\mathbf{g}, t)$ the number of mini-tasks in the randomly chosen buffer in QS $\overline{\mathcal{S}}_N^{(L)}$ at time t (here, each server is chosen with probability $1/N$). Furthermore, let $\kappa^{(L)}(\mathbf{g},t)$ denote the number of mini-tasks in $\overline{\mathcal{L}}^{(L)}$ at time t. The solution $\mathbf{y}(\mathbf{g},t)$ to (2.17), (2.18) gives the tail probabilities of random variable $\kappa^{(L)}(\mathbf{g},t)$.

When $L\to\infty$, the limiting $M/GI/1$ queue is denoted $\overline{\mathcal{L}}$; it is governed by the linear FDE

$$\dot{y}(x,t)=\frac{\partial y}{\partial x}(x,t)+\lambda\theta_0\int_0^\infty dB(z)\big[y(x-z,t)-y(x,t)\big],\ t\geq 0,\ x\geq 0, \qquad (2.19)$$

with the initial-boundary value

$$y(x,0)=g(x),\ x>0,\ \ y(x,t)=1,\ t\geq 0,\ x\leq 0, \qquad (2.20)$$

with $\mathbf{g}\in\mathcal{V}$. Let $\kappa(\mathbf{g},t)$ denote the (random) waiting time in the QS $\overline{\mathcal{L}}$.

Let $\prec_{\mathbf{c}}$ denote the ordering between real-valued random variables: $\xi_1\prec_{\mathbf{c}}\xi_2$ if $\mathsf{E}(\xi_1-z)_+\leq\mathsf{E}(\xi_2-z)_+\forall z\in\mathbf{R}$. Cf. [24].

Theorem 5 *$\forall t\geq 0$, and $L\geq 1$, $\psi^{(L)}(\mathbf{g},t)\prec_{\mathbf{c}}\kappa^{(L)}(\mathbf{g},t)\prec_{\mathbf{c}}\kappa(\mathbf{g},t)$.*

Furthermore, $\forall t\geq 0$ and $L\geq 1$, $\kappa^{(L)}\to\kappa(\mathbf{g},t)$ as $L\to\infty$. See again [24].

3 Limiting equations: the discrete case

In this section we prove Theorem 3. The proof of this theorem is done in parallel with the method used in [30]. For simplicity, we set $L=1$ and omit the superscript L from the notation.

Consider the truncated version of problem (2.9):

$$\begin{aligned}\dot{u}_k^{(K)}(t) &= u_{k+1}^{(K)}(t)-u_k^{(K)}(t) \\ &\quad +\lambda\sum_{l\geq 1}b(l)\left[h(u_{k-l}^{(K)}(t))-h(u_k^{(K)}(t))\right],\ t\geq 0,\ 1\leq k\leq K,\end{aligned} \qquad (3.1)$$

$$\begin{aligned}u_k^{(K)}(0) &= g_k,\ 1\leq k\leq K,\\ u_k^{(K)}(t) &= 1,\ t\geq 0,\ k\leq 0,\\ u_k^{(K)}(t) &= g_{K+1},\ t\geq 0,\ k\geq K+1.\end{aligned} \qquad (3.2)$$

Lemma 3.1 *$\forall\mathbf{g}\in\overline{\mathcal{U}}$ and $K>l^0=\min\left[l:\ b(l)>0\right]$, the (unique) solution $\mathbf{u}^{(K)}(t)=(u_k^{(K)}(t))$ to (3.1)–(3.2) belongs to $\overline{\mathcal{U}}$, $\forall t\geq 0$.*

Proof. Problem (3.1)–(3.2) comprises K ordinary differential equations and the RHS of these equations is C^2. This guarantees the existence and uniqueness of the (global) solution $\mathbf{u}^{(K)}(t)$, as well as its continuous dependence on the initial value $\mathbf{g}$. To check that $\mathbf{u}^{(K)}(t) \in \overline{\mathcal{U}}$, it suffices to verify the monotonicity in k of $u_k^{(K)}(t)$, $\forall\, t \geq 0$. Owing to the continuity in the initial value, it is enough to assume that $g_{k+1} < g_k$, $\forall\, k \geq 1$ and verify that the same (strict) inequalities hold for $u_k^{(K)}(t)$, $\forall\, t \geq 0$, $k \leq K+1$. For simplicity, the superscript (K) in the notation $u_k^{(K)}(t)$ will be omitted till the end of the proof of the lemma.

Assume that $t^0 > 0$ is the first time when one of the inequalities fails. As $g_{K+1} < 1$, one of two possibilities occurs: either a) $\exists\, k$, $1 < k \leq K$, such that $u_{k-1}(t^0) = u_k(t^0) > u_{k+1}(t^0)$, or b) $u_k(t^0) \equiv g_{K+1}$, $1 \leq k \leq K$.

In case a), we compare

$$\dot{u}_{k-1}(t^0) = \lambda \sum_{l \geq 1} b(l) \left[h\left(u_{k-1-l}(t^0)\right) - h\left(u_{k-1}(t^0)\right) \right] \tag{3.3}$$

and

$$\dot{u}_k(t^0) = \left[u_{k+1}(t^0) - u_k(t^0) \right] + \lambda \sum_{l \geq 1} b(l) \left[h\left(u_{k-l}(t^0)\right) - h\left(u_k(t^0)\right) \right] \tag{3.4}$$

The summand $u_{k+1}(t^0) - u_k(t^0)$ in the RHS of (3.4) is strictly negative. Furthermore, $\forall\, l$,

$$h\left(u_{k-1-l}(t^0)\right) - h\left(u_{k-1}(t^0)\right) \geq h\left(u_{k-l}(t^0)\right) - h\left(u_k(t^0)\right).$$

Thus, $\dot{u}_{k-1}(t^0) > \dot{u}_k(t^0)$ which leads to the contradiction.

On the other hand, in case b) we have

$$\dot{u}_k(t^0) = \lambda \sum_{l \geq k} b(l) \big[h(1) - h\big(g(K+1)\big) \big], \ 1 \leq l \leq K.$$

With $K > l^0$ this implies that $\dot{u}_1(t^0) > \dot{u}_K(t^0)$ which again leads to the contradiction. □

Lemma 3.2 *$\forall \mathbf{g}, \mathbf{g}' \in \overline{\mathcal{U}}$ with $g_k \geq g'_k$, $k \geq 1$, and $K, K' > l^0 = \min\left[l :\ b(l) > 0 \right]$, the solutions $\mathbf{u}^{(K)}(\mathbf{g}, t)$ and $\mathbf{u}^{(K')}(\mathbf{g}', t)$ to the corresponding truncated problems (3.1)–(3.2) obey $u_k^{(K)}(\mathbf{g}, t) \geq u_k^{(K')}(\mathbf{g}', t)$ $\forall t \geq 0$, $k \geq 1$.*

Proof. Arguing as before, it suffices to check that if $g_k > g'_k$, $k \geq 1$, then the strict inequalities hold, $\forall t > 0$, between $u_k^{(K)}(\mathbf{g}, t)$ and $u_k^{(K')}(\mathbf{g}', t)$. Again assume that $t^0 > 0$ is the first time when one of the inequalities breaks, and let $\overline{k}$, $1 \leq \overline{k} \leq K, K'$, be the maximal value for which $u_{\overline{k}}^{(K)}(\mathbf{g}, t^0) = u_{\overline{k}}^{(K')}(\mathbf{g}', t^0)$. Then it is easily seen that $\dot{u}_{\overline{k}}^{(K)}(\mathbf{g}, t^0) > \dot{u}_{\overline{k}}^{(K)}(\mathbf{g}', t^0)$, which is a contradiction. □

Lemma 3.3 $\forall\, \mathbf{g} \in \overline{\mathcal{U}}$ *and* $t > 0$, $\exists$ *the limit*

$$\lim_{K \to \infty} \mathbf{u}^{(K)}(\mathbf{g}, t) = \mathbf{u}(\mathbf{g}, t), \tag{3.5}$$

giving a unique solution in $\overline{\mathcal{U}}$ *to problem (2.9).*

Proof. The existence of the limit in (3.5) and the fact that $\mathbf{u}(\mathbf{g}, t) \in \overline{\mathcal{U}}$ follows from the monotonicity property established in Lemma 3.2. From the form of the convergence it follows that $\mathbf{u}(\mathbf{g}, t)$ satisfies the integral version of problem (2.9), while the Lipschitz property of the RHS guarantees the uniqueness. □

Lemmas 3.1–3.3 immediately imply

Lemma 3.4 $\forall \mathbf{g}, \mathbf{g}' \in \overline{\mathcal{U}}$ *with* $g'_k \geq g''_k$, $k \geq 1$, *the solutions* $\mathbf{u}(\mathbf{g}', t)$ *and* $\mathbf{u}(\mathbf{g}'', t)$ *to (2.9) obey* $\mathbf{u}_k(\mathbf{g}, t) \geq \mathbf{u}_k(\mathbf{g}', t)$ $\forall t \geq 0$ *and* $k \geq 1$.

An important role in the future analysis is played by the quantity

$$v_k(\mathbf{u}) = \sum_{l \geq k} u_l, \quad k \geq 1, \ \mathbf{u} \in \mathcal{U}, \tag{3.6}$$

and, given a solution $\mathbf{u}(\mathbf{g}, t)$ to problem (2.9), we set $v_k(\mathbf{g}, t) = v_k\big(\mathbf{u}(\mathbf{g}, t)\big)$.

Lemma 3.5 *Assume that* $\mathbf{g} \in \mathcal{U}$. *Then, for* $\forall t > 0$, *the solution* $\mathbf{u}(\mathbf{g}, t)$ *to problem (2.9) belongs to* $\mathcal{U}$. *Moreover,*

$$\dot{v}(\mathbf{g}, t) \leq \lambda \theta_1 \sum_{l \geq 1} l b(l) - u_1(t), \tag{3.7}$$

where θ_1 *is defined in (2.4).*

Proof. We have:

$$\dot{u}_k(t) \leq \lambda \theta_1 \sum_{l \geq 1} b(l) \big[u_{k-l}(t) - u_k(t)\big] + u_{k+1}(t) - u_k(t).$$

Summing up over $k \geq 1$ yields (3.7). □

Lemma 3.6 (i) *Problem (2.11) has in $\mathcal{U}$ a unique solution,* $\mathbf{a}$.

(ii) *If* $g_k \le a_k$, $k \ge 1$, *or* $g_k \ge a_k$, $k \ge 1$, *then* $\mathbf{u}(\mathbf{g}, t) \to \mathbf{a}$ *as* $t \to \infty$.

(iii) $\lim_{t\to\infty} \mathbf{u}(\mathbf{g}, t) = \mathbf{a}$, $\forall \mathbf{g} \in \mathcal{U}$.

Proof. (i) It can be checked straight away that the element $\mathbf{a}$ with the values a_k, $k \ge 1$, recursively calculated from the equation

$$a_k = \lambda \sum_{l\ge 1} b(l) \sum_{1\le z\le l} h\big(a_{k-z}\big), \ k \ge 1, \tag{3.8}$$

with the boundary condition $a_k = 1$, $k \le 0$, gives a solution to (2.11) and belongs to $\overline{\mathcal{U}}$. Note that $a_1 = \lambda h(1) \sum_{\lambda\ge 1} b(l) l$. We are going to check that $\mathbf{a} \in \mathcal{U}$; the uniqueness will then follow from assertion (iii).

In turn, assertion (iii) follows from (ii) and Lemma 3.4, as we can replace the initial values $g(x)$ by max $\big[g(x), a(x)\big]$ or min $\big[g(x), a(x)\big]$.

So, let us check that $\mathbf{a} \in \mathcal{U}$. Take $T \ge 1$ and assess the sum $\sum_{1\le k\le T} a_k$. Then,

$$\begin{aligned}
\sum_{1\le k\le T} a_k &= \lambda h(1) \sum_{1\le k\le T} \sum_{l\ge k} b(l)(l-k) + \lambda \sum_{1\le k\le T} \sum_{1\le z\le k-1} h\left(a_{k-z}\right) \sum_{l\ge z} b(l) \\
&= \lambda h(1) \sum_{1\le k\le T} \sum_{l\ge 1} l b(l+k) + \lambda \sum_{1\le r\le T} h\left(a_r\right) \sum_{1\le z\le T-r} \sum_{l\ge z} b(l) \\
&\le \lambda h(1) \sum_{l\ge 1} l \sum_{k\ge l+1} b(k) + \lambda \sum_{1\le r\le T} h\left(a_r\right) \sum_{z\ge 1} \sum_{l\ge z} b(l) \\
&\le \lambda h(1) \sum_{k\ge 1} b(k) \frac{k(k-1)}{2} + \lambda\theta_0 \sum_{1\le r\le T} a_r \sum_{k\ge 1} b(k) k.
\end{aligned}$$

In other words,

$$\Big(1 - \lambda\theta_0 \sum_{l\ge 1} b(l) l\Big) \sum_{1\le k\le T} a_k \le \frac{1}{2} \lambda h(1) \sum_{l\ge 1} b(l) l(l-1),$$

i.e.,

$$\sum_{1\le k\le T} a_k \le \frac{1}{2} \lambda h(1) \sum_{l\ge 1} b(l) l(l-1) \Big(1 - \lambda\theta_0 \sum_{l'\ge 1} b(l') l'\Big)^{-1}. \tag{3.9}$$

As the RHS of (3.9) does not depend on T, we get that the series $\sum_{k\geq 1} a_k$ converges and

$$\sum_{k\geq 1} a_k \leq \frac{1}{2}\lambda h(1)\sum_{l\geq 1} b(l)l(l-1)\Big(1-\lambda\theta_0\sum_{l'\geq 1} b(l')l'\Big)^{-1}. \tag{3.10}$$

(ii) First, we show that, under the condition of this assertion, $v_1(\mathbf{g},t)$ remains bounded uniformly in $t\geq 0$ (and so are $v_k(\mathbf{g},t)$, $k>1$). In fact, if $g_k\leq a_k$, $k\geq 1$, then, by virtue of Lemma 3.4, $u_k(\mathbf{g},t)\leq a_k$, $k\geq 1$, and thus $v_1(\mathbf{g},t)\leq v(\mathbf{a})$, $t\geq 0$. On the other hand, if $g_k\geq a_k$, $k\geq 1$, then

$$\dot{v}_1(\mathbf{g},t) = -u_1(\mathbf{g},t)+\lambda h(1)\sum_{l\geq 1} b(l)l = -u_1(\mathbf{g},t)+\ a_1\leq 0.$$

The rest of the proof of assertion (ii) does not differ from the corresponding part of the proof of Lemma 7(c) from [30]. Namely, if $g_k\geq a_k$, $k\geq 1$, then the convergence $\mathbf{u}(\mathbf{g},t)\to\mathbf{a}$ follows from the relations

$$\int_0^\infty ds\big(u_k(s)-a_k\big)<\infty,\quad k\geq 1, \tag{3.11}$$

whereas in the case $g_k\leq a_k$, $k\geq 1$, it follows from

$$\int_0^\infty ds\big(a_k-u_k(s)\big)<\infty,\quad k\geq 1. \tag{3.12}$$

The last relations are checked similarly, and we discuss, say, the first of them.

We use the induction in k, starting with $k=1$. Here, $\forall t\geq 0$,

$$\begin{aligned} -v_1(\mathbf{g},t)+v_1(\mathbf{g},0) &= -\int_0^t ds\,\dot{v}_1(\mathbf{g},s) = \int_0^t ds\,\big(u_1(\mathbf{g},s)-\lambda h(1)\big)\\ &= \int_0^t ds\,\big(u_1(\mathbf{g},s)-a_1\big). \end{aligned}$$

As $t\to\infty$, the LHS remains bounded, and thus the integral in the RHS tends to a finite limit as $t\to\infty$. Assume that the integral (3.11) converges for $k\leq K-1$. Then

$$\begin{aligned} -v_K(\mathbf{g},t)+v_K(\mathbf{g},0) &= -\int_0^t ds\,\dot{v}_K(\mathbf{g},s)\\ &= \int_0^t ds\Big[u_K(\mathbf{g},s)-\lambda\sum_{l\geq 1} b(l)\sum_{0\leq j<l} h\big(u_{K-1-j}(s)\big)\Big]. \end{aligned}$$

In view of (3.8), we can re-write the last integral as

$$\int_0^t ds\Big[\big(u_K(\mathbf{g},s)-a_K\big)-\lambda\sum_{l\geq 1}b(l)\sum_{0\leq j<l}\big[h\big(u_{K-1-j}(s)\big)-h\big(a_{K-1-j}(s)\big)\big]\Big].$$

By the induction hypothesis, this equals

$$\int_0^t ds\big(u_K(\mathbf{g},s)-a_K\big)+J_K(t)$$

where $J(t)$ has a finite limit as $t\to\infty$. Thus, (3.11) converges for $k=K$. □

Theorem 3 can be straightforwardly deduced from the above lemmas. Following the plan of paper [30], in the rest of Section 2 we establish auxiliary Lemmas 3.7–3.10 used in the proof of Theorem 5. In the course of the exposition we use an elementary bound on the solution of a linear system of ODEs (cf. Proposition 1 from [30], Sect. 3).

Lemma 3.7 *Consider an infinite linear system*

$$\dot{z}_k(t)=\sum_{i=0}^{\infty}A_{k,i}(t)z_i(t)+B_k(t),\quad k\geq 1,\ t\geq 0,$$

and let the following inequalities be valid: $\sum_{i\geq 1}^{\infty}|A_{k,i}(t)|\leq A$, $|B_k(t)|\leq B_0e^{Bt}$, $|z_k(0)|\leq C$, *with* $B_0, C\geq 0$, $0<A<B$. *Then the solution obeys*

$$|z_k(t)|\leq C\exp(At)+\frac{B_0}{B-A}\big(\exp(Bt)-\exp(At)\big),\quad k\geq 1.$$

Applying Lemma 3.7, we obtain

Lemma 3.8 *The solution* $\mathbf{u}(\mathbf{g},t)$ *to problem (2.9) obeys, for* $t\geq 0,\ k,i,j\geq 1$,

$$\Big|\frac{\partial}{\partial g_j}u_k(\mathbf{g},t)\Big|\ \leq\ \exp\big[(2+2\theta_1)t\big], \tag{3.13}$$

$$\Big|\frac{\partial^2}{\partial g_i\partial g_j}u_k(\mathbf{g},t)\Big|\ \leq\ \frac{\lambda\theta_2}{1+\lambda\theta_1}\exp\big[(2+2\theta_1)t\big]\left(\exp\big[(2+2\theta_1)t\big]-1\right). \tag{3.14}$$

We also consider the linear system (cf. (2.17))

$$\dot{y}_k(t) = y_{k+1}(t) - y_k(t) + \lambda\theta_0 \sum_{l\geq 1} b(l)\big[y_{k-l}(t) - y_k(t)\big],\ t\geq 0,\ k\geq 1, \tag{3.15}$$

with one of the following initial-boundary values:

$$y_k(0) = g_k,\ k\geq 1,\ u_k(t) = 1,\ t\geq 0, k\leq 0, \tag{3.16}$$

or

$$y_k(0) = g_k,\ k\geq 1,\ u_k(t) - u_1(t) = 1,\ t\geq 0, k\leq 0, \tag{3.17}$$

The solution to (3.15), (3.16), (3.17) is denoted below by $\mathbf{y}(\mathbf{g}, t)$.

Let $\mathcal{Y}$ denote the set of the sequences $\mathbf{y} = (y_k,\, k\in\mathbf{Z})$ with

$$1 = \cdots = y_{-1} - y_0 = y_0 - y_1 \geq y_1 - y_2 \geq \cdots \geq 0,\quad \lim_{k\to\infty} y_k = 0. \tag{3.18}$$

A direct corollary of Lemmas 3.1–3.5 is

Lemma 3.9 *$\forall$ $\mathbf{g}\in\mathcal{Y}$ and $t\geq 0$, there exists in $\mathcal{Y}$ a unique solution $\mathbf{y}^{(0)}(\mathbf{g},t) = \big(y^{(0)}_k(\mathbf{g},t)\big)$, to (3.15), (3.17). This solution can be represented in the form $y^{(0)}_k(\mathbf{g},t) = \sum_{j\geq k} y'_j(t)$ where $\mathbf{y}' = \big(y'_k(t)\big)$ is the solution to* (3.15), *(3.16), and where the initial condition reads $y'_k(0) = g_k - g_{k+1}$, $k\geq 1$.*

In Lemma 3.10 we discuss the relations between problems (3.15), (3.16) and (3.17), and the system of linear differential inequalities

$$\dot{w}(t) \leq w_{k+1}(t) - w_k(t) + \lambda\theta_0 \sum_{l\geq 1} b(l)\big[w_{k-l}(t) - w_k(t)\big],\ \ t\geq 0,\ k\geq 1, \tag{3.19}$$

completed with the boundary values

$$w_k(t) - w_1(t) = 1,\ t\geq 0,\ k\leq 0. \tag{3.20}$$

Lemma 3.10 *Let $\mathbf{w}(t) = \big(w_k(t)\big)\in\mathcal{Y}$, $t\geq 0$, satisfy (3.19), (3.20). Let $\mathbf{y} = \big(y_k(t)\big)$, $t\geq 0$, be the solution to (3.15), (3.17), with $g_k\geq w_k(0)$, $k\geq 1$. Then, $\forall t>0$, $y_k(t)\geq w_k(t)$, $k\geq 1$.*

The proof of Lemma 3.10 is similar to that of Lemma 14 from [30], Sect. 3, and we omit it from the paper.

4 Limiting equations: the continuous case

We now pass to the proof of Theorem 3. It is convenient to work with an auxiliary Cauchy problem

$$\begin{aligned}\frac{\partial\varphi}{\partial t}(z,t) &= \lambda h(1)\int_{z-t}^{\infty} dB(y)\\ &\quad+\lambda\int_0^{(z-t)-} dB(y)\,h\big(\varphi(z-y,t)\big)-\lambda h\big(\varphi(z,t)\big),\ \ 0<t<z, \qquad (4.1)\\ \varphi(0,z) &= g(z),\ \ z>0.\end{aligned}$$

The corresponding integral equation is

$$\varphi(z,t)=G(z,t)+\big(\mathbf{G}\varphi\big)(z,t),\ \ 0<t<z, \tag{4.2}$$

where

$$G(z,t)=g(z)+\lambda h(1)\int_0^t ds\int_{z-s}^{\infty} dB(y),\ \ 0<t<z, \tag{4.3}$$

and the integral operator $\mathbf{G}$ is given by

$$\big(\mathbf{G}\varphi\big)(z,t)=\lambda\int_0^t ds\Big[\int_0^{(z-s)-} dB(y)\,h\big(\varphi(z-y,s)\big)-\lambda h\big(\varphi(z,s)\big)\Big],\ \ 0<t<z, \tag{4.4}$$

It is easy to see that solutions $\mathbf{u}(\mathbf{g},t)=\big(u(\mathbf{g},x,t)\big)$ and $\Phi(\mathbf{g},t)=\varphi(\mathbf{g},z,t)$ to (2.10) and (4.1) are tied via the equality

$$u(x,t)=\begin{cases}\varphi(t+x,t), & x>0,\ t\geq 0,\\ 1, & x\leq 0,\ t\geq 0.\end{cases} \tag{4.5}$$

Correspondingly, the properties of solution $\mathbf{u}(\mathbf{g},t)$ are translated into properties of $\varphi(\mathbf{g},t)$; in what follows, we repeatedly refer to this fact by writing $\varphi(\mathbf{g},t)\in\overline{\mathcal{V}}$, $\varphi(\mathbf{g},t)\in\mathcal{V}$, etc. A weak solution to (4.1) is defined as a solution to (4.2).

Lemma 4.1 *$\forall\mathbf{g}\in\overline{\mathcal{V}}$, there exists a unique weak solution $\Phi(\mathbf{g},s)=\big(\varphi(\mathbf{g},z,s)\big)$ to problem (4.1). Furthermore, this solution belongs to $\overline{\mathcal{V}}$ $\forall t\geq 0$. Finally, if $\mathbf{g}$ is C^1 and B is absolutely continuous w.r.t. to Lebesgue measure, with a continuous derivative $dB(y)/dy$, $y>0$, then $\Phi(\mathbf{g},t)$ is the strong solution.*

Proof. We use the standard (Picard) scheme of subsequent approximations. Set

$$\begin{aligned}\varphi^0(z,t) &= G(z,t),\\ \varphi^r(z,t) &= G(z,t)+\big(\mathbf{G}\varphi^{r-1}\big)(z,t),\ \ 0\le t<z,\ \ r=1,2,\dots\ ,\end{aligned}\tag{4.6}$$

and

$$\varepsilon_r(t)=\sup_{0\le s\le t}\sup_{s<z}\big|\varphi^r(z,s)-\varphi^{r-1}(z,s)\big|\,,\ \ t\ge 0, r\ge 1.\tag{4.7}$$

A straightforward calculation shows that

$$\varepsilon_1(t)\le 2\lambda\theta_0\big(t+\lambda h(1)t^2/2\big).\tag{4.8}$$

Furthermore, for $t\ge 0$, $r\ge 1$,

$$\begin{aligned}&\big|\varphi^{r+1}(z,t)-\varphi^r(z,t)\big|\\ &\quad\le\ \lambda\theta_1\int_0^t dt_1\bigg[\int_{z-t_1}^\infty dB(y)\,\big|\varphi^r(z-y,t_1)-\varphi^{r-1}(z-y,t_1)\big|\\ &\qquad\qquad +\big|\varphi^r(z,t_1)-\varphi^{r-1}(z,t_1)\big|\bigg]\\ &\quad\le\ 2\lambda\theta_1\int_0^t dt_1\varepsilon_r(t_1).\end{aligned}\tag{4.9}$$

The last inequality implies that

$$\varepsilon_{r+1}(t)\le 2\lambda\theta_1\int_0^t dt_1\varepsilon_r(t_1),\ \ t\ge 0,\ r\ge 1,$$

and after iterating

$$\begin{aligned}\varepsilon_{r+1}(t)\ &\le\ (2\lambda\theta_1)^r\int_0^t dt_1\int_0^{t_1}dt_2\cdots\int_0^{t_{r-1}}dt_r\varepsilon_1(t_r)\\ &\le\ \big(2\lambda\theta_0\big)^{r+1}(t+\lambda h(1)t^2/2)t^r/r!,\ \ t\ge 0,\ r\ge 1.\end{aligned}\tag{4.10}$$

Therefore, $\forall T>0$, the φ^r's converge as $r\to\infty$, uniformly in the strip $0\le t\le T$, $z>t$. Plainly, the limiting function obeys (4.2).

The uniqueness is again established by a standard argument: suppose that $\Phi'(\mathbf{g},t)=\big(\varphi'(\mathbf{g},z,t)\big)$ and $\Phi''(\mathbf{g},t)=\big(\varphi''(\mathbf{g},z,t)\big)$ are two solutions to (4.2) and set

$$\bar\varepsilon(t)=\sup_{z>t}\big|\varphi'(\mathbf{g},z,t)-\varphi''(\mathbf{g},z,t)\big|.$$

Then $\bar{\varepsilon}(t) \leq 1$ and

$$\left|\varphi'(\mathbf{g}, z, t) - \varphi''(\mathbf{g}, z, t)\right| \leq 2\lambda\theta_1 \int_0^t dt_1\, \bar{\varepsilon}(t_1), \quad t \geq 0.$$

The last inequality yields

$$\bar{\varepsilon}(t) \leq \int_0^t dt_1\, \bar{\varepsilon}(t_1), \quad t \geq 0,$$

which, after r iterations gives

$$\bar{\varepsilon}(t) \leq \int_0^t dt_1 \int_0^{t_1} dt_2 \ldots \int_0^{t_r} dt_{r+1} \bar{\varepsilon}(t_{r+1}) \leq t^r/r!, \quad t \geq 0.$$

As r can be made arbitrarily large, we conclude that $\bar{\varepsilon}(t) = 0$, i.e., $\Phi'(\mathbf{g}, t) = \Phi''(\mathbf{g}, t)$, $t \geq 0$.

The key moment of the proof that $\Phi(\mathbf{g}, t)$ belongs to $\overline{\mathcal{V}}$ is to check the monotonicity properties. First let us assume that

$$y^0 = \inf\left[y > 0 : \; B(y) > 0\right] > 0 \tag{4.11}$$

and consider function $\varphi(z, t)$ for $0 \leq t < z < y^0$ (argument $\mathbf{g}$ is omitted when convenient). Here, $\varphi(z, t)$ obeys, according to (4.1),

$$\frac{\partial \varphi}{\partial t}(z, t) = \lambda\big(h(1) - h(\varphi(z, t))\big), \quad \varphi(0, z) = g(z), \tag{4.12}$$

which is an ordinary differential equation (variable z is simply a parameter labeling the initial value $g(z)$). A straightforward comparison argument, for the vector field $(t, \varphi) \mapsto h(1) - h(\varphi)$ determined by the RHS of (4.12) (in the strip $t \geq 0$, $0 \leq \varphi \leq 1$, which forms the phase plane of the dynamical system (4.12)) shows that $\varphi(z, t)$ increases with $g(z)$, which gives that if $0 \leq t < z' < z'' < y^0$ then

$$\varphi(\mathbf{g}, z', t) \geq \varphi(\mathbf{g}, z'', t), \tag{4.13}$$

and if $0 \leq t < z < y^0$ and $g'(z) \geq g''(z)$ then

$$\varphi(\mathbf{g}', z, t) \geq \varphi(\mathbf{g}'', z, t), \tag{4.14}$$

i.e., both the monotonicity in z and in $\mathbf{g}$. In particular, setting $\varphi(\mathbf{g}, z - y^0, t) = 1$, we have, for $0 \leq t < z < y^0$,

$$\varphi(\mathbf{g}, z, t) \leq \varphi(\mathbf{g}, z - y^0, t). \tag{4.15}$$

Next, assume that the above monotonicity properties (4.13)–(4.14) are established when, respectively, $0 \leq t < z' < z'' < Ky^0$ and $0 \leq t < z < Ky^0$, $g'(z) \geq g''(z)$, where K is a natural number. Consider function $\varphi(z,t)$ in the region $0 \leq t < z$, $Ky^0 \leq z < (K+1)y^0$. Here, again according to (4.1), $\varphi(z,t)$ obeys an ordinary differential equation in variable t:

$$\frac{\partial \varphi}{\partial t}(z,t) = H(z,t) - \lambda h\big(\varphi(z,t)\big), \tag{4.16}$$

where

$$H(z,t) = \lambda h(1) \int_{z-t}^{\infty} dB(y) + \lambda \int_{0}^{(z-t)-} dB(y) h\big(\varphi(z-y,t)\big) \tag{4.17}$$

is considered as a known quantity. The vector field of dynamical system (4.17) is given by $(t,\varphi) \mapsto H(z,t) - \lambda h(\varphi)$ and defined again in the strip $t \geq 0$, $0 \leq \varphi \leq 1$. As before, the direct comparison argument shows that $\varphi(z,t)$ satisfies the same inequalities (4.13)–(4.15) in the region under examination (the fact that the vector field is in general not continuous does not matter as we deal with weak solutions). Thus, by induction, (4.13)–(4.15) hold when $0 \leq t < z$.

To get rid of assumption (4.11), we perform an additional limiting procedure, by setting

$$B_\eta(y) = \mathbf{1}\{y > \eta\} B(y), \quad \eta \to 0+. \tag{4.18}$$

Denote by $\Phi_\eta(\mathbf{g},t) = (\varphi_\eta(\mathbf{g},z,t))$ the solution to (4.2) for the distribution function B_η (in general, the subscript η will be used for denoting the entries of problem (4.2) for B_η). Clearly, the corresponding Picard approximants $\varphi^r_\eta(z,t)$, $r \geq 0$, converge to $\varphi^r(z,t)$ (see (4.6)) as $\eta \to 0$, but to establish the convergence $\varphi_\eta(\mathbf{g},z,t) \to \varphi(\mathbf{g},z,t)$ we need a uniform bound. Set

$$\widetilde{\varepsilon}_r(t,\eta) = \sup_{0 \leq s \leq t,\ s<z} \big|\ \varphi^r(z,s) - \varphi^r_\eta(z,s)\ \big|,\ t \geq 0. \tag{4.19}$$

Then

$$\widetilde{\varepsilon}_0(t,\eta) = \sup_{0 \leq s \leq t,\ s<z} \big|G(z,s) - G_\eta(z,s)\big| \leq \lambda h(1)\eta t. \tag{4.20}$$

Furthermore, after a straightforward calculation, we obtain that

$$\begin{aligned}\left|\varphi^r(z,t)-\varphi^r_\eta(z,t)\right| &\leq \left|G(z,t)-G_\eta(z,t)\right|+\left|\left(\mathbf{G}\varphi^{r-1}\right)(z,t)-\left(\mathbf{G}\varphi^{r-1}_\eta\right)(z,t)\right| \\ &\quad +\left|\left(\mathbf{G}\varphi^{r-1}_\eta\right)(z,t)-\left(\mathbf{G}_\eta\varphi^{r-1}_\eta\right)(z,t)\right| \\ &\leq \lambda t h(1)\eta+2\lambda\theta_1\int_0^t dt_1\widetilde{\varepsilon}_{r-1}(t_1,\eta)+\lambda\theta_0\eta,\end{aligned}$$

which yields the bound

$$\widetilde{\varepsilon}_r(t,\eta)\leq 2\lambda\theta_1\int_0^1 dt_1\left(\widetilde{\varepsilon}_{r-1}(t_1,\eta)+\eta\right).$$

After $r-1$ iterations we get

$$\begin{aligned}\widetilde{\varepsilon}_r(t,\eta) &\leq (2\lambda\theta_1)^{r-1}\int_0^1 dt_1\int_0^{t_1}dt_2\cdots\int_0^{t_{r-1}}dt_r\widetilde{\varepsilon}_0(t_r,\eta)+\eta\sum_{1\leq j<r}(2\lambda\theta_1 t)^j/j! \\ &\leq \eta\exp\left(2\lambda\theta_1 t\right). \qquad (4.21)\end{aligned}$$

The RHS of (4.21) does not depend on r: this leads to the bound

$$\left|\varphi(\mathbf{g},z,t)-\varphi_\eta(\mathbf{g},z,t)\right|\leq\eta\exp\left(2\lambda\theta_1 t\right). \qquad (4.22)$$

Thus, $\Phi(\mathbf{g},t)$ inherits the monotonicity properties which leads to the conclusion that $\Phi(\mathbf{g},t)\in\overline{\mathcal{V}}$.

Finally, if $g(x)$ is smooth in x and B is Lebesgue absolutely continuous and has a continuous Radon−Nicodym derivative, then the weak solution $\Phi(\mathbf{g}t)$ is in fact strong. This completes the proof of Lemma 4.1. □

Lemma 4.2 *If* $\mathbf{g}\in\mathcal{V}$ *then* $\Phi(\mathbf{g},t)\in\mathcal{V}\ \forall t>0$.

Proof. Assume that $\mathbf{g}\in\mathcal{V}$. We want to check that $\forall t>0$, $\Phi(\mathbf{g},t)\in\mathcal{V}$. Take $Z>0$ and assess the integral

$$\int_t^Z dz\,\varphi(\mathbf{g},z,t)=\int_t^Z dz\,G(z,t)+\int_t^Z dz\int_0^t ds\,\left(\mathbf{G}\varphi\right)(z,s). \qquad (4.23)$$

Under the agreement that $\varphi(z,t) = 1$ for $t \geq 0$, $z \leq t$, the RHS of (4.23) does not exceed

$$\int_0^\infty dz\, g(z) + \lambda \int_0^t ds \int_0^\infty dB(y) \int_0^Z dz \big[h\big(\varphi(z-y,s)\big) - h\big(\varphi(z,s)\big)\big]$$
$$= \int_0^\infty dz g(z) + \lambda \int_0^t ds \int_0^\infty dB(y) \int_0^y dw \big[h\big(\varphi(-w,s)\big) - h\big(\varphi(Z-w,s)\big)\big]$$
$$\leq \int_0^\infty dz g(z) + \lambda\theta_0 t \int_0^\infty dB(y) y.$$

The last expression is finite and does not depend on Z. Thus,

$$\int_t^\infty dz\, \varphi(\mathbf{g},z,t) \leq \int_0^\infty dz\, g(z) + \lambda\theta_0 t \int_0^\infty dB(y)\, y. \tag{4.24}$$

□

Assertion (a) of Theorem 3 follows directly from Lemmas 4.1 and 4.2. Furthermore, the argument used in the proof of Lemma 4.1 allows us to establish

Lemma 4.3 *$\forall \mathbf{g}, \mathbf{g}' \in \overline{\mathcal{V}}$ with $g'(x) \geq g''(z)$, $z > 0$, the solutions $\Phi(\mathbf{g}',t)$ and $\Phi(\mathbf{g}'',t)$ to (4.1) obey $\varphi(\mathbf{g},z,t) \geq \varphi(\mathbf{g}',x,t)\ \forall\ t \geq 0,\ t < z$.*

Lemma 4.4 (i) *Problem (2.12) has in $\mathcal{V}$ a unique weak solution. If B is absolutely continuous w.r.t. to the Lebesgue measure, with continuous derivative $dB(y)/dy$, $y > 0$, then this solution is strong.*

(ii) *If $g(x) \leq a(x)$, $x > 0$, or $g(x) \geq a(x)$, $x > 0$, then $\mathbf{u}(\mathbf{g},t) \to \mathbf{a}$ as $t \to \infty$.*

(iii) $\lim\limits_{t\to\infty} \mathbf{u}(\mathbf{g},t) = \mathbf{a}$, $\forall \mathbf{g} \in \mathcal{U}$.

Proof. (i) Consider problem (2.12) with the additional condition $\lim_{x\to\infty} a(x) = 0$. Then the corresponding integral equation reads:

$$a(x) = \lambda \int_x^\infty dy \int_0^\infty dB(z) \left[h\big(a(y-z)\big) - h\big(a(y)\big)\right] = A(x) + (\mathbf{A}a)(x),\ x > 0, \tag{4.25}$$

with the boundary condition $a(x) = 1$, $x \leq 0$. Here

$$A(x) = \lambda h(1) \int_x^\infty dB(y)\,(y-x),\ x > 0, \tag{4.26}$$

and

$$(\mathbf{A}a)(x) = \lambda \int_{0+}^{x} dy \int_{y}^{\infty} dB(z)\, h(a(x-y)).\ x > 0, \tag{4.27}$$

This is a second kind Fredholm equation which again can be solved by the Picard approximation method. The uniqueness of the solution can also be proved by a standard argument, like the one used in the proof of Lemma 4.1. Furthermore, if B is Lebesgue absolutely continuous and has a continuous Radon – Nicodym derivative $dB(y)/dy$, $y > 0$, then $a(x)$ is C^1 in $x > 0$, i.e., it gives the strong solution to (2.12).

Note that, similarly to the discrete case, $a(0+) = \lambda h(1) \int_0^\infty dB(y)\, y$. Also, $\varphi(\mathbf{a}, z, t) = a(z - t)$, $0 \le t < z$.

We have to verify that $\mathbf{a} \in \overline{\mathcal{V}}$. As in the proof of Lemma 4.1, the key point here is to verify the monotonicity of $a(x)$ in x. The method is again similar to the one used in the proof of Lemma 4.1: first we assume that (4.11) holds and establish that $a(x)$ does not increase with x by a direct inspection. Then condition (4.11) is removed by using approximation B_η (see (4.18)).

It remains to check that $\mathbf{a} \in \mathcal{V}$. The argument below is similar to that from the proof of Lemmas 4.2 and 3.6(i). However, we give the sketch of the proof here (although omit purely technical details). Take $X > 0$ and assess the integral

$$\int_{0+}^{X} dx\, a(x) = \int_0^X dx\, A(x) + \int_0^X dx\, (\mathbf{A}a)(x). \tag{4.28}$$

The RHS of (4.28) does not exceed

$$\lambda h(1) \int_0^\infty dB(y)\, y^2/2 + \lambda\theta_0 \int_0^\infty dB(y)', y \int_0^X dx\, a(x),$$

which gives that

$$\int_0^X dx\, a(x) \le \frac{1}{2}\lambda h(1) \int_0^\infty dB(y)\, y^2 \Big(1 - \lambda\theta_0 \int_0^\infty dB(y')\, y'\Big)^{-1}.$$

The last expression is finite and does not depend on X. Thus,

$$\int_0^\infty dx a(x) \le \frac{1}{2}\lambda h(1) \int_0^\infty dB(y) y^2 \Big(1 - \lambda\theta_0 \int_0^\infty dB(y') y'\Big)^{-1}. \tag{4.29}$$

Observe that, as in the discrete case, assertion (iii) follows from (ii) and Lemma 4.3. Thus, to complete the proof of Lemma 4.4, it remains to prove assertion (ii).

(ii) As before, denote

$$v(x, \mathbf{g}) = \int_x^\infty dz\, g(z), \quad \mathbf{g} \in \mathcal{U}, \tag{4.30}$$

and set $v(x, \mathbf{g}, t) = v\big(x, \Phi(\mathbf{g}, t)\big)$, for the solution $\Phi(\mathbf{g}, t)$ to problem (4.1). For $x = 0$, the argument 0 in the notation $v(0, \mathbf{g})$ and $v(0, \mathbf{g}, t)$ is omitted. Bound (4.24) simply reads

$$v\big(\Phi(\mathbf{g}, t)\big) \leq v(\mathbf{g}, t) + \lambda\theta_0 t \int_{+0}^\infty dB(y)\, y.$$

Our first aim, however, is to show that, under the condition of assertion (ii), $(\mathbf{g}, t)$ remains bounded uniformly in $t \geq 0$. If $g(x) \leq a(x)$, $x > 0$, then, by virtue of Lemma 4.3, $v(\mathbf{g}, t) \leq v(\mathbf{a})$, $\forall t \geq 0$. If, on the other hand, $g(x) \geq a(x)$, $x > 0$, then

$$\dot{v}(\mathbf{g}, t) = v(\mathbf{g}) + \int_0^t ds\, \big(-\varphi(s+, s) + a(0+)\big) \leq v(\mathbf{g}).$$

The rest of the proof of assertion (ii) does not differ from the discrete case. □

Assertion (b) of Theorem 3 follows from Lemma 4.4 straight away. This completes the proof of Theorem 4.

5 Convergence to the limiting equations

In this section we establish Theorems 1, 4 and 5. It is convenient to begin with a series of auxiliary lemmas (see Lemmas 5.1–5.3). As before, let $q_k(t)$ denote the fraction of the number of servers in QS $\overline{\mathcal{S}}_N^{(L)}$ at time $t \geq 0$ with the queue-size $k \geq 1$, given that the initial state of the corresponding MP is $\mathbf{g} \in \mathcal{U}_N^{(L)}$. Set

$$w_{k,N}(t, \mathbf{g}) = \mathbf{T}_N^{(L)}(t) v_k(\mathbf{g}). \tag{5.1}$$

Here $v(\mathbf{g})$ is defined in (3.6).

Lemma 5.1 *Let $\mathbf{y}(\mathbf{g}, t) = \big(y_k(\mathbf{g}, t)\big)$ be the solution to (3.15), (3.17). Then, $\forall \mathbf{g} \in \mathcal{U}_N^{(L)}$, $t > 0$ and $k \geq 1$,*

$$w_{k,N}(t, \mathbf{g}) \leq v_k\big(\mathbf{y}(\mathbf{g}, t)\big). \tag{5.2}$$

Proof. It is easy to see that $\forall \mathbf{g} = (g_k) \in \mathcal{U}$,

$$\begin{aligned}
\mathbf{A}_N^{(L)} v_k(\mathbf{g}) &= L\lambda \sum_{l\geq 1} b^{(L)}(l) h\big(g_{k-l}\big) - g_k \\
&= L\lambda h(1) \sum_{l\geq k} b^{(L)}(l) + L\lambda \sum_{1\leq l\leq k-1} b^{(L)}(l) h\big(g_{k-l}\big) - g_k \\
&\leq L\lambda\theta_0 \sum_{l\geq 1} b^{(L)}(l) g_{k-l} - g_k \\
&\leq L\lambda\theta_0 \sum_{l\geq 1} b^{(L)}(l)\big(v_{k-l}(\widetilde{\mathbf{g}}) - v_{k+1-l}(\widetilde{\mathbf{g}})\big) - v_k(\widetilde{\mathbf{g}}) + v_{k+1}(\widetilde{\mathbf{g}}) \\
&\leq L\lambda\theta_0 \Big(\sum_{l\geq 1} b^{(L)}(l)(v_{k-l}(\widetilde{\mathbf{g}}) - v_k(\widetilde{\mathbf{g}})\Big) - v_k(\widetilde{\mathbf{g}}) + v_{k+1}(\widetilde{\mathbf{g}}). \qquad (5.3)
\end{aligned}$$

Here, $\widetilde{\mathbf{g}}$ stands for the sequence $(\widetilde{g}_k)$ with $\widetilde{g}_k = g_k$, $k \geq 1$, and $\widetilde{g}_k = \widetilde{g}_{k+1} + 1$, $k \leq 0$.

Bound (5.3) allow us to use Lemma 3.10 which completes the proof (see the proof of Lemma 16 from [30]). □

Lemma 5.2 *Let $\mathbf{g} \in \mathcal{V}$. Then the solution $\mathbf{u}(\mathbf{g}, t)$ to (2.10) and the solution $\mathbf{y}(\mathbf{g})$ to (2.19), (2.20) obey*

$$v\big(x, \mathbf{u}(\mathbf{g}, t)\big) \leq v\big(x, \mathbf{y}(\mathbf{g}, t)\big). \qquad (5.4)$$

The proof of Lemma 5.2 is similar to that of Lemma 5.1, and we omit it.

The following assertion is proved exactly as Theorem 3 from [30].

Lemma 5.3 *Assume that $\mathbf{g} \in \mathcal{V}$ and $\mathbf{g}_{N,L} \in \mathcal{U}_N^{(L)}$, $N, L \geq 1$, the sequence $\mathbf{g}_{N,L} \to \mathbf{g}$ as $N, L \to \infty$ and the series $v\left(\mathbf{g}_{N,L}\right) = \sum_{k\geq 1} \left(g_{N,L}\right)_k$ converges uniformly in N, L. Then, uniformly in t within a bounded interval,*

$$\lim_{N,L\to\infty} \mathbf{T}_N^{(L)} v(t) \left(g_{N,L}\right) = v\left(\mathbf{u}(\mathbf{g}, t)\right). \qquad (5.5)$$

Lemmas 5.1–5.3 enable us to give a quick proof of Theorem 5.

Proof of Theorem 5. The proof follows that of Theorem 4 from [30]. It is plain that the quantity $w_{k,N}(t, \mathbf{g}) \equiv w_{k,N}^{(L)}(t, \mathbf{g})$ introduced in (5.1) obeys

$$w_{k,N}^{(L)}(t, \mathbf{g}) = \mathsf{E}_{\mathbf{g}}\big(\psi(t) - (k-1)\big)_+, \quad g \in \mathcal{U}, \ t \geq 0, \ k \geq 1, \qquad (5.6)$$

whereas $v_k\big(\mathbf{y}^{(L)}(\mathbf{g},t)\big)$ and $v\big(x,\mathbf{y}(\mathbf{g},t)\big)$ obey

$$v_k\big(\mathbf{y}^{(L)}(\mathbf{g},t)\big) = \mathsf{E}_{\mathbf{g}}\big(\kappa^{(L)} - (k-1)\big)_+, \ \ \mathbf{g}\in\mathcal{U},\ t\geq 0,\ k\geq 1, \tag{5.7}$$

and

$$v_k\big(x,\mathbf{y}(\mathbf{g},t)\big) = \mathsf{E}_{\mathbf{g}}\big(\kappa(t) - x\big)_+, \ \ \mathbf{g}\in\mathcal{V},\ t,x\geq 0. \tag{5.8}$$

Representations (5.6)–(5.8) immediately lead to the assertion of Theorem 5. □

Proof of Theorem 1. The proof follows that of Theorem 5a) from [30]. As the MPs under consideration have countably many states, it suffices to check that

1. $\forall C>0$, the set of functions $\big\{\mathbf{g}\in\overline{\mathcal{U}}_N^{(L)}\colon\ v(\mathbf{g})\leq C\big\}$ is bounded in metric δ (see (2.3)) on space $\overline{\mathcal{U}}_N^{(L)}$.
2. $\forall\mathbf{g}\in\mathcal{U}_N^{(L)}$,

$$\sup_{t\geq 0} v\left(\mathbf{T}_N^{(L)}(t)\mathbf{g}\right) < \infty. \tag{5.9}$$

Assertion 1 is straightforward. As to Assertion 2, it follows from Theorem 5. □

The above argument immediately implies

Lemma 5.4 $\forall L,N\geq 1$,

$$\mathsf{E}_\pi\, v = \int \pi_N^{(L)}(d\mathbf{g})v(\mathbf{g}) < \infty, \tag{5.10}$$

where, as before, $v(\mathbf{g})$ is the function defined in (3.6).

We now pass to the proof of Theorem 4.

Denote by $\Pi_N^{(L)}$ the projection $\mathrm{C}(\overline{\mathcal{U}}_N^{(L)}) \to \mathrm{C}(\overline{\mathcal{U}}_N)$ (or, equivalently, $\mathrm{C}(\overline{\mathcal{V}}) \to \mathrm{C}(\overline{\mathcal{U}}_N)$) given by

$$\Pi_N^{(L)} f(\mathbf{g}) = f(\widehat{\mathbf{g}}^{(L)}), \tag{5.11}$$

where $\mathbf{g}=(g_k)\in\overline{\mathcal{U}}_N$, $\widehat{\mathbf{g}}_N^{(L)} = (\widehat{u}_N^{(L)}(x))\in\overline{\mathcal{U}}_N^{(L)}$, and

$$\widehat{g}^{(L)}(x) = g_k \ \text{ for } \ x\in\big((k-1)/L, k/L\big],\ k\geq 1. \tag{5.12}$$

Given $L, T \geq 1$, denote by $\mathrm{C}_T^{(L)}(\overline{\mathcal{V}})$ the set of functions $f \in \mathrm{C}(\overline{\mathcal{V}})$ for which $f(\mathbf{g})$ depends only on the integral mean values $L\int_{(k-1)/L}^{k/L} dy\, g(y)$ with $k \leq T$ (rather than on the whole $\mathbf{g} \in \mathrm{C}(\overline{\mathcal{V}})$). Elements from $\mathrm{C}_T^{(L)}(\overline{\mathcal{V}})$ are identified with functions of T variables g_k, $k = 1, \ldots, T$, on the set $1 \geq \mathrm{g}_1 \geq \mathrm{g}_2 \cdots \geq \mathrm{g}_T \geq 0$. Furthermore, let $\mathrm{D}_T^{(L)}(\overline{\mathcal{V}})$ denote the set of functions $f \in \mathrm{C}_T^{(L)}(\overline{\mathcal{V}})$ which, under the above identification, are C^2-functions of g_k, $k = 1, \ldots, T$, and obey

$$\sup_{1\leq j,k\leq T}\ \sup_{\mathbf{g}\in\overline{\mathcal{V}}}\left[\left|\frac{\partial f(\mathbf{g})}{\partial \mathrm{g}_k}\right| + \left|\frac{\partial^2 f(\mathbf{g})}{\partial \mathrm{g}_j \partial \mathrm{g}_k}\right|\right] < \infty. \tag{5.13}$$

It is plain that the union $\mathrm{D}(\overline{\mathcal{V}}) = \cup_{L,T\geq 1} \mathrm{D}_T^{(L)}(\overline{\mathcal{V}})$ is dense in $\mathrm{C}(\overline{\mathcal{V}})$. We will use set $\mathrm{D}(\overline{\mathcal{V}})$ for proving the convergence in Theorem 5.

The semi-group $\mathbf{T}_N^{(L)}(t)$, $t \geq 0$, and its generator $\mathbf{A}_N^{(L)}$ (see (2.13) and (2.8)) act in space $\mathrm{C}(\overline{\mathcal{U}}_N^{(L)})$. Their limiting counterparts are operators $\mathbf{T}(t)$, $t \geq 0$, and $\mathbf{A}$ acting in $\mathrm{C}(\overline{\mathcal{V}})$. Namely, $\mathbf{T}(t)$ is given by

$$\mathbf{T}(t)f(g) = f\big(\mathbf{u}(\mathbf{g}, t)\big),\ \ \mathbf{g} \in \overline{\mathcal{V}}, \tag{5.14}$$

where $\mathbf{u}(\mathbf{g}, t)$ is the solution to (2.10). The action of $\mathbf{A}$ on $\mathrm{D}_T^{(L)}(\overline{\mathcal{V}})$ is given by

$$\mathbf{A}f(g) = \sum_{k=1}^{K} \frac{\partial f}{\partial \mathrm{g}_k}(\mathbf{g})\Big[L(g_{k+1} - g_k) + \lambda \sum_{l\geq 1} b^{(L)}(l)\big(h(g_{k-l}) - h(g_k)\big)\Big],\ \ \mathbf{g} \in \overline{\mathcal{V}}. \tag{5.15}$$

Lemma 5.5 *The set* $\mathrm{D}(\overline{\mathcal{V}})$ *is a core for operator* $\mathbf{A}$.

Proof. The proof of Lemma 5.5 is standard (cf. the proof of Lemma 15 from [30]) and is based upon using Lemma 3.8. To avoid the repetition, we omit it from the paper. □

Proof of Theorem 4(a). The proof of relations (2.14) and (2.15) is similar, and we focus on one of them, say (2.15). The argument that follows is again similar to the one from [30, pp. 24-25], and we will omit some technical details. By using the MPs convergence theorems (see, e.g., [6, Ch. 1, Theorem 6.1]), it suffices to check that, $\forall f \in \mathrm{D}(\overline{\mathcal{V}})$,

$$\lim_{N,L\to\infty}\ \sup_{\mathbf{g}\in\overline{\mathcal{U}}_N^{(L)}} \left|\mathbf{A}_N^{(L)} f(\mathbf{g}) - \mathbf{A}f(\mathbf{g})\right| = 0, \tag{5.16}$$

which can be done as in [30]. □

Proof of Theorem 4(b). Here, the argument follows that of the proof of Theorem 5 from [30]. A direct corollary of Theorem 4 is

Lemma 5.6 *There exists a unique measure on $\mathcal{V}$ that is invariant with respect to the maps $\mathbf{g} \mapsto \mathbf{u}(\mathbf{g}, t)$, $t \geq 0$. This measure is concentrated at the fixed point $\mathbf{a}$.*

As $\overline{\mathcal{V}}$ is a compact space, the set of the probability measures on $\overline{\mathcal{V}}$ is compact. On the other hand, owing to Theorem 4, any limit point π of the sequence of distributions $\pi_N^{(L)}$ such that $\pi(\mathcal{V}) = 1$ is an invariant measure for the maps $\mathbf{g} \to \mathbf{u}(\mathbf{g}, t)$, $t \geq 0$. By Lemma 5.6, it is concentrated at the point $\mathbf{a} \in \mathcal{V}$. Thus, it suffices to check that any limit point for $\pi_N^{(L)}$ is concentrated on $\mathcal{V}$.

Due to Lemma 5.1, the invariant distributions $\pi_N^{(L)}$ has the property that, $\forall k \geq 1$,

$$\begin{aligned} \mathsf{E}_{\pi_N^{(L)}}\, g_k &= L\lambda \sum_{l\geq 1} b^{(L)}(l)\, \mathsf{E}_{\pi_N^{(L)}} \big(h(g_{k-l})\big) \\ &\leq L\lambda\theta_0 L \sum_{l\geq 1} b^{(L)}(l)\, \mathsf{E}_{\pi_N^{(L)}}\, g_{k-l}. \end{aligned}$$

This inequality leads to the bound (cf. (3.10)):

$$\sum_{k\geq 1} \mathsf{E}_{\pi_N^{(L)}}\, g_k \leq \frac{1}{2} L\lambda h(1) \sum_{l\geq 1} b(l) l(l-1) \Big(1 - L\lambda\theta_0 \sum_{l'\geq 1} b(l')l'\Big)^{-1}. \tag{5.17}$$

Owing to Fatou's Lemma, (5.17) implies that

$$\int_0^\infty dx\, \mathsf{E}_\pi\, g(x) \leq \frac{1}{2} L\lambda h(1) \sum_{l\geq 1} b(l) l(l-1) \Big(1 - L\lambda\theta_0 \sum_{l'\geq 1} b(l')l'\Big)^{-1},$$

which in turn leads to the desired assertion. □

6 Examples and generalizations

As before, we set in what follows $L = 1$ and omit the superscript (L) from the notation. We begin this section with a discussion of several examples of the networks and investigation into their performance in the limit as N tends to ∞. The first model is described by the following routing policy.

Model 6.1. *The task chooses randomly m queues and then selects the s-th shortest one.*

This model was mentioned in Sections 1 and 2. Here, we note that if $s < m$ and the distribution $b(l)$ or distribution function B has a finite range (or, more generally, the tail of b or B decreases in a super-exponential fashion), the invariant solution $\mathbf{a}$ has the property that $a(x)$ decreases super-exponentially, as $x \to \infty$. This follows directly from formula (3.8) and equation (4.25) for the case under consideration.

This example shows that selecting a short queue leads to a reduction of the typical queue size. Surprisingly, in a 'dual' model, where a dispatcher selects a task from the longest queue, among m randomly chose queues, the typical queue size decreases much slower. E.g.,

Model 6.1*. *Each task arriving in the Poisson flow of rate λN is placed in a randomly selected queue, out of the total of N queues processed by a single server. Suppose that, after processing a task, the server chooses at random two queues and then selects the longest one among them. Each task has an exponential service time of mean $1/N$.*

Following the same argument as in the preceding sections, one can derive the following limiting equations as $N \to \infty$:

$$\dot{u}_k(t) = \lambda\big(u_{k-1}(t) - u_k(t)\big) + \big(1 - u_k(t)\big)^2 - \big(1 - u_{k+1}(t)^2\big),\ k > 1, \quad u_0(t) = 1, \tag{6.1}$$

$$u_k(0) = g_k,\ k > 1.$$

System (6.1) is dual to (1.4) in a natural sense. By using methods of Section 3, it is possible to check the existence and uniqueness of a solution $\mathbf{u} = (u_k(t))$ to (6.1) belonging to space $\overline{\mathcal{U}}$, as well as the existence of a unique invariant solution $\mathbf{a} = (a_k) \in \mathcal{U}$ and the convergence $\mathbf{u}(t) \to \mathbf{a}$. The invariant solution has the following form:

$$a_1 = \lambda_1, \quad \lambda a_{k-1} = 1 - (1 - a_k)^2,\ k > 1,$$

where λ_1 is the solution of

$$\lambda_1^2 - 2\lambda_1 + \lambda = 0.$$

The non-overload condition reads $\lambda < 1$, and the asymptotics of a_k for large k is $a_k = c_0(\lambda/2)^{c_1 k}$.

A generalization of Model 6.1 is as follows:

Model 6.2. *The task chooses randomly m queues out of which it selects $M \leq m$, shortest ones, to each of which its identical copy is simultaneously sent.*

Here, function $h \equiv h^{m,M}$ is given by

$$h(x) = \sum_{s=1}^{M} h_{m,s}(x), \tag{6.2}$$

where $h_{m,s}$ is defined in (2.2). The non-overload condition takes the form $\lambda\ell < 1/M$. For invariant solution, we have, in the discrete case,

$$a_k = c_0 \big(\lambda\ell\big)^{c_1(m-M+1)^k},$$

and in the continuous case

$$a(x) = c_0 \big(\lambda\ell\big)^{c_1(m-M+1)^x},$$

where

$$\ell = \sum_{l\geq 1} b(l)l \ \text{ or } \ \int_0^\infty dB(y)y,$$

depending on the setting. Constants $c_0,\ c_1 > 0$ do not depend on k or x.

Not surprisingly, the rate of decrease of a_k is determined by the parameter $m-M$, not m.

The pair of examples 6.3–6.4 below formally speaking is not included in the above setup, but can be treated in a similar way after minor modifications. In these examples we assume that the service time of a task is exponentially distributed, with mean value one. Consequently, we analyse the difference equations only.

Model 6.3. *The arriving task subsequently chooses at random a test queue. Each time the decision is taken, depending on whether the size of the chosen queue is $>$ or $\leq K$, where K is a fixed number. If the queue size is $\leq K$ the task joins it, otherwise it checks the next queue. If, after m trials, all chosen queues have size $> K$, the task joins the last queue, regardless of its size.*

The limiting equation reads

$$\begin{aligned} \dot{u}_k(t) &= \lambda\Big(\sum_{j=0}^{m-1} u_K(t)^j\Big)\big(u_{k-1}(t) - u_k(t)\big) \\ &\quad + u_{k+1}(t) - u_k(t),\ 1 \leq k \leq K,\ \ u_0(t) = 1, \qquad (6.3) \\ \dot{u}_k(t) &= \lambda\big(u_K(t)\big)^{m-1}\big(u_{k-1}(t) - u_k(t)\big) + u_{k+1}(t) - u_k(t),\ \ k > K, \qquad (6.4) \end{aligned}$$

In this model, the invariant solution $\mathbf{a} = (a_k)$ behaves for $k > K$ as $a_k = a_K(\lambda a_K^{m-1})^{k-K}$, where $0 < a_K < 1$. Note that even for $K = 1$, $m = 2$, we have $a_k = \lambda^{2k-1}$. In other words, a slight change of the routing policy, comparing with linear model (1.4), reduces the tail of the invariant queue-size distribution.

Model 6.4. *The task proceeds as before, with the only proviso that if, after m trials, all chosen queues have size $> K$, the task joins the shortest queue, among m chosen ones.*

The limiting equation has the form

$$\begin{aligned}\dot{u}_k(t) &= \lambda\Big(\sum_{j=0}^{m-1} u_K(t)^j\Big)\big(u_{k-1}(t) - u_k(t)\big) \\ &\qquad + u_{k+1}(t) - u_k(t),\ 1 \le k \le K,\ \ u_0(t) = 1, \end{aligned} \tag{6.5}$$

$$\dot{u}_k(t) = \lambda\big[\big(u_{k-1}(t)\big)^m - u_k(t)^m\big] + u_{k+1}(t) - u_k(t),\ \ k > K, \tag{6.6}$$

The invariant solution $\mathbf{a} = (a_k)$ decreases super-exponentially.

In the case $m = 2$, Models 6.3 and 6.4 were discussed in [28].

Equations (6.3)–(6.5) are not of form (2.9). However, the above scheme needs only minor modification. An important fact is that quantity $v_1(\mathbf{g}, t)$ (see (3.6)) in both models obeys

$$\dot{v}_1(t) = \lambda - u_1(t). \tag{6.7}$$

The existence and uniqueness of a global solution, in $\overline{\mathcal{U}}$, of the Cauchy problem for equations (6.3)–(6.6) can be established by the same argument as in Section 3, due to an obvious monotonicity properties.

To check the existence, in $\mathcal{U}$, of a invariant solution, we need the non-overload condition $\lambda < 1$. At first, one solves the invariant equations for $k > K$, expressing a_k, $k > K$, as functions of a_K. The remaining equations, for a_k, $1 \le k \le K$, result in an equation for a_K. For definiteness, let us discuss in detail Model 6.3 only. Here, assuming that $a_K = 1$, the K-th equation reads

$$\lambda\Big(\sum_{j=0}^{m-1} a_K^j\Big)(a_{K-1} - a_K) + a_{K+1} - a_K < 0.$$

On the other hand, assuming that $a_K = 0$, we have $a_k = (\lambda^k - \lambda^K)/(1-\lambda^K)$, $k \le K$, and $a_k = 0$, $k > K$, and from the K-th equation we get

$$\lambda\Big(\sum_{j=0}^{m-1} a_K^j\Big)(a_{K-1} - a_K) + a_{K+1} - a_K > 0.$$

Therefore, by continuity, there exists a_K, $0 < a_K < 1$, for which equations (6.3)–(6.4) are valid for all $k \ge 1$.

To check the uniqueness, in $\mathcal{U}$, of an invariant solution $\mathbf{a}$, one uses monotonicity. Again, consider Model 6.3. Suppose, there are two values of a_K, $a_K^{(1)} > a_K^{(2)}$, determining different invariant solutions. Then $1 - a_1^{(1)} = 1 - a_1^{(2)}$, $a_1^{(1)} - a_2^{(1)} > a_1^{(2)} - a_2^{(2)}$, $a_2^{(1)} - a_3^{(1)} > a_2^{(2)} - a_3^{(2)}$, ... , and thus $a_K^{(1)} < a_K^{(2)}$. The contradiction obtained completes the proof.

The proof of the convergence $\mathbf{u}(t) \to \mathbf{a}$ follows the line of Lemma 3.6. We again can assume that for the initial date $\mathbf{g} = (g_k) \in \mathcal{U}$, either $g_k \ge a_k$, $k \ge 1$, or $g_k \le a_k$, $k \ge 1$. As in the proof of Lemma 3.6, we get that $\pm \int_0^\infty \big(a_1 - u_1(t)\big)\, dt < \infty$. Consider the derivative

$$\begin{aligned} \dot v_2(\mathbf{g}, t) &= \big(u_1(t) - a_1\big) - \Big(\sum_{j=0}^{m-1} u_K(t)^j\Big)(1 - u_1(t)) \\ &\quad + \Big(\sum_{j=0}^{m-1} a_K^j\Big)(1 - a_1) - \big(u_2(t) - a_2\big) \\ &= \big(u_1(t) - a_1\big) - \Big(\sum_{j=0}^{m-1} a_K^j\Big)(a_1 - u_1(t)) \\ &\quad - \Big(\sum_{j=0}^{m-1} (u_K(t)^j - a_K^j)\Big)(1 - u_1) - \big(u_2(t) - a_2\big). \end{aligned}$$

The integral over t of LHS of this chain of equalities converges, and so do the integrals of the $\big(a_1 - u_1(t)\big)$–terms in the RHS. Therefore, the integrals of last two terms also converge (these terms are both of the same sign). Thus, $u_2(t) \to a_2$ as $t \to \infty$, and $u_K(t) \to a_K$. As in Lemma 3.6, the argument is completed by induction.

Model 6.4 is considered in a similar way.

The analysis of the constructions performed in Sections 3, 4 leads to the conclusion that a more general form of the functional equations is allowed. Fix a function h of three variables x, x' and $\mathbf{u}$, with non-negative values. Here $x > 0$, $-\infty < x' < x$

and $\mathbf{u} \in \overline{\mathcal{V}}$ (in the discrete set-up we assume that $\mathbf{u} \in \overline{\mathcal{U}}$). Function h is supposed to satisfy the following conditions (against argument $\mathbf{u}$):

- *Monotonicity*: if $\mathbf{u}' \geq \mathbf{u}''$ then $h(x, x', \mathbf{u}') \geq h(x, x', \mathbf{u}'')$.
- *Locality*: There exist $T, T' \geq 0$ such that if $u'(z) = u''(z)$, $z > x + T$, then $h(x, x', \mathbf{u}') = h(x, x', \mathbf{u}'')$, and $h(x, x', \mathbf{u}) = h(u(x'))$ for $x, x' > T'$. Also $h(x, x', \mathbf{u}) = 0$ if $u(x) = 0$, $x > 0$, $x' < x$.
- *Smoothness*:

$$\begin{aligned}\theta_0 &= \sup\left[\frac{h(x,x',\mathbf{u})}{u(x)} : x > 0,\ -\infty < x' \leq x,\ \mathbf{u} \in \overline{\mathcal{V}}\right] < \infty,\\ \theta_1 &= \sup\left[\frac{\left|h(x,x',\mathbf{u}') - h(x,x',\mathbf{u}'')\right|}{\int_0^x |u'(z) - u''(z)|\,dz} : x > 0,\ -\infty < x' \leq x,\right.\\ &\qquad\qquad \left.\mathbf{u}', \mathbf{u}'' \in \overline{\mathcal{V}},\ \mathbf{u}' \neq \mathbf{u}''\right] < \infty.\end{aligned}$$

  $\forall L > 0$, the restriction of h to $\mathbf{N}^{(L)} \times \mathbf{N}^{(L)} \times \overline{\mathcal{U}}^{(L)}$ is a C^2-function of variables $u(l/L)$, $l \geq 1$, and

$$\begin{aligned}\theta_2 &= \sup\left[\frac{\partial^2}{\partial u(l_1/L)\partial u(l_2/L)} h(x,x',\mathbf{u}) : x, x' \in \mathbf{N}^{(L)},\ x \geq 1/L,\right.\\ &\qquad\qquad \left.-\infty < x' \leq x,\ \mathbf{u} \in \overline{\mathcal{U}}^{(L)},\ L > 0\right] < \infty.\end{aligned}$$

- *Conservation*: this condition is on the pair $h, \{b(l)\}$ or $h, \{B\}$, depending on the set-up. In the discrete case,

$$\sum_{k\geq 1}\sum_{l\geq 1} b(l) h(k-l, k, \mathbf{u}) - \sum_{k\geq 1} h(k, k, \mathbf{u}) = 1,$$

  and in the continuous case,

$$\int_0^\infty dx \int_0^\infty dB(y) h(x-y, x, \mathbf{u}) - \int_0^\infty h(x, x, \mathbf{u}) = 1.$$

The whole scheme of Sections 3 and 4 works under these conditions; the argument of Section 5 remains intact. Models 6.3 and 6.4 fall in this more general category.

Acknowledgments The authors would like to thank Professors I.G.B.S. Adan, O. Boxma, G. Fayolle, F.I. Karpelevich, F.P. Kelly, J.L. Lewis, V.A. Malyshev, M.S. Pinsker and M. Sidi for support and stimulating discussions. The first authors wants also to thank the projects HIPERCOM and MEVAL at INRIA-Rocquencourt for their kind hospitality.

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Éditeur
INRIA, Domaine de Voluceau, Rocquencourt, BP 105, 78153 LE CHESNAY Cedex (France)
http://www.inria.fr
ISSN 0249-6399