Mathematical Study of very High Voltage Power Networks III: The Optimal AC Power Flow Problem

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N ° 3268

Septembre 1997

THÈME 4

Rapport de recherche

ISSN 0249-6399

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Thème 4 — Simulation et optimisation
de systèmes complexes
Projet PROMATH

Rapport de recherche n° 3268 — Septembre 1997 — 18 pages

Abstract: This paper shows how to apply the perturbation theory for nonlinear programming problems to the study of the optimal power flow problem. The latter is the problem of minimizing losses of active power over a very high voltage power networks. In this paper, the inverse of the reference voltage of the network is viewed as a small parameter. We call this scheme the very high voltage approximation.

After some proper scaling, it is possible to formulate a limiting problem, that does not satisfy the Mangasarian-Fromovitz qualification hypothesis. Nevertheless, it is possible to obtain under natural hypotheses the second order expansion of losses and first order expansion of solutions. The latter is such that the computation of the active and reactive parts are decoupled. We also obtain the high order expansion of the value function, solution and Lagrange multiplier, assuming that interactions with the ground are small enough. Finally we show that the classical direct current approximation may be justified using the framework of very high voltage approximation.

Key-words: Electrical networks, very high voltage, active-reactive decoupling, asymptotic analysis, optimization with perturbations, sensitivity analysis, expansion of solutions, constrained optimization.

(Résumé : tsvp)

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Etude mathématique des réseaux en très haute tension III : Le problème du transit optimal de puissance en courant alternatif

Résumé : Cet article montre comment appliquer la théorie de la perturbation en programmation non linéaire au problème du transit optimal de tension. Ce dernier est le problème de minimisation des pertes actives sur un réseau de très haute tension. Dans cet article, l'inverse de la tension de référence du réseau est vu comme un petit paramètre. Ce schéma est appelé approximation des très haute tensions.

Après une mise à l'échelle convenable, il est possible de formuler un problème limite, qui ne satisfait pas l'hypothèse de qualification Mangasarian et Fromovitz. Néanmoins, il est possible d'obtenir sous des hypothèses naturelles le développement au second ordre des pertes, et le développement au premier ordre des solutions. Ce dernier est tel que le calcul se décompose suivant les parties actives et réactives. Nous obtenons aussi le développement de degré quelconque des pertes, solutions et multiplicateur de Lagrange en supposant les interactions avec le sol assez faibles. Finalement nous montrons comment l'approximation classique du courant continu peut être justifiée dans le cadre de l'approximation des très haute tensions.

Mots-clé : Réseau électrique, très haute tension, découplage actif-réactif, analyse asymptotique, optimisation avec perturbations, analyse de sensibilité, développement des solutions, optimisation avec contraintes.

AMS subject classifications 26B10, 41A58, 78A99, 90B10

1 Introduction

This paper shows how to apply some results in the theory of nonlinear programming problems subject to perturbations (see e.g. the review paper [6]) in order to conduct an analysis of a significant real world problem, namely the optimal power flow problem (OPFP). Rougly speaking, this is the problem of minimizing the losses of energy over an alternating current (AC) network, given some bounds on the inputs of active and reactive power at the nodes of the network. (A more precise description of the problem is given in section 2.) This is a classical problem for high voltages electrical networks, see e.g. [2].

This paper is the third in a series dedicated to the mathematical study of this problem. The point of view adopted in the three papers is to make an asymptotic analysis of the problem by assuming the "average" voltage to be very high. This of course is a natural hypothesis for high voltage networks. Part I [4] is a preliminary step where is discussed a similar problem, but in a direct current (DC) setting. Part II [3] presents an asymptotic analysis for the power flow problem (without optimization), i.e. when the input of active and reactive power at the nodes is a given data. In part III (the present paper) we discuss the problem that motivated the two other papers, and that is the most important from a practical point of view, that is the optimal power flow problem in an AC setting.

Our results are as follows. We show that the limiting problem has a unique and explicit solution. The critical cone at this solution is reduced to 0, while the associated set of Lagrange multipliers is the sum of a particular Lagrange multiplier and of the set of singular multipliers. Then, using the theory of [1, 5, 11] the first order expansion of solutions (and second order expansion of value function, i.e. of losses of energy) is obtained through the computation of the solution of an associated quadratic problem. The latter may be decomposed into two independent subproblems, one for the active part and one for the reactive part. Each of these subproblems has an interpretation in a DC setting.

Then under stronger hypotheses we show that the strong regularity framework [10] applies in an indirect way. More precisely, the directional strong regularity condition introduced in [4] is satisfied whenever the interaction with the ground is small enough. This allows the use of results in [7] that give the power series expansion of the value function, solution and Lagrange multiplier.

The paper is organized as follows. The next section sets the optimal power flow problem. Section 3 is devoted to the study of the limit problem. In section 4 we establish the uniform boundedness of solutions to the family of perturbed problems. We compute the first order expansion of solutions in section 5, and the high order expansions in section 6. Finally the conclusion discusses the results and some open problems.

Table of notations

$g_{k\ell} + jh_{k\ell}$ admittance of capacitive lines ($g_{k\ell} \geq 0$, $h_{k\ell} \geq 0$).

$\overline{J}_k$ input of current at node k.
$j := \sqrt{-1}$ basis of imaginary numbers.
p: vector of $\mathbb{R}^n$ such that $\sum_i p_i = 1$, support of corrector of input of active power.
P_k, Q_k nominal input of active power, and reactive power, at node k.
$\overline{I}_{k\ell}$ current sent from node k toward node l.
$\mathbb{R}_+$, $\mathbb{R}_{++}$ set of nonnegative, positive real numbers.
$\mathcal{S} = \{1, \cdots, n\}$ set of nodes of the network, of cardinal $n = |\mathcal{S}|$.
$\mathcal{S}_V$ set of nodes over which V is given; $\mathcal{S}_V \neq \emptyset$.
$\mathcal{S}_Q$ set of nodes over which the reactive power equation is satisfied; $\mathcal{S}_Q = \mathcal{S} \backslash \mathcal{S}_V$.
$T^a_{k\ell}$, $T^r_{k\ell}$ active and reactive power flow sent from k on line $k\ell$.
$\overline{V}_k$, V_k, θ_k voltage, modulus and phase of voltage, at node k.
$u = (\varepsilon, \eta, g, h)$ vector of small parameters.
V^D scaled value of V over $\mathcal{S}_V$. Vector of components equal to 1.
$y_{k\ell}e^{-j\xi_{k\ell}}$ admittance of inductive line $k\ell$; $y_{k\ell} \geq 0$, $\xi_{k\ell} \in [\pi/4, \pi/2]$.
$\eta_{k\ell} := \pi/2 - \xi_{k\ell}$ ($\eta_{k\ell} \in [0, \pi/4]$).
ν corrector of input of active power.
$y_{k\ell}e^{-j\xi_{k\ell}}$ admittance of inductive line $k\ell$; $y_{k\ell} \geq 0$, $\xi_{k\ell} \in [\pi/4, \pi/2]$.
$\eta_{k\ell} := \pi/2 - \xi_{k\ell}$ ($\eta_{k\ell} \in [0, \pi/4]$).
$\pi_{k\ell}$ losses of active power on line $k\ell$.

Let (P) be an optimization problem. By $F(P)$, $S(P)$ and val(P) we denote the set of feasible solutions, optimal solutions and optimal value of problem (P).

2 Setting of the optimal power flow problem

2.1 Basic equations

This subsection recalls the classical model for high voltage electrical networks. The set $\mathcal{S}$ of nodes is labeled from 1 to n. Nodes are related by arcs, that represent a physical electrical line. In the model, the physical line composed of three elementary electrical lines. The *inductive* one relates nodes k and ℓ. Its admittance is $y_{k\ell}e^{-j\xi_{k\ell}}$, with $y_{k\ell} = y_{\ell k} \geq 0$ and $\xi_{k\ell} = \xi_{\ell k} \in [\pi/4, \pi/2]$. The two *capacitive* lines relate k and ℓ to the ground (the ground node does not appear explicitly in the model). These capacitive lines model the interaction between the physical line and the ground. Their admittance is $g_{k\ell} + jh_{k\ell}$, with $g_{k\ell} = g_{\ell k} \geq 0$ and $h_{k\ell} = h_{\ell k} \geq 0$. (note that the frequency is fixed once for all, and therefore does not appear explicitly in the equations.)

We denote by $\bar{I}_{k\ell}$ the *current sent from node k on line $k\ell$*, defined as the sum of the currents in the inductive line and the capacitive line related to node k. By Ohm's law, the current in each of these lines is the product of admittance by difference of voltages. Therefore, the expression of $\bar{I}_{k\ell}$ is

$$\bar{I}_{k\ell} = (\bar{V}_k - \bar{V}_\ell)y_{k\ell}e^{-j\xi_{k\ell}} + \bar{V}_k(g_{k\ell} + jh_{k\ell}). \tag{1}$$

Note that this formulation takes into account the case when there is no line between nodes k and ℓ: then $y_{k\ell} = g_{k\ell} = h_{k\ell} = 0$, and we may set $\xi_{k\ell}$ to $\pi/2$. If $y_{k\ell} \neq 0$ we say that there exists an *effective* line between k and ℓ. In general, $\bar{I}_{k\ell} + \bar{I}_{\ell k} \neq 0$ because of the capacitive lines. We assume that the network is *connected*, in the sense that, between any two nodes, there exists a path in the graph composed of effective lines linking these nodes.

The *power flow* sent from node k on line $k\ell$ is defined as $\bar{V}_k \bar{I}^*_{k\ell}$. With (1), we get

$$\bar{V}_k \bar{I}^*_{k\ell} = V_k(V_k - V_\ell e^{j(\theta_k - \theta_\ell)}) y_{k\ell} e^{j\xi_{k\ell}} + V_k^2 (g_{k\ell} - j h_{k\ell}). \tag{2}$$

The resistance of the inductive line being usually small, it is useful to make the change of parameter $\xi_{k\ell} = \pi/2 - \eta_{k\ell}$, where $\eta_{k\ell}$ is small and nonnegative. The expression of the real and imaginary parts of power flow, called the *active and reactive power flow*, are

$$T^a_{k\ell} \quad := \quad V_k^2 (y_{k\ell} \sin \eta_{k\ell} + g_{k\ell}) + y_{k\ell} V_k V_\ell \sin(\theta_k - \theta_\ell - \eta_{k\ell}), \tag{3}$$

$$T^r_{k\ell} \quad := \quad V_k^2 (y_{k\ell} \cos \eta_{k\ell} - h_{k\ell}) - y_{k\ell} V_k V_\ell \cos(\theta_k - \theta_\ell - \eta_{k\ell}). \tag{4}$$

Let $\overline{J}_k$ be the *input of current at node k*. By Kirchhoff's law, $\overline{J}_k = \sum_{\ell \neq k} \overline{I}_{k\ell}$. The *amount of power injected at node k*, defined as $\bar{V}_k \overline{J}^*_k$, is therefore equal to the sum of power flow sent from node k on the lines of the network. In particular, we have:

$$\mathcal{R}e(\bar{V}_k \overline{J}^*_k) = \sum_{\ell \neq k} T^a_{k\ell}\,; \quad \mathcal{I}m(\bar{V}_k \overline{J}^*_k) = \sum_{\ell \neq k} T^r_{k\ell}.$$

Let $(\mathcal{S}_V, \mathcal{S}_Q)$ be a partition of $\mathcal{S}$; i.e., $\mathcal{S}_V \cup \mathcal{S}_Q = \mathcal{S}$ and $\mathcal{S}_V \cap \mathcal{S}_Q = \emptyset$. We assume that $\mathcal{S}_V \neq \emptyset$. Let p be a n-dimensional vector such that $\sum_{k \in \mathcal{S}} p_k = 1$. The *power input equations* are

$$\nu p_k + \sum_{\ell \neq k} T^a_{k\ell} = P_k,\ k \in \mathcal{S}; \quad p^T \theta \quad = \quad 0, \tag{5}$$

$$\sum_{\ell \neq k} T^r_{k\ell} = Q_k,\ k \in \mathcal{S}_Q; \quad V \quad = \quad \frac{V^D}{\sqrt{\varepsilon}} \text{ on } \mathcal{S}_V. \tag{6}$$

In these equations, the scalar parameter ε takes into account the large values of voltages over the network, V^D is a vector over $\mathcal{S}_V$ with all components equal to 1, and T^a and T^r are the functions of V and θ defined in (3)-(4). The power flow problem is, given P and Q, to solve these equations with unknowns V, θ and ν. (The latter is a scalar variable.) The power flow equations being invariant w.r.t. a translation of phases, the purpose of the second constraint is to recover a unique solution. Instead we could have fixed the phase at a certain node, e.g. $\theta_1 = 0$. However, writing $p^T\theta = 0$ eases the analysis of these equations. The input of active power is $P - \nu p$, viewed as a difference between a nominal value and a corrector term. The latter is necessary, as seen later, in order to check (at least in some cases) that the above system is well-posed, see also [3]. For instance, if p is the first basis vector, relation (5) amounts to fix θ_1 to 0 and to fix ν so that the active power equation at node 1 is always satisfied.

It is convenient to scale the voltage, corrector term and power flow by making the change of variable

$$\tilde{V} := \sqrt{\varepsilon} V, \quad \tilde{\nu} := \varepsilon \nu, \quad \tilde{T}^a_{k\ell} := \varepsilon T^a_{k\ell}, \quad \tilde{T}^r_{k\ell} := \varepsilon T^r_{k\ell}. \tag{7}$$

Eliminating T^a and T^r in (5)-(6), we can state the power flow equations in term of the scaled variables:

$$\begin{aligned}
\tilde{\nu} p_k + \sum_{\ell \neq k} \left(\tilde{V}_k^2 (y_{k\ell} \sin \eta_{k\ell} + g_{k\ell}) + y_{k\ell} \tilde{V}_k \tilde{V}_\ell \sin(\theta_k - \theta_\ell - \eta_{k\ell}) \right) &= \varepsilon P_k, \; k \in \mathcal{S}, && (8)\\
p^T \theta &= 0, && (9)\\
\sum_{\ell \neq k} \left(\tilde{V}_k (y_{k\ell} \cos \eta_{k\ell} - h_{k\ell}) - y_{k\ell} \tilde{V}_\ell \cos(\theta_k - \theta_\ell - \eta_{k\ell}) \right) &= \varepsilon \frac{Q_k}{\tilde{V}_k}, \; k \in \mathcal{S}_Q && (10)\\
\tilde{V} &= V^D \text{ on } \mathcal{S}_V. && (11)
\end{aligned}$$

Note that this system is similar to (5)-(6), except for the variable ε that appears now as multiplying the inputs of power. That is, the scaling changes "high voltages" into "small inputs of power".

Since we want to study networks with high values of voltages at all nodes, there is no harm in dividing the reactive power flow equations by the value of the voltage at the corresponding node, as we did above. We call *active* (resp. *reactive*) *part* of (8)-(11) the two first (resp. two last) equations.

We assume throughout the paper that the parameter $u := (\varepsilon, \eta, g, h)$ satisfies

$$u = (\varepsilon, \eta, g, h) \in A := \mathbb{R}_+ \times [0, \pi/4]^{n^2} \times \mathbb{R}_+^{n^2} \times \mathbb{R}_+^{n^2}.$$

We regard u as the set of small parameters. Assuming (η, g, h) to be small is a classical hypothesis in the field of high voltage electrical networks. Indeed, considering η as small amounts to say that the resistance of the inductive line is small w.r.t. its imaginary part. Neglecting g and h amounts to neglect the capacitive lines. Since $\varepsilon^{-1/2}$ measures the order of magnitude of value of the reference voltage of the network, it is natural to assume it to be small for high voltages networks.

The variable $(\tilde{\nu}, \theta, \tilde{V})$ are restricted to the set $\mathbb{R} \times \Theta \times \mathcal{V}$, where

$$\begin{aligned}
\Theta &:= \{\theta \in \mathbb{R}^n \; ; \; \theta_k - \theta_\ell \in [-\pi/4, \pi/4]; \; \forall k, \ell \in \mathcal{S}\}, && (12)\\
\mathcal{V} &= \{V \in \mathbb{R}^n \; ; \; V_k > 0, \; \forall k = 1, \cdots, n\}. && (13)
\end{aligned}$$

The above relations imply

$$\theta_k - \theta_\ell - \eta_{k\ell} \in [-\pi/2, \pi/2], \; \forall k, \ell \in \mathcal{S}. \tag{14}$$

Remark. Actually it suffices to restrict $\theta_k - \theta_\ell$ to $[-\pi/4, \pi/4]$ whenever $y_{k\ell} > 0$, because otherwise $\theta_k - \theta_\ell$ does never appear in the power flow equations. We keep the above definition of Θ for the sake of simplicity.

The *loss of energy*, or *loss of active power* $\pi_{k\ell}$ on line $k\ell$ being by the definition equal to the sum of active power flow sent on line $k\ell$, its expression is

$$\begin{array}{rcl} \pi_{k\ell} & := & T^a_{k\ell} + T^a_{\ell k}, \\ & = & (V_k^2 + V_\ell^2)(y_{k\ell}\sin\eta_{k\ell} + g_{k\ell}) + y_{k\ell}V_kV_\ell[\sin(\theta_k - \theta_\ell - \eta_{k\ell}) + \sin(\theta_\ell - \theta_k - \eta_{k\ell})], \\ & = & (V_k^2 + V_\ell^2)(y_{k\ell}\sin\eta_{k\ell} + g_{k\ell}) - 2y_{k\ell}V_kV_\ell\sin\eta_{k\ell}\cos(\theta_k - \theta_\ell), \\ & = & g_{k\ell}(V_k^2 + V_\ell^2) + y_{k\ell}(V_k - V_\ell)^2\sin\eta_{k\ell} + 2y_{k\ell}V_kV_\ell(1 - \cos(\theta_k - \theta_\ell))\sin\eta_{k\ell}. \end{array}$$

Equivalently,

$$\pi_{k\ell} = \frac{g_{k\ell}}{\varepsilon}(\tilde V_k^2 + \tilde V_\ell^2) + y_{k\ell}(\tilde V_k - \tilde V_\ell)^2\frac{\sin\eta_{k\ell}}{\varepsilon} + 2y_{k\ell}\tilde V_k\tilde V_\ell(1 - \cos(\theta_k - \theta_\ell))\frac{\sin\eta_{k\ell}}{\varepsilon}.$$

The last expression allows to check that, as is expected, $\pi_{k\ell} \geq 0$ (since $g_{k\ell} \geq 0$, $y_{k\ell} \geq 0$, and $\eta_{k\ell} \in [0, \pi/4]$ by hypothesis). The terms on the right-hand side of the above display may be interpreted as the loss of power over the capacitive line, a term related to the difference of voltages but not to phases, and a last term primarily related to the difference of phases. If $g = 0$ and $\eta = 0$, there is no resistance on the network, and therefore no loss of active power.

The discussion of the power flow problem in [3] implies the following. If P and Q are given data, or more generally belong to some compact set, and if V^D is constant over $\mathcal{S}_V$, then

$$\|\tilde V\| = O(1), \quad \tilde V_k - \tilde V_\ell = O(\|u\|), \quad \theta_k - \theta_\ell = O(\|u\|).$$

It follows that the sum of losses of active power over the network satisfies

$$\sum_{k<\ell} \pi_{k\ell} = \varepsilon^{-1}\left(O(\|g\|) + O(\|\eta\| \cdot \|u\|^2)\right).$$

This rough estimate shows that if $u \to 0$ in A along some path, then the sum of losses of active power over the network remains bounded if $\|u\| = O(\varepsilon)$, and is of order $O(\varepsilon)$ if in addition $\|g\| = O(\varepsilon^2)$. This motivates the choice of the path $\varepsilon \to u(\varepsilon)$ to be done later on.

2.2 The optimal power flow problem

This is the problem of minimizing the sum of losses of active power over the network, subject to the power flow equations and some given bounds on the inputs of power flow. We establish now the notation for the latter. The bound constraints for P and Q are

$$\check P_i \leq P_i \leq \hat P_i \quad \text{and} \quad \check Q_i \leq Q_i \leq \hat Q_i, \quad 1 \leq i \leq n,$$

where $\check{P}_i$, $\hat{P}_i$, $\check{Q}_i$ and $\hat{Q}_i$ are given vectors of $\mathbb{R}^n$. The optimization problem is as follows (where $\pi_{k\ell}$ is expressed as a function of $(\theta, \tilde{V}, u)$) :

$$(OPFP_u) \qquad \operatorname*{Min}_{(\tilde{\nu},\theta,\tilde{P},\tilde{V},\tilde{Q})} \sum_{k<l} \pi_{k\ell} \qquad \text{subject to}$$

$$\begin{aligned}
\tilde{\nu} p_k + \sum_{\ell \neq k} \left(\tilde{V}_k^2 (y_{k\ell} \sin \eta_{k\ell} + g_{k\ell}) + y_{k\ell} \tilde{V}_k \tilde{V}_\ell \sin(\theta_k - \theta_\ell - \eta_{k\ell}) \right) &= P_k,\ k \in \mathcal{S}, && (15)\\
p^T \theta &= 0, && (16)\\
\varepsilon \check{P} \leq \tilde{P} &\leq \varepsilon \hat{P}, && (17)\\
\sum_{\ell \neq k} \left(\tilde{V}_k (y_{k\ell} \cos \eta_{k\ell} - h_{k\ell}) - y_{k\ell} \tilde{V}_\ell \cos(\theta_k - \theta_\ell - \eta_{k\ell}) \right) &= \frac{Q_k}{\tilde{V}_k},\ k \in \mathcal{S}_Q && (18)\\
\tilde{V} &= V^D \text{ on } \mathcal{S}_V, && (19)\\
\varepsilon \check{Q} \leq \tilde{Q} &\leq \varepsilon \hat{Q}. && (20)
\end{aligned}$$

We call *active* (resp. *reactive*) *constraints* of problem $(OPFP_u)$ the three first (resp. three last) constraints.

Although the value function and solution of an optimization problem are seldom differentiable with respect to the parameter, it is often possible to make an analysis along a path (see e.g. [6]). We choose a parametric path of the form

$$u(\varepsilon) := (\varepsilon, \eta(\varepsilon), g(\varepsilon), h(\varepsilon)),$$

where $\eta(\cdot)$, $g(\cdot)$ and $h(\cdot)$ are twice differentiable functions of ε that vanish at 0. We write the expansion of $\eta(\cdot)$ as

$$\eta(\varepsilon) = \varepsilon \eta^1 + \frac{\varepsilon^2}{2} \eta^2 + o(\varepsilon)^2,$$

and adopt similar conventions for $g(\cdot)$ and $h(\cdot)$. We assume in the sequel of the paper that

$$\tilde{\eta}^1_{k\ell} > 0 \quad \text{whenever} \quad k\ell \quad \text{is an effective line.} \qquad (21)$$

(Remember that a line is said to be effective if $y_{k\ell} > 0$.) In some statements we will assume that $\tilde{g}^1$ is "close" enough to 0. This choice of a parametric path is related to the estimates of the sum of losses in the previous section.

We can write the loss on line $k\ell$ as

$$\pi_{k\ell} == \pi^0_{k\ell} + r^\varepsilon_{k\ell},$$

where $\pi^0_{k\ell}$ is a zero-th order term and $r^\varepsilon_{k\ell}$ is a remainder term such that $r^0_{k\ell} = 0$, the expression of these terms being

$$\pi^0_{k\ell} \quad := \quad (\tilde{V}_k^2 + \tilde{V}_\ell^2) g^1_{k\ell} + (\tilde{V}_k - \tilde{V}_\ell)^2 y_{k\ell} \eta^1_{k\ell} + 2 y_{k\ell} \tilde{V}_k \tilde{V}_\ell \eta^1_{k\ell} (1 - \cos(\theta_k - \theta_\ell)),$$

$$r^{\varepsilon}_{k\ell} := \left(\frac{g(\varepsilon)}{\varepsilon} - g^1_{k\ell}\right)(\tilde{V}_k^2 + \tilde{V}_\ell^2) + y_{k\ell}\left(\frac{\sin\eta_{k\ell}(\varepsilon)}{\varepsilon} - \eta^1\right)\left((\tilde{V}_k - \tilde{V}_\ell)^2 + 2\tilde{V}_k\tilde{V}_\ell(1 - \cos(\theta_k - \theta_\ell))\right).$$

3 Study of the limit problem

Setting the small parameter u to 0 in problem $(OPFP_u)$, we get the *limit optimization problem*

$$(OPFP_0) \qquad \operatorname*{Min}_{(\tilde{\nu},\theta,\tilde{P},\tilde{V},\tilde{Q})} \sum_{k<l} \pi^0_{k\ell} \qquad \text{subject to}$$

$$\tilde{\nu} p_k + \sum_{\ell\neq k} y_{k\ell}\tilde{V}_k\tilde{V}_\ell \sin(\theta_k - \theta_\ell) = \tilde{P}_k,\ k \in \mathcal{S}, \quad p^T\theta = 0, \quad 0 \le \tilde{P} \le 0, \tag{22}$$

$$\sum_{\ell\neq k} y_{k\ell}\left(\tilde{V}_k - \tilde{V}_\ell\cos(\theta_k - \theta_\ell)\right) = \frac{\tilde{Q}_k}{\tilde{V}_k},\ k \in \mathcal{S}, \quad \tilde{V} = V^D \text{ on } \mathcal{S}_V, \quad 0 \le \tilde{Q} \le 0. \tag{23}$$

Obviously the inequality constraints are equivalent to $\tilde{P} = 0$ and $\tilde{Q} = 0$, but we need to keep them into the original form in order to apply the theory of nonlinear programming with perturbation.

Lemma 3.1 *Problem $(OPFP_0)$ has a unique solution $(\tilde{\nu}^0, \theta^0, \tilde{P}^0, \tilde{V}^0, \tilde{Q}^0)$, with $\tilde{\nu}^0 = 0$, $\theta^0 = 0$, $\tilde{P}^0 = 0$, $\tilde{Q}^0 = 0$, and $\tilde{V}^0 = 1$ over $\mathcal{S}$.*

*Proof.*Let $(\tilde{\nu}^0, \theta^0, \tilde{P}^0, \tilde{V}^0, \tilde{Q}^0)$ be feasible for $(OPFP_0)$. Obviously $\tilde{P}^0 = \tilde{Q}^0 = 0$. Summing the active part of the power equations over $\mathcal{S}$, we find that

$$\tilde{\nu}^0 = \tilde{\nu}^0 \sum_{k\in\mathcal{S}} p_k = - \sum_{k,\ell\in\mathcal{S}} y_{k\ell}\tilde{V}^0_k\tilde{V}^0_\ell \sin(\theta^0_k - \theta^0_\ell) = 0.$$

Let k be a node at which θ^0 attains its maximum; then

$$\sum_{\ell\neq k} y_{k\ell}\tilde{V}_k\tilde{V}_\ell \sin(\theta^0_k - \theta^0_\ell) = 0.$$

Since $\theta^0 \in \Theta$, we have $\sin(\theta^0_k - \theta^0_\ell) \ge 0$ over $\mathcal{S}$. Each term of the above sum being nonnegative, must be equal to 0. Therefore, θ^0 has the same value at all neighboring nodes (i.e. nodes connected by an effective line). The network being connected, by an induction argument it follows that θ^0 has the same value over all $\mathcal{S}$. Since $p^T\theta^0 = 0$, and $\sum_{k\in\mathcal{S}} p_k = 1$, we have $\theta^0 = 0$.

Considering now the reactive part of the power equations, we find that $\tilde{V}^0$ is solution of the DC linear equation

$$\tilde{V} = V^D \text{ on } \mathcal{S}_V; \quad \sum_{\ell \neq k} y_{k\ell}(\tilde{V}_k - \tilde{V}_\ell) = 0,\ k \in \mathcal{S}_Q. \tag{24}$$

Since the network is connected, the unique solution of this system of equations is $\tilde{V}^0 = 1$ over $\mathcal{S}$. ∎

In order to compute the *critical cone*, and the *set of Lagrange multipliers*, associated with $(OPFP_0)$, we formulate the problem obtained by linearizing with respect to the optimization variable $x = (\tilde{\nu}, \theta, \tilde{P}, \tilde{V}, \tilde{Q})$ the cost function and constraints at the solution, i.e.

$$(LOPFP_0) \qquad \operatorname*{Min}_{(d\tilde{\nu}, d\theta, d\tilde{P}, d\tilde{V}, d\tilde{Q})} 2\sum_{k<l} \tilde{g}^0_{k\ell}\left(d\tilde{V}_k + d\tilde{V}_\ell\right) \qquad \text{subject to}$$

$$(d\tilde{\nu})p_k + \sum_{\ell \neq k} y_{k\ell}(d\theta_k - d\theta_\ell) = d\tilde{P}_k,\ k \in \mathcal{S}, \quad p^T d\theta = 0, \quad 0 \le d\tilde{P} \ \le \ 0, \tag{25}$$

$$\sum_{\ell \neq k} y_{k\ell}\left(d\tilde{V}_k - d\tilde{V}_\ell\right) = d\tilde{Q}_k,\ k \in \mathcal{S}, \quad d\tilde{V} = 0 \text{ on } \mathcal{S}_V, \quad 0 \le d\tilde{Q} \ \le \ 0. \tag{26}$$

It is known that the solution set of the above problem, whenever it is non empty, is equal to the the critical cone. Also, whenever the set of Lagrange multipliers is non empty, its recession set is the set of *singular multipliers*, see e.g. [6]. If λ is a Lagrange or singular multiplier for problem $(OPFP_u)$, we denote its components corresponding to the six constraints of $(OPFP_u)$ as

$$\lambda = \left(\lambda^\nu, \lambda^\theta, \lambda^P, \lambda^V, \lambda^{V^D}, \lambda^Q\right),$$

with the convention that $\lambda^P \in \mathbb{R}^{2n}_+$, the first (resp. last) n components corresponding to the inequality $-\tilde{P} + \varepsilon\check{P} \le 0$ (resp. $\tilde{P} - \varepsilon\hat{P} \le 0$), and similarly for λ^Q.

Lemma 3.2 (i) *The linearized problem has the unique feasible point, and therefore the unique solution* $\left(d\tilde{\nu}, d\theta, d\tilde{P}, d\tilde{V}, d\tilde{Q}\right) = 0$. *(In other words, the critical cone is reduced to 0.)*
(ii) *The set* Λ_0*, of Lagrange multipliers associated with* $x^0 = (\tilde{\nu}^0, \theta^0, \tilde{P}^0, \tilde{V}^0, \tilde{Q}^0)$ *for problem* $(OPFP_0)$*, is non empty and unbounded. More precisely, we have*

$$\Lambda_0 = \bar{\lambda} + \Lambda_0^\infty,$$

where $\bar{\lambda} = \left(\bar{\lambda}^\nu, \bar{\lambda}^\theta, \bar{\lambda}^P, \bar{\lambda}^V, \bar{\lambda}^{V^D}, \bar{\lambda}^Q\right)$ *is the Lagrange multiplier defined by*

$$(\bar{\lambda}^\nu, \quad \bar{\lambda}^\theta, \quad \bar{\lambda}^P, \quad \bar{\lambda}^Q) = 0,$$

$\bar\lambda^V$, defined over $\mathcal{S}_Q$, and that we extend by 0 over $\mathcal{S}_V$, is the unique solution of the linear DC problem

$$\sum_{\ell\neq k} y_{k\ell}(\bar\lambda^V_k - \bar\lambda^V_\ell) = -2\sum_{\ell\neq k} g^1_{k\ell}, \text{ for all } k\in\mathcal{S}_Q; \quad \bar\lambda^V = 0 \quad \text{on } \mathcal{S}_V, \tag{27}$$

$\bar\lambda^{V^D}$ is such that

$$\bar\lambda^{V^D}_k = -2\sum_{\ell\neq k}\left(\tilde g^0_{k\ell} + y_{k\ell}\lambda^V_\ell\right), \quad k\in\mathcal{S}_V. \tag{28}$$

and the recession set Λ_0^∞ is of the form

$$\begin{aligned}\Lambda_0^\infty \;=\; & \Big\{\lambda = \left(\lambda^\nu,\lambda^\theta,\lambda^P,\lambda^V,\lambda^{V^D},\lambda^Q\right);\ \lambda\neq 0;\ \left(\lambda^\nu,\lambda^\theta,\lambda^P,\lambda^Q\right)=0;\\ & \lambda^P_k = \lambda^P_{n+k};\ \lambda^Q_k = \lambda^Q_{n+k};\ 1\le k\le n\Big\}.\end{aligned}$$

Proof.(i) Let $(d\tilde\nu, d\theta, d\tilde P, d\tilde V, d\tilde Q)$ be a feasible point of the linearized problem. Then clearly $d\tilde P = d\tilde Q = 0$. Since $d\tilde V$ is solution of the well-posed linear DC problem (26) with zero right hand side, we have $d\tilde V = 0$. Also $(d\tilde\nu, d\theta)$ is zero since it is solution of a well-posed linear DC problem (22) with zero right hand side. This proves (i).
(ii) The set Λ_0 of Lagrange multipliers is the set of solutions of the dual to the linearized problem. Since the latter has a value 0, the dual has solutions, i.e. Λ_0 is non empty. On the other hand, since the inequality constraints cannot be strictly satisfied, the Mangasarian-Fromovitz qualification hypothesis does not hold. Therefore Λ_0 is unbounded.

Let $\lambda\in\Lambda_0$. Computing the derivative of the Lagragian function of the linearized problem with respect to $d\tilde\nu$ and $d\tilde\theta$, we obtain

$$\sum_k \lambda^\nu_k p_k = 0; \quad \lambda^\theta p_k + \sum_{\ell\neq k} y_{k\ell}(\lambda^\nu_k - \lambda^\nu_\ell) \;=\; 0.$$

Summing the second equation over k, we obtain that $0 = \lambda^\theta\sum_k p_k = \lambda^\theta$. It follows then from the second equation that λ^ν has a constant value over $\mathcal{S}$. By the first equation this value is 0.

The derivative of the Lagrangian function with respect to $\tilde V_k$, $k\in\mathcal{S}_Q$, is

$$\sum_{\ell\neq k} y_{k\ell}(\lambda^V_k - \lambda^V_\ell) \;=\; -2\sum_{\ell\neq k}\tilde g_{k\ell}, \ k\in\mathcal{S}_Q,$$

where in the last equation we have extended λ^V, (a priori defined over $\mathcal{S}_Q$) by 0 over $\mathcal{S}$. This equation uniquely determines λ^V, since we know that λ^V is null outside $\mathcal{S}_Q$. Note that, since the right hand side is nonpositive, we have $\lambda^V\le 0$.

Derivating now the Lagrangian function with respect to $\tilde V_k$, $k\in\mathcal{S}_V$, and using $\lambda^V = 0$, we obtain (28).

Finally derivating now the Lagrangian function with respect to $\tilde P$, and $\tilde Q$, we obtain the general expression for a Lagrange multiplier, from which the result follows. ∎

4 Uniform boundedness of solutions

In order to apply some general results of the theory of nonlinear programming with perturbations, we need to check that the solution set of $(OPFP_{u(\varepsilon)})$ is, for small enough $\varepsilon > 0$, non empty and uniformly bounded. This result will be established in this section. A preliminary step is to obtain an upper estimate of the value function of $(OPFP_{u(\varepsilon)})$, based on the property of *directional qualification* [9]. For this purpose we need to distinguish which of the inequality constraints on $\tilde{P}$ and $\tilde{Q}$ are in fact equality constraints. That is, denote

$$I_P := \{1 \le i \le n;\ \check{P}_i = \hat{P}_i\},\quad J_P := \{1, \cdots, n\} \backslash I_P,$$

and define I_Q and J_Q accordingly. Then we may reparametrize the inequality constraints of $(OPFP_u)$ as

$$\left\{ \begin{array}{ll} \tilde{P}_{I_P} = \varepsilon \hat{P}_{I_P}; & \varepsilon \check{P}_{J_P} \le \tilde{P}_{J_P} \le \varepsilon \hat{P}_{J_P}. \\ \tilde{Q}_{I_Q} = \varepsilon \hat{Q}_{I_Q}; & \varepsilon \check{Q}_{J_Q} \le \tilde{Q}_{J_Q} \le \varepsilon \hat{Q}_{J_Q}. \end{array} \right. \tag{29}$$

There is a simple relation between the Lagrange multipliers in this new formulation of the constraints and those in the old formulation, that we do not need to explicit.

Let us now formulate the linear programming problem obtained by linearizing the cost function and constraints both with respect to to the optimization variables and the small variable ε along the path $u(\varepsilon)$, at the solution of the limit problem:

$$(LOPFP_1) \qquad \underset{(d\tilde{\nu}, d\theta, d\tilde{P}, d\tilde{V}, d\tilde{Q})}{\mathrm{Min}} \sum_{k<l} \left(2g^1_{k\ell} \left(d\tilde{V}_k + d\tilde{V}_\ell \right) + \frac{1}{2} g^2_{k\ell} \right) \qquad \text{subject to}$$

$$\begin{aligned}
(d\tilde{\nu})p_k + \sum_{\ell \ne k} g^1_{k\ell} + \sum_{\ell \ne k} y_{k\ell}(d\theta_k - d\theta_\ell) &= d\tilde{P}_k,\ k \in \mathcal{S}, && (30)\\
p^T d\theta = 0, \quad d\tilde{P}_{I_P} = \check{P}_{I_P}, \quad \check{P}_{I_P} \le d\tilde{P}_{I_P} &\le \hat{P}_{I_P}, && (31)\\
-\sum_{\ell \ne k} h^1_{k\ell} + \sum_{\ell \ne k} y_{k\ell} \left(d\tilde{V}_k - d\tilde{V}_\ell \right) &= d\tilde{Q}_k,\ k \in \mathcal{S}, && (32)\\
d\tilde{V} = 0 \text{ on } \mathcal{S}_V, \quad d\tilde{Q}_{I_Q} = \check{Q}_{I_Q}, \quad \check{Q}_{J_Q} \le d\tilde{Q}_{J_Q} &\le \hat{Q}_{J_Q}. && (33)
\end{aligned}$$

Note that the only difference between this problem and $(LOPFP_0)$ lies in the bound constraints for $d\tilde{P}$ and $d\tilde{Q}$, and the constant terms related to g^1 and h^1 in the power equations. Note also that the set of solutions of $(LOPFP_1)$ has the following product form:

$$S(LOPFP_1) = S_a(LOPFP_1) \times S_r(LOPFP_1), \tag{34}$$

where subscripts a and r again refer to the active and reactive part,

$$S_a(LOPFP_1) := \{(d\tilde{\nu}, d\theta, d\tilde{P}) :\ (30) - (31) \text{ hold}\},$$

and $S_r(LOPFP_1)$ is the set of solutions of the problem of minimizing with respect to $(d\tilde{V}, d\tilde{Q})$ the cost function of $(LOPFP_1)$ subject to (32)-(33). It is easily checked that, due to the bounds on $d\tilde{P}$ and $d\tilde{Q}$, the solution set of $(LOPFP_1)$ is non empty and bounded.

Lemma 4.1 *Problem $(OPFP_1)$ satisfies the following directional qualification hypothesis [9]:*
(i) *The gradients of equality constraints (w.r.t. the optimization variables) are linearly independent,*
(ii) *There exists a feasible direction for the linearized problem $(LOPFP_1)$ that satisfies strictly every inequality constraint*

*Proof.*Set $d\tilde{P} := \frac{1}{2}(\check{P} + \hat{P})$, and $d\tilde{Q} := \frac{1}{2}(\check{Q} + \hat{Q})$. Then we may find $(d\tilde{\nu}, d\tilde{\theta}, d\tilde{V})$ such that the linearized power equations are satisfied and $d\tilde{V} = 0$ on $\mathcal{S}_V$, since this reduces to some well-posed linear DC equations. This proves (ii). For proving (i) it suffices to notice that the equality constraints of $(LOPFP_0)$ with an arbitrary right hand side can be solved by fixing $d\tilde{P}_{I_P}$, $d\tilde{Q}_{I_Q}$, and $d\tilde{V}$ to their prescribed value, $d\tilde{P}_{J_P}$ and $d\tilde{Q}_{J_Q}$ arbitrarily, and then solving the other equations with respect to $(d\tilde{\nu}, d\tilde{\theta}, d\tilde{V}_{\mathcal{S}_Q})$. The latter reduces again to some well-posed linear DC equations. ∎

As a consequence of the directional qualification property we get the following estimate, see [8, 9]:

Lemma 4.2 *For small enough $\varepsilon > 0$, problem $(OPFP_{u(\varepsilon)})$ is feasible and the following upper estimate of the value function holds:*

$$\text{val}\,(OPFP_{u(\varepsilon)}) \leq \text{val}\,(OPFP_0) + \varepsilon\,\text{val}\,(LOPFP_1) + o(\varepsilon).$$

It follows in particular from this lemma that val $(OPFP_{u(\varepsilon)}) \leq O(\varepsilon)$. We can now state the main result of the section.

Theorem 4.1 *The set of solutions of $(OPFP_{u(\varepsilon)})$ is non empty and uniformly bounded for small enough $\varepsilon > 0$.*

*Proof.*By the upper estimate of lemma 4.2, the set

$$F_1(OPFP_{u(\varepsilon)}) := \left\{ x \in F(OPFP_{u(\varepsilon)}); \quad \sum_{k<l} \pi_{k\ell}(x, u(\varepsilon)) \leq \text{val}\,(OPFP_0) + 1 \right\}$$

is non empty for small enough $\varepsilon > 0$. We prove that $F_1(OPFP_{u(\varepsilon)})$ is bounded. Indeed θ belongs to the bounded set Θ and the inequality constraints of $(OPFP_{u(\varepsilon)})$ imply that $\tilde{P}$ and $\tilde{Q}$ are bounded. As a consequence of the active power equation, $\tilde{\nu}$ is bounded. On the other hand, since $\tilde{\eta}^1_{k\ell} > 0$ whenever $k\ell$ is an effective line, we have for small enough $\varepsilon > 0$

$$\pi_{k\ell} \geq \frac{1}{2} \sum_{k<l} \tilde{\eta}^1_{k\ell} (\tilde{V}_k - \tilde{V}_\ell)^2.$$

Combining with (21), the upper estimate for the cost function and the fact that $V = V^D$ on $\mathcal{S}_V$, we obtain that $\tilde{V} \to \tilde{V}^0$. It follows that $F_1(OPFP_{u(\varepsilon)})$ is bounded, and passing to the limit in the power equations, we deduce that any sequence x_k of feasible solutions for $(OPFP_{u(\varepsilon_k)})$, with $\varepsilon_k \to 0$, converges in fact to the solution of $(OPFP_0)$.

This implies in particular that for given $\varepsilon > 0$ small enough, $F_1(OPFP_{u(\varepsilon)})$ is bounded. and the set of feasible voltages is bounded from below by a positive constant. Then we can prove the existence of a solution for $(OPFP_{u(\varepsilon)})$ by passing to the limit in a minimizing sequence. The conclusion follows. ∎

5 First order expansion of solutions

The theory of first order expansion of solutions of nonlinear programs involves the Hessian of the Lagrangian function of problem $(OPFP_u)$, evaluated at $u = 0$, along a direction in $S(LOPFP_1) \times du$, where we denote

$$du = u'(0) = (1, \eta^1, g^1, h^1)^T.$$

Taking into account the structure of the set of Lagrange multipliers, we find that its expression is

$$\begin{aligned}
H(dx, du) \quad := \quad & \sum_{k<l} \Big[2g^1_{k\ell} \left((d\tilde{V}_k)^2 + (d\tilde{V}_\ell)^2 \right) + g^2_{k\ell} \left(d\tilde{V}_k + d\tilde{V}_\ell \right) \\
& + 2y_{k\ell} \left(d\tilde{V}_k - d\tilde{V}_\ell \right)^2 \eta^1_{k\ell} + 2y_{k\ell} \left(d\theta_k - d\theta_\ell \right)^2 \eta^1_{k\ell} \Big] \\
& + \sum_{k \in \mathcal{S}_Q} \lambda^V_k \sum_{\ell \neq k} \Big[-d\tilde{V}_k dh_{k\ell} + y_{k\ell} \left(\left(d\theta_k - d\theta_\ell - \eta^1_{k\ell} \right)^2 - \left(\eta^1_{k\ell} \right)^2 \right) \Big] \\
& + \sum_{k \in \mathcal{S}_Q} \lambda^V_k d\tilde{Q}_k d\tilde{V}_k.
\end{aligned}$$

For later use we need the expression of the Hessian w.r.t. the optimization variables only:

$$\begin{aligned}
H(dx, 0) \quad := \quad & 2 \sum_{k<\ell} \Big[g^1_{k\ell} \left((d\tilde{V}_k)^2 + (d\tilde{V}_\ell)^2 \right) + 2y_{k\ell} \left(d\tilde{V}_k - d\tilde{V}_\ell \right)^2 \eta^1_{k\ell} \\
& + y_{k\ell} \left(d\theta_k - d\theta_\ell \right)^2 \eta^1_{k\ell} + y_{k\ell} (\bar{\lambda}^V_k + \bar{\lambda}^V_\ell)(d\theta_k - d\theta_\ell)^2 \Big] \\
& \sum_{k \in \mathcal{S}_Q} \lambda^V_k d\tilde{Q}_k d\tilde{V}_k.
\end{aligned}$$

Note that, in case $g^1_{k\ell} = 0$ for all k and ℓ, then $\bar{\lambda}^V = 0$ and in that case $H(dx, 0) = H_0(dx)$, where

$$H_0(dx) \quad := \quad 2 \sum_{k<\ell} y_{k\ell} \eta^1_{k\ell} \left(\left(d\tilde{V}_k - d\tilde{V}_\ell \right)^2 + \left(d\theta_k - d\theta_\ell \right)^2 \right).$$

The problem of minimizing the Hessian of Lagrangian over the set of solutions of the linearized problem $(LOPFP_1)$ is

$$(SP) \qquad \operatorname*{Min}_{dx} H(dx, du); \quad dx \in S(LOPFP_1).$$

Note that $H(dx, du)$ can be split into an active part (not depending on $d\tilde{V}$ and $d\tilde{Q}$) and a reactive part (not depending on $d\tilde{\nu}$, $d\theta$ and $d\tilde{P}$). Reminding that, by (34), the set $S(LOPFP_1)$ has a product form, we observe that the above problem splits into two independent subproblems, one for the active part and the other for the reactive part.

We have checked that the directional qualification hypothesis holds. Since the critical cone is reduced to 0, the standard second order sufficient conditions (positivity of the curvature of Hessian of Lagrangian for at least one Lagrange multiplier, along any nonzero critical direction) hold automatically. This, combined with lemma 4.1 and theorem 4.1, implies by [1, 5, 11] that

Theorem 5.1 (i) *The value function of* $(OPFP_{u(\varepsilon)})$ *has the following expansion:*

$$\text{val}\,(OPFP_{u(\varepsilon)}) = \text{val}\,(OPFP_0) + \varepsilon\ \text{val}\,(LOPFP_1) + \varepsilon^2 \text{val}\,(SP) + o(\varepsilon^2).$$

(ii) *Let* $x(\varepsilon)$ *be a* $o(\varepsilon^2)$ *solution of* $(OPFP_{u(\varepsilon)})$*. Then any limit point of* $\varepsilon^{-1}(x(\varepsilon)-x^0)$ *belongs to* $S(SP)$*. Conversely, with any* $dx \in S(SP)$ *we may associate a path* $x(\varepsilon) = x^0 + \varepsilon dx + o(\varepsilon)$ *such that* $x(\varepsilon)$ *is a* $o(\varepsilon^2)$ *solution of* $(OPFP_{u(\varepsilon)})$*.*

The above theorem implies that, if (SP) has a unique solution dx, and $x(\varepsilon)$ is a $o(\varepsilon^2)$ solution of $(OPFP_{u(\varepsilon)})$, then we have the first order expansion for approximate solutions

$$x(\varepsilon) = x^0 + \varepsilon dx + o(\varepsilon).$$

However, due to the contribution of the Hessian of constraints, the function $H(\cdot, \cdot)$ may be nonconvex and we cannot prove the uniqueness of solutions of (SP), except under some additional assumption that we set now.

The *enlarged critical cone* (see e.g. [7, Section 4]) is defined as the set of directions dx in the space of optimization variables for which each scalar constraint function of the form $g(x, u)$ is such that $D_x g(x^0, 0)dx = 0$ if either this is an equality constraint or there exists a Lagrange multiplier with a corresponding nonzero component.

Theorem 5.2 *Fix all data of the problem except* $g(\cdot)$*. Assume that the restriction of* p *to* I_P *is not 0. If* g^1 *is close enough to 0 (in particular if* $g^1 = 0$*), then* (i) $H_0(\cdot)$ *is positive definite over the enlarged critical cone, and* (ii) (SP) *has a unique solution.*

Proof.(i) It follows from (27) that, once all data of the problem are fixed except $g(\cdot)$, then when $g^1 \to 0$, we have $\bar{\lambda}^V \to 0$, and therefore

$$H(dx, 0) \to H_0(dx).$$

It is clear that $H_0(\cdot)$ is semi definite positive. Let us check that it is definite positive over the linear space spanned by $F(LOPFP_1)$. If dx belongs to the latter, then $d\tilde{P}_{I_P} = 0$, $d\tilde{Q}_{I_Q} = 0$ and $d\tilde{V} = 0$ on $\mathcal{S}_V$. If in addition dx belongs to the kernel of $H_0(\cdot)$, then (since $\eta^1_{k\ell} \neq 0$ whenever $k\ell$ is an effective line) $d\tilde{V}$ and $d\theta$ are constant over $\mathcal{S}$. Since $\mathcal{S}_V \neq \emptyset$ and $p^T\theta = 0$, we have $d\tilde{V} = 0$ and $d\theta = 0$. From the active (resp. reactive) power flow equation, it follows that $d\tilde{P} - d\tilde{\nu}p$ (resp. $d\tilde{Q}$) is null over $\mathcal{S}$ (resp. $\mathcal{S}_Q$). Finally, let $k_0 \in I_P$ be such that $p_{k_0} \neq 0$. Then since $d\tilde{P}_{k_0} = 0$, we have $d\tilde{\nu} = 0$, hence $d\tilde{P} = 0$.
(ii) This is an easy consequence of point (i). ∎

6 High order expansion of the solution

Under the assumptions of theorem 5.2, we can check that the local solution of $(OPFP_{u(\varepsilon)})$ is unique, associated with a unique Lagrange multiplier, and obtain its power expansion. The proof is based on an abstract result [4, Theorem 4.3], where the high order expansion of the local solution and Lagrange multiplier is obtained by reducing to the strong regularity framework and then using [7, Theorem 4.13]. Theorem 4.3 of [4] uses two hypotheses, introduced in [4], that we discuss now.

We say that the *directional linear independence qualification condition* holds if the linearized problem $(LOPFP_1)$ is feasible, and the hypothesis of linear independence of gradients of constraints holds at each solution of (SP). It is easy to check that linear independence of gradients of constraints holds in fact at each feasible point of $(LOPFP_1)$, hence the directional linear independence qualification condition holds. It follows from this hypothesis that with the linearized problem $(LOPFP_1)$ is associated a unique dual solution, i.e. a unique Lagrange multiplier.

The *strong directional second-order condition* assumes that with the linearized problem $(LOPFP_1)$ is associated a unique Lagrange multiplier (dual solution) such that, for any nonzero direction of the enlarged critical cone, the associated curvature of the Hessian of Lagrangian along that direction is positive. By theorem 5.2, this hypothesis is satisfied whenever g^1 is close enough to 0. Applying [4, Theorem 4.3], we obtain the following result:

Theorem 6.1 *Assume that all $(\eta(\varepsilon), g(\varepsilon), h(\varepsilon))$ are analytic functions. Fix all data of the problem except $g(\cdot)$. Assume that the restriction of p to I_P is not 0. If g^1 is close enough to 0, then for $\varepsilon > 0$ small enough, the local solution of $(OPFP_{u(\varepsilon)})$ is unique, associated with a unique Lagrange multiplier, and has a power series expansion.*

By [7], the coefficient of the first term of the power series is the unique solution of (SP) and associated Lagrange multiplier, and the other terms can be computed as solutions of quadratic problems and associated Lagrange multiplier.

7 The direct current approximation

The framework presented here allows a rigorous study of some classical approximations, like the direct current approximation (DCA) that we present now. The DCA consists in assuming two kind of hypotheses:

- On the parameters of the problem: neglect the capacitive lines, and assume η to be "small".

- On the optimization variables: assume V to be constant over the network, (say, after scaling, equal to 1) and differences of phases to be "small".

By "small" it is meant here that the nonlinear functions of θ and η may be linearized. These hypotheses allow to reduce $(OPFP_u)$ to the simpler problem

$$(DCA) \qquad \operatorname*{Min}_{(\tilde{\nu},\theta,\tilde{P})} \sum_{k<l} y_{k\ell}\eta^1_{k\ell}(\theta_k - \theta_\ell)^2 \qquad \text{subject to}$$

$$\tilde{\nu} p_k + \sum_{\ell \neq k} y_{k\ell}(\theta_k - \theta_\ell) = P_k,\ k \in \mathcal{S}, \qquad p^T\theta = 0, \quad \check{P} \leq \tilde{P} \leq \hat{P}.$$

It is enough to observe that, in the framework of the high voltage approximation, this is the problem whose solution gives the first order expansion of solution, in the case $g^1 = 0$. Our scheme gives a rigorous basis to this computation. Note that it gives also an expression for the first order variation of voltages, that is not present in the direct current approximation.

8 Conclusion

The high voltage approximation combines the classical approximation that resistors and interaction between ground and lines are small, with the hypothesis that the nominal voltage is large. Combining this approximation with the theory of nonlinear programming with perturbations, it is possible to obtain an expansion of the value function and solution of the problem. In addition, the quadratic subproblem that allows to compute the expansion of value function and solution splits into two independent subproblems, related to the active (reps. reactive) variables. This asymptotic analysis justifies in particular the direct current approximation.

Some of our hypotheses are somewhat restrictive and deserve further extensions. For instance we have assumed for simplicity that V^D is constant over $\mathcal{S}_V$. This property is not really needed from the mathematical point of view and is too restrictive for practical problems. A possible extension is to fix the average value of V^D and to bound its variations.

Another useful, but more difficult, extension is to include in the analysis the voltage regulators, that may be modelized by a certain monotonous relation between the voltage and input of reactive power at each node.

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Éditeur
INRIA, Domaine de Voluceau, Rocquencourt, BP 105, 78153 LE CHESNAY Cedex (France)
http://www.inria.fr
ISSN 0249-6399