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Patrick Joly, Leila Rhaouti

INSTITUT NATIONAL DE RECHERCHE EN INFORMATIQUE ET EN AUTOMATIQUE

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Patrick Joly, Leila Rhaouti *

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Abstract: In this paper, we are interested in a mathematical analysis of a simplified 1D mechanical model for the interaction between a mallet and an elastic membrane. First, using elementary mathematical techniques, we study quite precisely the case where the mallet is in interaction with a rigid body. This analysis allows us to understand the influence of physical parameters on the mallet behavior. Then, we show how the results of this study can be extended to the case where the mallet is in interaction with a membrane.

Key-words: Wave equation - Ordinary Differential Equations - Non-linear interaction - Musical Acoustics - Contact problem

(Résumé : tsvp)

* INRIA, Domaine de Voluceau, B.P.105 78153 Le Chesnay cedex, France.

Unité de recherche INRIA Rocquencourt

Domaine de Voluceau, Rocquencourt, BP 105, 78153 LE CHESNAY Cedex (France)

Téléphone : (33) 01 39 63 55 11 – Télécopie : (33) 01 39 63 53 30

Analyse d'un modèle simplifié pour l'interaction non linéaire membrane-maillet

Résumé : Dans ce rapport, nous nous intéressons à l'analyse mathématique d'un modèle mécanique 1D simplifié de l'interaction entre un maillet et une membrane élastique. Dans un premier temps, à l'aide de techniques mathématiques élémentaires, nous menons une étude assez fine du cas où le maillet est en contact avec un corps rigide. Cette analyse permet notamment de comprendre l'influence des paramètres physiques sur le mouvement du maillet. Dans un deuxième temps, nous montrons que les résultats de cette étude peuvent s'étendre au cas où le maillet est en contact avec une membrane.

Mots-clé : Equations des ondes - Equations Différentielles Ordinaires - Interaction non linéaire - Acoustique Musicale - Problème de contact

1 Introduction - Position of the problem

In this paper, we are interested in a mathematical analysis of a mechanical model for the interaction between a mallet and an elastic membrane. Such a model occurs in the mathematical description of the behavior of a kettle-drum. The model given below is taken from an article by A. Chaigne and A. Ramdane (see [3]). Let us consider a plane elastic membrane, denoted Σ, that is supposed to be located in the plane $z = 0$. We make the usual assumption of small displacements and small deformations, which allows us to use a linear elasticity model. The movement of the membrane can be described by the distribution $w(x, y, t)$ of the normal displacement of the membrane, where $(x, y) \in \Sigma$ denotes a point of the membrane and $t > 0$ is time. The mallet is assimilated to a deformable solid whose shape is a sphere of radius δ when it is not deformed. When the mallet is compressed, we assume that its shape and its movement can completely be described by the knowledge of the distance $\tilde{u}(t) < \delta$ between the center of gravity of the mallet and the contact point. $\tilde{u}$ varying with the time, we shall say that the mallet is in a phase of compression if $\tilde{u}$ decreases and in a phase of decompression if $\tilde{u}$ increases. The mallet can be in contact with another body (rigid or perfectly elastic). We shall suppose that this contact remains punctual and shall denote by C the contact point, G denoting the center of gravity of the mallet. During the contact, the mallet can be compressed under the action of the other body.

Remark 1.1 *All what we say has a physical meaning if and only if $\tilde{u}(t)$ remains strictly positive. However, the mathematical model we shall use, will allow $\tilde{u}(t)$ to become negative for some times. This will not be possible if the initial data of the evolution problem are adequately chosen.*

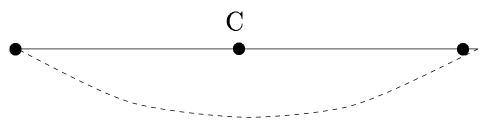

Figure 1.1: The movement of the membrane

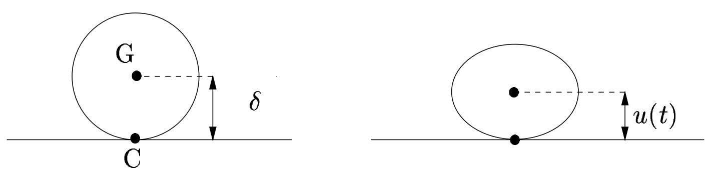

Figure 1.2: The undeformed mallet The deformed mallet

In which follows, we are interested in the interaction between the mallet and the membrane. We assume that the contact is punctual and that the contact point $(x_0, y_0) \in \Sigma$ does not change with time. We also suppose that the movement of the center of gravity of the mallet lies at a vertical line $x = x_0, y = y_0$. Of course, if $u(t)$ denotes the vertical z-component of G at time t, we see that (see figure 1.3) :

- if $u(t) \geq w(x_0, y_0, t) + \delta$, the mallet is not deformed
- if $u(t) < w(x_0, y_0, t) + \delta$, the mallet is deformed and $\tilde{u}(t) = u(t) - w(x_0, y_0, t)$

The mechanical interaction between the mallet and the membrane is characterized by a vertical force $F(t)$ (which is positive and applied at point G if one considers the action of the membrane on the mallet and applied at point C if one considers the action of the mallet on the membrane). The intensity of the force is completely defined by $\tilde{u}(t)$, more precisely :

$$(1.1) \quad \left| \begin{array}{ll} \text{(i)} & \text{if } \tilde{u}(t) \geq \delta \quad F(t) = 0 \\ \text{(ii)} & \text{if } \tilde{u}(t) < \delta \quad F(t) = K\varphi(\tilde{u}(t)) + R\dfrac{d}{dt}\varphi(\tilde{u}(t)) \end{array} \right.$$

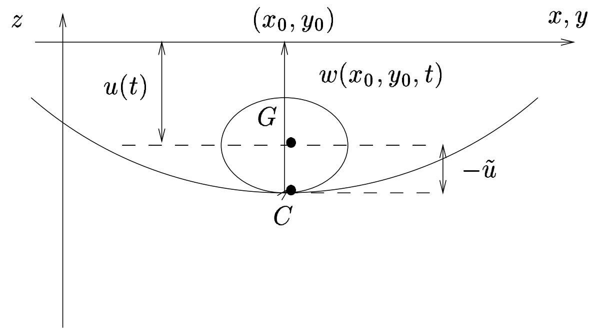

Figure 1.3: The interaction

where $\varphi(\tilde{u})$ is the function defined by :

(1.2) $$\varphi(\tilde{u}) = [(\delta - \tilde{u})^+]^p \quad \text{, for some } p \geq 1$$

and where K and R are two real positive constants

$$\begin{cases} \bullet \ K > 0 & \text{is the stiffness coefficient} \\ \bullet \ R \geq 0 & \text{is the resistive coefficient} \end{cases}$$

When $R = 0$, (1.1) expresses in particular that the interaction of the force F increases when $\tilde{u}$ decreases. When $R > 0$, the force does not only depend on the compression of the mallet (characterized by $\tilde{u}$), but also on the velocity of this compression : the additional force is positive if $\tilde{u}$ decreases in time and negative if $\tilde{u}$ is a decreasing function of time. Moreover, quicker is the compression (or the decompression) of the mallet, stronger is the intensity of the additional force.

Remark 1.2 *We see that the function $\varphi(\tilde{u})$ is a decreasing function of $\tilde{u}$ which remains defined for $\tilde{u} < 0$.*

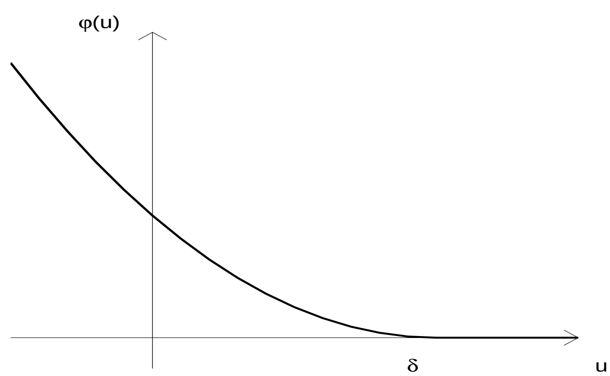

Figure 1.4: The Graph of φ

A more realistic model would probably consist to consider a function φ which tends to $+\infty$ when $\tilde{u}$ tends to 0. This would prevent that $\tilde{u}(t)$ becomes negative in the evolution problem.

Finally, we assume that the membrane is perfectly elastic and homogeneous, which means in particular that there is no damping term in the mathematical model. Time and space scalings will be chosen in such a way that the equation governing w is the wave equation with velocity 1 and the mass of the mallet is equal to 1.

The resulting model writes :

$$
(1.3) \qquad \begin{cases} \dfrac{\partial^2 w}{\partial t^2} - \Delta w = -\eta F(t)\delta(x-x_0, y-y_0, z-z_0) \quad (x,y) \in \Sigma\ ,\ t>0 \quad (\eta > 0) \\ u'' = F(t) \\ F(t) = K\varphi\left(u(t) - w(x_0,y_0,t)\right) + R\dfrac{d}{dt}\left[\varphi\left(u(t) - w(x_0,y_0,t)\right)\right] \end{cases}
$$

plus initial conditions, for instance :

$$
(1.4) \qquad \begin{cases} w(x,y,0) = \dfrac{\partial w}{\partial t}(x,y,0) = 0 \\ u(0) = \delta \\ u'(0) = -v_0 \quad (v_0 > 0) \end{cases}
$$

which express that the system is initially at rest and that, at time $t = 0$, the mallet strikes the membrane with a velocity equal to v_0. To close the system, we also need boundary conditions, namely

$$
(1.5) \qquad w(x,y,t) = 0 \quad (x,y) \in \partial\Sigma\ ,\ t > 0
$$

which express that the membrane is fixed along its boundary.

Remark 1.3 • *To be able to get finite energy solutions, it is necessary to regularize the δ-function and to replace the first and third equation of (1.3) by*

$$
\begin{cases} \dfrac{\partial^2 w}{\partial t^2} - \Delta w = -\eta F(t)\delta^\varepsilon(x-x_0, y-y_0, z-z_0) \\ F(t) = K\varphi\left(u(t) - \displaystyle\int_\Sigma w\,\delta^\varepsilon dxdy\right) + R\dfrac{d}{dt}\left[\varphi\left(u(t) - \displaystyle\int_\Sigma w\,\delta^\varepsilon dxdy\right)\right] \end{cases}
$$

where δ^ε is a smooth function with support in the ball $\{(x-x_0)^2 + (y-y_0)^2 \leq \varepsilon^2\}$ and satisfying

$$
\begin{cases} \delta^\varepsilon \geq 0 \\ \displaystyle\int_\Sigma \delta^\varepsilon dxdy = 1 \end{cases}
$$

• *System ((1.3),(1.4),(1.5)) describes the movement of the membrane in the vacuum. To get a more realistic model, it would be necessary to take into account the external fluid and its interaction with the membrane via the pression in the fluid (see [7]).*

In this paper, we wish to make a mathematical study of a 1D version of the model. This means that we replace the membrane by a line. Moreover, we shall assume (this is not necessarily a strong restriction, taking into account the finite velocity of propagation) that the line is infinite (we thus work on the real line $\mathbb{R}$), which permits us in particular to get ride of the boundary condition (1.5). Finally, in dimension 1, we no longer need to work with a regularized δ-function : this is due to the fact that δ belongs to $H^{-1}(\mathbb{R})$. Therefore, we shall consider the following non-linear initial value problem :

$$
\text{(1.6)}\quad
\begin{array}{|l}
\text{Find} \left\{\begin{array}{lcll} w(x,t) & : & \mathbb{R}\times\mathbb{R}^+ & \rightarrow \mathbb{R} \\ u(t) & : & \mathbb{R}^+ & \rightarrow \mathbb{R} \end{array}\right. \\
\text{Such that} \\
\left\{\begin{array}{l}
\dfrac{\partial^2 w}{\partial t^2} - \dfrac{\partial^2 w}{\partial x^2} = -\eta F(t)\delta(x) \quad x\in\mathbb{R}\,,\ t>0 \\
u'' = F(t) \\
F(t) = K\varphi\left(u(t)-w(0,t)\right) + R\dfrac{d}{dt}\left[\varphi\left(u(t)-w(0,t)\right)\right] \quad t>0 \\
w(x,0) = \dfrac{\partial w}{\partial t}(x,0) = 0 \quad x\in\mathbb{R} \\
u(0) = \delta \\
u'(0) = -v_0 \quad (v_0>0)
\end{array}\right.
\end{array}
$$

where φ is defined by (1.2). In fact, most of the results we are going to obtain will be valid for more general functions $\varphi(u)$ provided that the following assumptions are satisfied :

$$
\text{(1.7)}\quad
\left\{\begin{array}{l}
\varphi(u)\geq 0\,,\ \forall u\in\mathbb{R} \text{ and } \varphi(u)=0\,,\ \forall u\geq\delta \\
\varphi(u) \text{ is Lipschitz continuous} \\
\varphi(u) \text{ is convex and decreasing}
\end{array}\right.
$$

For some results, we shall be let to add the assumption :

$$
\text{(1.8)}\quad \varphi\in C^1(\mathbb{R}-\{\delta\})
$$

It is easy to verify that the function φ defined by (1.2) satisfies (1.7) and (1.8). The function φ is even C^1 as soon as $p>1$, which corresponds to the realistic case. For $p=1$, $\varphi(u)=(\delta-u)^+$ is almost linear. We shall refer to this case as the 'linear case'. This is the only case when one is able to get explicit solutions. But this case is not realistic since physically, in the membrane-mallet interaction, p is always greater than $3/2$ (see [2]).

A first step of the analysis consists in looking at the even simpler case where the membrane is no longer elastic but perfectly rigid : we are then led to study the movement of the mallet which strikes a rigid body (a table for instance) which amounts to consider the ordinary differential equation :

$$
\text{(1.9)}\quad
\left\{\begin{array}{l}
u'' = F(t) \\
F(t) = K\varphi(u) + R\dfrac{d}{dt}\varphi(u) \quad t>0 \\
u(0) = \delta \\
u'(0) = -v_0
\end{array}\right.
$$

The study of this evolution equation is in fact the basis of this paper. It is the object of the second section in which we first consider the equation with $R=0$ (i.e. without any damping) and then analyze the influence of R on the behavior of the solution. We show in particular that only two distinct behaviors are possible, depending on the value of the coefficient R. Then, in section 3, we show how the analysis of system (1.6) can be reduced to the one of an equation analogous to (1.9), which leads us to a qualitative description of system (1.6).

2 The movement of the mallet on a rigid body

Throughout this section, we shall be interested in the study of the differential equation (1.9), assuming that function φ satisfies (1.7). Note that, as $\operatorname{supp}\varphi \subset]-\infty,\delta]$, we can introduce the function :

$$\psi(u) = \int_u^{+\infty} \varphi(t)dt \tag{2.1}$$

which is a $C^{1,1}$ function satisfying $\psi' = -\varphi$, so that we easily deduce from (1.7) the following properties

$$\begin{cases} \psi(u) \text{ is convex positive} \\ \psi(u) = 0 \text{ for } u \geq \delta \\ \psi(u) \to +\infty \text{ when } u \to -\infty \end{cases} \tag{2.2}$$

For instance, if φ is given by formula (1.2), then :

$$\psi(u) = \frac{[(\delta - u)^+]^{p+1}}{p+1} \tag{2.3}$$

and the graph of ψ is similar to the one of φ (see figure 2.3). Note that ψ is a nothing but a potential connected to the force F.

2.1 The case $R = 0$

We first state an existence and uniqueness result.

Theorem 2.1 *Under assumptions (1.7), problem (1.9) admits a unique global solution :*

$$u \in C^2(\mathbb{R})$$

and one has energy conservation :

$$E(u,t) = E(u,0) = \frac{1}{2}v_0^2 \quad \textit{where} \quad E(u,t) = \frac{1}{2}\left|u'\right|^2 + K\psi(u) \tag{2.4}$$

Proof.

1. Introducing $v = u'$, we can rewrite (1.9) as a first order system :

$$\frac{d}{dt}\begin{bmatrix} u \\ v \end{bmatrix} = F\begin{bmatrix} u \\ v \end{bmatrix} \tag{2.5}$$

where we have set

$$F\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} v \\ K\varphi(u) \end{bmatrix} \tag{2.6}$$

Then it suffices to remark that, under assumption, F is locally Lipschitz-continuous in $\mathbb{R}^2$. Then the local Cauchy-Lipschitz theorem applies and we deduce that (1.9) admits a unique local maximal solution

$$u \in C^2([0, T_{\max}])$$

with the alternative :

$$\left| \begin{array}{ll} (1) & T_{\max} = +\infty : \text{The solution is global} \\ (2) & \lim\limits_{t \nearrow T_{\max}} \left\{ |u'(t)|^2 + |u(t)|^2 \right\} = +\infty \end{array} \right.$$

2. Multiplying (1.9) by u' for $t < T_{\max}$, we get :

$$\forall t < T_{\max} \quad \frac{d}{dt} E(u,t) = 0$$

which means that

$$\frac{1}{2}\left|u'(t)\right|^2 + K\psi(u(t)) = \frac{1}{2}v_0^2 \quad \forall t < T_{\max} \tag{2.7}$$

As $\psi(u) \geq 0$, (2) cannot occur and thus the solution is global.

□

In our next theorem, we give a precise description of the behavior of the solution :

Theorem 2.2 *Let's assume that φ satisfies (1.7) and (1.8). There exists a unique time $t_m > 0$, such that $u'(t_m) = 0$. The function $u(t)$ is strictly convex in $(0, t_m)$, strictly decreasing in $[0, t_m)$ and strictly increasing in $(t_m, +\infty)$. Moreover,*

$$\left| \begin{array}{llll} \textit{(i)} & \forall t \leq t_m & , & u(2t_m - t) = u(t) \\ \textit{(ii)} & \forall t \geq 2t_m & , & u(t) = \delta + v_0(t - 2t_m) \end{array} \right.$$

Proof. Let us introduce the set :

$$\mathcal{E} = \{t \geq 0 \text{ such that } \forall s \,/\, 0 < s \leq t \;,\; u'(s) < 0\}$$

which is clearly an interval of $\mathbb{R}^+$ with lower bound 0.

1. $t_m = \sup \mathcal{E} < +\infty$.

   If not, the function $u(t)$ would be decreasing on $\mathbb{R}^+$. From the energy identity, we know that $\psi(u(t)) < v_0^2/2K$. Since $\psi(u) \to +\infty$ when $u \to -\infty$, this proves that $u(t)$ is bounded from below and thus converges to some limit value $u_\infty < \delta$ when t goes to $+\infty$. Therefore, $\lim\limits_{t\to+\infty} \varphi(u(t)) = \varphi(u_\infty) > 0$. The equality

$$u'' = K\varphi(u)$$

   shows then that

$$u'' > K\varphi(u_\infty) > 0 \quad \text{for } t \text{ large enough}$$

   This would implies that

$$u'(t) \to +\infty \quad \text{when } t \to +\infty$$

   which contradicts the facts that $u'(t) \leq 0 \quad \forall t$.

2. $u'(t_m) = 0$

   Indeed, if $u'(t_m) < 0$, u' would remain negative in some interval $[t_m, t_m + \epsilon]$, which would contradicts the definition of t_m.

3. Conclusion

   To conclude the proof, it suffices to check that knowing u in $[0, t_m]$, one completes the solution by setting

$$\left\{ \begin{array}{ll} u(2t_m - t) = u(t) & \forall t \leq t_m \\ u(t) = \delta + v_0(t - 2t_m) & \forall t \geq 2t_m \end{array} \right.$$

   The main points are the following facts :

   - Equation $u'' = K\varphi(u)$ is reversible in time
   - $\varphi(u) = 0$ for $u \geq \delta$
   - The solution is unique

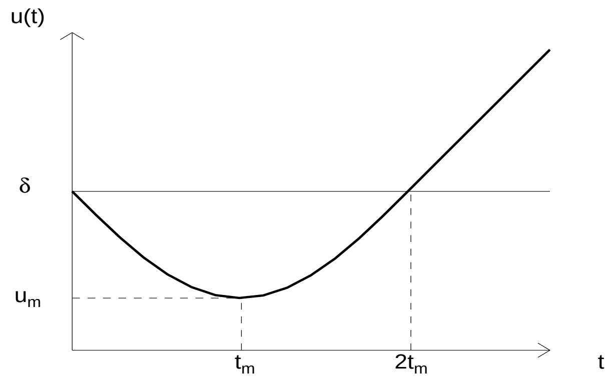

Figure 2.1: Graph of $u(t)$

The details are left to the reader.

□

We give in figure 2.1, the typical graph of the variation of $u(t)$:

The movement of the mallet can also be analyzed in the phase plane (u, v), with $v = u'$. The trajectories of the point $M(t) = (u(t), v(t))$ belongs to the line $v^2 + 2K\psi(u) = v_0^2$. We have represented in figure 2.2, the level lines of the function $(u, v) \to v^2 + 2K\psi(u)$ when φ is given by (1.9) and indicated the trajectories of the point $M(t)$.

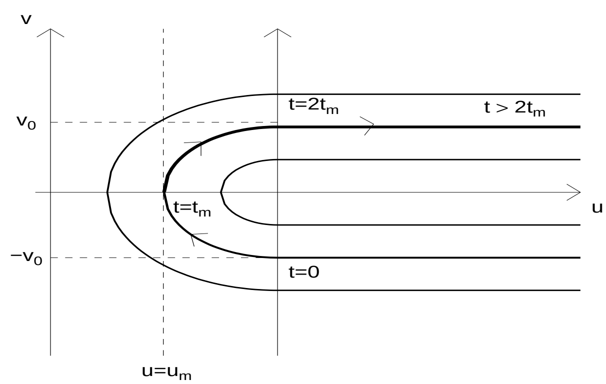

Figure 2.2: The movement in the phase plane

The 'linear case' $\varphi(u) = (\delta - u)^+$ is of particular interest because one can get an explicit formula for $u(t)$:

$$\left\{ \begin{array}{lcll} u(t) & = & \delta - \dfrac{v_0}{\sqrt{K}} \sin(\sqrt{K}t) & \text{for } t \leq \dfrac{\pi}{\sqrt{K}} \\ u(t) & = & \delta + v_0 \left(t - \dfrac{\pi}{\sqrt{K}} \right) & \text{for } t > \dfrac{\pi}{\sqrt{K}} \end{array} \right.$$

In this case, note that t_m is given by

$$t_m = \frac{\pi}{\sqrt{2K}}$$

while the minimum value taken by u is :

$$u_m = u(t_m) = \delta - \frac{v_0}{\sqrt{K}}$$

Remark 2.1 *In the case* $p = 1$ *:*

- *The curvilinear part of the graph $u(t)$ is one half of one arch of sinusoïd and the iso-energy lines in the phase plane in the region $u \leq \delta$ are nothing but semi-circles centered at point $(u = \delta, v = 0)$.*
- *One see that u_m decreases with v_0, and increases with K. The time t_m is also a decreasing function of K but is independent of v_0.*

For a general case φ, it is not possible to get an explicit solution. However, we can give a quasi explicit representation of the solution by remarking that, in the interval $[0, t_m]$, $u(t)$ is nothing but the inverse of the function $T_1(u) \;:\; [u_m, \delta] \rightarrow [0, t_m]$ defined by :

$$T_1(u) = \sqrt{2} \int_u^{\delta} \frac{ds}{[2K\psi(s) - v_0^2]^{1/2}} \tag{2.8}$$

where u_m is the unique solution of the equation :

$$\psi(u_m) = \frac{v_0^2}{2K} \tag{2.9}$$

Formula (2.8) is directly deduced from identity (2.4).

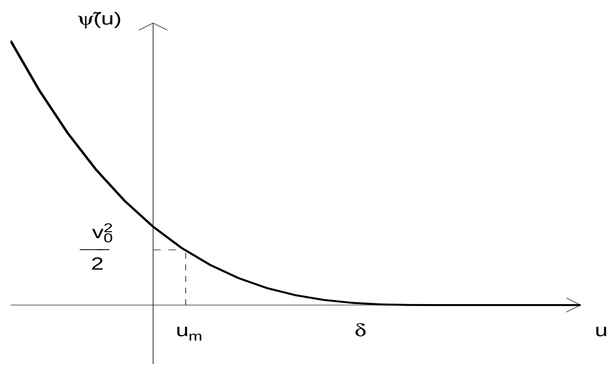

Figure 2.3: Construction of u_m

From the properties of ψ, we infer that u_m is a decreasing function of v_0 and an increasing function of K, results which are in accordance with the physical intuition.
Finally, as $u_m = u(t_m)$, t_m is given by :

$$t_m = \sqrt{2} \int_{u_m}^{\delta} \frac{ds}{[2K\psi(s) - v_0^2]^{1/2}} \tag{2.10}$$

The dependence of t_m with respects to K and v_0 in the general case is more delicate to analyze and we have not succeeded in doing it.

When φ is given by (1.2), with $p = 2$, and $p = 3$, one can remark that u can be expressed as inverse of elliptic functions (see [1]). Indeed from 2.8, we get :

$$\tag{2.11} \left| \begin{aligned} T_1(u) &= \sqrt{2} \int_u^{\delta} \frac{ds}{[v_0^2 - \frac{2K}{p+1}(\delta - s)^{p+1}]^{1/2}} = \sqrt{\frac{p+1}{K}} \int_u^{\delta} \frac{ds}{[\frac{v_0^2(p+1)}{2K} - (\delta - s)^{p+1}]^{1/2}} \\ &= \sqrt{\frac{p+1}{K}} \int_0^{\delta - u} \frac{ds}{[\frac{v_0^2(p+1)}{2K} - s^{p+1}]^{1/2}} = \sqrt{\frac{p+1}{K}} \left(\frac{(p+1)v_0^2}{2K} \right)^{\frac{1}{p+1} - \frac{1}{2}} \int_0^{\xi(\delta - u)} \frac{ds}{[1 - s^{p+1}]^{1/2}} \\ &= \left(\frac{p+1}{K} \right)^{\frac{1}{p+1}} \left(\frac{2}{v_0^2} \right)^{\frac{1}{2} - \frac{1}{p+1}} \int_0^{\xi(\delta - u)} \frac{ds}{[1 - s^{p+1}]^{1/2}} \end{aligned} \right.$$

where

$$\xi = \left(\frac{2K}{(p+1)v_0^2} \right)^{\frac{1}{p+1}}$$

When $p = 2$ or $p = 3$, $\displaystyle\int_0^{\xi(\delta-u)} \frac{ds}{[1-s^{p+1}]^{1/2}}$, is nothing but an elliptic integral and can thus be expressed in terms of elementary functions. The inversion of this first kind integral gives u as an elliptic function. More precisely, for $p = 2$, u is a Weierstrass'elliptic function, and for $p = 3$, u is a Jacobian elliptic function.

Moreover, when φ is given by (1.2), for any p, one can compute explicitly the time t_m. Indeed, since

$$\psi(u_m) = \frac{v_0^2}{2K},$$

we get :

(2.12) $$\xi(\delta - u_m) = 1$$

Joining (2.11) and (2.12), we get

$$\left| \begin{aligned} t_m &= \left(\frac{p+1}{K}\right)^{\frac{1}{p+1}} \left(\frac{2}{v_0^2}\right)^{\frac{1}{2}-\frac{1}{p+1}} \int_0^1 \frac{dy}{(y^{p+1}-1)^{1/2}} \\ &= \left(\frac{p+1}{K}\right)^{\frac{1}{p+1}} \left(\frac{2}{v_0^2}\right)^{\frac{1}{2}-\frac{1}{p+1}} \frac{1}{p+1} \int_0^1 u^{\frac{1}{p+1}-1}(1-u)^{1/2-1} du \\ &= \left(\frac{p+1}{K}\right)^{\frac{1}{p+1}} \left(\frac{2}{v_0^2}\right)^{\frac{1}{2}-\frac{1}{p+1}} \frac{1}{p+1}\, \beta\left(\frac{1}{p+1}, \frac{1}{2}\right) \end{aligned} \right.$$

β is known as the β-function and can be related to the gamma function by the next formula (see [6] p.15)

$$\beta\left(\frac{1}{p+1}, \frac{1}{2}\right) = \frac{\Gamma(\frac{1}{2})\,\Gamma(\frac{1}{p})}{\Gamma(\frac{1}{2}+\frac{1}{p})}$$

In this case, it is obvious that t_m is a decreasing function of K and of v_0.

Remark 2.2 *The case where φ is given by (1.2) is typical of the situation where :*

$$\varphi \in C^\infty(\mathbb{R} - \{\delta\})$$

In such case, u will have the regularity :

$$u \in C^\infty([0, 2t_m) \cup (2t_m, +\infty))$$

The degree of smoothness of u at $t = 2t_m$, depends then on the degree of regularity of φ at δ. For instance, if $\varphi(u)$ is given by (1.2) with $k \le p < k+1$, $k \in \mathbf{N}^*$, u is of class C^{k+1} at time $t = 2t_m$.

2.2 The case $R \neq 0$

Curiously, the existence and the uniqueness is not as straightforward as in the case $R = 0$. If we wished to apply Cauchy-Lipschitz theorem to (1.9) as in the proof of theorem, we would have to consider the function :

(2.13) $$F\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} v \\ K\varphi(u) + R\varphi'(u)v \end{bmatrix}$$

The trouble then comes from the fact that the function φ is not necessarily smooth (think to the case where φ is given by (1.2) with $1 \le p < 2$).

We shall say that $u(t)$ is a solution of (1.9) in $[0, T]$ if u has the regularity

(2.14) $$u \in W^{2,\infty}(0, T)$$

and satisfies the equality

(2.15) $$u'' = K\varphi(u) + R\varphi'(u)u'$$

almost everywhere in $[0, T]$ (or equivalently in $\mathcal{D}'(0, T)$).

Remark 2.3 *Note that, as soon as φ is C^1, the solution is of class C^2.*

To overcome the difficulty set up by F, we remark that if $u \in W^{2,\infty}(0,T)$ is a solution of (1.9) then (u,w) where $w(t) = \int_0^t \varphi(u(s))ds$ is a classical C^1-solution of the system :

$$\left\{ \begin{array}{l} u' = R\varphi(u) + Kw + v_0 \\ \\ w' = \varphi(u) \end{array} \right. \tag{2.16}$$

with the initial conditions

$$\left\{ \begin{array}{l} u(0) = \delta \\ \\ w(0) = 0 \end{array} \right. \tag{2.17}$$

Reciprocally, if $(u,w) \in C^1(0,T) \times C^1(0,T)$ is a solution of ((2.16),(2.17)) on $[0,T]$, then $u \in W^{2,\infty}(0,T)$ is a solution of (1.9), in the sense of distributions. Therefore, looking for a solution at equation (1.9) amounts to solving ((2.16),(2.17)).

Theorem 2.3 *Let's assume that φ satisfies (1.7. Problem (1.9) has a unique global solution :*

$$u \in W^{2,\infty}$$

which satisfies, in the sense of the distribution

$$\frac{d}{dt}\left\{\frac{1}{2}\left|u'\right|^2 + K\psi(u)\right\} = R\,\varphi'\,(u))\left|u'\right|^2 \tag{2.18}$$

Proof.

1. Since the function G

$$G\begin{pmatrix} u \\ \\ w \end{pmatrix} = \begin{pmatrix} R\varphi(u) + Kw + v_0 \\ \\ \varphi(u) \end{pmatrix} \tag{2.19}$$

   is locally Lipschitz-continuous in $\mathbb{R}^2$, we know that problem (2.16,2.17) has a unique maximal solution :

$$(u,w) \in C^1(0,T_{\max}) \times C^1(0,T_{\max}) \tag{2.20}$$

   with the alternative

$$\left| \begin{array}{ll} (1) & T_{\max} = +\infty : \text{ The solution is global} \\ \\ (2) & \lim\limits_{t \nearrow T_{\max}} \left\{\left|u'(t)\right|^2 + |u(t)|^2\right\} = +\infty \end{array} \right.$$

2. We know that $u \in W^{2,\infty}([0,T_{\max}])$ and satisfies almost everywhere

$$u'' = K\varphi(u) + R\varphi'(u)u' \tag{2.21}$$

   Multiplying this equation by u', we get :

$$\frac{d}{dt}\left\{\frac{1}{2}\left|u'\right|^2 + K\psi(u)\right\} = R\varphi'(u)\left|u'\right|^2$$

   In particular

$$\frac{1}{2}\left|u'\right|^2 + K\psi(u) = \frac{1}{2}v_0^2 + R\int_0^t \varphi'(u)\left|u'\right|^2 ds$$

   As φ is decreasing, $\varphi' \leq 0$ almost everywhere, which means that :

$$\frac{1}{2}\left|u'(t)\right|^2 + K\psi(u(t)) \leq \frac{1}{2}v_0^2 \quad \forall t < T_{\max} \tag{2.22}$$

   As ψ is positive, this inequality says that u', and thus u, remains bounded in finite time, which shows that the solution is global.

□

Our next theorem will say that only two situations are possible for the behavior of $u(t)$.

Theorem 2.4 *Let's assume here that φ satisfies (1.7) and (1.8). There exists a unique time $t_m(R)$ such that $u'(t_m(R)) = 0$. The function $u(t)$ is strictly decreasing and strictly convex from 0 to $t_m(R)$. For $t > t_m(R)$, $u(t)$ is strictly increasing and one has the following alternative*

i) $\lim_{t \to +\infty} u(t) = \delta$

ii) $\exists !\ t_c(R) > 2t_m(R)$ *such that* $u(t_c(R)) = \delta$ *and*

$$u(t) = \delta + u'(t_c(R))(t - t_c(R)) \quad \textit{for } t > t_c(R)$$

where $u'(t_c(R)) > 0$

Proof.

Step 1 : Let us introduce

$$\mathcal{E} = \{t \geq 0 \text{ such that } \forall s \,,\, 0 < s \leq t \ \ , \ \ u'(s) < 0\}$$

which is clearly an interval of $\mathbb{R}^+$ with lower bound 0. If $\mathcal{E}$ were not bounded, $u(t)$ would be decreasing on $\mathbb{R}^+$. Since $\psi(u(t)) \leq \frac{1}{2}v_0^2$, and $\psi(u) \to +\infty$ when $u \to -\infty$, we know that $u(t)$ remains bounded from below. Therefore $u(t)$ converges to some limit value $u_\infty < \delta$. From the equality

$$u'' = K\varphi(u) + R\varphi^{'}(u)u'$$

we deduce, since $u' < 0$ and $\varphi'(u) < 0$ that :

$$u'' \geq K\varphi(u)$$

This means that for t large enough, $u'' \geq K\varphi(u_\infty) > 0$. Then u' would tend to $+\infty$ when t tends to $+\infty$, which contradicts the fact that it remains negative. Therefore $\mathcal{E}$ is bounded and we can introduce :

$$t_m(R) = \sup \mathcal{E}$$

which clearly satisfies :

$$\begin{cases} \forall t < t_m(R) \ , \ u'(t) < 0 \\ u'(t_m(R)) = 0 \\ \ u \text{ is strictly convex on } (0, t_m(R)] \quad (\text{ since } u''(t) = K\varphi(u) + R\varphi^{'}(u)u' \text{ and } u' < 0 \ ) \end{cases}$$

Step 2 : Let us introduce

$$\mathcal{F} = \{t > t_m(R) \text{ such that } \forall s \,,\, t_m(R) < s \leq t \ \ , \ \ u(s) < \delta\}$$

which is an interval of $\mathbb{R}^+$ with lower bound $t_m(R)$. Let us first show that $u'(t) > 0$ for $t \in \mathcal{F}$ and $t \neq t_m(R)$. Let us introduce

$$\mathcal{D} = \{t \in \{\mathcal{F} \text{ such that } u'(t) \leq 0\}$$

Let us assume that $\mathcal{D} \neq \emptyset$ and let t^* be $\inf \mathcal{D}$. A $t = t_m(R)$, u verifies : $u''(t_m(R)) = K\varphi\,(u_m(R))$ where $u_m(R) = u(t_m(R))$. As $\varphi\,(u_m(R)) > 0$, u' is strictly positive in some interval $(t_m(R), t_m(R) + \tau)$. Therefore, we would have, using definition of t^* :

$$\begin{cases} \forall t_m(R) < t < t^* \ , \ u'(t) > 0 \\ u'(t^*) = 0 \ \text{ and } \ u(t^*) < \delta \end{cases}$$

From the equation satisfied by u, we get

$$u''(t^*) = K\varphi(u(t^*)) > 0 \quad (u(t^*) < \delta) \tag{2.23}$$

while, as $\frac{1}{t-t^*}[u'(t) - u'(t^*)] < 0$ for $t_m(R) < t < t^*$, we get, taking the limit when t tends to t^* :

$$u''(t^*) \leq 0 \tag{2.24}$$

We thus have a contradiction which proves that $\mathcal{D}$ is empty and thus that $u' > 0$ on $\mathcal{F}$.

step 3 : Let us consider two cases

(a) $\sup \mathcal{F} = +\infty$

In this case, $u(t)$ is increasing on $[t_m(R), +\infty)$. Therefore, $u(t)$ converges, when t tends to $+\infty$, to some limit value $u_\infty \leq \delta$. If $u_\infty < \delta$, integrating (1.9) between 0 and t, the inequality :

$$u'(t) = K\int_0^t \varphi(u(s))ds + R\varphi(u(t)) - v_0$$

proves that

$$u'(t) \simeq K\varphi(u_\infty)t \quad (t \to +\infty)$$

which would contradicts the fact that u remains bounded ($\varphi(u_\infty) > 0$). Therefore $u_\infty = \delta$.

(b) $t_c(R) = \sup \mathcal{F} < +\infty$

In this case, we obviously have

$$\begin{cases} \bullet \ u(t) \text{ is strictly increasing on } [t_m(R), t_c(R)] \\ \bullet \ u(t_c(R)) = \delta \end{cases}$$

Note that necessarily $u'(t_c(R)) > 0$. Indeed, as $u'(t) > 0$ in $(t_m(R), t_c(R))$, $u'(t_c(R)) \geq 0$. If $u'(t_c(R))$ were equal to 0, we would have

$$u(t_c(R)) = \delta \ , \ u'(t_c(R)) = 0$$

Then by uniqueness for the Cauchy problem, u would be identically equal to δ on $[0, t_c(R)]$, which is not the case. Finally, one easily checks that one extends the solution for $t > t_c(R)$ with :

$$u(t) = \delta + u'(t_c(R))(t - t_c(R))$$

This corresponds to the case (ii) of the theorem.

□

Remark 2.4 •

- *The fact that $t_c(R) > 2t_m(R)$ is a particular case of a more general result we shall prove later (see Theorem 2.5).*

- *Once again if one makes the assumption*

$$\varphi \in C^\infty(\mathbb{R} - \{\delta\})$$

then one has the regularity results :

$$\begin{cases} u \in C^\infty(\mathbb{R}^+) & \textit{in the case (i)} \\ u \in C^\infty(\mathbb{R}^+ - \{t_c(R)\}) & \textit{in the case (ii)} \end{cases}$$

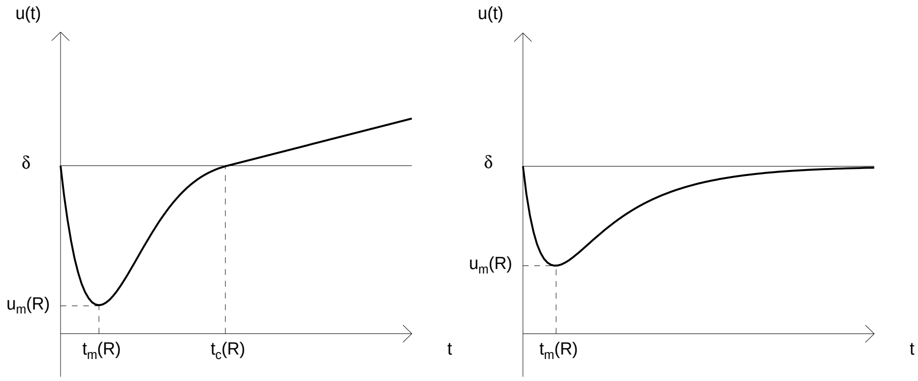

Figure 2.4: Possible graphs of $u(t)$. case (ii) (left) and case (i) (right)

We illustrate in figure 2.4 the two possible scenarii for the behavior of $u(t)$ In what follows, we shall set :

(2.25) $$t_c(R) = +\infty \quad \text{if case (i) occurs}$$

Setting as in figure 2.4, $u_m(R) = u(t_m(R)) = \min\limits_{t \geq 0} u(t)$, we clearly have the property :

(2.26)
$$\left| \begin{array}{l} \forall u \in [u_m(R), \delta] \ , \ \exists! \ (T_R^1(u), T_R^2(u)) \in [0, t_m(R)] \times [t_m(R), t_c(R)] \ \text{ such that} \\ u(T_R^1(u)) = u(T_R^2(u)) = u \end{array} \right.$$

We illustrate this property in figure 2.5.

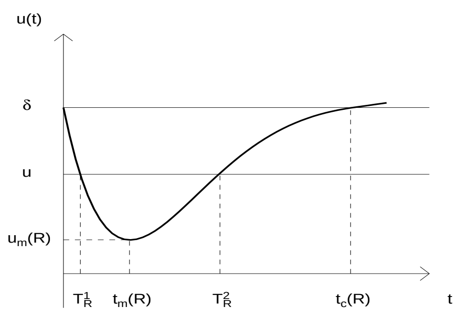

Figure 2.5: Construction of $T_R^1(u)$ and $T_R^2(u)$

We thus define two functions T_R^1 and T_R^2 which have the following properties (see also figure 2.6).

(2.27)

- $(T_R^1(u), T_R^2(u)) \in C^0((u_m(R), \delta])^2$
- $T_R^1(u)$ is bijective and decreasing from $[u_m(R), \delta]$ in $[0, t_m(R)]$
- T_R^2 is bijective and increasing from $[u_m(R), \delta]$ in $[t_m(R), t_c(R)]$
- $\lim\limits_{u \searrow u_m(R)} (T_R^1)'(u) = - \lim\limits_{u \searrow u_m(R)} (T_R^2)'(u) = -\infty$

We have the following properties

Theorem 2.5 *$T_R^1(u)$ and $T_R^2(u)$ satisfy*

(i) $\forall \overline{u} \in [u_m(R), \delta] \quad u'(T_R^1(\overline{u})) < 0 \ , \ u'(T_R^2(\overline{u})) > 0$

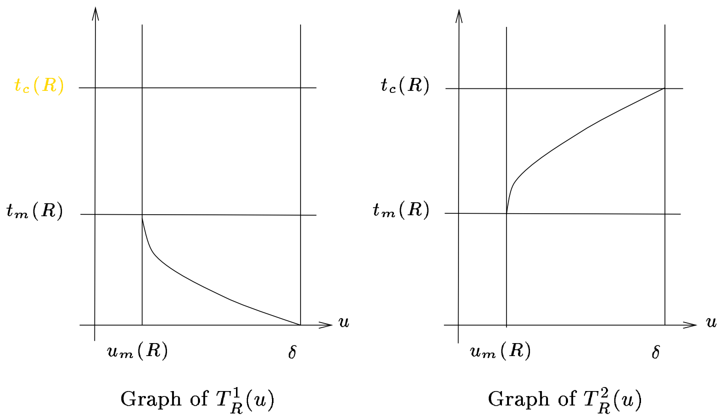

Figure 2.6: The Variations of $T_R^1(u)$ and $T_R^2(u)$

(ii) $\forall \overline{u} \in [u_m(R), \delta] \quad \left|u'(T_R^2(\overline{u}))\right| < \left|u'(T_R^1(\overline{u}))\right|$

(iii) $T_R^1(\overline{u}) - t_m(R) > t_m(R) - T_R^1(\overline{u})$. *In particular, for* $\overline{u} = \delta$, $t_c(R) > 2t_m(R)$.

Proof.

- (i) is obvious
- For (ii), we use the energy identity

$$\frac{d}{dt}\left\{\frac{1}{2}\left|u'\right|^2 + K\psi(u)\right\} = R\varphi^{'}(u)\left|u'\right|^2$$

that we integrate between $T_R^1(\overline{u})$ and $T_R^2(\overline{u})$ to obtain

$$\frac{1}{2}\left\{\left|u'(T_R^2(\overline{u}))\right|^2 - \left|u'(T_R^1(\overline{u}))\right|^2\right\} = R\int_{T_R^1(\overline{u})}^{T_R^2(\overline{u})} \varphi^{'}(u)\left|u'\right|^2 du < 0$$

- For (iii), we simply write that

$$T_R^2(\overline{u}) - t_m(R) = \int_{t_m(R)}^{T_R^2(\overline{u})} (T_R^2(s))'(s)ds$$

Using the change of variable $u = T_R^2(s)$ and the fact that $T_R^2(s)$ is the inverse of u in $[t_m(R), t_c(R)]$, we get

(2.28) $$T_R^2(\overline{u}) - t_m(R) = \int_{u_m(R)}^{\overline{u}} \frac{du}{u'(T_R^2(u))}$$

With the same method, we also obtain

(2.29) $$T_R^1(\overline{u}) - t_m(R) = \int_{u_m(R)}^{\overline{u}} \frac{du}{u'(T_R^1(u))}$$

Therefore (iii) is a direct consequence of (i) and (ii).

□

Remark 2.5 *To establish (2.28) and (2.29), we have implicitly used the formula :*

(2.30) $$(T_R^j(\overline{u}))' = \frac{1}{u'(T_R^j(\overline{u}))} \quad j = 1, 2$$

Of course it would be interesting to determine for what values of R, one is in case (i) and for what values of R, one is in case (ii).

Remark 2.6 *The fact that u is a C^1 function of R and t is a consequence of general theorems about solutions of differential equations depending on a parameter (see [5] and [4]). Note that here, equation (1.9) depends smoothly on R.*

Section 2.1 says that for $R = 0$, one is in case (ii). By a continuity argument, it is clear that for R small enough, one expects to be in case (ii). This can be stated more precisely thanks to our next theorem. In order to state our result, let us consider the function

$$G(R) = 2R\varphi(u_m(R)) \tag{2.31}$$

and introduce the set

$$\mathcal{R} = \{R \geq 0 \text{ such that } G(R) < v_0\} \tag{2.32}$$

Since $v_0 > 0$ and $G(0) = 0$, it is clear that $\mathcal{R}$ is not empty and contains some interval $[0, R_*)$, $R_* > 0$.

Theorem 2.6 *When $R \in \mathcal{R}$, one is in case (ii)*

Proof. We are going to prove that assuming that $t_c(R) = +\infty$, then necessarily $R \geq R^*$. First note that

$$u'(t) + v_0 = K \int_0^t \varphi(u(s))ds + R\varphi(u(t)) \tag{2.33}$$

(we simply integrate (1.9) between 0 and t). Taking the limit of (2.33) when t tends to $+\infty$, we get (do not forget that we assume to be in case (i))

$$v_0 = K \int_0^{+\infty} \varphi(u(t))dt \tag{2.34}$$

On the other hand, by energy identity

$$\frac{1}{2}\left|u'(t)\right|^2 + K\psi(u(t)) = R\int_0^t \varphi'(u)\left|u'\right|^2 ds + \frac{1}{2}v_0^2 \tag{2.35}$$

Taking the limit of (2.35), when t tends to $+\infty$, we get

$$\frac{1}{2}v_0^2 = -R\int_0^{+\infty} \varphi'(u)\left|u'\right|^2 dt \tag{2.36}$$

Thanks to an integration by parts, and to the properties of φ ($\varphi(\delta) = 0$), this equality also writes

$$\frac{1}{2}v_0^2 = R\int_0^{+\infty} \varphi(u)u'' dt$$

That is to say, using equation (1.9)

$$\frac{1}{2}v_0^2 = RK\int_0^{+\infty} \varphi(u)^2 dt + R^2\int_0^{+\infty} \varphi'(u)\varphi(u)u' dt$$

which leads to

$$\frac{1}{2}v_0^2 = RK\int_0^{+\infty} \varphi(u)^2 dt \tag{2.37}$$

We know that $u(t) \geq u_m(R)$ and that $\varphi(u)$ is positive and decreasing. This implies that

$$v_0^2 \leq G(R)\,K\int_0^{+\infty} \varphi(u)dt \tag{2.38}$$

That is to say, thanks to (2.34)

$$G(R) \geq v_0 \tag{2.39}$$

This concludes the proof of the theorem. □

Remark 2.7 • *Since $\mathcal{R}$ contains an interval of the form $[0, R^*)$, this results shows that one is in the case (ii) if R is small enough.*

- *The set $\mathcal{R}$ depends of course on v_0 and K. We shall give more details of $\mathcal{R}$ in section 2.3.*

It is natural to introduce the two sets

$$\left| \begin{array}{l} \mathcal{R}_1 = \{R \geq 0 \text{ such that } t_c(R) = +\infty\} \\ \\ \mathcal{R}_2 = \{R \geq 0 \text{ such that } t_c(R) < +\infty\} \end{array} \right. \tag{2.40}$$

Theorem 2.6 expresses that $\mathcal{R}_2 \supset \mathcal{R} \supset [0, R^*)$. More generally, we know that

$$\left\{ \begin{array}{l} \mathcal{R}_1 \text{ is a closed subset of } R^+ \\ \\ \mathcal{R}_2 \text{ is an open subset of } R^+ \end{array} \right. \tag{2.41}$$

To show this, it suffices to consider $u(t)$ as a function of two variables denoted by $u(R, T)$ and to remark that saying that R belongs to this set $\mathcal{R}_2$ amounts to say that the equation $u(R, t) = 0$ has a solution $t = t_c(R) > 0$. Now, if $\overline{R} \in \mathcal{R}_2$, we know by theorem 2.4 that

$$\frac{\partial u}{\partial t}(\overline{R}, t_c(\overline{R})) \equiv u'(t_c(\overline{R}))$$

is strictly positive. Therefore, thanks to the implicit function theorem, we know that equation $u(R, t) = \delta$ has a unique solution in a neighborhood of $t_c(\overline{R})$, provided that $|R - \overline{R}|$ is small enough. This proves that $\mathcal{R}_2$ is open.
One can wonder if $\mathcal{R}_2 = \mathbb{R}^+$. The following result proves that it is not the case, at least in general.

Theorem 2.7 *When φ is given by (1.2) with $p = 1$,*

$$\left\{ \begin{array}{l} \mathcal{R}_1 = [2\sqrt{K}, +\infty] \\ \\ \mathcal{R}_2 = [0, 2\sqrt{K}) \end{array} \right. \tag{2.42}$$

Proof. It suffices to remark that when $p = 1$, (1.9) can be solved explicitly. We omit the details. The solution is given by

(i) If $R > 2\sqrt{K}$, the solution is given by :

$$u(t) = \delta - 2v_0(R^2 - 4K)^{-\frac{1}{2}} \sinh\left((R^2 - 4K)^{\frac{1}{2}} \frac{t}{2}\right) e^{-R\frac{t}{2}}$$

(ii) If $R = 2\sqrt{K}$, the solution is given by :

$$u(t) = \delta - v_0 t e^{-R\frac{t}{2}}$$

(iii) If $R < 2\sqrt{K}$, the solution is given by :

$$\left\{ \begin{array}{ll} u(t) = \delta - 2v_0(4K - R^2)^{-\frac{1}{2}} \sin\left((4K - R^2)^{\frac{1}{2}} \frac{t}{2}\right) e^{-R\frac{t}{2}} & \text{If } t \leq t_c(R) \\ \\ u(t) = \delta + v_0 e^{-R\frac{t_c(R)}{2}}(t - t_c(R)) & \text{If } t \geq t_c(R) \end{array} \right.$$

where $t_c(R)$ is given by

$$t_c(R) = \frac{2\pi}{(4K - R^2)^{\frac{1}{2}}}$$

□

Remark 2.8 *For $p = 1$,*

- *surprisingly, the sets $\mathcal{R}_1$ and $\mathcal{R}_2$ only depends on K, not on v_0.*
- *one can compute in this case the quantities $t_m(R)$, $u_m(R)$, $t_c(R)$. Let us define*

$$A = \left|R^2 - 4K\right|^{\frac{1}{2}}$$

The results are the following

(i) If $R > 2\sqrt{K}$

$$\left| \begin{array}{ll} t_m(R) & = \log\left(\dfrac{A-R}{A+R}\right) \\ u_m(R) & = \delta - 2\dfrac{v_0}{A}\left(\left(\dfrac{R+A}{R-A}\right)^{\left(\frac{A-R}{2A}\right)} - \left(\dfrac{R+A}{R-A}\right)^{-\left(\frac{A+R}{2A}\right)}\right) \\ t_c(R) & = +\infty \end{array} \right.$$

(ii) If $R = 2\sqrt{K}$

$$\left| \begin{array}{ll} t_m(R) & = \dfrac{2}{R} = \dfrac{1}{\sqrt{K}} \\ u_m(R) & = \delta - 2\dfrac{v_0}{Re} \\ t_c(R) & = +\infty \end{array} \right.$$

(ii) If $R < 2\sqrt{K}$

$$\left| \begin{array}{ll} t_m(R) & = \dfrac{2}{A}\arctan\left(\dfrac{A}{R}\right) \\ u_m(R) & = \delta - 2\dfrac{v_0}{A}\sin\left(\arctan\left(\dfrac{A}{R}\right)\right) \\ t_c(R) & = \dfrac{2\pi}{A} \end{array} \right.$$

This result suggests us the following conjecture that we should be able to prove at least if φ 'looks like' to the linear case

Conjecture 2.1 *There exists a strictly critical value of R, namely R_c, depending on v_0 and K such that :*

(2.43)
$$\left\{ \begin{array}{l} \mathcal{R}_1 = [R_c, \infty] \\ \mathcal{R}_2 = [0, R_c) \end{array} \right.$$

2.3 On the influence of R

In this section, we would like first to analyze the influence of R on the minimal value $u_m(R)$ of $u(t)$.

Theorem 2.8 *The function $u_m(R)$ is an increasing function of R*

Proof. We consider u as a function of two variables R and t and, in the sequel, $u(R,t)$ or $u(t)$ will denote the same quantity. We introduce the R-derivative of u :

(2.44)
$$w(R,t) = \frac{\partial u}{\partial R}(R,t)$$

$w(R,t)$ will also be often written $w(t)$ for simplicity.

Step 1 : Characterization of $w(t)$

It is easy to see that $w(t)$ is the unique solution of the differential equation with variable coefficients :

$$(2.45) \qquad \begin{cases} w'' = K\varphi'(u)w + R\dfrac{d}{dt}\left[\varphi'(u)w\right] + \varphi'(u)u' \\ w(0) = w'(0) = 0 \end{cases}$$

Step 2 : Two identities

We introduce the adjoint state $q(t)$ defined as the unique solution of the linear equation (see ([8]):

$$(2.46) \qquad q'' = \varphi'(u)\left[Kq - Rq'\right]$$

which satisfied the final conditions at time $t = t_m(R)$

$$(2.47) \qquad \begin{cases} q(t_m(R)) = 0 \\ q'(t_m(R)) = 1 \end{cases}$$

Multiplying (2.46) by u', we get :

$$(2.48) \qquad q''u' = K\varphi'(u)qu' - R\varphi'(u)q'u'$$

On the other hand, multiplying (1.9) by q', we get :

$$(2.49) \qquad u''q' = K\varphi(u)q' + R\varphi'(u)q'u'$$

By addition of (2.48) and (2.49), we finally obtain :

$$(2.50) \qquad \frac{d}{dt}\left[u'q' - Kq\varphi(u)\right] = 0$$

which is our first identity. Multiplying (2.45) by q and (2.46) by w, we obtain, after subtraction of the two resulting equalities :

$$qw'' - wq'' = R\left[\frac{d}{dt}[\varphi'(u)w] + \varphi'(u)wq'\right] + \varphi'(u)u'q'$$

Integrating this equation between 0 and t, leads to our second identity :

$$(2.51) \qquad w'(t)q(t) - q'(t)w(t) = R\varphi'(u(t))q(t)w(t) + \int_0^t \varphi'(u)u'p\,ds$$

Step 3 : The quantity $w(t_m(R))$ is strictly positive.

Writing (2.51) at $t = t_m(R)$ gives, since $q(t_m(R))$=0

$$(2.52) \qquad w(t_m(R)) = -\int_0^{t_m(R)} \varphi'(u)u'q\,ds$$

On the other, it is easy to prove that the function $q(t)$ is strictly increasing on $(0, t_m(R))$. If it were not the case, there would exist some times t satisfying $q'(t) = 0$, $0 < t < t_m(R)$. Let t^* be the greatest of such times. We thus have, since $q'(t_m(R)) = 1 > 0$

$$\begin{cases} q'(t^*) = 0 \text{ and } t^* < t < t_m(R) \Rightarrow q'(t) > 0 \\ q(t^*) < 0 \end{cases}$$

By (2.50), we know that the quantity $(u'q' - Kq\varphi(u))$ is constant. Looking at $t = t_m(R)$, we deduce

$$u'q' = Kq\varphi(u) \quad \forall t \tag{2.53}$$

In particular at $t = t^*$, we get :

$$\varphi(u(t^*))q(t^*) = 0$$

Which is impossible since $q(t^*) < 0$ and :

$$t^* \in (0, t_m(R)) \Rightarrow u(t^*) < \delta \Rightarrow \varphi(u(t^*)) > 0$$

Therefore q is strictly increasing on $(0, t_m(R))$., which means in particular, since $q(t_m(R)) = 0$

$$q(t) < 0 \ \ , \ \ \forall t < t_m(R) \tag{2.54}$$

On the other hand, we already know (cf theorem 2.4) that

$$0 < t < t_m(R) \ \ \Rightarrow \ \ u'(t) < 0 \text{ and } \varphi'(u(t)) < 0 \tag{2.55}$$

Joining (2.52),(2.54),(2.55) proves that $w(t_m(R)) > 0$.

<u>Step 4</u> : Conclusion of the proof : $\frac{du_m}{dR} = w(t_m(R))$

By definition of $t_m(R)$ and $u_m(R)$, we have

$$\begin{cases} u(R, t_m(R)) = u_m(R) \\ \dfrac{\partial u}{\partial t}(R, t_m(R)) = 0 \end{cases}$$

Differentiating the first equation, with respect to R and using the second one, leads to

$$\frac{du_m}{dR}(R) = \frac{\partial u}{\partial R}(R, t_m(R)) = w(t_m(R))$$

□

This result permits us to make precise a result concerning the set $\mathcal{R}$ introduced in section (2.2).

Corollary 2.1 *$\mathcal{R}$ contains the interval $[0, R^*)$ where R^* is defined by $R^* = \dfrac{2v_0}{\varphi(u_m)}$ with $\psi(u_m) = \dfrac{v_0^2}{2K}$.*

Proof. From theorem 2.8, we know that

$$u_m(R) \le u_m(0) = u_m$$

Therefore $\varphi(u_m(R)) \le \varphi(u_m)$ and thus $G(R) \le 2R\varphi(u_m)$. Consequently

$$R \le R^* \ \ \Rightarrow \ \ G(R) < v_0 \ \ \Rightarrow \ \ R \in \mathcal{R}$$

□

Remark 2.9 *When φ is given by (1.2), we know that*

$$\frac{(\delta - u_m(R))^{p+1}}{p+1} = \frac{v_0^2}{2K}$$

so that

$$\varphi(u_m(R)) = \left(\frac{p+1}{2}\frac{v_0^2}{K}\right)^{\frac{p}{p+1}}$$

We thus get :

$$R^* = 2v_0\left(\frac{2}{p+1}\frac{K}{v_0^2}\right)^{\frac{p}{p+1}}$$

Of course, it would also be interesting to have informations about the variations of $t_m(R)$ with respect to R. On the basis of numerical computations and of the results obtained in the case $p = 1$ (see remark (2.11)), we have the following conjecture :

Conjecture 2.2 *The function $t_m(R)$ is a decreasing function of R*

Remark 2.10 *Let us indicate below the difficulties one encounters when one tries to prove the conjecture using the method used for proving theorem 2.8. First note that we have :*

$$\frac{\partial^2 u}{\partial t^2}(R, t_m(R))\frac{dt_m}{dR}(R) + \frac{\partial^2 u}{\partial R \partial t}(R, t_m(R)) = 0$$

On the other hand, using (1.9) :

$$\frac{\partial^2 u}{\partial t^2}(R, t_m(R)) = K\varphi(u_m(R)) > 0$$

while, by definition of w :

$$\frac{\partial^2 u}{\partial R \partial t}(R, t_m(R)) = w'(t_m(R))$$

so that

(2.56) $$\frac{dt_m}{dR}(R) = -\left[K\varphi(u_m(R))\right]^{-1} w'(t_m(R))$$

which shows that we only have to study the sign of $w'(t_m(R))$. Let us introduce the adjoint state satisfying equation (2.46) together with the final conditions :

$$\begin{cases} q(t_m(R)) = 1 \\ q'(t_m(R)) = -R\varphi(u_m(R)) \end{cases}$$

in which case one verifies that

(2.57) $$w'(t_m(R)) = \int_0^{t_m(R)} \varphi'(u)u'q\,dt$$

On would conclude if one could prove that q remains positive in $[0, t_m(R)]$. This property seems to be true but difficult to show. Note for instance that the identity

(2.58) $$u'q' - Kq\varphi(u) = -K\varphi(u_m(R))$$

does not provide any insight on the sign of $q(0)$. In fact from (2.58), one deduce that if $q'(t) = 0$ then

$$q(t) = \frac{\varphi(u)}{\varphi(u_m(R))} > 1$$

and $q''(t) = \phi'(u)[Kq] < 0$. This proves that the only possible local extremum of $q(t)$ is a maximum. From this property, we deduce that $q(t)$ remains positive between 0 and $t_m(R)$ if $q(0) \geq 0$. The problem is thus reduced to the determination of the sign of $q(0)$, which remains for us an open question ...

Now, we assume that R belongs to $\mathcal{R}_2$ and consider the variations of :

$$u'(t_c(R)) = v(t_c(R))$$

as a function of R, i.e. the variations of the velocity of the mallet when it quits the table.

Theorem 2.9 *The function $u'(t_c(R))$ is a positive and decreasing function of R in each connected component of $\mathcal{R}_2$.*

Proof. It is analogous to the proof of theorem 2.8

Step 1 : Caracterisation of $\frac{d}{dR}v(t_c(R))$

We have

$$\frac{d}{dR}[v(t_c(R))] = w'(t_c(R)) \quad \text{where } w = \frac{du}{dR}$$

Indeed, using once again the same notation for $u(t)$ and $u(R,t)$, we have :

$$v(t_c(R)) = \frac{du}{dR}(R, t_c(R))$$

Therefore, after differentiation with respects to R, we get :

$$\frac{d}{dR}[v(t_c(R))] = w'(t_c(R)) + u''(t_c(R)).\frac{dt_c}{dR}(R)$$

But thanks to (1.9) :

$$u''(t_c(R)) = K\varphi(u(t_c(R))) - R\varphi'(u(t_c(R)))u'(t_c(R)) = 0$$

since $u(t_c(R)) = \delta$ and $\varphi(\delta) = \varphi'(\delta) = 0$.

Step 2 : A formula for $w'(t_c(R))$

We introduce the adjoint state $q(t)$ defined by :

(2.59) $$q'' = \varphi'(u)\,[Kq - Rq']$$

and the final conditions at time $t = t_c(R)$

(2.60) $$\begin{cases} q(t_c(R)) = 1 \\ q'(t_c(R)) = 0 \end{cases}$$

Therefore, with the same manipulations as in the proof of theorem 2.8, we have :

(2.61) $$w'(t_c(R)) = \int_0^{t_c(R)} \varphi'(u)u'q dt$$

The difference with theorem 2.8 is that $\varphi'(u)u'$ has no constant sign on $[0, t_c(R)]$. However, as $\varphi'(u)$ has a constant sign, it suffices to look at the sign of the product $u'q$.

Step 3 : The function $u'q$ remains positive in $[0, t_c(R)]$.

We know (cf proof of theorem 2.8) that the function $u'q' - Kq\varphi(u)$ is constant. Looking at $t = t_c(R)$, we get :

(2.62) $$u'q' - Kq\varphi(u) = 0$$

For proving that $u'q$ remains positive, we are going to prove that q is positive for $t \geq t_m(R)$ and negative for $t \leq t_m(R)$.

a) $q(t_m(R)) = 0$.

   Indeed, it suffices to write (2.62) at $t = t_m(R)$.

b) q is strictly increasing in $(t_m(R), t_c(R))$.

   If not q' would vanish for some $t^* \in (t_m(R), t_c(R))$. Then by (2.62), we would have :

$$q(t^*) = q'(t^*) = 0$$

   then q would vanish identically which is not the case.

c) q is strictly decreasing in $(0, t_m(R))$.

As for the case b), we know that q' cannot vanish for $0 < t < t_m(R)$. Therefore, $q(t)$ is monotonous in $[0, t_m(R)]$. On the other hand, writing (2.62) at $t = 0$ gives :

(2.63) $$q'(0) = 0 \quad (\text{ and } q'(t_m(R)) = 0)$$

If q were decreasing, q would be positive in $[0, t_m(R)]$ then $q'' = \varphi'(u)\,[Kq - Rq']$ would be negative in $[0, t_m(R)]$ which means that q' would decrease (strictly since it cannot remain constant) between 0 and $t_m(R)$ and since, (thanks to equation (2.63)) $q'(0) = 0$, q' would be strictly positive on $[0, t_m(R)]$, which contradicts the fact that q is decreasing.

a), b) and c) prove that qu' remains positive, which concludes the proof of the theorem.

□

It would also be interesting to have information on the variations of $t_c(R)$ as a function of R when R belongs to $\mathcal{R}_2$. In fact, we have the following conjecture (verified in the case $p = 1$ (see remark 2.11)):

Conjecture 2.3 *In each* $R \in \mathcal{R}_2$, $\dfrac{dt_c}{dR}(R) > 0$

If we were able to prove this result, we would then deduce that one of the following propositions is true :

(i) $\mathcal{R}_1 = \emptyset$

(ii) $\exists R_c \;/\; \mathcal{R}_1 = [R_c, +\infty] \;,\; \mathcal{R}_2 = [0, R_c]$

which is almost conjecture 2.1. It is easy to check that proving that $\dfrac{dt_c}{dR}(R) > 0$ is equivalent to proving that :

(2.64) $$w(t_c(R)) < 0$$

Indeed, $t_c(R)$ is characterized by :

$$u(R, t_c(R)) = 0$$

Differentiating this equality with respect to R leads to :

$$u'(t_c(R))\frac{dt_c}{dR}(R) + w(t_c(R)) = 0$$

As we know that $u'(t_c(R)) > 0$, we clearly have :

$$\frac{dt_c}{dR}(R) > 0 \iff w(t_c(R)) < 0$$

Unfortunately, we did not succeed in proving (2.64). The difficulty we encounter is analogous to the one we meet for proving that $w'(t_m(R)) < 0$ (cf remark 2.10). We indicate below the difficulties one encounters when one tries to prove the conjecture 2.3. Let us introduce the adjoint state satisfying equation (2.46) together with the final conditions :

$$\begin{cases} q(t_c(R)) = 0 \\ q'(t_c(R)) = 1 \end{cases}$$

in which case one verifies that

(2.65) $$w(t_c(R)) = -\int_0^{t_c(R)} \varphi'(u)u'q\,dt$$

On would easily conclude if one could prove that $u'q$ remains negative in $[0, t_c(R)]$. This property is not true, since the identity

(2.66) $$u'q' - Kq\varphi(u) = u'(t_c(R)) \quad \forall t$$

shows that $q(t_m(R))$ is strictly negative.

In fact from (2.66), one deduce that if $q'(t) = 0$ then $q(t) = -\dfrac{\varphi(u)}{u'(t_c(R))} < 0$ and $q''(t) = \phi'(u)[Kq] > 0$. This proves that the only possible local extremum of $q(t)$ is a minimum. One can remark that

$$\int_0^{t_c(R)} \varphi'(u)u'q dt = \int_\delta^{u_m(R)} \varphi'(u)[q(T_R^1(u)) - q(T_R^2(u))]du$$

which shows that we only have to study the sign of $q(T_R^1(u)) - q(T_R^2(u))$. The problem is thus reduced to prove that $q(T_R^1(u)) - q(T_R^2(u))$ remains negative, which remains for us an open question ...

Remark 2.11 *In the case $p = 1$, when φ is given by (1.2), conjectures 2.2 and 2.3 are fortunately verified. Indeed, differentiating the explicit values of $t_m(R)$ given in remark 2.8 leads to :*

$$\left| \begin{array}{ll} R > 2\sqrt{K} : & \dfrac{dt_m}{dR}(R) = -\dfrac{2}{A} < 0 \\ R = 2\sqrt{K} : & \dfrac{dt_m}{dR}(R) = -\dfrac{1}{R^2} < 0 \\ R < 2\sqrt{K} : & \dfrac{dt_m}{dR}(R) = \dfrac{2}{A^3}\left(R\arctan\left(\dfrac{A}{R}\right) - A\right) \leq \dfrac{2}{A^3}\left(R\dfrac{A}{R} - A\right) = 0 \end{array} \right.$$

And differentiating $t_c(R)$ with respect to R in the case $R < \sqrt{K}$ leads to

$$R < 2\sqrt{K} : \quad \frac{dt_c}{dR}(R) = \frac{1}{2}\frac{R\pi}{A^3} > 0$$

3 The coupled mallet-membrane model

In this section, we are interested in finite energy solutions of the system

$$\text{(3.1)} \qquad \left\{ \begin{array}{ll} \dfrac{\partial^2 w}{\partial t^2}(x,t) - \dfrac{\partial^2 w}{\partial x^2}(x,t) = -\eta F(t)\delta(x) & x \in \mathbb{R}\,,\ t > 0 \\ u''(t) = F(t) & t > 0 \\ F(t) = K\varphi(u(t) - w(0,t)) + R\dfrac{d}{dt}\varphi(u(t) - w(0,t)) & \\ u(0) = \delta & \\ u'(0) = -v_0 & \\ w(x,0) = \dfrac{\partial w}{\partial t}(x,0) = 0 & \end{array} \right.$$

The energy of the system is being defined as :

$$\text{(3.2)} \qquad E(u,w;t) = \frac{1}{2}\left|u'(t)\right|^2 + K\psi(u(t) - w(0,t)) + \frac{1}{2\eta}\int\left(\left|\frac{\partial w}{\partial t}(t)\right|^2 + \left|\frac{\partial w}{\partial x}(t)\right|^2\right)dx$$

3.1 A reduction of the problem

Let us introduce the function :

$$\text{(3.3)} \qquad \left| \begin{array}{ll} W(t) = w(0,t) & \text{if } t \geq 0 \\ W(t) = 0 & \text{if } t < 0 \end{array} \right.$$

Then the solution of

$$\text{(3.4)} \qquad \frac{\partial^2 w}{\partial t^2}(x,t) - \frac{\partial^2 w}{\partial x^2}(x,t) = -\eta F(t)\delta(x) \quad x \in \mathbb{R}, t > 0$$

necessarily satisfies (see [9] p 47), since $w(x,0) = \frac{\partial w}{\partial t}(x,0) = 0$

$$(3.5) \qquad \left| \begin{array}{ll} w(x,t) = W(t-x) & \text{if } x > 0 \\ w(x,t) = W(t+x) & \text{if } x < 0 \end{array} \right.$$

(3.4) also implies that $\frac{\partial w}{\partial x}(0^+,t) = \frac{\partial w}{\partial x}(0^-,t) = \eta F(t)$, which we can combine with (3.5) to deduce :

$$(3.6) \qquad W'(t) = -\eta \frac{F(t)}{2}$$

Taking into account that $u'' = F$, we easily get :

$$(3.7) \qquad u' + \frac{2}{\eta} W = -v_0$$

which shows that (u, W) is solution of the differential system :

$$(3.8) \qquad \left\{ \begin{array}{l} u' + \frac{2}{\eta} W = -v_0 \\ u'' = K\varphi(u-W) + R\frac{d}{dt}\varphi(u-W) \end{array} \right.$$

which allows us the following theorem

Theorem 3.1 *(i) If (u,w) is a finite energy solution of (3.1), then setting $W(t) = w(0,t)$, (u,W) is a solution of (3.8).*

(ii) Reciprocally, if (u,W) is a solution of (3.8), then setting $w(x,t) = w(t - signe(x)x)$, (u,w) is a finite energy solution of (3.1).

We have thus reduced the analysis of (3.1) to a simple ordinary differential equation since eliminating W in (3.8), we simply get :

$$(3.9) \qquad u'' = K\varphi\left(u + \frac{\eta}{2}(v_0 + u')\right) + R\frac{d}{dt}\varphi\left(u + \frac{\eta}{2}(v_0 + u')\right)$$

We are going to see in the next two sections, how we can go back to section 2 to study this equation.

3.2 The case R=0

In fact, it is useful to reintroduce the function :

$$(3.10) \qquad \tilde{u} = u - W$$

which says to us in particular that :

$$\left\{ \begin{array}{ll} \bullet \ \text{if } \tilde{u} \le \delta & \text{, there is a contact} \\ \bullet \ \text{if } \tilde{u} > \delta & \text{, there is no contact} \end{array} \right.$$

The first equation of (3.8) then writes :

$$(3.11) \qquad u' + \frac{2}{\eta}(u - \tilde{u}) = -v_0$$

And the second one, since $R = 0$:

$$(3.12) \qquad u'' = K\varphi(\tilde{u})$$

From (3.11), we get :

$$(3.13) \qquad \frac{\eta}{2}\frac{d^3 u}{dt^3} + u'' = \tilde{u}''$$

That is to say, using (3.12)

$$(3.14)\qquad \begin{cases} \tilde{u}'' = K\varphi(\tilde{u}) + \dfrac{K\eta}{2}\dfrac{d}{dt}\varphi(\tilde{u}) \\ \tilde{u}(0) = \delta \quad \tilde{u}'(0) = -v_0 \end{cases}$$

We see that $\tilde{u}$ satisfies the simple mallet equation (1.9) with an artificial friction term corresponding to $\overline{R} = K\eta/2$, term which is only due to the interaction with the membrane ! We can thus transform the results of section 2 about equation (1.9) into a theorem about system (3.8).

Theorem 3.2 *System (3.1) admits a unique solution :*

$$(u, W) \in C^2(\mathbb{R}^+) \times C^2(\mathbb{R}^+; H^1(\mathbb{R}))$$

Denoting $\tilde{u}$ the unique global solution of (3.14), the solution is given by

$$(3.15)\qquad \left| \begin{array}{l} \bullet\ u(t) = \delta e^{-\frac{2}{\eta}t} - v_0\left(1 - \eta\dfrac{e^{-\frac{2}{\eta}t}}{2}\right) + \dfrac{2}{\eta}\displaystyle\int_0^t e^{\frac{2}{\eta}(s-t)}\tilde{u}(s)ds \\ \bullet\ w(x,t) = W(t - |x|) \ ,\ \ 0 < x < t \\ \qquad \textit{where } W \textit{ is the causal function defined by } W(t) = u(t) - \tilde{u}(t) \quad \textit{for } t > 0 \end{array} \right.$$

This solution satisfies the energy identities :

$$(3.16)\qquad \frac{d}{dt}E(u, w; t) = 0$$

$$(3.17)\qquad \frac{d}{dt}\left\{\frac{1}{2}\left|u'(t)\right|^2 + \psi(u(t) - w(0,t))\right\} = -\frac{1}{2}\left|u''(t)\right|^2$$

$$(3.18)\qquad \frac{d}{dt}\left\{\frac{1}{2\eta}\int\left|\frac{\partial w}{\partial t}(t)\right|^2 + \frac{1}{2\eta}\int\left|\frac{\partial w}{\partial x}(t)\right|^2\right\} = \frac{1}{2}\left|u''(t)\right|^2$$

Proof. This existence theorem is a direct consequence of the previous computations and theorem 3.1. Formula (3.15) is a direct consequence of (3.11), (3.10), (3.3). Let us now give a direct proof of identity (3.16). Multiplying the wave equation by $\dfrac{\partial w}{\partial t}$, leads, after an integration in x, to :

$$(3.19)\qquad \frac{d}{dt}\left[\frac{1}{2\eta}\int\left|\frac{\partial w}{\partial t}(t)\right|^2 + \frac{1}{2\eta}\int\left|\frac{\partial w}{\partial x}(t)\right|^2 dx\right] = -F(t)\frac{\partial w}{\partial t}(0,t)$$

On the other hand, since :

$$u''(t) = K\varphi(u(t) - w(0,t))$$

We have, after multiplying by $u'(t) - \dfrac{\partial w}{\partial t}(0,t)$

$$(3.20)\qquad \frac{d}{dt}\left[\frac{1}{2}\left|u'(t)\right|^2 + K\psi(u(t) - w(0,t))\right] - \frac{\partial w}{\partial t}(0,t)u''(t) = 0$$

Then adding (3.19) and (3.20), we get, since $u''(t) = F(t)$:

$$(3.21)\qquad \frac{d}{dt}\left[\frac{1}{2\eta}\int\left|\frac{\partial w}{\partial t}(t)\right|^2 + \frac{1}{2\eta}\int\left|\frac{\partial w}{\partial x}(t)\right|^2 dx + \frac{1}{2}\left|u'\right|^2 + K\psi(u(t) - w(0,t))\right] = 0$$

which is nothing but (3.16). Note that since we know (cf (3.6)) that $F(t) = -2\dfrac{1}{\eta}\dfrac{\partial w}{\partial t}(0,t) = u''(t)$, (3.19) and (3.20) can be rewritten :

$$\left| \begin{array}{l} \dfrac{d}{dt}\left[\dfrac{1}{2}\left|u'\right|^2 + K\psi(u(t) - w(0,t))\right] = -2\left|\dfrac{\partial w}{\partial t}(0,t)\right|^2 \\ \dfrac{d}{dt}\left[\dfrac{1}{2}\displaystyle\int\left|\dfrac{\partial w}{\partial t}(t)\right|^2 + \dfrac{1}{2}\displaystyle\int\left|\dfrac{\partial w}{\partial x}(t)\right|^2 dx\right] = 2\dfrac{1}{\eta}\left|\dfrac{\partial w}{\partial t}(0,t)\right|^2 \end{array} \right.$$

which is nothing but (3.17) and (3.18). □

Remark 3.1 *If we define :*

$$\left|\begin{array}{ll} \bullet \ \dfrac{1}{2\eta}\displaystyle\int\left|\frac{\partial w}{\partial t}(t)\right|^2 + \frac{1}{2\eta}\int\left|\frac{\partial w}{\partial x}(t)\right|^2 dx & \textit{as the energy of the membrane} \\ \bullet \ \dfrac{1}{2}\left|u'(t)\right|^2 + K\psi(u(t) - w(0,t)) & \textit{as the energy of the mallet} \end{array}\right.$$

Therefore identities (3.17) and (3.18) express that, while the total energy remains constant, there is a continuous transfer of energy from the mallet to the membrane. This is another way to understand the apparition of the absorption term in the equation for $\tilde{u}$.

In the next preliminary, we recall the results we have concerning the behavior of $\tilde{u}$. This preliminary is nothing but the transcription of theorem 2.4, since $\tilde{u}$ satisfies (3.14).

Preliminary 3.1 *Let's assume here that φ satisfies (1.7) and (1.8). There exists a unique time $t_m(\overline{R})$ such that $\tilde{u}'(t_m(\overline{R})) = 0$. The function $\tilde{u}(t)$ is strictly decreasing and strictly convex from 0 to $t_m(\overline{R})$. For $t > t_m(\overline{R})$, $\tilde{u}(t)$ is strictly increasing and one has the following alternative*

i) $\lim\limits_{t\to+\infty} \tilde{u}(t) = \delta$

ii) $\exists!\ t_c(\overline{R}) > 2t_m(\overline{R})$ *such that* $\tilde{u}(t_c(\overline{R})) = \delta$ *and*

$$\tilde{u}(t) = \delta + \tilde{u}'(t_c(\overline{R}))(t - t_c(\overline{R})) \quad \textit{for } t \geq t_c(\overline{R})$$

where $\tilde{u}'(t_c(\overline{R})) > 0$

From the properties of $\tilde{u}$, one can get a description of the behavior of u and W.

Theorem 3.3 *Let's assume here that φ satisfies (1.7) and (1.8). Let us define as in section 2,*

$$\left|\begin{array}{l} \overline{\mathcal{R}}_1 = \{\overline{R} = K\eta/2 \geq 0 \ \textit{such that}\ t_c(\overline{R}) = +\infty\} \\[2ex] \overline{\mathcal{R}}_2 = \{\overline{R} = K\eta/2 \geq 0 \ \textit{such that}\ t_c(\overline{R}) < +\infty\} \end{array}\right.$$

One has the following alternative :

i) if $\overline{R} \in \overline{\mathcal{R}}_1$

- *u is strictly convex and strictly decreasing in* $]0, +\infty[$ *and*

$$\lim_{t\to+\infty} u(t) = \delta - \frac{v_0\eta}{2}$$

- *W is strictly decreasing* $]0, +\infty[$ *and*

$$\lim_{t\to+\infty} W(t) = -\frac{v_0\eta}{2}$$

ii) if $\overline{R} \in \overline{\mathcal{R}}_2$

- *There exists a unique time $t_m(\overline{R}) < t_m^*(\overline{R}) < t_c(\overline{R})$ such that $u'(t_m^*(\overline{R})) = 0$. u is strictly decreasing from 0 to $t_m^*(\overline{R})$. For $t > t_m^*(\overline{R})$, u is strictly increasing and :*

$$u(t) = \delta - \frac{v_0\eta}{2} - \frac{\eta}{2}u'(t_c(\overline{R})) + u'(t_c(\overline{R}))(t - t_c(\overline{R})) \quad \textit{for } t \geq t_c(\overline{R})$$

  where $u'(t_c(\overline{R})) = \tilde{u}'(t_c(\overline{R})) > 0$.
  Moreover, u is strictly convex in $]0, t_c(\overline{R})]$.

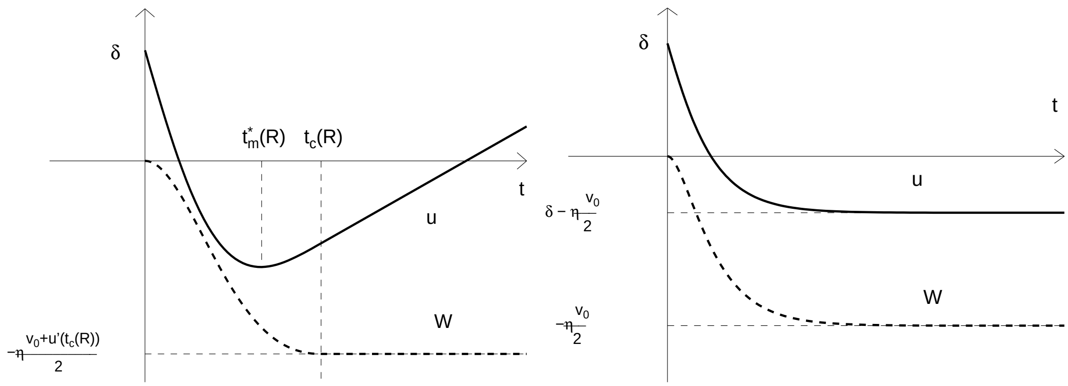

Figure 3.1: Possible graphs of $u(t)$ and $W(t)$. case (ii) (left) and case (i) (right)

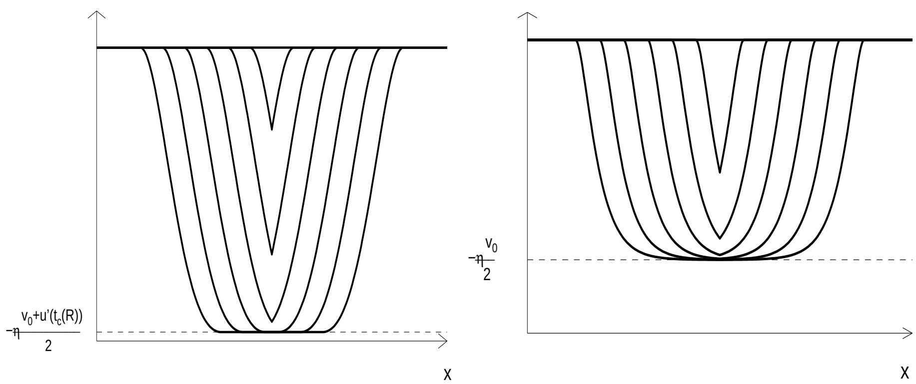

Figure 3.2: Look of the membrane at different times. case (ii) (left) and case (i) (right)

- *W is strictly decreasing from 0 to $t_c(\overline{R})$ and*

$$W(t) = -\frac{v_0\eta}{2} - \frac{\eta}{2}u'(t_c(\overline{R})) \quad \textit{(W remains constant)} \quad \textit{for } t \geq t_c(\overline{R})$$

We illustrate in figure 3.1 the two possible scenarii for the behavior of $u(t)$ and $W(t)$ We give in figure 3.2 the corresponding looks of the displacement of the membrane
Proof. This theorem is a direct consequence of the preliminary.

i) if $\overline{R} \in \overline{\mathcal{R}}_1$

In this case, thanks to the preliminary, we know that $\tilde{u}(t) < \delta$ for all $t > 0$, $\lim_{t\to+\infty} \tilde{u}(t) = \delta$, and $\lim_{t\to+\infty} \tilde{u}'(t) = 0$.

- Let us recall that :
(3.22) $$u'' = K\varphi(\tilde{u})$$
As $\tilde{u}(t) < \delta$ for all $t > 0$, we immediately deduce that u is strictly convex in $]0, +\infty[$. As $\lim_{t\to+\infty} \tilde{u}(t) = \delta$, (3.22) implies that :
(3.23) $$\lim_{t\to+\infty} u''(t) = 0$$
From (3.11), after derivation in time, we get

(3.24) $$u'' + \frac{2}{\eta}(u' - \tilde{u}') = 0$$

Joining (3.23) and (3.24), we get :

$$\lim_{t\to+\infty} u' = \lim_{t\to+\infty} \tilde{u}' = 0$$

To conclude on the decrease of u, we have obviously :

$$u'(0) = -v_0 < 0\ ,\ u \text{ is strictly convex in }]0,+\infty[\ ,\ \lim_{t\to+\infty} u' = 0 \quad \Rightarrow \quad \text{u is strictly decreasing in }]0,+\infty[$$

Concerning the asymptotic behavior of u, using once again (3.11), we get :

$$\lim_{t\to+\infty} u = \lim_{t\to+\infty} \tilde{u} - \frac{v_0\eta}{2} = \delta - \frac{v_0\eta}{2}$$

– Let us recall that :

(3.25) $$W'(t) = -\frac{K\eta}{2}\varphi(\tilde{u})$$

As $\tilde{u}(t) < \delta$ for all $t > 0$, we immediately deduce that W is strictly decreasing in $]0,+\infty[$. As $W = u - \tilde{u}$, we easily get :

$$\lim_{t\to+\infty} W = -\frac{v_0\eta}{2}$$

ii) if $\overline{R} \in \overline{\mathcal{R}}_2$

In this case, thanks to the preliminary, we know that there exists a unique time $t_c(\overline{R}) > 2t_m(\overline{R})$ such that $\tilde{u}(t_c(\overline{R})) = \delta$, $\tilde{u}(t) < \delta$ for $t < t_c(\overline{R})$ and $\tilde{u}'(t_c(\overline{R})) > 0$. Using (3.22), we immediately deduce that u is strictly convex in $]0, t_c(\overline{R})[$. But since

(3.26) $$u' = \tilde{u}' - \frac{K\eta}{2}\varphi(\tilde{u})$$

we have :

$$\left| \begin{array}{l} u'(0) = -v_0 \\ \\ u'(t_c(\overline{R})) = \tilde{u}'(t_c(\overline{R})) > 0 \end{array} \right.$$

which means in particular, since u is strictly convex in $]0, t_c(\overline{R})[$, that there exists a time $t_m^*(\overline{R})$ in $]0, t_c(\overline{R})[$ such that $u'(t_m^*(\overline{R})) = 0$. On the other hand, we know that there exists $t_m(\overline{R})$ such that $\tilde{u}'(t_m(\overline{R})) = 0$. At this time using (3.26), we get :

$$u'(t_m(\overline{R})) = -\frac{K\eta}{2}\varphi(\tilde{u}(t_m(\overline{R}))) < 0$$

We thus necessarily have $t_m(\overline{R}) < t_m^*(\overline{R})$. The proof of the remaining points of the theorem is left to the reader.

□

Remark 3.2 *In the case $p = 1$, when φ is given by (1.2), using the previous theorem, we have the following alternative concerning the behavior of the mallet :*

$K\eta^2 \geq 16$: *the mallet keeps sticked to the membrane*

$K\eta^2 < 16$: *the mallet comes up*

3.3 The case $R \neq 0$

We proceed as in section 3.2, introducing :

(3.27) $$\tilde{u} = u - W$$

We still have

(3.28) $$\tilde{u}'' = \frac{\eta}{2}\frac{d^3u}{dt^3} + u''$$

Using :

$$u'' = K\varphi(\tilde{u}) + R\frac{d}{dt}\varphi(\tilde{u}) \tag{3.29}$$

we get

$$\frac{d^2}{dt^2}\left[\tilde{u} - \frac{R\eta}{2}\varphi(\tilde{u})\right] = K\varphi(\tilde{u}) + \left(R + \frac{K\eta}{2}\right)\frac{d}{dt}\varphi(\tilde{u}) \tag{3.30}$$

This equation suggests us to introduce the function :

$$G_R(u) = u - \frac{R\eta}{2}\varphi(u) \tag{3.31}$$

The properties of function φ imply that :

$$\begin{cases} \bullet\ G_R(u) \text{ is Lipschitz-continuous} \\ \bullet\ u \to G_R(u) \text{ is strictly increasing} \\ \bullet\ G_R \text{ is a bijection from } \mathbb{R} \text{ to } \mathbb{R} \\ \bullet\ G_R(u) = u \text{ for } u > \delta \end{cases} \tag{3.32}$$

A typical graph of the function $u \to G_R(u)$ is given in 3.3 (when φ is given by (1.2)). We can thus introduce its inverse function $H_R = G_R^{-1}$, which has exactly the same properties (3.32). In fact, one can write :

$$H_R(u) = u + \frac{R\eta}{2}\tilde{\varphi}(u)$$

where function $\tilde{\varphi}$ has the same properties (1.7) as function φ.

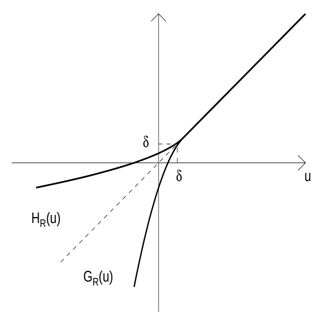

Figure 3.3: Functions G_R and H_R

Equation (3.30) suggests us the change of unknown function

$$\overline{u} = G_R(\tilde{u}) \Leftrightarrow \tilde{u} = H_R(\overline{u})$$

and leads to introduce :

$$\varphi_R(u) = \varphi \circ H_R(u)$$

Then the equation satisfied by $\overline{u}$ can be written :

$$\overline{u}'' = K\varphi_R(\overline{u}) + \left(R + \frac{K\eta}{2}\right)\frac{d}{dt}\varphi_R(\overline{u}) \tag{3.33}$$

That is to say, exactly (1.9) if one replaces φ by φ_R and R by $\overline{R} = R + \frac{K\eta}{2}$. Knowing $\overline{u}$, one knows $\tilde{u}$ by $\tilde{u} = H_R(\overline{u})$, thus u by $u = \tilde{u} - \frac{\eta}{2}u' - \frac{\eta}{2}v_0$ and then W by $W = u - \tilde{u}$. Note that, from a qualitative point of view, the effect of the coupling between the mallet and the membrane can be represented by :

- a change in the friction coefficient : $R \to \overline{R} = R + \dfrac{K\eta}{2}$.
- a change in the non-linearity $\varphi \to \varphi_R$.

Remark 3.3 • *If one considers the case where*

$$\varphi(u) = [(\delta - u)^+]^p \qquad \textit{with } p > 1$$

It is not difficult to prove that, for u close to δ :

$$H_R(u) = u + \frac{R\eta}{2}\left[(\delta - u)^+\right]^p + o(|\delta - u|^p)$$

which means that $\tilde{\varphi}(u) \simeq \varphi(u)$ when $u \to \delta$. One then checks that :

$$\varphi_R(u) = \left[(\delta - u)^+\right]^p - p\frac{R\eta}{2}\left[(\delta - u)^+\right]^{2p-1} + o(|\delta - u|^{2p-1})$$

which shows that for u sufficiently close to δ, the difference between $\varphi_R(u) - \varphi(u)$ is very small. On the other hand, when $u \to -\infty$, one sees that :

$$H_R(u) \simeq -\left(\frac{2}{R\eta}|u|\right)^{\frac{1}{p}}$$

which proves that φ_R behaves linearly at $-\infty$.

$$\varphi_R(u) \simeq -\frac{2}{R\eta}u \quad , u \to -\infty$$

- *In the case $p = 1$, one easily checks that*

$$H_R(u) = u + \frac{R\eta}{2 + R\eta}(\delta - u)^+$$

(in other words $\tilde{\varphi} = \dfrac{2}{2 + R\eta}(\delta - u)^+ = \dfrac{2}{2 + R\eta}\varphi(u)$) and that

$$\varphi_R(u) = \left(1 - \frac{R\eta}{2 + R\eta}\right)(\delta - u)^+$$

which means that one can once again compute explicitly the solution of the problem.

We proceed as in section 3.2. Let us first recall the results we have concerning the behavior of $\overline{u}$ in the next preliminary. This preliminary is nothing but the transcription of theorem 2.4, since $\overline{u}$ satisfies (3.33).

Preliminary 3.2 *Let's assume here that φ satisfies (1.7) and (1.8). There exists a unique time $t_m(\overline{R})$ such that $\overline{u}'(t_m(\overline{R})) = 0$. Function $\overline{u}(t)$ is strictly decreasing and strictly convex from 0 to $t_m(\overline{R})$. For $t > t_m(\overline{R})$, $\overline{u}(t)$ is strictly increasing and one has the following alternative*

i) $\lim_{t \to +\infty} \overline{u}(t) = \delta$

ii) $\exists!\ t_c(\overline{R}) > 2t_m(\overline{R})$ *such that* $\overline{u}(t_c(\overline{R})) = \delta$ *and*

$$\overline{u}(t) = \delta + \overline{u}'(t_c(\overline{R}))(t - t_c(\overline{R})) \quad \textit{for } t > t_c(\overline{R})$$

where $\overline{u}'(t_c(\overline{R})) > 0$

Theorem 3.4 *Let's assume here that φ satisfies (1.7) and (1.8). Let us define*

$$\left|\begin{array}{l} \overline{\mathcal{R}}_1 = \{\overline{R} = R + K\eta/2 \geq 0 \textit{ such that } t_c(\overline{R}) = +\infty\} \\ \\ \overline{\mathcal{R}}_2 = \{\overline{R} = R + K\eta/2 \geq 0 \textit{ such that } t_c(\overline{R}) < +\infty\} \end{array}\right.$$

One has the following alternative :

i) if $\overline{R} \in \overline{\mathcal{R}}_1$

- *u is strictly convex in* $]0, t_m(\overline{R})]$ *and strictly decreasing in* $]0, +\infty[$ *and*

$$\lim_{t \to +\infty} u(t) = \delta - \frac{v_0 \eta}{2}$$

- *W is strictly decreasing* $]0, t_m(\overline{R})]$ *and*

$$\left| \begin{array}{l} W(t) > -\dfrac{v_0 \eta}{2} \quad \forall t \\ \lim\limits_{t \to +\infty} W(t) = -\dfrac{v_0 \eta}{2} \end{array} \right.$$

ii) if $\overline{R} \in \overline{\mathcal{R}}_2$

- *There exists a unique time* $t_m(\overline{R}) < t_m^*(\overline{R}) < t_c(\overline{R})$ *such that* $u'(t_m^*(\overline{R})) = 0$. *u is strictly decreasing from* 0 *to* $t_m^*(\overline{R})$. *For* $t > t_m^*(\overline{R})$, *u is strictly increasing and :*

$$u(t) = \delta - \frac{v_0 \eta}{2} - \frac{\eta}{2} u'(t_c(\overline{R})) + u'(t_c(\overline{R}))(t - t_c(\overline{R})) \quad \text{for } t \geq t_c(\overline{R})$$

where $u'(t_c(\overline{R})) = \tilde{u}'(t_c(\overline{R})) = \overline{u}'(t_c(\overline{R})) > 0$.
Moreover, u is strictly convex from $]0, t_m(\overline{R})[$.

- *W is strictly decreasing from* 0 *to* $t_m(\overline{R})$ *and*

$$\left| \begin{array}{l} W(t) < -\dfrac{v_0 \eta}{2} \quad \forall t > t_m^*(\overline{R}) \\ W(t) = -\dfrac{v_0 \eta}{2} - \dfrac{\eta}{2} u'(t_c(\overline{R})) \quad (W \text{ remains constant}) \quad \text{for } t > t_c(\overline{R}) \end{array} \right.$$

We illustrate in figure 3.1 the two possible scenarii for the behavior of $u(t)$ and $W(t)$ We give in figure 3.5 the

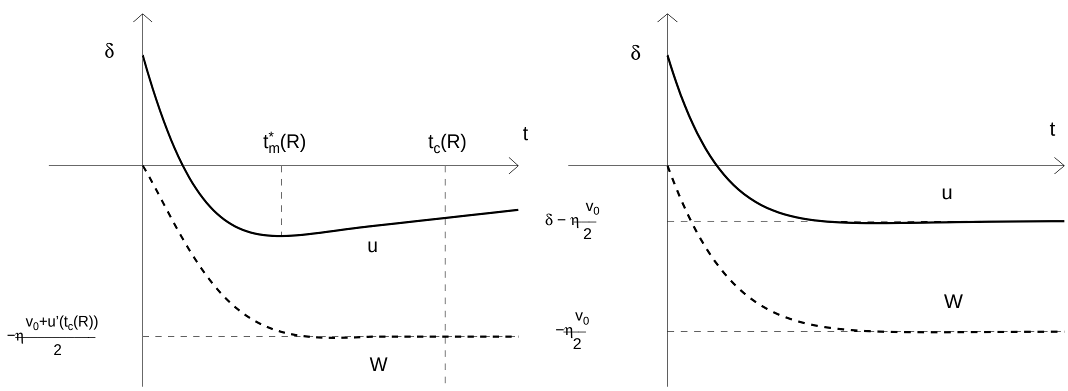

Figure 3.4: Possible graphs of $u(t)$ and $W(t)$. case (ii) (left) and case (i) (right)

corresponding looks of the displacement of the membrane
Proof. It is easy to deduce from the properties of $\varphi(u)$ that G_R is strictly increasing and concave . Consequently, H_R is strictly increasing and convex, and since $\tilde{u} = H_R(\overline{u})$, $\tilde{u}$ has exactly the same monoticity and the same convexity properties as $\overline{u}$ in $]0, t_c(\overline{R})[$. Moreover as H_R is written

$$H_R(u) = u + \frac{R\eta}{2} \tilde{\varphi}(u)$$

and $\tilde{\varphi}(\delta) = 0$, one can easily conclude that $\tilde{u}(t) = \overline{u}(t)$ for all $t \geq t_c(\overline{R})$.

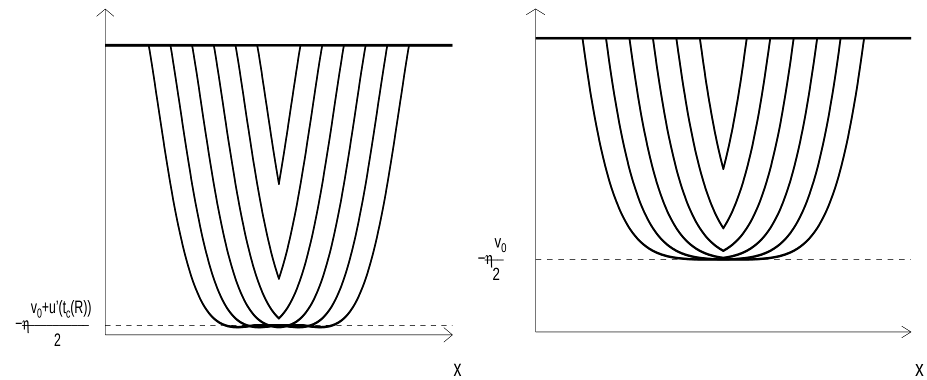

Figure 3.5: Look of the membrane at different times. case (ii) (left) and case (i) (right)

i) if $\overline{R} =\in \overline{\mathcal{R}}_1$

- In this case, as $\lim\limits_{t\to+\infty} \tilde{u} = \delta$ and $\lim\limits_{t\to+\infty} \tilde{u}' = 0$, joining (3.11) and (3.24), we get the following limits for u and u' :

  $$\left| \begin{array}{l} \lim\limits_{t\to+\infty} u = \delta - \dfrac{v_0\eta}{2} \\ \lim\limits_{t\to+\infty} u' = 0 \end{array} \right.$$

  We thus have :

  $$u'(0) = -v_0 \quad \text{and} \quad \lim_{t\to+\infty} u' = 0$$

  In $[0, t_m(\overline{R})]$, u is obviously convex, since it verifies (3.9) and $\tilde{u}' < 0$. At $t = t_m(\overline{R})$, where $\tilde{u}'(t_m(\overline{R})) = 0$, from (3.24), we deduce that :

  $$u'(t_m(\overline{R})) = -\frac{K\eta}{2}\varphi(\tilde{u}(t_m(\overline{R}))) < 0$$

  Then necessarily if there exists a time where $u' = 0$, then this time is greater than $t_m(\overline{R})$. We are going to prove that it is not possible. Let us then suppose that there exists a time $t_1 > t_m(\overline{R})$ satisfying $u'(t_1) = 0$. At this time, from (3.24), we get :

  $$u''(t_1) = \frac{2}{\eta}\tilde{u}'(t_1) > 0 \quad \text{since } t_1 > t_m(\overline{R})$$

  We would then have :

  $$\begin{cases} u'(t_1) = 0 \quad \text{and} \quad u''(t_1) > 0 \quad \text{and} \quad u(t_1) = -\dfrac{v_0\eta}{2} + \tilde{u}(t_1) < -\dfrac{v_0\eta}{2} + \delta \\ \lim\limits_{t\to+\infty} u = -\dfrac{v_0\eta}{2} + \delta \end{cases}$$

  This implies that it should exist a time $t_2 > t_1$ such that

  $$u'(t_2) = 0 \quad \text{and} \quad u''(t_2) < 0$$

  which is not possible since necessarily $u''(t_2) > 0$. We have proved that u' is always strictly negative and u is thus strictly decreasing in $[0, +\infty]$.
- Concerning the behavior of W, from (3.7), we know that :

  $$W(t) = -\frac{v_0\eta}{2} - \frac{\eta}{2}u'(t)$$

  We immediately deduce that W is strictly decreasing from 0 to $t_m(\overline{R})$ and that $W(t) > v_0\eta/2$ for all t since $u'(t) < 0$.

ii) if $\overline{R} \in \overline{\mathcal{R}}_2$

In this case, it is not difficult to prove that u' verifies :

$$u'(t) = \tilde{u}'(t) = \tilde{u}'(t_c(\overline{R})) > 0 \quad \text{for } t \geq t_c(\overline{R})$$

That is to say $u'(t) = \tilde{u}'(t_c(\overline{R})) > 0$ for all $t \geq t_c(\overline{R})$. We thus have :

$$u'(0) = -v_0 < 0 \quad \text{and} \quad u'(t_c(\overline{R})) > 0$$

It is then obvious that there exists a time $t_m^*(\overline{R})$ such that $u'(t_m^*(\overline{R})) = 0$. This time satisfies that $t_m^*(\overline{R}) > t_m(\overline{R})$ (the demonstration is the same as in the previous theorem) and

$$u''(t_m^*(\overline{R})) > 0$$

This proves that u' is strictly positive in some interval $(t_m^*(\overline{R}), t_m^*(\overline{R}) + \tau)$. We are now going to prove that u' remains strictly positive after $t_m^*(\overline{R})$. If it were not the case, there would exist some times t satisfying $u'(t) = 0$, $t > t_m^*(\overline{R})$. Let t^* be the smallest of such times. We have already seen that t^* also satisfies $u''(t^*) > 0$. On the other hand, for all $t_m^*(\overline{R}) < t < t^*$, we have $u'(t) > 0$, what implies that

$$\frac{1}{t - t^*}\left[u'(t) - u'(t^*)\right] < 0 \qquad \text{for } t_m^*(\overline{R}) < t < t^*$$

We get, taking the limit when t tends to t^* :

$$u''(t^*) \leq 0 \tag{3.34}$$

We thus have a contradiction which proves that u' remains strictly positive for all $t > t_m^*(\overline{R})$.
The remaining points of the theorem concerning the behavior of W are left to the reader.

□

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Unit´e de recherche INRIA Lorraine, Technopôle de Nancy-Brabois, Campus scientifique,
615 rue du Jardin Botanique, BP 101, 54600 VILLERS LÈS NANCY
Unit´e de recherche INRIA Rennes, Irisa, Campus universitaire de Beaulieu, 35042 RENNES Cedex
Unit´e de recherche INRIA Rhône-Alpes, 655, avenue de l'Europe, 38330 MONTBONNOT ST MARTIN
Unit´e de recherche INRIA Rocquencourt, Domaine de Voluceau, Rocquencourt, BP 105, 78153 LE CHESNAY Cedex
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