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Abstract: In this paper we study the asymptotic behavior of the tail of the stationary backlog distribution in a single server queue with constant service capacity c, fed by the so-called "$M/G/\infty$ input process" or "Cox input process". Asymptotic lower bounds are obtained for any distribution G and asymptotic upper bounds are derived when G is a subexponential distribution. We find the bounds to be tight in some instances, e.g., G corresponding to either the Pareto or lognormal distribution and $c - \rho < 1$, where ρ is the arrival rate to the buffer.

Key-words: Asymptotic self-similar process; Long-range dependence; Subexponential distributions; Pareto distribution; Large deviations; Queues.

(Résumé : tsvp)

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Comportement Asymptotique d'un Multiplexeur Alimenté par un Processus à Mémoire Longue

Résumé : Nous nous intéressons dans cet article au comportement asymptotique de la distribution de probabilité complémentaire de la charge (W) d'une file d'attente à serveur unique de capacité c, alimentée par un processus d'arrivée de type "$M/G/\infty$", dit aussi "processus de Cox". Des bornes asymptotiques inférieures et supérieures sont obtenues quand la distribution de probabilité G est sous-exponentielle. Dans le cas où G suit une loi de Pareto ou une loi *lognormal* le comportement asymptotique de $P(W > x)$ peut être calculé lorsque $c - \rho < 1$, où ρ est le taux des arrivées.

Mots-clé : Processus auto-similaire; Processus à mémoire longue; Loi sous-exponentielle; Loi de Pareto; Grandes déviations; Files d'attente.

1 Introduction

The recent discovery [18, 22, 30] that traffic in networks possess long-range time dependencies that cannot be easily captured by Poisson-based models has motivated queueing theorists to propose and analyze new queueing models that capture these dependencies. One such model that has received attention is a buffer with server having rate c fed by an $M/G/\infty$ input process where G is heavy-tailed (e.g., [1, 13, 20, 27]). This is of interest because of its versatility, i.e., the dependencies over different time-scales can be controlled by varying the tail behavior of G.

In this paper we consider the model introduced by Parulekar and Makowski [27]. A discrete-time single-server queue (called the multiplexer) with infinite waiting room and with service capacity c is fed by an integer-valued process $\{b_t,\, t \in \mathbb{N}\}$. The r.v. b_t is defined as the number of busy servers at time t in an M/G/∞ queue with arrival intensity $\lambda > 0$ and i.i.d. service times $\{\sigma_n\}_n$ with common probability distribution function (p.d.f.) $G(x) = P(\sigma_n \leq x)$ and finite mean $\overline{\sigma}$. An appealing feature of the (stationary version of the) input process $\{b_t,\, t \in \mathbb{N}\}$ is that it is a long-range dependent process [3] for some well-chosen *subexponential* p.d.f.'s G (see Section 2).

Let Q_t be the queue-length at the multiplexer at time t. Then, Q_t satisfies the Lindley's equation $Q_{t+1} = \max(0, Q_t + b_t - c)$ for all $t \in \mathbb{N}$, with $Q_0 = 0$. Let Q be the stationary queue-length under the stability condition $c > \rho := \lambda\overline{\sigma}$ (see Section 2). The aim of this paper is to study the behavior of $\log P(Q > x)$ and of $P(Q > x)$ for large x. More precisely, we show that there exist positive and finite constants θ_1 and θ_2 such that

$$-\theta_1 \leq \liminf_{x\to\infty} \frac{\log\, P(Q > x)}{-\log \overline{G_1}(x)} \leq \limsup_{x\to\infty} \frac{\log\, P(Q > x)}{-\log \overline{G_1}(x)} \leq -\theta_2. \tag{1}$$

The lower bound in (1) holds for any p.d.f. G whereas the upper bound holds for any *subexponential* p.d.f. G (to be defined in Section 2). Here G_1 is defined as

$$G_1(x) := \frac{1}{\overline{\sigma}} \int_0^x \overline{G}(u)\, du, \quad x \geq 0 \tag{2}$$

and $\overline{F}(x) = 1 - F(x)$ for any probability distribution F. We also show that the bounds in (1) are tight (i.e. $\theta_1 = \theta_2$) when G is Pareto or lognormal (see Corollary 4.1), provided that $c - \rho < 1$. In the following the bounds in (1) will be referred to as *large deviations* bounds. Asymptotic upper and lower bounds for $P(Q > x)$ are also obtained.

Large deviations bounds were obtained in [29] in the case when G is short-tailed. Duffield observed in [13] that the approach in [27], based on the Gärtner-Ellis theorem, cannot be used to derive large deviations *lower* bounds for heavy-tailed G. By refining Theorem 2.2 in [14] and by using results in [28] Duffield was able to obtain the following large deviations *upper* bound (see [13])

$$\limsup_{x\to\infty} \frac{\log\, P(Q > x)}{\log x} \leq 1 - (\alpha - 1)(c - \rho) \tag{3}$$

in the case of the Pareto distribution $\overline{G}(x) \sim x^{-\alpha}$. An asymptotic lower bound for $P(Q > x)$ was obtained by Jelenkovic and Lazar [20] in the case when $c - \rho < 1$ and under a technical condition on G_1 (see comment after the proof of Proposition 3.2).

In this paper we propose an alternative to the approach based on the Gärtner-Ellis theorem that will yield asymptotic lower and upper bounds. We will observe that the large deviations bounds are tight for a number of subexponential distributions when $c-\rho < 1$ and that, in the case of G Pareto, the large deviations upper bound that can be derived from (1) (see Proposition 4.1) is tighter than that of Duffield when $c - \rho \leq \alpha/(\alpha - 1)$; otherwise Duffield's is tighter.

Other models have been proposed for modeling the effects of long-range dependence in arrival processes on buffer occupancy statistics. These include fractional brownian motion [14, 25], fractional gaussian noise [27], and a finite population of on-off sources where the on state holding times are characterized by heavy-tailed distributions [8, 6, 10, 20, 23] (see [7] for a survey paper on fluid queues with long-tailed activity periods).

The rest of the paper is structured as follows. Section 2 contains a characterization of the stationary behavior of the $M/G/\infty$ input process and the definition and characterization of the family of subexponential distributions. Asymptotic lower and upper bounds are established in Sections 3 and 4 respectively. Concluding remarks on the superposition of independent M/G/∞ input processes are given in Section 5.

2 Preliminaries

The lemma below gives a useful characterization of the stationary behavior of the input process $\{b_t,\, t \in \mathbb{N}\}$. We will assume that customers entering the M/G/∞ queue begin their service upon arrival (see Remark 2.1).

Lemma 2.1 *The distribution of the sequence $\{b_{t+k},\, t \in \mathbb{N}\}$ converges monotonically for $k \to \infty$ to that of a proper stationary and ergodic sequence $\{b^t,\, t \in \mathbb{N}\}$ such that*

$$b^t \stackrel{\text{st}}{=} \sum_{j=0}^{b^0} I\left(\hat{\sigma}_j > t\right) + \sum_{s=0}^{t-1} \sum_{s \leq T_j < s+1} I(\sigma_j > t - T_j), \quad t \in \mathbb{N} \tag{4}$$

where

(i) $0 \leq T_1 \leq T_2 \leq \cdots$ *are the successive jump times of a Poisson process with intensity λ, independent of the service times $\{\sigma_n,\, n = 1, 2, \ldots\}$;*

(ii) b^0 *is a Poisson r.v. with parameter $\rho := \lambda\, \overline{\sigma}$;*

(iii) conditioned on the event $\{b^0 = k\}$, $k \geq 1$, the r.v.'s $\{\hat{\sigma}_1, \ldots, \hat{\sigma}_k\}$ are i.i.d. with common p.d.f. G_1 as defined in (2), namely,

$$P\left(\hat{\sigma}_1 \leq x_1, \ldots, \hat{\sigma}_k \leq x_k \,|\, b^0 = k\right) = \prod_{j=1}^{k} G_1(x_j).$$

Further, the r.v.'s $\{T_j, \sigma_j,\ j = 1, 2, \ldots\}$ *are independent of the r.v.'s* $\{b^0, \hat{\sigma}_j,\ j = 1, 2, \ldots\}$.

The proof of this lemma follows from [5, Chapter 6] and [32, pp. 160-162] (see also [27]). The interpretation of (4) is the following: given that the $M/G/\infty$ queue is in steady-state at time $t = 0$, the first sum in the r.h.s. gives the number of busy servers at time $t > 0$ among all servers busy at time 0; the second sum gives the number of servers that became busy at time s, $0 \leq s \leq t-1$, and that are still busy at time t.

Assume that $\rho < c$. Since the process $\{b_{t+k},\ t \in N\}$ converges to the stationary and ergodic process $\{b^t,\ t \in \mathbb{N}\}$ (see Lemma 2.1) then it is well-known (see e.g. [5, Theorem 6, p. 12]) that there exists a proper r.v. Q such that

$$P(Q > x) = \lim_{t\to\infty} P(Q_t > x) = P\left(\sup_{t\in\mathbb{N}}\left(\sum_{s=0}^{t-1} b^{-s} - ct\right) > x\right), \quad x \in \mathbb{N} \tag{5}$$

where $\{b^t,\ -\infty < t < \infty\}$ is a stationary and ergodic process obtained by supplementing $\{b^t,\ t \in \mathbb{N}\}$. We will however prefer the following representation for the stationary queue length distribution:

$$P(Q > x) = P\left(\sup_{t\in\mathbb{N}}\left(\sum_{s=0}^{t-1} b^{s} - ct\right) > x\right), \quad x \in \mathbb{N}, \tag{6}$$

which follows from (5) together with the property that the number of busy servers in a stationary M/G/∞ queue is a reversible stochastic process [21, Theorem 3.11].

The rest of this paper is devoted to the computation of asymptotic lower and upper bounds for $P(Q > x)$. Particular attention will be devoted to the case when the p.d.f. G of the service times is *subexponential*. Recall that a probability distribution F on $[0, \infty)$ is subexponential ($F \in \mathcal{S}$) if $\overline{F^{\star 2}}(x) \sim 2\,\overline{F}(x)$ where $F^{\star 2}$ denotes the 2nd convolution of F with itself, namely, $F^{\star 2}(x) = \int_0^\infty F(x-u)\,F(du)$. As usual, the notation $f(x) \sim g(x)$ stands for $\lim_{x\to\infty} f(x)/g(x) = 1$ and $f(x) = o(g(x))$ stands for $\lim_{x\to\infty} f(x)/g(x) = 0$. The class of subexponential distributions was introduced by Chistakov [9] and contains lognormal, Pareto and Weibull distributions (see Section 3), among others. A probability distribution F on $[0, \infty)$ belongs to the class $\mathcal{D}$ of dominated-variation distributions if $\limsup_{x\to\infty} \overline{F}(x)/\overline{F}(2x) < \infty$ and to the class $\mathcal{L}$ of long-tailed distributions if $\lim_{x\to\infty} \overline{F}(x-y)/\overline{F}(x) = 1$ for all $y \in (-\infty, \infty)$.

For any p.d.f. F on $[0, \infty)$ with finite expectation μ, (i.e. $\mu := \int_0^\infty u\,F(du) < \infty$), define the integrated tail distribution F_1 by

$$F_1(x) := \frac{1}{\mu}\int_0^x \overline{F}(u)\,du, \quad x \geq 0.$$

Note that G_1 in (2) is the integrated tail distribution of σ_n.

The next lemma reports basic properties of subexponential probability distributions.

Lemma 2.2 *The following statements hold:*

(a) $\mathcal{D} \cap \mathcal{L} \subset \mathcal{S} \subset \mathcal{L}$ **[16, 19]**;

(b) If F *has finite expectation and if* $F \in \mathcal{D}$ *then* $F_1 \in \mathcal{D} \cap \mathcal{L}$ **[16]**.

(c) If $F \in \mathcal{S}$ *and* $G(x) \sim C\,F(x)$ *where* $0 < C < \infty$*, then* $G \in \mathcal{S}$ **[33]**.

In particular, we see from properties (a) and (b) that if $F \in \mathcal{D} \cap \mathcal{L}$ and if F has finite expectation then $F, F_1 \in \mathcal{S}$.

We conclude this section by pointing out an interesting feature (already observed in [27, p. 1455]) of the process $\{b^t,\, t \in \mathbb{N}\}$ defined in (4). First, it has been shown in [12, formula (5.39)] that $\mathrm{cov}(b^t, b^{t+h}) = \rho\,\overline{G_1}(h)$ for all $t, h \in \mathbb{N}$. Therefore, the stationary process $\{b^t,\, t \in \mathbb{N}\}$ will be long-range dependent [3] if $\sum_{h=0}^{\infty} \overline{G_1}(h) = \infty$, which will occur, for instance, when G is Pareto (i.e. $\overline{G}(x) \sim x^{-\alpha}$) with parameter $1 < \alpha < 2$.

Remark 2.1 *By taking integer-valued service times our model reduces to that in [27]. This follows from the fact that in the case of integer-valued service times the number of busy servers at time* $t+1$ *is the same whether customers entering the* $M/G/\infty$ *queue in* $(t, t+1)$ *begin their service upon arrival (as in our model) or begin their service at time* $t+1$ *(as in [27]).*

3 Lower Bounds

The following representation of $A(0,t) := \sum_{s=0}^{t-1} b^s$ will prove useful:

$$\begin{aligned} A(0,t) &= \sum_{s=0}^{t-1} b^s \\ &= \sum_{s=0}^{t-1} \sum_{j=1}^{b^0} I\left(\hat{\sigma}_j > s\right) + \sum_{s=0}^{t-1} \sum_{k=0}^{s-1} \sum_{k \le T_j < k+1} I(\sigma_j > s - T_j) \\ &= \sum_{j=1}^{b^0} \sum_{s=0}^{t-1} I\left(\hat{\sigma}_j > s\right) + \sum_{k=0}^{t-2} \sum_{k \le T_j < k+1} \sum_{s=k+1}^{t-1} I(\sigma_j > s - T_j) \\ &= \sum_{j=1}^{b^0} \min\left(\lceil \hat{\sigma}_j \rceil, t\right) + \sum_{k=0}^{t-2} \sum_{k \le T_j < k+1} \sum_{s=k+1}^{t-1} I(\sigma_j > s - T_j). \end{aligned} \tag{7}$$

The first sum in the r.h.s. of (7) gives the total number of customers arriving to the multiplexer in $[0, t)$ generated by all servers in the infinite-server queue busy at time 0; the second sum gives the total number of customers arriving to the multiplexer in $(0, t)$ generated by all servers in the infinite-server queue that become active at time $1, 2, \ldots, t-1$. Set

$$a_0(t) \;:=\; \sum_{j=1}^{b^0} \min\left(\lceil \hat{\sigma}_j \rceil, t\right) \tag{8}$$

$$a_s(t) \quad := \quad \sum_{s-1 \le T_j < s} \sum_{i=s}^{t-1} I(\sigma_j > i - T_j) \tag{9}$$

so that

$$A(0,t) = \sum_{s=0}^{t-1} a_s(t). \tag{10}$$

The following asymptotic lower bound for log $P(Q > x)$ holds:

Proposition 3.1 (Large deviations lower bound)
For any p.d.f. G,

$$\liminf_{x\to\infty} \frac{\log P(Q > x)}{-\log \overline{G_1}(x)} \ge -\inf_{\beta>0} \left\{ (\lfloor c - \rho + \beta \rfloor + 1) \limsup_{x\to\infty} \frac{\log \overline{G_1}(x)}{\log \overline{G_1}(\beta x)} \right\}. \tag{11}$$

Proof. Fix $\beta > 0$, $\epsilon > 0$, and define $\gamma := c - \rho + \beta + \epsilon$. Note that $\gamma > 0$ under the stability condition $c > \rho$.

We have

$$\begin{aligned}
\liminf_{x\to\infty} \frac{\log P(Q > x)}{-\log \overline{G_1}(x)} &= \liminf_{t\to\infty} \frac{\log P(Q > \beta t)}{-\log \overline{G_1}(\beta t)} \\
&\ge \liminf_{t\to\infty} \frac{\log P(A(0,t) - ct > \beta t)}{-\log \overline{G_1}(\beta t)} \qquad (12) \\
&\ge \liminf_{t\to\infty} \frac{-1}{\log \overline{G_1}(\beta t)} \log P\left(a_0(t) \ge \gamma t, \sum_{s=1}^{t-1} a_s(t) > (\rho - \epsilon) t \right) \\
&= \liminf_{t\to\infty} \frac{-1}{\log \overline{G_1}(\beta t)} \left[\log P(a_0(t) \ge \gamma t) + \log P\left(\sum_{s=1}^{t-1} a_s(t) > (\rho-\epsilon)t \right) \right] \qquad (13) \\
&\ge \liminf_{t\to\infty} \frac{\log P(a_0(t) \ge \gamma t)}{-\log \overline{G_1}(\beta t)} + \liminf_{t\to\infty} \frac{-1}{\log \overline{G_1}(\beta t)} \log P\left(\sum_{s=1}^{t-1} a_s(t) > (\rho-\epsilon)t \right). \qquad (14)
\end{aligned}$$

Inequality (12) follows from $P(Q > x) \ge P(A(0,t) - ct > x)$ (see (6)); (13) is a consequence of the independence of the r.v.'s $a_0(t)$ and $\sum_{s=1}^{t} a_s(t)$ (see Lemma 2.1); (14) comes from the inequality $\liminf_n (a_n + b_n) \ge \liminf_n a_n + \liminf_n b_n$.

Let us now focus on the first limit in the r.h.s. of (14). We have for $t > 0$

$$\begin{aligned}
P(a_0(t) \ge \gamma t) &= P\left(\sum_{j=1}^{b^0} \min\left(\lceil \hat{\sigma}_j \rceil, t\right) \ge \gamma t \right) \\
&\ge \sum_{k=\lceil \gamma \rceil}^{\infty} P\left(\sum_{j=1}^{k} \min(\hat{\sigma}_j, t) \ge \gamma t \,|\, b^0 = k \right) P(b^0 = k) \qquad (15)
\end{aligned}$$

$$\begin{aligned} &\geq \sum_{k=\lceil\gamma\rceil}^{\infty} P\left(\hat{\sigma}_1 > t, \cdots, \hat{\sigma}_{\lceil\gamma\rceil} > t \,|\, b^0 = k\right) P(b^0 = k) \\ &= \overline{G_1}(t)^{\lceil\gamma\rceil}\, P(b^0 \geq \lceil\gamma\rceil) \qquad (16) \end{aligned}$$

where (16) follows from Lemma 2.1(iii).

Since $P(b^0 \geq \lceil\gamma\rceil) > 0$ (see Lemma 2.1(ii)) we deduce from (16) that

$$\liminf_{t\to\infty} \frac{\log P(a_0(t) \geq \gamma t)}{-\log \overline{G_1}(\beta t)} \geq -\lceil\gamma\rceil \limsup_{t\to\infty} \frac{\log \overline{G_1}(t)}{\log \overline{G_1}(\beta t)}. \qquad (17)$$

Let us show that the second limit in the r.h.s. of (14) is 0. We see from the definition of $A(0,t)$ and from (8)-(10) that

$$\sum_{s=1}^{t-1} a_s(t) \geq \sum_{s=0}^{t-1} b^s - \sum_{j=1}^{b^0} \lceil\hat{\sigma}_j\rceil. \qquad (18)$$

On the other hand, the stationarity and ergodicity of the sequence $\{b^t,\, t \in \mathbb{N}\}$ together with $\rho = E[b^0] < \infty$ (see Lemma 2.1) yields

$$\lim_{t\to\infty} \frac{1}{t} \sum_{s=0}^{t-1} b^s = \rho \quad \text{a.s.} \qquad (19)$$

from ergodic theory (see e.g. [31, Chapter V]). We therefore deduce from (18)-(19) that

$$\liminf_{t\to\infty} \frac{1}{t} \sum_{s=1}^{t-1} a_s(t) \geq \rho \quad \text{a.s.} \qquad (20)$$

since $\sum_{j=1}^{b^0} \hat{\sigma}_j < \infty$ a.s. by Lemma 2.1.

Combining [24, Proposition I-4-3] together with (20) yields

$$1 \geq \liminf_t P\left(\sum_{s=1}^{t-1} a_s(t) > (\rho-\epsilon)t\right) \geq P\left(\liminf_t \left\{\sum_{s=1}^{t-1} a_s(t) > (\rho-\epsilon)t\right\}\right) = 1 \qquad (21)$$

which entails that

$$\liminf_{t\to\infty} \frac{-1}{\log \overline{G_1}(\beta t)} \log P\left(\sum_{s=1}^{t-1} a_s(t) > (\rho-\epsilon)t\right) = 0. \qquad (22)$$

In summary, we have shown that (cf. (14), (17), (22))

$$\begin{aligned} \liminf_{x\to\infty} \frac{\log P(Q > x)}{-\log \overline{G_1}(x)} &\geq -\inf_{\beta>0,\epsilon>0} \left\{\lceil c-\rho+\beta+\epsilon\rceil \limsup_{t\to\infty} \frac{\log \overline{G_1}(t)}{\log \overline{G_1}(\beta t)}\right\} \\ &\geq -\inf_{\beta>0} \left\{(\lfloor c-\rho+\beta\rfloor + 1) \limsup_{t\to\infty} \frac{\log \overline{G_1}(t)}{\log \overline{G_1}(\beta t)}\right\} \end{aligned}$$

which completes the proof. ∎

It is worth noting that the lower bound in (11) is never trivial as it is always larger than or equal to $-(\lfloor c-\rho\rfloor+2)$ that is obtained for $\beta=1$.

The next result proposes asymptotic lower bounds for $P(Q>x)$.

Proposition 3.2 (Asymptotic lower bound)
For any p.d.f. G,

$$\liminf_{x\to\infty}\frac{P(Q>x)}{\overline{G_1}(x)^{\lfloor c-\rho\rfloor+1}} \geq \sup_{0<\beta<1+\lfloor c-\rho\rfloor-(c-\rho)} \liminf_{x\to\infty}\left(\frac{\overline{G_1}(x)}{\overline{G_1}(\beta x)}\right)^{\lfloor c-\rho\rfloor+1}\left(1-\sum_{k=0}^{\lfloor c-\rho\rfloor}\frac{\rho^k}{k!}e^{-\rho}\right). \tag{23}$$

If $G_1\in\mathcal{S}$ then

$$\liminf_{x\to\infty}\frac{P(Q>x)}{\overline{G_1}(x)^{\lfloor c-\rho\rfloor+1}} \geq (\delta_0+\rho) \sup_{\substack{\epsilon>0,\beta>0\\ \lceil c-\rho+\beta+\epsilon\rceil=r_0}} \liminf_{x\to\infty}\left(\frac{\overline{G_1}((c-\rho+\beta+\epsilon)x/r_0)}{\overline{G_1}(\beta x)}\right)^{r_0} \tag{24}$$

with $\delta_0:=e^{-\rho}\left(\sum_{l=1}^{l_0-1} l^{r_0}\sum_{i=lr_0}^{(l+1)r_0-1}\rho^i/i!-\sum_{l=1}^{l_0r_0-1}\rho^l/(l-1)!\right)$, $r_0:=\lfloor c-\rho\rfloor+1$, $l_0:=\min\{l\geq 1\,:\,l^{r_0}\geq(l+1)r_0-1\}$. *In particular, when $c-\rho<1$ then*

$$\liminf_{x\to\infty}\frac{P(Q>x)}{\overline{G_1}(x)}\geq\rho \sup_{\substack{\epsilon>0,\beta>0\\ \beta+\epsilon<1-(c-\rho)}} \liminf_{x\to\infty}\frac{\overline{G_1}((c-\rho+\beta+\epsilon)x)}{\overline{G_1}(\beta x)}. \tag{25}$$

Proof. The proof of (23) follows the same line of arguments as that of Proposition 3.1. Define $\gamma:=c-\rho+\beta+\epsilon$. Let $0<\beta<1+\lfloor c-\rho\rfloor-(c-\rho)$ and pick $\epsilon>0$ small enough so that $\lceil\gamma\rceil=\lfloor c-\rho\rfloor+1$.

In direct analogy with the derivation of (14) and by using (16) and (21) we get

$$\begin{aligned}\liminf_{x\to\infty}\frac{P(Q>x)}{\overline{G_1}(x)^{\lfloor c-\rho\rfloor+1}} &\geq \liminf_{t\to\infty}\frac{P(a_0(t)>\gamma t)}{\overline{G_1}(\beta t)^{\lfloor c-\rho\rfloor+1}} &(26)\\ &\geq \liminf_{t\to\infty}\left(\frac{\overline{G_1}(t)}{\overline{G_1}(\beta t)}\right)^{\lfloor c-\rho\rfloor+1} P(b^0\geq\lfloor c-\rho\rfloor+1) &(27)\end{aligned}$$

for all $0<\beta<1+\lfloor c-\rho\rfloor-(c-\rho)$, from which (23) follows.

Assume now that $G_1\in\mathcal{S}$. Straightforward manipulations in (15) yield

$$\begin{aligned}&P(a_0(t)>\gamma t)\\ &\geq \sum_{l=1}^{\infty}\sum_{m=0}^{\lceil\gamma\rceil-1}P\left(\sum_{j=1}^{l\lceil\gamma\rceil+m}\min(\hat\sigma_j,t)\geq\gamma t\,\middle|\,b^0=l\lceil\gamma\rceil+m\right)P(b^0=l\lceil\gamma\rceil+m)\\ &\geq \sum_{l=1}^{\infty}\sum_{m=0}^{\lceil\gamma\rceil-1}P\left(\sum_{j=1}^{l\lceil\gamma\rceil}\min(\hat\sigma_j,t)\geq\gamma t\,\middle|\,b^0=l\lceil\gamma\rceil+m\right)P(b^0=l\lceil\gamma\rceil+m)\end{aligned}$$

$$\geq \sum_{l=1}^{\infty} \sum_{m=0}^{\lceil\gamma\rceil-1} P\left(\sum_{j=1}^{l} \min(\hat{\sigma}_j, t) \geq \frac{\gamma}{\lceil\gamma\rceil} t \,\middle|\, b^0 = l\lceil\gamma\rceil + m\right)^{\lceil\gamma\rceil} P(b^0 = l\lceil\gamma\rceil + m). \quad (28)$$

It is shown in Lemma A.1 in Appendix A that $P\left(\sum_{j=1}^{l} \min(\hat{\sigma}_j, t) \geq \theta t \,|\, b^0 = k\right) \sim l\,\overline{G_1}(\theta t)$ for all $\theta \in (0,1]$, $k \geq l$. By applying Fatou's lemma to (28) and by using the latter result we obtain

$$\liminf_{t\to\infty} \frac{P(a_0(t) > \gamma t)}{\overline{G_1}(\beta t)^{\lceil\gamma\rceil}} \quad \geq \quad \eta \sum_{l=1}^{\infty} l^{\lceil\gamma\rceil} P(l\lceil\gamma\rceil \leq b^0 \leq (l+1)\lceil\gamma\rceil - 1) \quad (29)$$

with $\eta := \liminf_{t\to\infty} \left(\frac{\overline{G_1}(\gamma t/\lceil\gamma\rceil)}{\overline{G_1}(\beta t)}\right)^{\lceil\gamma\rceil}$.

Define $l_0 = \min\left\{l \geq 1 : l^{\lceil\gamma\rceil} \geq (l+1)\lceil\gamma\rceil - 1\right\}$. Observe that $l^{\lceil\gamma\rceil} \geq (l+1)\lceil\gamma\rceil - 1$ for all $l \geq l_0$ since the mapping $l \to l^{\lceil\gamma\rceil} - (l+1)\lceil\gamma\rceil + 1$ is non-decreasing. The above and (29) yield

$$\liminf_{t\to\infty} \frac{P(a_0(t) > \gamma t)}{\overline{G_1}(\beta t)^{\lceil\gamma\rceil}} \geq \eta \left(\sum_{l=1}^{l_0-1} l^{\lceil\gamma\rceil} P(l\lceil\gamma\rceil \leq b^0 \leq (l+1)\lceil\gamma\rceil - 1) + \sum_{l=l_0\lceil\gamma\rceil}^{\infty} l\,P(b^0 = l)\right)$$

$$= \eta \left(\sum_{l=1}^{l_0-1} l^{\lceil\gamma\rceil} P(l\lceil\gamma\rceil \leq b^0 \leq (l+1)\lceil\gamma\rceil - 1) - \sum_{l=1}^{l_0\lceil\gamma\rceil-1} l\,P(b^0 = l) + \rho\right). \quad (30)$$

Substituting $\lceil\gamma\rceil$ for $\lfloor c-\rho\rfloor + 1$ in (30) yields (24). ∎

It is seen from Lemma A.2 in Appendix A that the supremum in the r.h.s. of (23) (resp. (24), (25)) is strictly positive if and only if $G_1 \in \mathcal{D}$. A sufficient condition for $G_1 \in \mathcal{D}$ is that $G \in \mathcal{D}$ (e.g. G Pareto) and G has finite expectation (see Lemma 2.2(b)).

When $c - \rho < 1$, Jelenkovic and Lazar [20, Theorem 11] have derived a tighter lower bound with the same decay function $\overline{G_1}(x)$ but with a larger coefficient. The bound in [20] holds provided that $L := \lim_{\delta\downarrow 1} \liminf_{x\uparrow\infty} \overline{G_1}(\delta x)/\overline{G_1}(x) > 0$ (Jelenkovic and Lazar [20] actually assume that $L = 1$ but this assumption can be weakened to $L > 0$; if so, then the coefficient of their lower bound in Theorem 11 has to be multiplied by L). Since $\overline{G_1}$ is non-increasing, it is easy to see from Lemma A.2 that $L > 0$ is equivalent to $G_1 \in \mathcal{D}$. Hence, the bounds in Proposition 3.2 and in [20] are non-trivial if and only if $G_1 \in \mathcal{D}$.

Corollary 3.1 *When $G_1 \in \mathcal{D}$ then*

$$\liminf_{x\to\infty} \frac{\log P(Q > x)}{-\log \overline{G_1}(x)} \geq -\lfloor c-\rho\rfloor - 1. \quad (31)$$

When Corollary 3.1 applies the lower bound in the r.h.s. of (31) is easier to compute than the lower bound in Proposition 3.1 but may not be as tight (for G Pareto both bounds in (11) and in (31) are the same as reported below).

We conclude this section by addressing the cases when G is (i) geometric, (ii) Pareto, (iii) Weibull, and (iv) lognormal.

(i) G is geometric. We have $G(r) = (1-q)\,q^{r-1}$ for $r = 1, 2, \ldots$ with $q := (\overline{\sigma}-1)/\overline{\sigma} \in (0,1)$. Hence, $\overline{G_1}(r) = q^r$ for $r = 1, 2, \ldots$.

Proposition 3.2 yields a trivial lower bound ($= 0$). From Proposition 3.1 we find

$$\liminf_{x\to\infty} \frac{1}{x} \log P(Q > x) \geq \log q \inf_{\beta>0} \frac{\lfloor c-\rho+\beta\rfloor + 1}{\beta} = \log q. \tag{32}$$

The r.h.s. of (32) follows from the inequalities

$$\frac{c-\rho+\beta+1}{\beta} \geq \frac{\lfloor c-\rho+\beta\rfloor+1}{\beta} \geq 1$$

together with $\lim_{\beta\to\infty}(c-\rho+\beta+1)/\beta = 1$.

(ii) G is Pareto. We have $\overline{G}(x) \sim x^{-\alpha}$ for some $\alpha > 1$. We assume that $\alpha > 1$ so that G has finite expectation $\overline{\sigma}$. We have

$$\overline{G_1}(x) \sim x^{-\alpha+1}/(\alpha-1). \tag{33}$$

From (24) we get

$$\liminf_{x\to\infty} \frac{P(Q>x)}{x^{(-\alpha+1)(\lfloor c-\rho\rfloor+1)}} \geq (\delta_0+\rho)\left(\frac{\lfloor c-\rho\rfloor+1}{c-\rho+1}\right)^{(\alpha-1)(\lfloor c-\rho\rfloor+1)}. \tag{34}$$

In particular, if $c-\rho<1$ then (cf. (25))

$$\liminf_{x\to\infty} \frac{P(Q>x)}{x^{-\alpha+1}} \geq \rho\left(\frac{1}{c-\rho+1}\right)^{\alpha-1}. \tag{35}$$

From (34) (or Proposition 3.1/ Corollary 3.1) we get

$$\liminf_{x\to\infty} \frac{1}{\log x} \log P(Q>x) \geq (-\alpha+1)\,(\lfloor c-\rho\rfloor+1). \tag{36}$$

(iii) G is Weibull. We have $\overline{G}(x) \sim e^{-x^\nu}$ for some $0<\nu<1$ and >0. Simple algebra yield

$$\overline{G_1}(x) \sim e^{-x^\nu}\,x^{1-\nu}/\nu. \tag{37}$$

Proposition 3.2 yields a trivial lower bound. By Proposition 3.1 we get (Corollary 3.1 does not apply since $G_1 \notin \mathcal{D}$)

$$\begin{aligned}\liminf_{x\to\infty} \frac{1}{x^\nu} \log P(Q>x) &\geq -\inf_{\beta>0} \frac{\lfloor c-\rho+\beta\rfloor+1}{\beta^\nu} \\ &= \begin{cases} -\min\left\{\dfrac{\lfloor c-\rho\rfloor+\lfloor a\rfloor}{(\lfloor a\rfloor-q)^\nu}; \dfrac{\lfloor c-\rho\rfloor+\lceil a\rceil}{(\lceil a\rceil-q)^\nu}\right\}, & \text{if } a\geq 1 \\ -\dfrac{\lfloor c-\rho\rfloor+1}{(1-q)^\nu}, & \text{if } a<1 \end{cases}\end{aligned} \tag{38}$$

with $a := (\nu\lfloor c-\rho\rfloor + q)/(1-\nu)$ and $q := c-\rho-\lfloor c-\rho\rfloor$.

[Hint for the derivation of (38): note that

$$\inf_{\beta>0} \frac{\lfloor c-\rho+\beta\rfloor+1}{\beta^\nu} = \min_{i=1,2,\ldots} \frac{\lfloor c-\rho\rfloor+i}{(i-q)^\nu}. \tag{39}$$

The mapping $g(x) := (\lfloor c-\rho\rfloor + x)/(x-q)^\nu$ being strictly decreasing in $(0,a)$ and strictly increasing in (a,∞), the mininum in (39) is reached when $\beta = \lfloor a\rfloor$ or when $\beta = \lceil a\rceil$ if $a \geq 1$ and when $\beta = 1$ if $a < 1$.]

(iv) G is lognormal. The p.d.f. G of a r.v. σ is *lognormal* if $\sigma \stackrel{\text{st}}{=} \exp(Y)$ where Y is a Gaussian r.v. with mean μ and variance δ^2. Then, $\overline{G}(x) \sim (2\pi)^{-1/2}\,(\delta/(\log x - \mu))\, e^{-(\log x-\mu)^2/(2\,\delta^2)}$. From this we get

$$\overline{G_1}(x) \sim \frac{\sigma^3\, x\, e^{-(\log x-\mu)^2/(2\,\delta^2)}}{\sqrt{2\pi}\,(\log x-\mu)^2}. \tag{40}$$

Proposition 3.2 yields a trivial lower bound. From Proposition 3.1 (Corollary 3.1 does not apply since $G_1 \notin \mathcal{D}$) we have

$$\liminf_{x\to\infty} \frac{1}{(\log x)^2}\, \log\, P(Q>x) \geq -\frac{\lfloor c-\rho\rfloor+1}{2\,\delta^2}. \tag{41}$$

4 Upper Bounds

We begin this section by stating two lemmas that will be used in the derivation of asymptotic upper bounds in the case when G and G_1 are subexponential probability distributions.

Lemma 4.1 *Let $F, F^1, \ldots, F^k$ be probability distributions such that $\overline{F^j}(x) \sim c_j\,\overline{F}(x)$, $c_j > 0$, for all $j = 1, 2, \ldots, k$. If $F \in \mathcal{S}$ then*

(a) $\overline{F^1 \star \cdots \star F^k}(x) \sim \sum_{j=1}^k c_j\,\overline{F}(x)$

(b) for each $\epsilon > 0$ there exists some constant $K_\epsilon < \infty$, independent of k, such that for all $x \geq 0$,

$$\overline{F^1 \star \cdots \star F^k}(x) \leq K_\epsilon\,(1+\epsilon)^k\,\overline{F}(x). \tag{42}$$

Proof. Statement (a) is due to Cline [11] and (b) to Athreya and Ney [2, p. 149]. ∎

Lemma 4.2 (Pakes [26]) *Consider a GI/GI/1 queue with i.i.d. service times $\{\sigma_n\}_n$ with common p.d.f. F and i.i.d. interarrival times $\{\tau_n\}_n$. Assume that $E[\sigma_n] < E[\tau_n]$.*

If F, $F_1 \in \mathcal{S}$, then

$$P\,(W>x) \sim \frac{E[\sigma_n]}{E[\tau_n]-E[\sigma_n]}\,\overline{F_1}(x)$$

where $W := \sup_{n\in\mathbf{N}}\left(\sum_{m=0}^{n-1}(\sigma_m-\tau_m)\right)$ is the stationary waiting time.

We are now in position to derive the following asymptotic upper bounds for $P(Q > x)$ and for $\log P(Q > x)$ when G and G_1 are in $\mathcal{S}$.

Proposition 4.1 (Upper bounds)
Assume that $G, G_1 \in \mathcal{S}$. Then,

$$\limsup_{x\to\infty} \frac{P(Q > x)}{\overline{G_1}(x)} \le \rho + \frac{\rho}{c-\rho}. \tag{43}$$

In particular, (43) implies that

$$\limsup_{x\to\infty} \frac{\log P(Q > x)}{-\log \overline{G_1}(x)} \le -1. \tag{44}$$

Proof. Define

$$a_0 = \sum_{j=1}^{b^0} (\hat{\sigma}_j + 1) \tag{45}$$

$$a_s = \sum_{j=1}^{v_{s-1}} (\sigma_j + 1), \quad s = 1, 2 \ldots \tag{46}$$

where $v_s := \sum_{j=1}^{\infty} I(s \le T_j < s+1)$ denotes the number of arrivals in the M/G/∞ queue in the interval of time $[s, s+1)$, $s \in \mathbb{N}$. Since the arrival process in this queue is Poisson with rate λ, $\{v_s,\, s \in \mathbb{N}\}$ constitutes an i.i.d. sequence of Poisson r.v's with parameter λ, namely, $P(v_s = k) = \lambda^k e^{-\lambda}/k!$ for all $k \in \mathbb{N}$.

From (8)-(9) and (45)-(46) we see that

(a1) $a_0(t) \le a_0$ and $a_s(t) \le a_s$ for all $s, t = 1, 2, \ldots$;

(a2) the r.v.'s a_s, $s = 1, 2, \ldots$ are i.i.d. and independent of the r.v. a_0.

To get the second inequality in (a1) observe from (9) that

$$a_s(t) \le \sum_{s-1 \le T_j < s} \sum_{l=0}^{t-1-s} I(\sigma_j > l) = \sum_{s-1 \le T_j < s} \min(\lceil \sigma_j \rceil, t-s) = \sum_{j=1}^{v_{s-1}} \min(\lceil \sigma_j \rceil, t-s) \le a_s.$$

We have, cf. (6), (10) and (a1),

$$\begin{aligned} P(Q > x) &= P\left(\sup_{t\in\mathbb{N}} \left(a_0(t) + \sum_{s=1}^{t-1} a_s(t) - ct\right) > x\right) \\ &\le P\left(a_0 + \sup_{t\in\mathbb{N}} \left(\sum_{s=1}^{t} a_s - ct\right) > x\right). \end{aligned} \tag{47}$$

Let us show that $P(a_0 > x) \sim \rho\,\overline{G_1}(x)$. From the definition of a_0 (see (45)) and Lemma 2.1(iii) we find

$$\lim_{x\to\infty} \frac{P\,(a_0 > x)}{\overline{G_1}(x)} = \lim_{x\to\infty} \left(\sum_{k=1}^{\lfloor x \rfloor} \frac{\overline{G_1^{\star k}}(x-k)}{\overline{G_1}(x)}\, P(b^0 = k) + \frac{P(b^0 > \lfloor x \rfloor)}{\overline{G_1}(x)} \right)$$

$$= \lim_{x\to\infty} \sum_{k=1}^{\infty} I(k \le \lfloor x \rfloor)\, \frac{\overline{G_1^{\star k}}(x-k)}{\overline{G_1}(x)}\, P(b^0 = k) \tag{48}$$

where (48) follows from the fact that b^0 decreases exponentially fast to 0 (hint: by Chernoff's bound $P(b^0 > x) \le \exp(\rho(\exp(\theta) - 1))\exp(-\theta x)$ for all $\theta > 0$, $x \ge 0$) which implies that $\lim_{x\to\infty} P(b^0 > \lfloor x \rfloor)/\overline{G_1}(x) = 0$ by [15, Lemma 1(b)]. By applying Lemma 4.1(a) with $F^j(x) = G_1(x-1)$ and $F \equiv G_1$ (note that this lemma applies with $c_j = 1$ since $\overline{F}^j(x)/\overline{F}(x) \sim 1$ since $G_1 \in \mathcal{L}$) we have $f_x(k) := I(k \le \lfloor x \rfloor)\,\overline{G_1^{\star k}}(x-k)/\overline{G_1}(x) \to k$ as $x \to \infty$ for each $k \ge 1$; furthermore, we see from Lemma 4.1(b) that there exists a constant $c_0 < \infty$, independent of k, such that $|f_k(x)| \le c_0\, 2^k$ for all $x \ge 0$, $k = 1, 2, \ldots$. Since $\sum_{k=1}^{\infty} 2^k P(b^0 = k) = e^{\rho}$ is finite we may therefore apply the bounded convergence theorem to the r.h.s. of (48) to finally get

$$\lim_{x\to\infty} \frac{P\,(a_0 > x)}{\overline{G_1}(x)} = \sum_{k=1}^{\infty} k\, P(b^0 = k) = \rho. \tag{49}$$

Let us now focus on $P\left(\sup_{t\in\mathbf{N}} \left(\sum_{s=1}^{t} a_s - ct\right) > x\right)$ when x is large.

Define $W = \sup_{t\in\mathbf{N}} \left(\sum_{s=1}^{t} a_s - ct\right)$. Since the r.v.'s a_s, $s = 1, 2, \ldots$ are i.i.d. it is seen that $P\,(W \le x)$ is the waiting time distribution in a stable (since $E[a_s] = \rho < c$) $D/GI/1$ queue with constant interarrival times c and service times $\{a_s\}_s$. By applying again Lemma 4.1 (with $F^j(x) = G(x-1)$, $F \equiv G$) and the bounded convergence theorem we obtain

$$\lim_{x\to\infty} \frac{P(a_s > x)}{\overline{G}(x)} = \sum_{k=0}^{\infty} \lim_{x\to\infty} \frac{\overline{G^{\star k}}(x-k)}{\overline{G}(x)}\, P(v_0 = k) \tag{50}$$

$$= \sum_{k=1}^{\infty} k\, P(v_0 = k) = \lambda. \tag{51}$$

Define $K(x) := P(a_s \le x)$. From $\overline{K}(x) \sim \lambda \overline{G}(x)$ (see (51)) and $\overline{K_1}(x) \sim \overline{G_1}(x)$ (which is an easy consequence of (51)) we may conclude from Lemma 2.2(c) that $K, K_1 \in \mathcal{S}$ since $G, G_1 \in \mathcal{S}$.

Therefore, Lemma 4.2 applies to this D/GI/1 queue (with $F = K$) to yield

$$P(W > x) \sim \frac{\rho}{c - \rho}\, \overline{G_1}(x). \tag{52}$$

The proof is concluded by applying Lemma 4.1(a) (with $F = G_1$, $F^1(x) = P(a_0 > x)$ and $F^2(x) = P(W > x)$) to the independent r.v.'s a_0 and W and by using (49) and (52). ∎

We conclude this section by specializing Proposition 4.1 to p.d.f.'s G that are (i) Pareto, (ii) Weibull, and (iii) lognormal. It is known that both G and G_1 belong to $\mathcal{S}$ when G is Pareto, Weibull or lognormal.

(i) G is Pareto. From (33) and (44) we get

$$\limsup_{x\to\infty} \frac{1}{\log x} \log P(Q > x) \le -\alpha + 1. \tag{53}$$

Also note that the bound in (53) is tighter than Duffield's corresponding bound (3) when $c - \rho \le \alpha/(\alpha - 1)$; otherwise Duffield's is tighter.

(ii) G is Weibull. From (37) and (44) we get

$$\limsup_{x\to\infty} \frac{1}{x^\nu} \log P(Q > x) \le -1. \tag{54}$$

(iii) G is lognormal. From (40) and (44) we get

$$\limsup_{x\to\infty} \frac{1}{(\log x)^2} \log P(Q > x) \le -\frac{1}{2\delta^2}. \tag{55}$$

We observe from (36), (53) and (41), (55) that the bounds are tight when $c - \rho < 1$:

Corollary 4.1 *Assume that $c - \rho < 1$. If G is Pareto then*

$$\lim_{x\to\infty} \frac{1}{\log x} \log P(Q > x) = -\alpha + 1 \tag{56}$$

and if G is lognormal then

$$\lim_{x\to\infty} \frac{1}{(\log x)^2} \log P(Q > x) = -\frac{1}{2\delta^2}. \tag{57}$$

5 Concluding Remarks

We conclude this paper by addressing the situation when the multiplexer is fed by N independent M/G/∞ input processes, with arrival rate λ_i and p.d.f. of the service times G^i for the system i ($i = 1, 2, \ldots, N$). Because the arrivals are Poisson this is equivalent to considering a single M/G/∞ queueing system with arrival intensity $\lambda := \sum_{i=1}^N \lambda_i$ and p.d.f. G of the service time given by $G(x) = \sum_{i=1}^N (\lambda_i/\lambda)\, G^i(x)$. All of the results in the paper therefore apply to this pair (λ, G). Of particular interest is the case when one p.d.f. of the service times, say G^1, has a heavier tail than the others, namely, $\overline{G^i}(x) = o(\overline{G^1}(x))$ for all

$i = 2, 3, \ldots, N$. Then, $\overline{G_1}(x) \sim (\lambda_1/\lambda)\,\overline{G_1^1}(x)$ and we may conclude from the results in Sections 3-4 that the source with the heaviest tail dominates the other sources. In particular, we see from (11) and (44) that

$$-\theta_1 \leq \liminf_{x\to\infty} \frac{\log P(Q > x)}{-\log \overline{G_1^1}(x)} \leq \limsup_{x\to\infty} \frac{\log P(Q > x)}{-\log \overline{G_1^1}(x)} \leq -1$$

where the upper bound holds if $G^1, G_1^1 \in \mathcal{S}$, with $\theta_1 := \inf_{\beta>0}\left\{h(\beta)\,\limsup_{x\to\infty} \dfrac{\log \overline{G_1^1}(x)}{\log \overline{G_1^1}(\beta x)}\right\}$, $h(\beta) := \lfloor c - \rho + \beta \rfloor + 1$ and $\rho = \sum_{i=1}^N (\lambda_i/\lambda)\int_0^\infty x\, G^i(dx)$.

A Appendix

Lemma A.1 *Assume that $G_1 \in \mathcal{S}$. Then, for every $k = 1, 2, \ldots$, $l \geq k$, and $0 < \theta \leq 1$,*

$$P\left(\sum_{j=1}^{k} \min(\hat\sigma_j, x) \geq \theta x \,\middle|\, b^0 = l\right) \sim k\,\overline{G_1}(\theta x). \tag{58}$$

Proof. Clearly, for all $k = 1, 2, \ldots$, $l \geq k$,

$$\limsup_{x\to\infty} \frac{P\left(\sum_{j=1}^k \min(\hat\sigma_j, x) \geq \theta x \,\middle|\, b^0 = l\right)}{\overline{G_1}(\theta x)} \leq \limsup_{x\to\infty} \frac{P\left(\sum_{j=1}^k \hat\sigma_j \geq \theta x \,\middle|\, b^0 = l\right)}{\overline{G_1}(\theta x)} = k$$

from Lemma 4.1(a) and Lemma 2.1(iii).

Let us now show that

$$\liminf_{x\to\infty} \frac{P\left(\sum_{j=1}^k \min(\hat\sigma_j, x) \geq \theta x \,\middle|\, b^0 = l\right)}{\overline{G_1}(\theta x)} \geq k \tag{59}$$

for all $k = 1, 2, \ldots$, $l \geq k$, which will conclude the proof.

Inequality (59) is true for $k = 1$, $l \geq 1$, since $P(\min(\hat\sigma_1, x) \geq \theta x \,|\, b^0 = l) = P(\hat\sigma_1 \geq \theta x \,|\, b^0 = l) = \overline{G_1}(\theta x)$ from Lemma 2.1(iii). Assume that (59) is true for $k = 1, 2, \ldots, n-1$, $l \geq k$, and let us show that it still holds for $k = n$, $l \geq k$.

We have

$$\begin{aligned} P\left(\sum_{j=1}^{n} \min(\hat\sigma_j, x) \geq \theta x \,\middle|\, b^0 = l\right) &\geq P\left(\sum_{j=1}^{n-1} \min(\hat\sigma_j, x) \geq \theta x \,\middle|\, b^0 = l\right) \\ &+ P(\min(\hat\sigma_n, x) \geq \theta x \,|\, b^0 = l) - P\left(\sum_{j=1}^{n-1} \min(\hat\sigma_j, x) \geq \theta x,\ \min(\hat\sigma_n, x) \geq \theta x \,\middle|\, b^0 = l\right) \end{aligned}$$

$$\geq \quad P\left(\sum_{j=1}^{n-1}\min(\hat{\sigma}_j, x) \geq \theta x \,\middle|\, b^0 = l\right) + \overline{G_1}(\theta x) - P\left(\sum_{j=1}^{n-1}\min(\hat{\sigma}_j, x) \geq \theta x \,\middle|\, b^0 = l\right)\overline{G_1}(\theta x) \tag{60}$$

where (60) follows from the conditional independence of the r.v.'s $\{\hat{\sigma}_j\}_j$ given b^0. The inequality (59) (with $k = n$) now follows from the induction hypothesis together with

$$\lim_{x\to\infty} P\left(\sum_{j=1}^{n-1}\min(\hat{\sigma}_j, x) \geq \theta x \,\middle|\, b^0 = l\right) = 0.$$

■

Lemma A.2 (Feller [17], Bingham et al. [4, Corollary 2.0.6])

Let F be a probability distribution. If $\liminf_{x\to\infty} \overline{F}(x)/\overline{F}(\delta_0 x) > 0$ *for some* $\delta_0 \in (0,1)$ *then* $\liminf_{x\to\infty} \overline{F}(x)/\overline{F}(\delta x) > 0$ *for all* $\delta \in (0,1)$.

As a consequence, $F \in \mathcal{D}$ *if and only if* $\liminf_{x\to\infty} \overline{F}(x)/\overline{F}(\delta x) > 0$ *for all* $\delta > 0$.

We give below a proof of this result for the sake of completeness.

Proof. Assume that there exists $\delta_0 \in (0,1)$ such that $\liminf_{x\to\infty} \overline{F}(x)/\overline{F}(\delta_0 x) > 0$. Fix $\delta > 0$. Since $\delta_0 < 1$ there exists $n \geq 1$ such that $\delta/\delta_0^n > 1$. From

$$\frac{\overline{F}(x)}{\overline{F}(\delta x)} = \left(\prod_{i=1}^{n} \frac{\overline{F}(\delta_0^{i-1} x)}{\overline{F}(\delta_0^{i} x)}\right)\frac{\overline{F}(\delta_0^n x)}{\overline{F}(\delta x)}$$

we readily deduce that

$$\liminf_{x\to\infty} \frac{\overline{F}(x)}{\overline{F}(\delta x)} \geq \left(\liminf_{x\to\infty} \frac{\overline{F}(x)}{\overline{F}(\delta_0 x)}\right)^n \liminf_{x\to\infty} \frac{\overline{F}(x)}{\overline{F}(\delta/\delta_0^n x)}. \tag{61}$$

The first factor in the r.h.s. of (61) is strictly positive from the definition of δ_0; the second factor too since $\overline{F}(x) \geq \overline{F}(\delta/\delta_0^n x)$ for all x. This proves the first statement.

Let us now prove the second statement. The "$\Leftarrow$" part is clearly true (take $\delta = 1/2$). The "$\Rightarrow$" part follows from the first statement by taking $\delta_0 = 1/2$. ■

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