Rare Events for Stationary Processes

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Abstract: Kielson (1979) and Aldous (1989) have given expressions for the asymptotics of the mean time until a rare event occurs. Here we extend these results beyond the Markovian setting using the theory for stationary point processes. We introduce two notions of asymptotic exponentiality and asymptotic independence and we study their implications on the asymptotics of the mean value of this hitting time under various probability measures.

Key-words: Rare event, stationary point process, exchange formula. *AMS 1980 subject classifications*: Primary 60G55; Secondary 60K25.

(Résumé : tsvp)

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Evénements Rares de Processus Stationnaires

Résumé : Kielson (1979) et Aldous (1989) ont donné des expressions pour le comportement asymptotique du temps moyen jusqu'à l'arrivée d'un événement rare. Dans cet article, nous généralisons ces résultats initialement prouvés dans un cadre markovien au cadre stationnaire au moyen de la théorie des processus ponctuels. Deux notions d'indépendance et d'exponentialité asymptotiques sont introduites qui permettent d'étudier le comportement asymptotique de ce temps moyen sous diverses lois de probabilité.

Mots-clé : Evénement rare, processus ponctuel stationnaire, formule d'échange. *Classification AMS 1980*: Primaire 60G55; Secondaire 60K25.

1 Introduction

Let X_t denote a discrete time Markov chain on a countable state space with stationary probability measure π and let α denote some fixed point called the origin. Define $R \geq 1$ to be the first return time to α having first made an excursion to a rarely visited set F (the *out of control set*), which does not contain the origin. Usually R will be composed of many returns back to α before the long excursion to F and then a last return trip to α.

Let $\tau(F)$ denote the time to hit F. Clearly $R \geq \tau(F)$ but since it takes so long to get to F, the extra return trip back to α is asymptotically negligible in comparison. This is clear from the following result due to Keilson (1979), where E_x denotes the law of the chain originating at point x.

Proposition 1

$$\lim_{\pi(F)\to 0} \frac{E_\alpha R}{E_\alpha \tau(F)} = 1. \tag{1.1}$$

There is also a simple expression for $E_\alpha R$ which is implicit in B17, Aldous (1989).

Proposition 2 *Let $f(x)$ denote the probability that, starting from $x \in F^c$, X hits α before F; that is*

$$f(x) = P_x(X \text{ hits } \alpha \text{ before } F) = P_x(\tau(\alpha) < \tau(F)). \tag{1.2}$$

Then

$$E_\alpha R = \left(\sum_{y\in F} \pi(y) \sum_{x\in F^c} K(y,x) f(x)\right)^{-1} = \left(\pi(\alpha) \sum_{x\neq\alpha} K(\alpha,x)(1-f(x))\right)^{-1}. \tag{1.3}$$

Proof: Consider the process

$$U_t = \chi\{X_t \in F, X_{t+k} \in F^c \text{ for } 1 \leq k \leq \tau(\alpha)\circ T_t\},$$

where T_t denotes the translation operator for the chain. The stationary mean value of U_t is given by

$$\sum_{y\in F} \pi(y) P_y(X_t \notin F \text{ for } 1 \leq t \leq \tau(\alpha))$$

(the probability $P_y(X_t \notin F$ for $1 \leq t \leq \tau(\alpha))$ is precisely $\sum_{x\in F^c} K(y,x)f(x)$).

Now consider cycles where the chain starts in α, eventually hits F and then returns to α. These cycles have the same distribution as R. The process U_t is a regenerative process for these cycles. Since U_t is equal to one exactly once per cycle, the stationary mean value of U_t is also equal to $1/E_\alpha R$. This gives the first equality. The second equality follows in the same way by analysing the point process

$$V_t = \chi\{X_t = \alpha, X_{t+k} \neq \alpha \text{ for } 1 \leq k \leq \tau(F)\circ T_t\}.$$

■

Combining Propositions 1 and 2 we get

Theorem 1

$$\lim_{\pi(F)\to 0} \Lambda E_\alpha \tau(F) = 1, \text{ where } \Lambda := \left(\sum_{y\in F} \pi(y) \sum_{x\in F^c} K(y,x) f(x) \right)$$

and where f is defined in Proposition 2.

The extension of these results to stationary processes (see Theorems 2, 3 and 4) and the application to Markov chains on a general state space (see Theorem 5 and Corollaries 3 and 4) are the main results of this paper.

2 Rare events for stationary processes

Let $(\Omega, \mathcal{F}, P)$ be a measurable space and let θ_t be a measure preserving flow on $(\Omega, \mathcal{F}, P)$. Let $\{X_t\}$ be a stochastic process defined on $(\Omega, \mathcal{F}, P)$, taking its values in some measurable space $(S, \mathcal{S})$. We assume that $\{X_t\}$ is θ_t-compatible, i.e. such that $X_t(\omega) = X_0(\theta_t(\omega))$, for all t, which implies its stationarity.

Let A be a set in $\mathcal{S}$, such that the $\{0,1\}$-valued, θ_t-compatible stochastic process $\chi(X_t \in A)$ is a.s. continuous, but for a denumerable set of discontinuities which admit no accumulation points. Then the discontinuity points of $\chi(X_t \in A)$ are of two types: entrance times into A (an entrance time into A is a time t such that $X_{t-\epsilon} \notin A$ for all $0 < \epsilon < a$, whereas $X_{t+\epsilon} \in A$ for all $0 < \epsilon < a$, for some $a > 0$ which may depend on ω and t) and exit times out of A (defined analogously).

Let $\{T_n^{\to A}\}$ denote the (non-decreasing) sequence of successive entrance times into A, and let $\{T_n^{A\to}\}$ denote the exit times out of A, with the convention that $T_0^H \leq 0$ and $T_1^H > 0$, where H is either $\{A \to\}$ or $\{\to A\}$.

The point processes $N^{\to A}$ with points $\{T_n^{\to A}\}$ and $N^{A\to}$ with points $\{T_n^{A\to}\}$, are both simple (i.e. a.s. without double points) and θ_t-compatible on $(\Omega, \mathcal{F}, P)$. These point processes may or may not have a finite intensity. Most often, our stochastic process X_t will be a continuous time, càd làg process with values in some topological space. In the discrete time case, we will assume that $X_{T_0^{\to A}}$ takes its value in A whereas $X_{T_0^{A\to}}$ takes its value in A^c, with similar definitions for other sets or point processes.

A set $A \in \mathcal{S}$ will be said to be *regular* for $\{X_t\}$ if the two key assumptions below are satisfied:

- the stochastic process $\chi(X_t \in A)$ is a.s. continuous, but for a denumerable set of discontinuities without accumulations;
- the point processes $N^{\to A}$ and $N^{A\to}$ both have a finite and positive intensity.

For any set A let

$$\tau(A) = \inf\{t > 0, X_t \in A\}. \tag{2.4}$$

Let A and F be two disjoint regular sets. Set A plays the same role as point α in the countable state space Markov case. We will consider several thinnings of the above stationary point processes defined as follows:

- The points of $N^{(A\rightarrow)F}$ consist of the subsequence of the exits $\{T_n^{A\rightarrow}\}$ out of A, after which $\{X_t\}$ hits F before A, namely we only keep the indices n such that
$$\tau(F) \circ \theta_{T_n^{A\rightarrow}} < \tau(A) \circ \theta_{T_n^{A\rightarrow}} = T_1^{\rightarrow A} \circ \theta_{T_n^{A\rightarrow}} \ ;$$
- The points of $N^{(A\rightarrow)A}$ consist of the subsequence of the exits $\{T_n^{A\rightarrow}\}$ out of A, after which $\{X_t\}$ hits A before F;
- The points of $N^{(\rightarrow A)F}$ consist of the subsequence of the entrances $\{T_n^{\rightarrow A}\}$ into A, after which, once $\{X_t\}$ has left A, it hits F before A, namely we only keep the indices n such that
$$\tau(F) \circ \theta_{T_n^{\rightarrow A}} < T_1^{\rightarrow A} \circ \theta_{T_n^{\rightarrow A}} = T_{n+1}^{\rightarrow A} - T_n^{\rightarrow A};$$
- The points of $N^{(\rightarrow A)A}$ consist of the subsequence of the entrances $\{T_n^{\rightarrow A}\}$ into A, after which, once $\{X_t\}$ has left A, it hits A before F.
- Finally, the points of $N^{F(\rightarrow A)}$ consist of the subsequence of the first entrances $\{T_n^{\rightarrow A}\}$ into A, after $\{X_t\}$ has left F, namely we only keep the indices n such that
$$T_n^{\rightarrow A} = T_1^{\rightarrow A} \circ \theta_{T_m^{\rightarrow F}} + T_m^{\rightarrow F} \text{ for some } m.$$
This last point process is the generalization of that of the cycles considered in the Markov case in the introduction.
- etc.

Of course, one can define symmetrical thinnings by exchanging the roles of A and F. Each of these point processes, a typical realization of which is exemplified in Figure 1, is defined on $(\Omega, \mathcal{F}, P)$, is θ_t-compatible, and has a finite intensity.

In what follows, the intensity of the point process N^H (with H being for instance $\{\rightarrow A\}$ or $\{(A \rightarrow)F\}$ or any other expressions of the same nature) will be denoted λ^H. Similarly, the Palm probability of P w.r.t. the point process N^H will be denoted P_0^H, and the corresponding expectation E_0^H.

2.1 General Case

2.1.1 Intensities

Throughout the paper, we will often make use of the exchange formula of Neveu (1983), which states that if $N_1 = \{T_n^1\}$ and $N_2 = \{T_n^2\}$ are two point processes compatible with respect to the same flow θ_t, then for any measurable $f : \Omega \mapsto I\!R_+$,

$$\lambda_1 E_1^0 f(\omega) = \lambda_2 E_2^0 \int_{[0,T_1^2)} f(\theta_x \omega)\, N_1(dx) \tag{2.5}$$

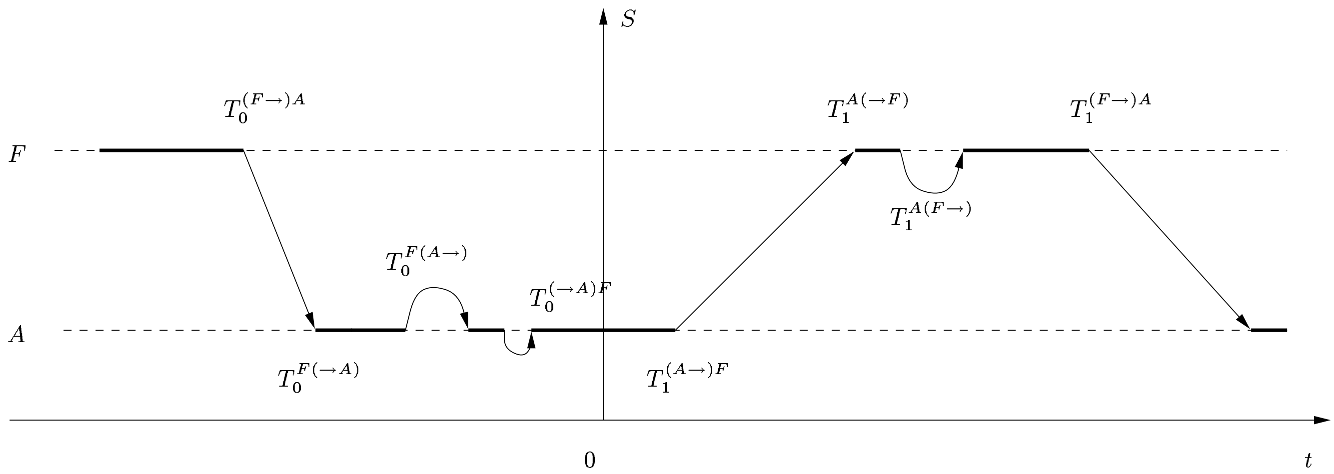

Figure 1:

(see Baccelli-Brémaud, Ch. 1 (3.4.2) for notation).

We start with a lemma which can be viewed as a stationary generalization of Formula (8.2.2) given in Theorem 8.2B in Keilson (1979).

Lemma 1 *For all regular sets A and F,*

$$\begin{aligned} E_0^{F(\to A)} T_1^{F(\to A)} &= \frac{E_0^{\to A}[T_1^{\to A}]}{P_0^{\to A}[T_{-1}^{\to A} < T_0^{\to F}]} \\ E_0^{(\to A)F} T_1^{(\to A)F} &= \frac{E_0^{A\to}[T_1^{A\to}]}{P_0^{A\to}[T_1^{A\to} > T_1^{\to F}]} . \end{aligned}$$

Proof: The exchange formula for $N_1 = N^{\to A}$ and $N_2 = N^{F(\to A)}$ gives

$$\begin{aligned} E_0^{F(\to A)}[1] &= \frac{E_0^{F(\to A)}[T_1^{F(\to A)}]}{E_0^{\to A}[T_1^{\to A}]} E_0^{\to A} \int_{[0,T_1^{\to A})} 1 \cdot N^{F(\to A)}(ds) \\ &= \frac{E_0^{F(\to A)}[T_1^{F(\to A)}]}{E_0^{\to A}[T_1^{\to A}]} P_0^{\to A}[T_{-1}^{\to A} < T_0^{\to F}]. \end{aligned}$$

This gives the first equality in the lemma. The second equality follows by the same reasoning applied to $E_0^{(\to A)F}[1]$. ∎

The above result is intuitively clear. A cycle of mean length $E_0^{(A\to)F}[T_1^{(A\to)F}]$ comprises many returns back to A before a successful excursion to F. The probability of such an excursion is $p := P_0^{A\rightarrow}[T_1^{A\rightarrow} > T_1^{\rightarrow F}]$. Hence we expect $1/p$ loops back to A each taking $E_0^{A\rightarrow}[T_1^{A\rightarrow}]$ time units. This gives the second equality in Lemma 1.

Lemma 2 *Under the foregoing assumptions*

$$\lambda^{A\rightarrow} = \lambda^{\rightarrow A}. \tag{2.6}$$

and

$$\lambda^{(F\rightarrow)A} = \lambda^{F(\rightarrow A)} = \lambda^{F(A\rightarrow)} = \lambda^{(\rightarrow A)F} = \lambda^{(A\rightarrow)F} = \lambda^{A(\rightarrow F)} = \lambda^{A(F\rightarrow)} = \lambda^{(\rightarrow F)A} \tag{2.7}$$

and the common value Λ *of the intensities in (2.7) is such that* $0 < \Lambda < \infty$.

Proof: The regularity of A imples $\lambda^{\rightarrow A} < \infty$. In order to prove the first equality (2.6), we apply the exchange formula to the point processes $N^{\rightarrow A}$ and $N^{A\rightarrow}$ and to the function $f = 1$, and we use the fact that between two consecutive points of $N^{\rightarrow A}$, there is exactly one point of $N^{A\rightarrow}$. In order to prove the first equality in (2.7), we apply the exchange formula to the point processes $N^{(F\rightarrow)A}$ and $N^{F(\rightarrow A)}$ and to the function $f = 1$, and we use the fact that between two consecutive points of $N^{F(\rightarrow A)}$, there is exactly one point of $N^{(F\rightarrow)A}$. The other equalities are obtained in the same way.

The fact that $\Lambda < \infty$ follows from the assumption that $\lambda^{\rightarrow A} < \infty$ and from the bound $\lambda^{F(\rightarrow A)} \leq \lambda^{\rightarrow A}$.

We now conclude the proof by showing that $\Lambda > 0$. Using the exchange formula for $N_1 = N^{\rightarrow F}$, $N_2 = N^{\rightarrow A}$ and $f = 1$, we obtain that

$$\lambda^{\rightarrow F} = \lambda^{\rightarrow A} E_0^{\rightarrow A}\left(N^{\rightarrow F}\left([0, T_1^{\rightarrow A})\right)\right).$$

This together with the assumptions that $\lambda^{\rightarrow F} > 0$ and $\lambda^{\rightarrow A} > 0$ (regularity) imply that

$$E_0^{\rightarrow A}\left[N^{\rightarrow F}\left([0, T_1^{\rightarrow A})\right)\right] > 0,$$

which in turn implies that

$$P_0^{\rightarrow A}\left(T_1^{\rightarrow F} < T_1^{\rightarrow A}\right) > 0.$$

From the second equality in Lemma 1, $\lambda^{(\rightarrow A)F} > 0$. ■

Remark 1 From Lemma 2, $E_0^{(A\rightarrow)F}[T_1^{(A\rightarrow)F}] = E_0^{F(\rightarrow A)}[T_1^{F(\rightarrow A)}]$ and $E_0^{A\rightarrow}[T_1^{A\rightarrow}] = E_0^{\rightarrow A}[T_1^{\rightarrow A}]$. Moreover, looking backwards in time we see

$$P_0^{\rightarrow A}[T_{-1}^{\rightarrow A} < T_0^{\rightarrow F}] = P_0^{A\rightarrow}[T_1^{A\rightarrow} > T_1^{\rightarrow F}]. \tag{2.8}$$

Therefore we see the left hand sides of the equations in Lemma 1 are both equal to Λ^{-1} and the numerator and denominator of the right hand sides are also equal.

■

2.1.2 Generalization of Keilson's asymptotic formula

For any set A let

$$\tau(A) = \inf\{t > 0, X_t \in A\}, \quad \tau^-(A) = \inf\{t > 0, X_{-t} \in A\}. \tag{2.9}$$

Theorem 2 *Let F_n be a sequence of sets of $\mathcal{S}$ which are regular w.r.t. $\{X_t\}$ and such that $\tau^-(F_n)$ tends to infinity in probability w.r.t. P as n goes to ∞. Then the following generalization of Keilson's asymptotic formula (1.1) holds:*

$$\frac{E_0^{F_n(\to A)}[\tau(F_n)]}{E_0^{F_n(\to A)}[T_1^{F_n(\to A)}]} = P[\tau^-(F_n) \geq \tau^-(A)] \to_{n\to\infty} 1, \tag{2.10}$$

so that

$$E_0^{F_n(\to A)}[\tau(F_n)] \sim \Lambda_n^{-1}.$$

Proof: The assumption that F_n and A are regular implies that the r.v.'s $T_1^{\to F_n}$ and $T_1^{F_n(\to A)}$ are integrable w.r.t. $P_0^{F_n(\to A)}$. Indeed we have

$$E_0^{F_n(\to A)}[T_1^{\to F_n}] < E_0^{F_n(\to A)}[T_1^{F_n(\to A)}] = \Lambda_n^{-1} < \infty,$$

where Λ_n is the real number defined in Lemma 2 for $F = F_n$. The time interval between $T_1^{\to F_n}$ and $T_1^{F_n(\to A)}$ viewed backwards in time gives

$$\begin{aligned} E_0^{F_n(\to A)}[T_1^{F_n(\to A)} - T_1^{\to F_n}] &= E_0^{F_n(\to A)} \int_0^{T_1^{F_n(\to A)}} \chi(\tau^-(F_n)\circ\theta_s < \tau^-(A)\circ\theta_s)ds \\ &= E_0^{F_n(\to A)}[T_1^{F_n(\to A)}] P[\tau^-(F_n) < \tau^-(A)], \end{aligned}$$

where the last equality follows from the Ryll-Nardzewski inversion formula. Consequently

$$\frac{E_0^{F_n(\to A)}[T_1^{F_n(\to A)} - T_1^{\to F_n}]}{E_0^{F_n(\to A)}[T_1^{F_n(\to A)}]} = P[\tau^-(F_n) < \tau^-(A)].$$

The proof is now concluded by using our assumption on the limiting behavior of $\tau^-(F_n)$ and the following inequality which holds for all positive real numbers c:

$$P[\tau^-(F_n) < \tau^-(A)] \leq P[\tau^-(F_n) < c] + P[\tau^-(A) > c].$$

■

Remark 2 The fact that

$$E_0^{F_n(\to A)}[T_1^{F_n(\to A)}] = (\Lambda_n)^{-1} = \left(\lambda_n^{(F_n\to)A}\right)^{-1},$$

which follows from Lemma 2, may be seen as a generalization of the expression obtained by Aldous in the Markov case which was recalled in Proposition 2.

■

Remark 3 It is easy to see that the assumptions

1. $\tau^{-}(F_n)$ tends to infinity in probability w.r.t. P as n goes to ∞ and
2. $\tau(F_n)$ tends to infinity in probability w.r.t. P as n goes to ∞,

are equivalent, so that Assumption 1, which is used in Theorem 2 can actually be replaced by Assumption 2. In order to prove this equivalence, note that 1. implies that

$$P[X(0) \in F_n] = P[\tau^{-}(F_n) = 0] \to 0 \text{ and that } P[T_0^{\to F_n} > -c] \leq P[\tau^{-}(F_n) < c] \to 0,$$

for all $c > 0$ as n goes to ∞. Using the fact that $-T_0^{\to F_n}$ and $T_1^{\to F_n}$ have the same law under P (see Baccelli-Brémaud, Ch. 1, §4.2), we get

$$\begin{aligned}
P[\tau(F_n) < c] &= P[\tau(F_n) < c, X(0) \in F_n] + P[\tau(F_n) < c, X(0) \notin F_n] \\
&= P[\tau(F_n) < c, X(0) \in F_n] + P[T_1^{\to F_n} < c, X(0) \notin F_n] \\
&= P[\tau(F_n) > c, X(0) \in F_n] - P[T_1^{\to F_n} < c, X(0) \in F_n] + P[T_1^{\to F_n} < c] \\
&= P[\tau(F_n) < c, X(0) \in F_n] - P[T_1^{\to F_n} < c, X(0) \in F_n] + P[T_0^{\to F_n} > -c] \\
&\leq P[X(0) \in F_n] + P[T_0^{\to F_n} > -c].
\end{aligned}$$

Therefore, under 1., $P[\tau(F_n) < c]$ tends to 0 for all $c > 0$ as n goes to ∞. The proof of the converse implication is similar.

■

Remark 4 Note that by the same type of arguments,

$$\begin{aligned}
E_0^{F_n(\to A)}[T_1^{(F_n\to)A} - T_1^{(A\to)F_n}] &= E_0^{F_n(\to A)} \int_0^{T_1^{F_n(\to A)}} \chi(\tau(F_n)\circ\theta_s < \tau(A)\circ\theta_s)ds \\
&= E_0^{F_n(\to A)}[T_1^{F_n(\to A)}]P[\tau(F_n) < \tau(A)].
\end{aligned}$$

Consequently

$$\frac{E_0^{F_n(\to A)}[T_1^{(F_n\to)A} - T_1^{(A\to)F_n}]}{E_0^{F_n(\to A)}[T_1^{F_n(\to A)}]} = P[\tau(F_n) < \tau(A)].$$

Again our assumption on the limiting behavior of $\tau^{-}(F_n)$ means the above ratio tends to 0. By Theorem 2 and the above calculation, we see that the expected time between $T_1^{(A\to)F_n}$ and $T_1^{F_n(\to A)}$ is negligable in the expected length of the round trip, $E_0^{F_n(\to A)}[T_1^{F_n(\to A)}]$.

■

Remark 5 Using similar arguments, it is easy to prove a set of asymptotic relations like for instance

$$\lim_{n\to\infty} \frac{E_0^{(F_n\to)A}[T_1^{\to F_n}]}{E_0^{(F_n\to)A}[T_1^{(F_n\to)A}]} = \lim_{n\to\infty} \frac{E_0^{(F_n\to)A}[\tau(F_n)]}{E_0^{(F_n\to)A}[T_1^{(F_n\to)A}]} = 1. \tag{2.11}$$

■

Remark 6 If the rare visits to F_n correspond to some undesired overload of a system then the above expectation $E_0^{F_n(\to A)}[T_1^{\to F_n}]$ represents the mean time until the next overload starting from the moment we have just recovered from the last one. See Meyn and Frater (1993) for a related discussion.

■

Remark 7 Note that (2.10) can also be viewed as a consequence of Little's law. It suffices to define $B := \{\tau^-(F_n) \geq \tau^-(A)\}$ and recall that $P(B) = \Lambda_n E_0^{F(\to A)}[W]$, where W is the waiting time in B.

■

Remark 8 Let X_t be as above. Define $Y_t = \{X_u(\theta_t(\omega))\}_{u\in\mathbb{R}}$, with values in the space of trajectories in $\mathcal{S}$ indexed by time u. The stochastic process $\{Y_t\}$ is also stationary, and the above framework can be used directly to obtain asymptotic properties of rare events for $\{Y_t\}$. In a Markovian setting, this allows one to handle the asymptotics of complex stopping times like those investigated in the paper by Kook and Serfozo (1993).

■

2.2 Asymptotic Exponentiality

Given (2.10), a natural question is whether the same result also holds when replacing the expectation w.r.t. the Palm measure $P_0^{F_n(\to A)}$ by the expectation w.r.t. P. It turns out that this is not always true; Theorem 3 below gives a first set of sufficient conditions for this to happen. We start with a preliminary result which gives a cycle representation of $E[\tau(F_n)]$ w.r.t. $E_0^{(F_n\to)A}$.

Lemma 3 *Under the foregoing assumptions, we have* $E[\tau(F_n)] = S_1 + S_2$ *with*

$$S_1 = \frac{E_0^{(F_n\to)A}\left[\left(T_1^{\to F_n}\right)^2\right]}{2\; E_0^{(F_n\to)A}\left[T_1^{(F_n\to)A}\right]}, \tag{2.12}$$

$$S_2 = \frac{1}{2}\lambda^{(F_n\to)F_n}\; E_0^{(F_n\to)F_n}\left[(T_1^{\to F_n})^2\right]. \tag{2.13}$$

where both sides of the equality $E[\tau(F_n)] = S_1 + S_2$ *may be finite or infinite.*

Proof: The Ryll-Nardzewski inversion formula gives

$$E[\tau(F_n)] = \frac{E_0^{F_n(\to A)}\left[\int_0^{T_1^{F_n(\to A)}} \tau(F_n)\circ\theta_s ds\right]}{E_0^{F_n(\to A)}[T_1^{F_n(\to A)}]} = S_1 + S_2 \ , \tag{2.14}$$

with

$$\begin{aligned} S_1 &= \frac{E_0^{F_n(\to A)}\left[\int_0^{T_1^{\to F_n}} \tau(F_n)\circ\theta_s ds + \int_{T_1^{(F_n\to)A}}^{T_1^{F_n(\to A)}} \tau(F_n)\circ\theta_s ds\right]}{E_0^{F_n(\to A)}[T_1^{F_n(\to A)}]} \\ S_2 &= \frac{E_0^{F_n(\to A)}\left[\int_{T_1^{\to F_n}}^{T_1^{(F_n\to)A}} \tau(F_n)\circ\theta_s ds\right]}{E_0^{F_n(\to A)}[T_1^{F_n(\to A)}]} \ . \end{aligned}$$

Let n_i denote the numerator of the expression defining S_i above.

We decompose the integral in n_2 into a sum of integrals corresponding to sojourns outside F_n which return to F_n without hitting A. We have

$$\begin{aligned} n_2 &= E_0^{F_n(\to A)}\left[\int_0^{T_1^{F_n(\to A)}} \chi(\tau(F_n)\circ\theta_s < \tau(A)\circ\theta_s)\frac{(T_1^{\to F_n}\circ\theta_s)^2}{2} N^{F_n\to}(ds)\right] \\ &= \frac{\lambda^{F_n\to}}{\Lambda_n} E_0^{F_n\to}\left[\frac{(\tau(F_n))^2}{2}\chi(\tau(F_n) < \tau(A))\right] , \end{aligned}$$

where we used the exchange formula between $P_0^{F_n(\to A)}$ and $P_0^{F_n\to}$ to get the last expression. Now using the relation

$$E_0^{(F_n\to)F_n}[U] = \frac{E_0^{F_n\to}[U\chi(\tau(F_n) < \tau(A))]}{P_0^{F_n\to}[\tau(F_n) < \tau(A)]}$$

(see Baccelli-Brémaud, Ch. 1 (5.2.3)), we finally obtain that

$$\begin{aligned} S_2 &= \frac{1}{2}\lambda^{F_n\to}\ P_0^{F_n\to}[\tau(F_n) < \tau(A)])E_0^{(F_n\to)F_n}\ \left[(\tau(F_n))^2\right] \\ &= \frac{1}{2}\lambda^{(F_n\to)F_n}\ E_0^{(F_n\to)F_n}\ \left[(\tau(F_n))^2\right] , \end{aligned}$$

where we used the exchange formula to show

$$\lambda^{(F_n\to)F_n} = \lambda^{F_n\to}\ P_0^{F_n\to}[\tau(F_n) < \tau(A)]). \tag{2.15}$$

As for S_1, we can rewrite

$$n_1 = E_0^{F_n(\to A)}\left[\int_{T_0^{(F_n\to)A}}^{T_1^{\to F_n}}\left(T_1^{\to F_n}-s\right)ds\right] = \frac{1}{2}E_0^{F_n(\to A)}\left[\left(T_1^{\to F_n}-T_0^{(F_n\to)A}\right)^2\right],$$

so that

$$S_1 = \frac{1}{2}\frac{E_0^{F_n(\to A)}\left[\left(T_1^{\to F_n}-T_0^{(F_n\to)A}\right)^2\right]}{E_0^{F_n(\to A)}\left[T_1^{F_n(\to A)}\right]} = \frac{1}{2}\frac{E_0^{(F_n\to)A}\left[\left(T_1^{\to F_n}\right)^2\right]}{E_0^{(F_n\to)A}\left[T_1^{(F_n\to)A}\right]},$$

where the last equality was obtained by using Lemma 2 to rewrite the denominator and the exchange formula between $E_0^{F_n(\to A)}$ and $E_0^{(F_n\to)A}$ to rewrite the numerator. ■

Theorem 3 *The assumptions are those of Theorem 2. In addition, we assume that the excursions of the process out of F_n which hit F_n before hitting A, are square integrable and such that the following rate condition*

$$\Lambda_n\ \lambda^{(F_n\to)F_n}\ E_0^{(F_n\to)F_n}\left[(\tau(F_n))^2\right] = o(1) \tag{2.16}$$

holds. In addition, we assume that the hitting time of F_n is asymptotically exponential in variance *w.r.t.* $P_0^{(F_n\to)A}$ *i.e.*

$$\lim_{n\to\infty}\frac{E_0^{(F_n\to)A}\left[(\tau(F_n))^2\right]}{\left(E_0^{(F_n\to)A}[\tau(F_n)]\right)^2} = 2. \tag{2.17}$$

Then we also have

$$E[\tau(F_n)] \sim \Lambda_n^{-1}. \tag{2.18}$$

Proof: We use the representation of $E[\tau(F_n)]$ given in Lemma 3. From (2.13), we have

$$\Lambda_n S_2 \quad = \quad \frac{1}{2}\lambda_n\ \lambda^{(F_n\to)F_n}\ E_0^{(F_n\to)F_n}\left[(\tau(F_n))^2\right]$$

which shows that the limiting value of $\Lambda_n S_2$ is zero in view of (2.16).

From (2.12)

$$\Lambda_n S_1 = \frac{1}{2}\frac{E_0^{(F_n\to)A}\left[\left(T_1^{\to F_n}\right)^2\right]}{\left(E_0^{(F_n\to)A}\left[T_1^{(F_n\to)A}\right]\right)^2},$$

so that

$$\lim_{n\to\infty} \Lambda_n E[\tau(F_n)] = \lim_{n\to\infty} \frac{1}{2} \frac{E_0^{(F_n\to)A}\left[\left(T_1^{\to F_n}\right)^2\right]}{\left(E_0^{(F_n\to)A}\left[T_1^{\to F_n}\right]\right)^2} = 1$$

as a direct consequence of (2.10) and (2.17). ■

Remark 9 Using (2.15) we see that a sufficient condition for (2.16) to hold is that

$$E_0^{(F_n\to)F_n}[(\tau(F_n))^2] < \infty, \quad \forall n \tag{2.19}$$

and

$$\lim_{n\to\infty} \left(\lambda^{\to F_n}\right)^2 E_0^{(F_n\to)F_n}[(\tau(F_n))^2] = 0. \tag{2.20}$$

This equation means that the 'rarity' of the entrances into F_n should dominate the possible growth of the length of the excursions from F_n back to F_n, when n tends to ∞.

■

Remark 10 In case the above asymptotic exponentiality does not hold, under (2.16), we still have the general property that

$$\lim_{n\to\infty} \Lambda_n E[T_1^{\to F_n}] = \lim_{n\to\infty} \frac{1}{2} \frac{E_0^{F_n(\to A)}\left[\left(T_1^{\to F_n}\right)^2\right]}{\left(E_0^{F_n(\to A)}\left[T_1^{\to F_n}\right]\right)^2} = \lim_{n\to\infty} \frac{1}{2} \frac{E_0^{F_n(\to A)}\left[(\tau(F_n))^2\right]}{\left(E_0^{F_n(\to A)}\left[\tau(F_n)\right]\right)^2}, \tag{2.21}$$

provided the last limit exists. In fact, we can say more: under the assumptions of Theorem 2 and the rate condition (2.16), the relation

$$\Lambda_n E[T_1^{\to F_n}] \sim 1 \tag{2.22}$$

is *equivalent* to the property that the hitting time of F_n is asymptotically exponential in variance.

■

2.3 Asymptotic Independence

The following lemma gives the first instance of another set of sufficient conditions for asymptotic formulas properties to hold based on asymptotic independence.

Lemma 4 *Assume that $E_0^{\to A}[T_1^{\to F_n}] < \infty$ and that*

$$E_0^{\to A}\left[T_1^{\to F_n}\chi\{T_0^{F_n\to} > T_{-1}^{\to A}\}\right] \quad\sim\quad E_0^{\to A}\left[T_1^{\to F_n}\right] P_0^{\to A}\left[T_0^{\to F_n} > T_{-1}^{\to A}\right]. \quad (2.23)$$

Then

$$E_0^{F_n(\to A)}\left[\tau(F_n)\right] \sim E_0^{\to A}\left[\tau(F_n)\right]. \quad (2.24)$$

Proof: Immediate from the relation

$$E_0^{F_n(\to A)}[\tau(F_n)] \quad = \quad \frac{E_0^{\to A}[\tau(F_n)\chi\{T_0^{F_n\to} > T_{-1}^{\to A}\}]}{P_0^{\to A}[T_0^{F_n\to} > T_{-1}^{\to A}]}.$$

■

In what follows, we will need another cycle representation of $E[\tau(F_n)]$, this time w.r.t. $E_0^{A\to}$.

Lemma 5 *Under the foregoing stationarity assumptions, we have*

$$E[\tau(F_n)\chi\{X_0 \in A\}] = \frac{E_0^{A\to}[|T_0^{\to A}| \cdot T_1^{\to F_n}]}{E_0^{A\to}[T_1^{A\to}]} + \frac{E_0^{A\to}[(T_0^{A\to})^2]}{2\, E_0^{A\to}[T_1^{A\to}]}, \quad (2.25)$$

where again both sides may either be finite or infinite. Let

$$J = \frac{1}{E_0^{A\to}[T_1^{A\to}]} E_0^{A\to} \int_{T_{-1}^{A\to}}^{T_0^{\to A}} (T_1^{\to F_n} - s)ds. \quad (2.26)$$

If $J < \infty$, then $E[\tau(F_n)\chi\{X_0 \notin A\}] < \infty$ and

$$E[\tau(F_n)\chi\{X_0 \notin A\}] = J - U_1 - U_2 - V_1 - V_2, \quad (2.27)$$

where the constants U_i and V_i are positive and finite and are defined as follows

$$U_1 E_0^{A\to}[T_1^{A\to}] = E_0^{A\to}\left[T_1^{\to F_n}\left(\sum_{p=\gamma}^{0}(T_p^{F_n\to} - T_p^{\to F_n})\right)\chi\{T_{-1}^{A\to} < T_0^{F_n\to}\}\right] \quad (2.28)$$

$$U_2 2\, E_0^{A\to}[T_1^{A\to}] = E_0^{A\to}\left[\sum_{p=\gamma}^{0}\left((T_p^{\to F_n})^2 - (T_p^{F_n\to})^2\right)\chi\{T_{-1}^{A\to} < T_0^{F_n\to}\}\right] \quad (2.29)$$

$$V_1 E_0^{A\to}[T_1^{A\to}] = E_0^{A\to}\left[T_1^{\to F_n}\left(T_\gamma^{\to F_n} - T_{-1}^{A\to} + \sum_{p=\gamma+1}^{0}(T_p^{\to F_n} - T_{p-1}^{F_n\to})\right)\chi\{T_{-1}^{A\to} < T_0^{F_n\to}\}\right] \tag{2.30}$$

$$V_2 E_0^{A\to}[T_1^{A\to}] = E_0^{A\to}\left[\left(|T_\gamma^{\to F_n}|(T_\gamma^{\to F_n} - T_{-1}^{A\to}) + \sum_{p=\gamma+1}^{0}|T_p^{\to F_n}|(T_p^{\to F_n} - T_{p-1}^{F_n\to})\right)\right. \tag{2.31}$$
$$\left.\chi\{T_{-1}^{A\to} < T_0^{F_n\to}\}\right].$$

Here $\gamma \leq 0$ denotes the index of the first point of $N^{\to F_n}$ in the interval $[T_{-1}^{A\to}, 0]$, on the event $\{T_{-1}^{A\to} < T_0^{F_n\to}\}$.

Proof: The first relation follows from the Ryll-Nardzewski inversion formula:

$$\begin{aligned}
E[T_1^{\to F_n}\cdot\chi\{X_0 \in A\}] &= \frac{1}{E_0^{A\to}[T_1^{A\to}]}E_0^{A\to}[\int_{T_{-1}^{A\to}}^{0} T_1^{\to F_n}\circ\theta_s\cdot\chi\{X_s \in A\}ds] \\
&= \frac{1}{E_0^{A\to}[T_1^{A\to}]}E_0^{A\to}[\int_{T_0^{\to A}}^{0}(T_1^{\to F_n} - s)ds] \\
&= \frac{1}{E_0^{A\to}[T_1^{A\to}]}E_0^{A\to}[-T_0^{\to A}\cdot T_1^{\to F_n} + \frac{1}{2}(T_0^{A\to})^2]\ .
\end{aligned}$$

Similarly, for the second relation, the Ryll-Nardzewski inversion formula gives

$$\begin{aligned}
E[\tau(F_n)\chi\{X_0 \notin A\}] &= \frac{1}{E_0^{A\to}[T_1^{A\to}]}E_0^{A\to}[\int_{T_{-1}^{A\to}}^{T_0^{\to A}}\tau(F_n)\circ\theta_s ds] \\
&= \frac{1}{E_0^{A\to}[T_1^{A\to}]}E_0^{A\to}\left[\int_{T_{-1}^{A\to}}^{T_0^{\to A}}(T_1^{\to F_n} - s)ds\chi\{T_{-1}^{A\to} > T_0^{F_n\to}\}\right] \\
&\quad + \frac{1}{E_0^{A\to}[T_1^{A\to}]}E_0^{A\to}\left[\int_{T_{-1}^{A\to}}^{T_0^{\to A}}\tau(F_n)\circ\theta_s ds\chi\{T_{-1}^{A\to} < T_0^{F_n\to}\}\right] \\
&= X + Y.
\end{aligned} \tag{2.32}$$

First, observe that $\tau(F_n)\circ\theta_s \leq T_1^{\to F_n} - s$ at any point s in the above integrals, so the finiteness assumption on J implies that $X < \infty$ and $Y < \infty$. We now rewrite Y as follows:

$$\begin{aligned}
Y &= \frac{1}{E_0^{A\to}[T_1^{A\to}]}E_0^{A\to}\left[\int_{T_0^{F_n\to}}^{T_0^{\to A}}(T_1^{\to F_n} - s)ds\chi\{T_{-1}^{A\to} < T_0^{F_n\to}\}\right] \\
&\quad + \frac{1}{E_0^{A\to}[T_1^{A\to}]}E_0^{A\to}\left[\int_{T_{-1}^{A\to}}^{T_\gamma^{\to F_n}}(T_\gamma^{\to F_n} - s)ds\chi\{T_{-1}^{A\to} < T_0^{F_n\to}\}\right]
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{E_0^{A\rightarrow}[T_1^{A\rightarrow}]}E_0^{A\rightarrow}\left[\sum_{p=\gamma+1}^{0}\int_{T_{p-1}^{F_n\rightarrow}}^{T_p^{\rightarrow F_n}}(T_p^{\rightarrow F_n}-s)ds\chi\{T_{-1}^{A\rightarrow}<T_0^{F_n\rightarrow}\}\right]\\
=\;& Y_1+Y_2+Y_3,
\end{aligned}
$$

where all three terms are necessarily finite. The quantity I defined as follows:

$$
I=\frac{1}{E_0^{A\rightarrow}[T_1^{A\rightarrow}]}E_0^{A\rightarrow}\left[\int_{T_{-1}^{A\rightarrow}}^{T_0^{F_n\rightarrow}}(T_1^{\rightarrow F_n}-s)ds\chi\{T_{-1}^{A\rightarrow}<T_0^{F_n\rightarrow}\}\right].
$$

This quantity is finite in view of our assumption on J. When adding and subtracting I to Y, this allows one to represent Y as

$$
Y\;=\;\frac{1}{E_0^{A\rightarrow}[T_1^{A\rightarrow}]}E_0^{A\rightarrow}\left[\int_{T_{-1}^{A\rightarrow}}^{T_0^{\rightarrow A}}(T_1^{\rightarrow F_n}-s)ds\chi\{T_{-1}^{A\rightarrow}<T_0^{F_n\rightarrow}\}\right]-Z,
$$

where the correction term Z is finite and equal to $I-Y_2-Y_3$. So in view of (2.33), (2.27) will be proved if we can show that $Z=U_1+U_2+V_1+V_2$.

We first consider the sum V of all the integrals in Z on the intervals $[T_{-1}^{A\rightarrow},T_\gamma^{\rightarrow F_n}]$ and $[T_{p-1}^{F_n\rightarrow},T_p^{\rightarrow F_n}]$, $p=\gamma+1,\ldots,0$. This gives the following result:

$$
\begin{aligned}
V\;=\;&\frac{1}{E_0^{A\rightarrow}[T_1^{A\rightarrow}]}E_0^{A\rightarrow}\left[\int_{T_{-1}^{A\rightarrow}}^{T_\gamma^{\rightarrow F_n}}(T_1^{\rightarrow F_n}-T_\gamma^{\rightarrow F_n})ds\chi\{T_{-1}^{A\rightarrow}<T_0^{F_n\rightarrow}\}\right]\\
&+\frac{1}{E_0^{A\rightarrow}[T_1^{A\rightarrow}]}E_0^{A\rightarrow}\left[\sum_{p=\gamma+1}^{0}\int_{T_{p-1}^{F_n\rightarrow}}^{T_p^{\rightarrow F_n}}(T_1^{\rightarrow F_n}-T_p^{\rightarrow F_n})ds\chi\{T_{-1}^{A\rightarrow}<T_0^{F_n\rightarrow}\}\right]\\
=\;&\frac{1}{E_0^{A\rightarrow}[T_1^{A\rightarrow}]}E_0^{A\rightarrow}\left[T_1^{\rightarrow F_n}\left(T_\gamma^{\rightarrow F_n}-T_{-1}^{A\rightarrow}+\sum_{p=\gamma+1}^{0}(T_p^{\rightarrow F_n}-T_{p-1}^{F_n\rightarrow})\right)\chi\{T_{-1}^{A\rightarrow}<T_0^{F_n\rightarrow}\}\right]\\
-\;&\frac{1}{E_0^{A\rightarrow}[T_1^{A\rightarrow}]}E_0^{A\rightarrow}\left[\left(T_\gamma^{\rightarrow F_n}(T_\gamma^{\rightarrow F_n}-T_{-1}^{A\rightarrow})+\sum_{p=\gamma+1}^{0}T_p^{\rightarrow F_n}(T_p^{\rightarrow F_n}-T_{p-1}^{F_n\rightarrow})\right)\chi\{T_{-1}^{A\rightarrow}<T_0^{F_n\rightarrow}\}\right]\\
=\;&V_1+V_2,
\end{aligned}
$$

which leads to the terms V_1 and V_2 defined in the lemma indeed.

Finally, we consider the sum U of the integrals in Z relative to the intervals $[T_p^{\rightarrow F_n},T_p^{F_n\rightarrow}]$ (only I contributes) which gives:

$$
U\;=\;\frac{1}{E_0^{A\rightarrow}[T_1^{A\rightarrow}]}E_0^{A\rightarrow}\left[\left(\sum_{p=\gamma}^{0}\int_{T_p^{\rightarrow F_n}}^{T_p^{F_n\rightarrow}}(T_1^{\rightarrow F_n}-s)ds\right)\chi\{T_{-1}^{A\rightarrow}<T_0^{F_n\rightarrow}\}\right]
$$

$$
\begin{aligned}
&= \frac{1}{E_0^{A\to}[T_1^{A\to}]} E_0^{A\to}\left[T_1^{\to F_n} \left(\sum_{p=\gamma}^{0} (T_p^{F_n \to} - T_p^{\to F_n}) \right) \chi\{T_{-1}^{A\to} < T_0^{F_n\to}\} \right] \\
&\quad - \frac{1}{2\ E_0^{A\to}[T_1^{A\to}]} E_0^{A\to}\left[\sum_{p=\gamma}^{0} \left((T_p^{F_n\to})^2 - (T_p^{\to F_n})^2 \right) \chi\{T_{-1}^{A\to} < T_0^{F_n\to}\} \right] \\
&= U_1 + U_2,
\end{aligned}
$$

and this leads to the terms U_1 and U_2. This concludes the proof of (2.27) since $Z = U + V$. ■

We are now in a position to give properties generalizing those of Lemma 4.

Lemma 6 *Suppose that $E_0^{\to A}[T_1^{\to F_n}]$ is finite for all n and tends to ∞. Assume that*

$$
E_0^{A\to}[T_1^{\to F_n} |T_0^{\to A}|] < \infty \tag{2.33}
$$

and that the following asymptotic property holds as $n \to \infty$:

$$
E_0^{A\to}[T_1^{\to F_n} T_0^{\to A}] \quad \sim \quad E_0^{A\to}[T_1^{\to F_n}] E_0^{A\to}[T_0^{\to A}]. \tag{2.34}
$$

If in addition

$$
E_0^{A\to}[(T_0^{\to A})^2] < \infty, \tag{2.35}
$$

then

$$
E[T_1^{\to F_n} \ |X_0 \in A] \sim E_0^{A\to}[T_1^{\to F_n}]. \tag{2.36}
$$

Proof: First note that under the above integrability assumptions, both sides of (2.25) are finite. Using (2.25), (2.35) and the fact that

$$
\frac{1}{E_0^{A\to}[T_1^{\to F_n}]} E_0^{A\to}[(T_0^{\to A})^2] \to 0
$$

(which follows from (2.36)), we obtain

$$
\begin{aligned}
E[T_1^{\to F_n} \cdot \chi\{X_0 \in A\}] &= \frac{1}{E_0^{A\to}[T_1^{A\to}]} E_0^{A\to}[-T_0^{\to A} \cdot T_1^{\to F_n} + \frac{1}{2}(T_0^{A\to})^2] \\
&\sim \frac{1}{E_0^{A\to}[T_1^{A\to}]} E_0^{A\to}[-T_0^{\to A}] \cdot E_0^{A\to}[T_1^{\to F_n}] \\
&= P(X_0 \in A) E_0^{A\to}[T_1^{\to F_n}].
\end{aligned}
$$

The last equality holds because of the Ryll-Nardzewski inversion formula. Dividing the above through by $P(X_0 \in A)$ gives (2.37). ■

Remark 11 By the exchange formula,

$$\begin{aligned} \lambda^{A\to} E_0^{A\to} \tau(F_n) &= \lambda^{\to A} E_0^{\to A} \int_0^{T_1^{\to A}} \tau(F_n) \circ \theta_s N^{A\to}(ds) \\ &= \lambda^{\to A} E_0^{\to A} \left[\tau(F_n) - T_1^{A\to} \right]. \end{aligned}$$

Therefore under the conditions of Lemma 6, $E_0^{A\to}\tau(F_n) \sim E_0^{\to A}\tau(F_n)$ since $\lambda^{A\to} = \lambda^{\to A}$. Moreover, by Lemma 6,

$$E[T_1^{\to F_n} \,|X_0 \in A] \sim E_0^{\to A}[T_1^{\to F_n}].$$

■

Remark 12 Note that the exchange formula also gives

$$E_0^{A\to}[(T_0^{\to A})^2] = E_0^{\to A}[\int_0^{T_1^{\to A}} (T_0^{\to A} \circ \theta_s)^2 N^{A\to}(ds)] = E_0^{\to A}[(T_1^{A\to})^2].$$

Hence we can check Condition (2.36) by verifying

$$E_0^{\to A}[T_1^{A\to}]^2 < \infty. \tag{2.37}$$

■

Remark 13 A sufficient condition for

$$E_0^{A\to} \tau(F_n) \to_{n\to\infty} \infty \tag{2.38}$$

is that $\tau(F_n)$ tends to infinity in probability w.r.t. P. This follows from the bound

$$\begin{aligned} E_0^{A\to} \tau(F_n) &= \frac{1}{\lambda^{A\to}} E \int_0^1 \tau(F_n) \circ \theta_s N^{A\to}(ds) \\ &\geq \frac{1}{\lambda^{A\to}} E[(\tau(F_n) - 1)\chi\{\tau(F_n) > \tau(A)\}\chi\{T_1^{A\to} < 1\}] \end{aligned}$$

where we assume, without loss of generality, that $P[T_1^{A\to} < 1] > 0$. The above lower bound is as big as we like because there exists an $\alpha > 0$ such that for all c and n large enough,

$$\begin{aligned} E[(\tau(F_n) - 1)\chi\{\tau(F_n) > \tau(A)\}\chi\{T_1^{A\to} < 1\}] &\geq (c-1)P[\tau(F_n) > c > \tau(A), T_1^{A\to} < 1] \\ &\geq (c-1)\alpha. \end{aligned}$$

■

Lemma 7 *Assume that the assumptions of Lemma 6 hold, that*

$$E_0^{A\to}[(T_1^{A\to})^2] < \infty, \quad E_0^{A\to}[(T_1^{\to F_n})^2] < \infty \ \forall n, \tag{2.39}$$

that $\tau(F_n)$ tends to ∞ in probability with respect to P, and that

$$\begin{aligned} E_0^{A\to}[T_1^{\to F_n} T_{-1}^{A\to} \chi\{T_{-1}^{A\to} < T_0^{F_n\to}\}] &\sim E_0^{A\to}[T_1^{\to F_n}] E_0^{A\to}[T_{-1}^{A\to} \chi\{T_{-1}^{A\to} < T_0^{F_n\to}\}] \\ E_0^{A\to}[T_1^{\to F_n} T_{-1}^{A\to} \chi\{T_{-1}^{A\to} > T_0^{F_n\to}\}] &\sim E_0^{A\to}[T_1^{\to F_n}] E_0^{A\to}[T_{-1}^{A\to} \chi\{T_{-1}^{A\to} > T_0^{F_n\to}\}]. \end{aligned} \tag{2.40}$$

Then

$$E[\tau(F_n) \mid X_0 \notin A] \sim E_0^{A\to}[\tau(F_n)]. \tag{2.41}$$

Proof: We use the representation of $E[\tau(F_n) | X_0 \notin A]$ given in Lemma 5. We show that under the above assumptions, each of the terms $U_i / E_0^{A\to} T_1^{\to F_n}$ and $V_i / E_0^{A\to} T_1^{\to F_n}$ vanishes when n tends to ∞.

Under the assumption that $\tau(F_n)$ tends to ∞ in probability with respect to P, we deduce from Remark 13 that $E_0^{A\to}[\tau(F_n)]$ tends to ∞ with n. We also deduce from Equation (2.8) and from the fact that Λ_n tends to 0 that $P_0^{A\to}[T_{-1}^{A\to} < T_0^{F_n\to}]$ tends to 0 when n tends to ∞.

Since under $P_0^{A\to}$ and on $\{T_{-1}^{A\to} < T_0^{F_n\to}\}$,

$$\begin{aligned} \sum_{p=\gamma}^{0} \left((T_p^{\to F_n})^2 - (T_p^{F_n\to})^2 \right) &\leq \sum_{p=\gamma}^{-1} \left((T_p^{\to F_n})^2 - (T_{p+1}^{\to F_n})^2 \right) + (T_0^{\to F_n})^2 - (T_0^{F_n\to})^2 \\ &\leq (T_{-1}^{A\to})^2, \end{aligned}$$

we have

$$\frac{U_2}{E_0^{A\to}[T_1^{\to F_n}]} \leq \frac{E_0^{A\to}\left[(T_{-1}^{A\to})^2\right]}{2\, E_0^{A\to}[T_1^{A\to}] E_0^{A\to}[T_1^{\to F_n}]}. \tag{2.42}$$

So this term tends to 0 as n goes to ∞ in view of (2.40).

In view of the bound

$$\sum_{p=\gamma}^{0} (T_p^{F_n\to} - T_p^{\to F_n}) \leq \left| T_{-1}^{A\to} \right|,$$

which is valid under $P_0^{A\rightarrow}$ and on $\{T_{-1}^{A\rightarrow} < T_0^{F_n\rightarrow}\}$ and of (2.41), we have

$$\begin{aligned}
\frac{U_1}{E_0^{A\rightarrow}[T_1^{\rightarrow F_n}]} &\leq \frac{E_0^{A\rightarrow}\left[T_1^{\rightarrow F_n}|T_{-1}^{A\rightarrow}|\chi\{T_{-1}^{A\rightarrow} < T_0^{F_n\rightarrow}\}\right]}{E_0^{A\rightarrow}[T_1^{A\rightarrow}]E_0^{A\rightarrow}[T_1^{\rightarrow F_n}]} \\
&\sim \frac{E_0^{A\rightarrow}[|T_{-1}^{A\rightarrow}|\chi\{T_{-1}^{A\rightarrow} < T_0^{F_n\rightarrow}\}]}{E_0^{A\rightarrow}[T_1^{A\rightarrow}]} \\
&\leq \frac{E_0^{A\rightarrow}[(T_{-1}^{A\rightarrow})^2]^{1/2}P_0^{A\rightarrow}[T_{-1}^{A\rightarrow} < T_0^{F_n\rightarrow}]^{1/2}}{E_0^{A\rightarrow}[T_1^{A\rightarrow}]},
\end{aligned}$$

where we used the Cauchy-Schwartz inequality. So this second term tends to 0 as n goes to ∞.

Using the bound

$$\left(T_\gamma^{\rightarrow F_n} - T_{-1}^{A\rightarrow} + \sum_{p=\gamma+1}^{0}(T_p^{\rightarrow F_n} - T_{p-1}^{F_n\rightarrow})\right)\chi\{T_{-1}^{A\rightarrow} < T_0^{F_n\rightarrow}\} \leq |T_{-1}^{A\rightarrow}|,$$

which is valid under $P_0^{A\rightarrow}$ and on $\{T_{-1}^{A\rightarrow} < T_0^{F_n\rightarrow}\}$, we get

$$\begin{aligned}
\frac{V_2}{E_0^{A\rightarrow}[T_1^{\rightarrow F_n}]} &\leq \frac{E_0^{A\rightarrow}\left[\left|T_\gamma^{\rightarrow F_n}\right|(T_0^{\rightarrow F_n} - T_{-1}^{A\rightarrow})\chi\{T_{-1}^{A\rightarrow} < T_0^{F_n\rightarrow}\}\right]}{E_0^{A\rightarrow}[T_1^{A\rightarrow}]E_0^{A\rightarrow}[T_1^{\rightarrow F_n}]} \\
&\leq \frac{E_0^{A\rightarrow}\left[(T_{-1}^{A\rightarrow})^2\right]}{E_0^{A\rightarrow}[T_1^{A\rightarrow}]E_0^{A\rightarrow}[T_1^{\rightarrow F_n}]},
\end{aligned} \tag{2.43}$$

and the conclusion is the same for this term.

Similarly, using (2.41) and Cauchy-Schwartz, we obtain

$$\frac{V_1}{E_0^{A\rightarrow}[T_1^{\rightarrow F_n}]} \leq \frac{E_0^{A\rightarrow}\left[(T_{-1}^{A\rightarrow})^2\right]^{1/2}P_0^{A\rightarrow}[T_{-1}^{A\rightarrow} < T_0^{F_n\rightarrow}]^{1/2}}{E_0^{A\rightarrow}[T_1^{A\rightarrow}]}.$$

So this last term also tends to 0 as n goes to ∞ since $P_0^{A\rightarrow}[T_{-1}^{A\rightarrow} < T_0^{F_n\rightarrow}]$ tends to 0.

Therefore, under the above assumptions, we get from (2.27) that

$$\begin{aligned}
E[\tau(F_n)\chi\{X_0 \notin A\}] &= \frac{1}{E_0^{A\rightarrow}[T_1^{A\rightarrow}]}\left\{E_0^{A\rightarrow}\left[T_1^{\rightarrow F_n}(T_0^{\rightarrow A} - T_{-1}^{A\rightarrow})\right] - \frac{1}{2}E_0^{A\rightarrow}\left[(T_0^{\rightarrow A})^2 - (T_{-1}^{A\rightarrow})^2\right]\right\} \\
&\qquad -U - V \\
&\sim \frac{1}{E_0^{A\rightarrow}[T_1^{A\rightarrow}]}\left\{E_0^{A\rightarrow}\left[T_1^{\rightarrow F_n}\right]E_0^{A\rightarrow}\left[T_0^{\rightarrow A} - T_{-1}^{A\rightarrow}\right]\right. \\
&\qquad \left. -\frac{1}{2}E_0^{A\rightarrow}\left[(T_0^{\rightarrow A})^2 - (T_{-1}^{A\rightarrow})^2\right]\right\} - U - V,
\end{aligned}$$

where we used (2.35) and the property

$$E_0^{A\rightarrow}[T_1^{\rightarrow F_n} T_{-1}^{A\rightarrow}] \sim E_0^{A\rightarrow}[T_1^{\rightarrow F_n}] E_0^{A\rightarrow}[T_{-1}^{A\rightarrow}],$$

which follows from (2.41). Therefore

$$\begin{aligned}
E[\tau(F_n)\chi\{X_0 \notin A\}] &= E_0^{A\rightarrow}\left[T_1^{\rightarrow F_n}\right] P\left[X_0 \notin A\right] - \frac{E_0^{A\rightarrow}\left[(T_0^{\rightarrow A})^2 - (T_{-1}^{A\rightarrow})^2\right]}{2E_0^{A\rightarrow}[T_1^{A\rightarrow}]} - U - V \\
&\sim E_0^{A\rightarrow}\left[T_1^{\rightarrow F_n}\right] \left(P\left[X_0 \notin A\right] - \frac{E_0^{A\rightarrow}\left[(T_0^{\rightarrow A})^2 - (T_{-1}^{A\rightarrow})^2\right]}{2E_0^{A\rightarrow}\left[T_1^{\rightarrow F_n}\right] E_0^{A\rightarrow}[T_1^{A\rightarrow}]} \right) \\
&\sim E_0^{A\rightarrow}\left[T_1^{\rightarrow F_n}\right] P\left[X_0 \notin A\right],
\end{aligned}$$

where we used (2.40). ■

We shall say that the hitting times of F_n are *asymptoticaly independent* w.r.t. $P_0^{A\rightarrow}$ if Conditions (2.23), (2.35), and (2.41) hold. Such an asymptotic independence is easy to check in the Markovian case using the conditional independence of the past and future given the present (see Corollary 5).

We summarize the results of this subsection in the following theorem:

Theorem 4 *Assume that*

$$E_0^{A\rightarrow}[(T_1^{A\rightarrow})^2] < \infty, \quad E_0^{A\rightarrow}[(T_1^{\rightarrow F_n})^2] < \infty \ \forall n,$$

that $\tau(F_n)$ tends to ∞ in probability with respect to P, asymptotic independence holds w.r.t. $P_0^{A\rightarrow}$. Then

$$\begin{aligned}
E[\tau(F_n)] &\sim E_0^{A\rightarrow}[\tau(F_n)] \sim E_0^{F_n(\rightarrow A)}[\tau(F_n)] \\
&\sim E[\tau(F_n) \mid X_0 \in A] \sim E[\tau(F_n) \mid X_0 \notin A] \sim (\Lambda_n)^{-1}.
\end{aligned} \tag{2.44}$$

2.4 Asymptotic independence, exponentiality and mixing

2.4.1 Relations between asymptotic independence and asymptotic exponentiality

Corollary 1 *Assume that*

- *$\tau(F_n)$ tends to infinity in probability w.r.t. P;*
- *the integrability condition (2.40) is satisfied;*
- *the rate condition (2.16) is satisfied.*

Then

1. *Asymptotic independence of the hitting times of* F_n *w.r.t.* $P_0^{A\to}$ *implies asymptotic exponentiality in variance w.r.t.* $P_0^{(F_n\to)A}$.
2. *If*

$$\Lambda_n \left(E_0^{F_n(A\to)} \left[(T_{-1}^{A\to})^2 \right] \right)^{\frac{1}{2}} = o(1), \tag{2.45}$$

then asymptotic exponentiality in variance w.r.t. $P_0^{F_n(A\to)}$ *implies the following formula which can be seen as another form of asymptotic independence:*

$$\frac{E_0^{A\to}\left[T_1^{\to F_n} |T_{-1}^{A\to}|\right]}{\Lambda_n^{-1} E_0^{A\to}[|T_{-1}^{A\to}|]} \sim 1. \tag{2.46}$$

Proof: The first property is immediate in view of Theorem 4 and of the last statement of Remark 10.

As for the second statement, assuming that asymptotic exponentiality holds, in view of the results of Theorem 3 and of Lemma 5, we have

$$\begin{aligned} 1 \sim \Lambda_n E[\tau(F_n)] &= \Lambda_n (E[\tau(F_n), X_0 \in A] + E[\tau(F_n), X_0 \notin A]) \\ &= \frac{\Lambda_n}{E_0^{A\to}[T_1^{A\to}]} E_0^{A\to}[\int_{T_{-1}^{A\to}}^0 (T_1^{\to F_n} - s) ds] - \Lambda_n (U_1 + U_2 + V_1 + V_2) \\ &\sim \frac{E_0^{A\to}[T_1^{\to F_n} |T_{-1}^{A\to}|]}{\Lambda_n^{-1} E_0^{A\to}[T_1^{A\to}]} - \Lambda_n (U_1 + V_1), \end{aligned}$$

where we used the bounds (2.43) and (2.44) to get rid of the terms U_2 and V_2. Using the definition of U_1 and V_1, we obtain

$$\begin{aligned} U_1 + V_1 &\le E_0^{A\to}\left[T_1^{\to F_n} |T_{-1}^{A\to}| \chi\{T_{-1}^{A\to} < T_0^{F_n\to}\}\right] \\ &= E_0^{F_n(A\to)}\left[T_1^{\to F_n} |T_{-1}^{A\to}|\right] P_0^{A\to}\left[T_{-1}^{A\to} < T_0^{F_n\to}\right]. \end{aligned}$$

Since the integrability condition (2.40) implies that $E_0^{F_n(A\to)}[(T_1^{\to F_n})^2] < \infty$ and $E_0^{F_n(A\to)}\left[(T_{-1}^{A\to})^2\right] < \infty$, we obtain from the last relation and from Cauchy Schwartz that

$$U_1 + V_1 \le \left(E_0^{F_n(A\to)} \left[(T_1^{\to F_n})^2 \right] \right)^{\frac{1}{2}} \left(E_0^{F_n(A\to)} \left[(T_{-1}^{A\to})^2 \right] \right)^{\frac{1}{2}} P_0^{A\to}\left[T_{-1}^{A\to} < T_0^{F_n\to}\right].$$

This, the fact that

$$P_0^{A\to}\left[T_{-1}^{A\to} < T_0^{F_n\to}\right] = P_0^{\to A}\left[T_{-1}^{A\to} < T_0^{F_n\to}\right],$$

together with Lemma 1 and Theorem 2, imply that

$$\begin{aligned}
U_1 + V_1 &\leq \left(E_0^{F_n(A\to)}\left[(T_1^{\to F_n})^2\right]\right)^{\frac{1}{2}}\left(E_0^{F_n(A\to)}\left[(T_{-1}^{A\to})^2\right]\right)^{\frac{1}{2}}\frac{E_0^{A\to}[|T_{-1}^{A\to}|]}{E_0^{F_n(A\to)}\left[T_1^{F_n(A\to)}\right]} \\
&\sim \left(E_0^{F_n(A\to)}\left[(T_1^{\to F_n})^2\right]\right)^{\frac{1}{2}}\left(E_0^{F_n(A\to)}\left[(T_{-1}^{A\to})^2\right]\right)^{\frac{1}{2}}\frac{E_0^{A\to}[|T_{-1}^{A\to}|]}{E_0^{F_n(A\to)}\left[T_1^{\to F_n}\right]} \\
&= \left(\frac{E_0^{F_n(A\to)}\left[(T_1^{\to F_n})^2\right]}{\left(E_0^{F_n(A\to)}\left[T_1^{\to F_n}\right]\right)^2}\right)^{\frac{1}{2}}\left(E_0^{F_n(A\to)}\left[(T_{-1}^{A\to})^2\right]\right)^{\frac{1}{2}} E_0^{A\to}[|T_{-1}^{A\to}|] \\
&\sim \sqrt{2}\left(E_0^{F_n(A\to)}\left[(T_{-1}^{A\to})^2\right]\right)^{\frac{1}{2}} E_0^{A\to}[|T_{-1}^{A\to}|],
\end{aligned}$$

where we used the asymptotic exponentiality assumption. Equation (2.47) follows immediately when making use of (2.46). ■

2.4.2 Relation with mixing and asymptotic exponentiality

The aim of this section is to show that the asymtotic independence properties of the previous section hold.

Assume the conditions of Theorem 2 hold. Each departure from A starts a new cycle. The past at the end of k^{th} cycle is described by $\mathcal{F}_k := \sigma(X_s; s \leq T_k^{A\to})$. The future beyond the m^{th} cycle is described by $\mathcal{F}^m := \sigma(X_s; s > T_m^{A\to})$. We now impose a strong mixing condition on the cycles relative to the Palm measure $P_0^{A\to}$. We say the cycles of departures from A are *strongly mixing* if, for any n,

$$\sup_{F,G} |P_0^{A\to}(F \cap G) - P_0^{A\to}(F)\cdot P_0^{A\to}(G)| \leq \beta_m,$$

where $F \in \mathcal{F}_k$ and $G \in \mathcal{F}^{k+m}$ and $\beta_m \to 0$ as $m \to \infty$.

Consider the sequence of rare events

$$E_{n,i} := \{T_i^{A\to} = T_m^{(A\to)F_n} \text{ for some } m\}.$$

These events called successes will occur if the process reaches F_n during a cycle which starts with a departure from A. Note that $E_{n,i} \in \mathcal{F}_{i+1}$. Define $N_n(s) := \sum_{0<i\leq s}\chi(E_{n,i})$. The process $N_n(s)$ counts the number of rare events in the first s (or integer part of s if s is a real number) cycles on the right of the origin.

Let $p_n := P_0^{A\rightarrow}(E_{n,1}) = P_0^{A\rightarrow}(T_1^{\rightarrow F_n} < T_1^{A\rightarrow})$. Note that by Remark 1

$$\Lambda_n^{-1} = \frac{E_0^{A\rightarrow}[T_1^{A\rightarrow}]}{p_n}.$$

For any of the point processes N^H of the preceding sections, we will use the following notation:

$$N^H(t) = N^H([0,t]), \quad t > 0.$$

Lemma 8 *Under the above strong mixing conditions, if $n \cdot p_n \rightarrow \lambda$, with $0 < \lambda < \infty$, then the process $M_n(t) := N_n(nt)$ converges to a Poisson process $L(t)$ with rate λ. More generally, the pair*

$$(\{M_n(t), 0 \leq t < \infty\}, N^{A\rightarrow}(nt)/nt)$$

converges weakly to $(\{L(t), 0 \leq t < \infty\}, \lambda^{A\rightarrow})$.

Proof: If the cycles of departures from A are strongly mixing, the events $\{E_{n,i}\}$ satisfy the long-range dependence condition $\Delta(E_{n,i})$ of Hüsler and Schmidt (1996), w.r.t. the measure $P_0^{A\rightarrow}$. This holds true because we can pick a sequence $m_n \rightarrow \infty$ such that $m_n \cdot p_n \sim \lambda m_n/n \rightarrow 0$. The local dependence condition $D'(E_{n,i})$ in this paper holds automatically since we only allow one success per cycle. The results follow from Theorem 1 in Hüsler and Schmidt (1996). ■

Notice that $N^{(A\rightarrow)F_n}(t) = N_n(N^{A\rightarrow}(t))$. We have

$$\begin{aligned} N^{(A\rightarrow)F_n}(nt) &= N_n(N^{A\rightarrow}(nt)) \\ &= N_n(nt \cdot \frac{N^{A\rightarrow}(nt)}{nt}) \\ &= M_n(t \cdot \frac{N^{A\rightarrow}(nt)}{nt}) \\ &\rightarrow L(\lambda^{A\rightarrow} t). \end{aligned}$$

We conclude that, relative to the measure $P_0^{A\rightarrow}$, $N^{(A\rightarrow)F_n}(nt)$ converges to a Poisson process with rate $\lambda\lambda^{A\rightarrow}$ or equivalently, $N^{(A\rightarrow)F_n}(t/\Lambda_n)$ converges to a Poisson process with rate 1.

Proposition 3 *Suppose the cycles started by departures from A are strongly mixing. Further suppose that for some ϵ,*

$$E_0^{A\rightarrow}[\left(T_0^{\rightarrow A}\right)^{2+\epsilon}] \quad \textit{is uniformly bounded in } n \tag{2.47}$$

and

$$E_0^{A\to}[\left(\Lambda_n T_1^{(F_n\to)A}\right)^2] \quad \textit{is uniformly bounded in } n. \tag{2.48}$$

Then

$$\Lambda_n E_0^{A\to}[T_1^{(A\to)F_n}] \to 1 \tag{2.49}$$

and

$$E_0^{A\to}[T_0^{\to A} \cdot T_1^{(A\to)F_n}] \sim E_0^{A\to}[T_0^{\to A}] E_0^{A\to}[T_1^{(A\to)F_n}]. \tag{2.50}$$

Proof: Under our hypotheses, $N^{(A\to)F_n}(t/\Lambda_n)$ converges to a Poisson process with rate 1. Next,

$$\{N^{(A\to)F_n}(t/\Lambda_n) = 0\} = \{T_1^{(A\to)F} > t/\Lambda_n\},$$

so $P_0^{A\to}(\Lambda_n T_1^{(A\to)F} > t) \to \exp(-t)$. By hypothesis, $E_0^{A\to}(\Lambda_n T_1^{(A\to)F})^2 < \infty$ uniformly in n, so we have uniform integrability and we get (2.50).

Pick some $m > 0$. Since $T_1^{(A\to)F_n}$ gets large as $n \to \infty$, the difference between

$$P_0^{A\to}(-T_0^{\to A} > s, \Lambda_n T_1^{(A\to)F_n} > t) \tag{2.51}$$

and

$$P_0^{A\to}(-T_0^{\to A} > s, \Lambda_n T_1^{(A\to)F_n} \circ T_m^{A\to} > t) \tag{2.52}$$

is less than

$$P_0^{A\to}(-T_1^{(A\to)F_n} < T_m^{A\to}) + P_0^{A\to}(\Lambda_n(T_1^{(A\to)F_n} - T_m^{A\to}) \le t < \Lambda_n T_1^{(A\to)F_n}).$$

The first of these two error terms tends to 0 as $n \to \infty$ by assumption. The second does also because $\Lambda_n T_1^{(A\to)F_n}$ converges weakly to an exponential distribution and $\Lambda_n T_m^{A\to} \to 0$.

Since $T_1^{(A\to)F_n} \circ T_m^{A\to}$ is in the future beyond the m^{th} cycle and $T_0^{\to A}$ is in the past of cycle 0, it follows by strong mixing that (2.53) differs from

$$P_0^{A\to}(-T_0^{\to A} > s) P_0^{A\to}(\Lambda_n T_1^{(A\to)F_n} \circ T_m^{A\to} > t)$$

by less than β_m. Now, as above, the difference between

$$P_0^{A\to}(\Lambda_n T_1^{(A\to)F_n} \circ T_m^{A\to} > t) \tag{2.53}$$

and

$$P_0^{A\to}(\Lambda_n T_1^{(A\to)F_n} > t) \tag{2.54}$$

is less than

$$P_0^{A\rightarrow}(T_1^{(A\rightarrow)F_n} < T_m^{A\rightarrow}) + P_0^{A\rightarrow}(\Lambda_n(T_1^{(A\rightarrow)F_n} - T_m^{A\rightarrow}) \leq t < \Lambda_n T_1^{(A\rightarrow)F_n}).$$

which again tends to 0 as $n \to \infty$.

We conclude that as $n \to \infty$

$$\limsup_{n\to\infty} |P_0^{A\rightarrow}(-T_0^{\rightarrow A} > s, \Lambda_n T_1^{(A\rightarrow)F_n} > t) - P_0^{A\rightarrow}(-T_0^{\rightarrow A} > s)P_0^{A\rightarrow}(\Lambda_n T_1^{(A\rightarrow)F_n} > t)| \leq \beta_m.$$

Since $\beta_m \to 0$ as $m \to \infty$,this means

$$\lim_{n\to\infty} |P_0^{A\rightarrow}(-T_0^{\rightarrow A} > s, \Lambda_n T_1^{(A\rightarrow)F_n} > t) = P_0^{A\rightarrow}(-T_0^{\rightarrow A} > s)\exp(-t). \tag{2.55}$$

We conclude that $(T_0^{\rightarrow A}, \Lambda_n T_1^{(A\rightarrow)F_n})$ converges jointly in distribution.

On the other hand

$$E_0^{A\rightarrow}|T_0^{\rightarrow A} \cdot \Lambda_n T_1^{(A\rightarrow)F_n}|^{1+\delta} \leq (E_0^{A\rightarrow}[T_0^{\rightarrow A}]^{2+\epsilon})^{1/(2+\epsilon)}(E_0^{A\rightarrow}[\Lambda_n T_1^{(A\rightarrow)F_n}]^2)^{1/2}$$

where $2/(1-\delta) = 2+\epsilon$ and this is uniformly bounded by hypothesis. We conclude, $T_0^{\rightarrow A} \cdot \Lambda_n T_1^{(A\rightarrow)F_n}$ is uniformly integrable so

$$E_0^{A\rightarrow}[T_0^{\rightarrow A} \cdot \Lambda_n T_1^{(A\rightarrow)F_n}] \to E_0^{A\rightarrow}[T_0^{\rightarrow A}].$$

The proof now follows since under the hypothesis of the proposition $\Lambda_n \sim E_0^{A\rightarrow}[T_1^{(A\rightarrow)F_n}]$. ■

Proposition 4 *Under the hypotheses of Proposition 3 and assuming*

$$E_0^{F_n(\rightarrow A)}(\Lambda_n \cdot N^{\rightarrow A}(T_1^{(A\rightarrow)F_n}))^2, \quad \textit{is uniformly bounded in } n, \tag{2.56}$$

we have

$$\Lambda_n E_0^{A\rightarrow}[T_1^{(F_n\rightarrow)A} - T_1^{(A\rightarrow)F_n})] \to 0.$$

It follows that $\Lambda_n E_0^{A\rightarrow} T_1^{\rightarrow F_n} \to 1$ *and that (2.35) holds.*

Proof: Using the same idea as Remark 11, we can use the exchange formula to get

$$\begin{aligned}
\lambda^{\rightarrow A} E_0^{A\rightarrow} T_1^{F_n(\rightarrow A)} &= \lambda^{\rightarrow A} E_0^{\rightarrow A}[\int_0^{T_1^{\rightarrow A}} T_1^{F_n(\rightarrow A)} \circ \theta_s N^{A\rightarrow}(ds)] \\
&= \lambda^{\rightarrow A} E_0^{\rightarrow A}\left[T_1^{F_n(\rightarrow A)} - T_1^{A\rightarrow}\right].
\end{aligned}$$

Therefore $E_0^{A\to}T_1^{F_n(\to A)} \sim E_0^{\to A}T_1^{F_n(\to A)}$ since $\lambda^{A\to} = \lambda^{\to A}$. Again, using the exchange formula we get

$$
\begin{aligned}
&\Lambda_n E_0^{\to A}[T_1^{(F_n\to)A} - T_1^{(A\to)F_n}] \\
&= \frac{\Lambda_n^2}{\lambda^{\to A}} E_0^{F_n(\to A)}[N^{\to A}(T_1^{(A\to)F_n})(T_1^{(F_n\to)A} - T_1^{(A\to)F_n})] \\
&\leq \frac{1}{\lambda^{\to A}}\left(E_0^{F_n(\to A)}[\Lambda_n N^{\to A}(T_1^{(A\to)F_n})]^2\right)^{1/2}\left(E_0^{F_n(\to A)}[\Lambda_n(T_1^{(F_n\to)A} - T_1^{(A\to)F_n})]^2\right)^{1/2}.
\end{aligned}
$$

However,

$$T_1^{(F_n\to)A} - T_1^{(A\to)F_n} \leq T_1^{F_n(\to A)}$$

and $E_0^{F_n(\to A)}[\Lambda_n T_1^{F_n(\to A)}]^2$ is uniformly bounded. Moreover

$$E_0^{F_n(\to A)}[\Lambda_n(T_1^{(F_n\to)A} - T_1^{(A\to)F_n})] \to 0.$$

Finally $E_0^{F_n(\to A)}[\Lambda_n N^{\to A}(T_1^{(A\to)F_n})]^2$ is uniformly bounded so

$$\Lambda_n E_0^{\to A}[T_1^{(F_n\to)A} - T_1^{(A\to)F_n})] \to 0$$

as $n\to\infty$. Next, $P_0^{\to A}$ almost surely, $T_1^{(A\to)F_n} < T_1^{\to F_n} \leq T_1^{(F_n\to)A}$, so we get $\Lambda_n E_0^{A\to}T_1^{\to F_n} \to 1$ by (2.50).

Moreover we can now replace $T_1^{(A\to)F_n}$ by $T_1^{\to F_n}$ on both sides of equation (2.51) as shown below. Let us first prove that

$$E_0^{A\to}[T_0^{\to A}\cdot\Lambda_n(T_1^{\to F_n} - T_1^{(A\to)F_n})] \to 0, \tag{2.57}$$

The fact that

$$\Lambda_n E_0^{\to A}[T_1^{(F_n\to)A} - T_1^{(A\to)F_n})] \to 0$$

implies that

$$T_0^{\to A}\cdot\Lambda_n(T_1^{\to F_n} - T_1^{(A\to)F_n})$$

converges to zero in distribution. Moreover

$$
\begin{aligned}
&\Lambda_n E_0^{A\to}|T_0^{\to A}\cdot(T_1^{\to F_n} - T_1^{(A\to)F_n})|^{1+\delta} \\
&\leq \left(E_0^{A\to}[T_0^{\to A}]^{2+\epsilon}\right)^{1/(2+\epsilon)}\left(E_0^{A\to}[\Lambda_n(T_1^{\to F_n} - T_1^{(A\to)F_n})]^2\right)^{1/2},
\end{aligned}
$$

where $2/(1-\delta) = 2+\epsilon$. This expression is uniformly bounded because $(T_1^{\to F_n} - T_1^{(A\to)F_n})$ is uniformly bounded by $T_1^{\to F_n}$ and $E_0^{A\to}[\Lambda_n T_1^{\to F_n}]^2$ is uniformly bounded by hypothesis. This completes the proof of (2.58).

Using (2.58) and (2.51), we get

$$\begin{aligned}
&\Lambda_n E_0^{A\to}[T_0^{\to A}\cdot T_1^{\to F_n}] \\
&= \Lambda_n E_0^{A\to}[T_0^{\to A}\cdot T_1^{(A\to)F_n}] + E_0^{A\to}[T_0^{\to A}\cdot \Lambda_n(T_1^{\to F_n} - T_1^{(A\to)F_n})] \\
&\sim \Lambda_n E_0^{A\to}[T_0^{\to A}] E_0^{A\to}[T_1^{(A\to)F_n}] \\
&= E_0^{A\to}[T_0^{\to A}].
\end{aligned}$$

Finally, it is easy to replace $T_1^{(A\to)F_n}$ by $T_1^{\to F_n}$ on right hand side of Equation (2.51) and this gives (2.35). ■

Remark 14 For a discrete time process, $N^{\to A}(T_1^{(F_n\to)A}) \le T_1^{(F_n\to)A}$. By the hypotheses of Proposition 3

$$E_0^{F_n(\to A)}[(\Lambda_n\cdot T_1^{(A\to)F_n})^2]$$

is uniformly bounded. Consequently (2.57) holds.

Exponentiality of $T_1^{\to F_n}$ follows also because,

$$\begin{aligned}
P_0^{A\to}(N^{(A\to)F_n}(t/\Lambda_n)=0) &\le P_0^{A\to}(\Lambda_n\cdot T_1^{\to F_n} > t) \\
&\le P_0^{A\to}(N^{(A\to)F_n}(\frac{1}{\Lambda_n}\left(t-\Lambda_n(T_1^{(F_n\to)A} - T_1^{(A\to)F_n})\right))=0).
\end{aligned}$$

Next,

$$P_0^{A\to}(N^{(A\to)F_n}(t/\Lambda_n)=0) \quad\to\quad P(L(t)=0)=\exp(-t)$$

and because

$$\Lambda_n E_0^{A\to}[T_1^{(F_n\to)A} - T_1^{(A\to)F_n})] \to 0,$$

we have

$$P_0^{A\to}(N^{(A\to)F_n}(\frac{1}{\Lambda_n}\left(t-\Lambda_n(T_1^{(F_n\to)A} - T_1^{(A\to)F_n})\right))=0) \to \exp(-t).$$

It follows that $\Lambda_n T_1^{\to F_n}$ is asymptotically exponentially distributed, under $P_0^{A\to}$.

Corollary 2 *Under the hypotheses of Proposition 3 and supposing (2.57) holds,*

$$E[T_1^{\to F_n}|X_0\in A] \sim E_0^{A\to}[T_1^{\to F_n}] \sim E_0^{\to A}[T_1^{\to F_n}] \sim E_0^{F_n(\to A)}[T_1^{\to F_n}] \sim \Lambda_n^{-1}.$$

Moreover $\Lambda_n T_1^{\to F_n}$ is asymptotically exponential with mean one, under $P_0^{A\to}$.

Proof: By Proposition 4, (2.35) holds. We also have (2.36) since (2.48) holds. Certainly if (2.49) and (2.36) hold then (2.34) holds. This gives the first asymptotic equivalence as a direct application of Lemma 6. The rest follow from Proposition 4. The convergence of $\Lambda_n T_1^{\to F_n}$ was shown above. ■

3 Rare events for Markov chains on general state spaces

Let X_t be a discrete time, ψ-irreducible, aperiodic, Harris recurrent Markov chain on a general state space $(S, \mathcal{S})$ with kernel K and stationary probability measure π (see Meyn and Tweedie (1993) pages 89 and 200 for the definitions of ψ-irreducibility and Harris recurrence). The shift operator defines a measure preserving flow on the doubly infinite product space $(S^\infty, \mathcal{S}^\infty, P_\pi)$. The flow is ergodic by Theorem 17.1.7 in Meyn and Tweedie (1993) and of course $\{X_t\}$ is compatible with this flow.

Lemma 9 *A set $A \in \mathcal{S}$ is regular for $\{X_t\}$ iff $1 > \pi(A) > 0$.*

Proof: Since time is discrete, A is regular iff $P_\pi[X_0 \in A, X_1 \in A^c] > 0$ (this corresponds to the assumption that $\lambda^{A\to} > 0$). Assume that $P_\pi[X_0 \in A, X_1 \in A^c] = 0$. Then

$$\bigcup_{n\geq 0} \{X_n \in A, X_{n+1} \in A^c\} = \emptyset, \quad P_\pi \ a.s.$$

and therefore $\{X_0 \in A\}$ implies that $X_n \in A$ for all n, but for a subset of Ω of measure 0. Thus, if $\pi(A) > 0$, the ergodic theorem implies that

$$1 = \lim_n \frac{1}{n} \sum_{i=1}^{n} \chi\{X_i \in A\} = \pi(A),$$

for almost all ω in the set $\{X_0 \in A\}$, which is of positive measure. This in turn implies that $\pi(A) = 1$. Therefore, $1 > \pi(A) > 0$ implies that A is regular. The converse implication follows from the bounds

$$\begin{aligned} P_\pi[X_0 \in A, X_1 \in A^c] &\leq \pi(A) \\ P_\pi[X_0 \in A, X_1 \in A^c] &\leq \pi(A^c). \end{aligned}$$

■

Below, we assume A is a measurable subset of in-control states while F_n is a measurable subset of out-of-control states with $\pi(A) > 0$ and $\pi(F_n) > 0$. We are again concerned with the time to exit A starting from initial distribution α and to pass through an indifference region before finally hitting F_n.

The Palm probability $P_0^{F_n(\to A)}$ corresponds to the law of the chain with initial probability β, where β is the invariant probability of the imbedded chain at the points of first return to

A after an excursion to F_n (which form a sequence of stopping times of the Markov chain). By the same arguments as those used in the proof of Proposition 2, the intensity of this imbedded chain is

$$\Lambda_n = P[X_0 \in F_n; T_1^{\to A} < T_1^{\to F_n}] = \left(\int_{y \in F_n} \pi(dy) \int_{x \in F_n^c} K(y, dx) f(x) \right) \leq \pi(F_n),$$

where $f(x)$ still denotes the probability of hitting A before F_n, starting from x. This intensity is positive as a corollary of Lemma 2, and so the Palm probability $P_0^{F_n(\to A)}$ is always well defined.

Note that $T_1^{\to F_n} \to \infty$ in probability w.r.t. P_π as $\pi(F_n) \to 0$. If this were not the case, then there would exist a time s such that $P_\pi(T_1^{\to F_n} \leq s) \geq \epsilon > 0$ for all n. This would mean there exists an m, $1 \leq m \leq s$ such that

$$\int \pi(dy) K^m(y, F_n) \geq \frac{\epsilon}{s} \text{ for all } n.$$

By stationarity then $\pi(F_n) \geq \epsilon/s$ and this is a contradiction.

The following corollary is then a direct application of Theorem 2.

Corollary 3 *Under the above assumptions*

$$E_\beta T_1^{\to F_n} \sim \Lambda_n^{-1}. \tag{3.58}$$

We now consider the time to hit F_n starting from an arbitrary probability α (like for instance $\alpha = \pi$). To study this second problem, we make assumptions that guarantee the existence of regeneration points. We assume that we can find a *small* set C such that $\psi(C) > 0$, a probability ν, a real $\delta > 0$ and an integer m, such that

$$P^m(x, E) \geq \delta \nu(E), \text{ for } x \in C, E \in \mathcal{S}.$$

Let $\sigma = \tau(C)$. We make the further assumptions:

$$E_\alpha \sigma < \infty \text{ and } E_x \sigma < c_1 \text{ for all } x \in C. \tag{3.59}$$

The splitting technique (described in Meyn and Tweedie (1993)) allows one to prove the following properties:

- The chain $\{X_t\}$ admits regeneration times $T_k,\ k = 1, 2, \ldots$.
- The segments of the sample path of $\{X_t\}$ between two regeneration times are independent, i.e. the segment $(X_t; 0 \leq t < T_1)$ is independent of the i.i.d. sequence of segments $(X_t; T_k \leq t < T_{k+1})_{k=1}^\infty)$.
- The k^{th} regenerative period ends with $X_{T_k - 1} \in C$ and for $k > 1$, the k^{th} regenerative period starts with $X_{T_{k-1}}$ having distribution ν.

- The k^{th} regenerative period has length $L_k := T_k - T_{k-1}$ of the form $L_k = \sum_{m=1}^{M} \sigma_m^k$, where σ_m^k are the lengths of successive cycle times back to C and M is a geometric random variable with parameter $1-\delta$ (i.e. $P_\nu(M=m) = (1-\delta)^{m-1}\delta$).

It follows from regenerative theory that $\pi(C)$ is the expected number of visits to C per regenerative cycle divided by the mean cycle length. That is $\pi(C) = (1/\delta)/E_\nu L_k$ or equivalently $E_\nu L_k = 1/(\delta\pi(C))$. By hypothesis,

$$E_\alpha L_1 \leq E_\alpha \sigma + c_1/\delta < \infty$$

since the mean return times to C are uniformly bounded by c_1 and the expected number of returns to C per cycle is $1/\delta$.

Since the regenerative cycles have finite mean, it is easy to extend Theorem 8.2B in Keilson (1979) to show:

Theorem 5 *Relative to a starting probability α satisfying Condition (3.60), $T_1^{\to F_n}/E_\alpha T_1^{\to F_n}$ converges in distribution to an exponential random variable with mean 1 as $\pi(F_n) \to 0$.*

We now show that asymptotic exponentiality holds in the present setting, and more precisely that Theorem 3 can be applied.

Let γ denote the invariant probability of the imbedded chain at entrance times into $B := F_n^c$.

Corollary 4 *If*

$$E_\pi \sigma^{2+\epsilon} < \infty \text{ and } E_x \sigma^{2+\epsilon} < c_2 \text{ for all } x \in C \tag{3.60}$$

and if for all n

$$g_n = E_\gamma \left[(\tau(F_n))^2 \;\mid T_1^{\to F_n} < \tau(A) \right] \tag{3.61}$$

is finite and such that

$$\limsup_{n\to\infty} \left(\int_{y \in F_n^c} K(y, F_n)\pi(dy) \right)^2 g_n = 0, \tag{3.62}$$

then

$$E_\pi T_1^{\to F_n} \sim \Lambda_n^{-1}. \tag{3.63}$$

To prove this corollary, we need the following technical lemma.

Lemma 10 *Let α be a probability measure and suppose*

$$E_\alpha \sigma^{2+\epsilon} < \infty \text{ and } E_x \sigma^{2+\epsilon} < c_2 \text{ for all } x \in C. \tag{3.64}$$

Then, when denoting $p_n = P_\nu[\tau(F) < L_1$, *we have*

$$\begin{aligned} E_\alpha\left[(T_1^{(F_n\to)A})^{2+\epsilon}\right] &= \mathcal{O}(p_n^{-(2+\epsilon)}) \\ \liminf_{n\to\infty} p_n E_\alpha T_1^{(A\to)F_n} &> 0. \end{aligned}$$

Proof: Making use of (3.65), we first check $E_\alpha L_1^{2+\epsilon} < \infty$ and $E_\nu L_k^{2+\epsilon} < \infty$ for $k \geq 2$. First,

$$E_\alpha L_1^{2+\epsilon} = E_\alpha\left[(\sigma + (L_1 - \sigma))^{2+\epsilon}\right] \leq 2^{1+\epsilon}\left(E_\alpha \sigma^{2+\epsilon} + E_\alpha E_{X_\sigma} L_1^{2+\epsilon}\right).$$

However, for $x \in C$,

$$\begin{aligned} E_x L_1^{2+\epsilon} &= E_x\left[\left(\sum_{k=1}^{M} \sigma_1^k\right)^{2+\epsilon}\right] \\ &\leq E_x\left[M^{1+\epsilon}\sum_{k=1}^{M}(\sigma_1^k)^{(2+\epsilon)}\right] \\ &\leq c_2 E_x M^{2+\epsilon} = C_2 < \infty. \end{aligned}$$

Consequently $E_\nu L_k^{2+\epsilon} < \infty$ for $k \geq 2$ and $E_\alpha L_1^{2+\epsilon} < \infty$.

Next let J denote the index of the first cycles such that X_t hits F_n during that cycle. Consequently,

$$\begin{aligned} E_\alpha\left[(T_1^{(F_n\to)A})^{2+\epsilon}\right] &\leq E_\alpha\left[(L_1 + \sum_{k=2}^{J} L_k)^{2+\epsilon}\right] \\ &\leq 2^{1+\epsilon}\left[E_\alpha(L_1)^{2+\epsilon} + E_\nu(\sum_{k=2}^{J} L_k)^{2+\epsilon}\right] \\ &\leq 2^{1+\epsilon}\left[E_\alpha(L_1)^{2+\epsilon} + E_\nu(J^{1+\epsilon}\sum_{k=2}^{J} L_k^{2+\epsilon})\right] \\ &\leq 2^{1+\epsilon}\left[E_\alpha L_1^{2+\epsilon} + C_2 E_\nu J^{2+\epsilon})\right]. \end{aligned}$$

Next,

$$E_\nu J^{2+\epsilon} = p'_n + (1 - p'_n)\sum_{k=2}^{\infty} k^{2+\epsilon}(1 - p_n)^{k-1} p_n,$$

where p'_n is the probability of hitting F_n during the first cycle and p_n is the probability of hitting F_n during any subsequent cycle. Since $\exp(-p_n) \geq 1 - p_n$ the above expression is bounded by

$$1 + \frac{1}{(1-p_n)^2}\int_0^\infty x^{2+\epsilon} p_n \exp(-p_n x)dx = 1 + \frac{1}{(1-p_n)^2}\frac{1}{p_n^{2+\epsilon}}\Gamma(3+\epsilon).$$

Consequently, as $n \to \infty$,

$$E_\alpha \left[(T_1^{(F_n \to)A})^{2+\epsilon} \right] = \mathcal{O}(p_n^{-(2+\epsilon)}).$$

On the other hand,

$$E_\alpha[T_1^{(A\to)F_n}] \geq E_\alpha L_1 + E_\nu \sum_{k=2}^{J-1} L_k = E_\alpha L_1 + \left(\frac{1}{p_n} - 2 \right) E_\nu L_2.$$

■

Proof of Corollary 4: This result is a corollary of Theorems 3 and 5. In the Markov case considered here

$$E_0^{(F_n\to)F_n}[(T_1^{\to F_n})^2] = g_n,$$

so that the finiteness of g_n implies (2.19). Similarly, using the fact that

$$\lambda^{\to F_n} = \int_{y \in F_n^c} K(y, F_n)\pi(dy),$$

we see that (3.63) is equivalent to (2.20), which in turn implies (2.16).

In order to be able to apply Theorem 3, we still have to check (2.17). This requires more than the result of Theorem 5 since we require the second moments of $T_1^{\to F_n}/E_\pi T_1^{\to F_n}$ to converge to the second moment of the exponential of mean 1, that is to 2. This will follow if we can establish that the sequence $(T_1^{\to F_n}/E_\pi T_1^{\to F_n})^2$ is uniformly integrable w.r.t. P_π. A sufficient condition for this is that $E_\pi(T_1^{\to F_n}/E_\pi T_1^{\to F_n})^{2+\epsilon}$ be uniformly bounded as $n \to \infty$, where $\epsilon > 0$ (see Theorem 4.5.2 in Chung (1974)). This follows from Lemma 10 with $\alpha = \pi$ because

$$E_\pi \left[(T_1^{\to F_n}/ET_1^{\to F_n})^{2+\epsilon} \right] \leq \mathcal{O}(1)$$

means $(T_1^{\to F_n}/ET_1^{\to F_n})^2$ is a uniformly integrable sequence. ■

We now conclude this section by showing that the current setting also allows one to apply Corollary 2.

Corollary 5 *Suppose A contains a small set C such that*

$$E_x \sigma^{2+\epsilon} < c \text{ for all } x \in A \tag{3.65}$$

where σ is the first return time to C. Further suppose that for all n, Condition (2.38) holds. Then

$$E_\pi(T_1^{\to F_n} | X_0 \in A) \sim \Lambda_n^{-1}. \tag{3.66}$$

Proof: We check the conditions of Corollary 2. We have assumed our chain is Harris recurrent relative to ϕ. Consequently, by Theorem 1 in Athreya and Pantula (1986), our chain is strongly mixing. Consequently the cycles started by departures from A are strongly mixing

Condition (2.57) holds automatically in the discrete time setting. Condition (2.48) follows from (3.66). Taking α to be the initial measure corresponding to the Palm measure $E_0^{A\rightarrow}$ in Lemma 10 we get

$$E_0^{A\rightarrow}[(T_1^{\rightarrow F_n})^{2+\epsilon}] = \mathcal{O}(p_n^{-(2+\epsilon)}).$$

On the other hand, taking α to be the Palm measure $E_0^{(A\rightarrow)F_n}$ we get

$$\liminf_{n\rightarrow\infty} p_n E_0^{(A\rightarrow)F_n}[T_1^{(A\rightarrow)F_n}] > 0$$

and this means $\liminf_{n\rightarrow\infty} p_n(\Lambda_n)^{-1} > 0$. It follows that $E_0^{A\rightarrow}[(\Lambda_n T_1^{\rightarrow F_n})^{2+\epsilon}]$ is uniformly bounded and so Condition (2.49) holds. This completes the verification of the conditions of Corollary 2. ■

If X_t is a Markov jump process with transition rate matrix $Q(x,y)$ and stationary probability measure π then the intensity of the point process $N^{F_n(\rightarrow A)}$ is

$$\Lambda_n := \left(\sum_{y\in F}\pi(y)\sum_{x\in F^c} Q(y,x)f(x)\right) = 1,$$

where again f is defined in Proposition 2. We can extend the above analysis to show the asymptotics of the mean time to hit F_n are given by Λ_n^{-1}.

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