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Calculating tangent sets to certain sets in functional spaces

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Abstract: We give necessary and sufficient conditions for a given element to be a member of the second order tangent set $T''_K(f,v)$ to the positive cone K in L^∞ . Since, in general $T''_K(f,v)$ may be empty we give conditions on functions f , v which ensure that the second tangent set is a cone. As an application of the results obtained we give a characterization of the elements of the first and second tangent set to the set $B = \{u \in W^{1,\infty}(\Omega) \,|\, |\nabla u|^2 \leq 1\}$.

Key-words: tangent cone, second order tangent sets, shape optimization

(Résumé : tsvp)

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Sur les ensembles tangents du premier et deuxième ordre de certains ensembles dans L^∞ et $W^{1,\infty}$

Résumé : On donne des conditions nécessaires et suffisantes pour qu'un élément donné appartienne à l'ensemble tangent du deuxième ordre du cône positif K dans L^∞. Comme, en général, on peut avoir $T''_K(f,v) = \emptyset$ on étudie des conditions sur f, v pour que $T''_K(f,v)$ soit un cône. Ensuite, on applique les résultats obtenus pour calculer les ensembles tangents d'ordre un et deux de l'ensemble $B = \{u \in W^{1,\infty}(\Omega) \,|\, |\nabla u|^2 \leq 1\}$.

Mots-clé : cône tangent, ensembles tangents du deuxième ordre, optimisation de formes

1 Introduction

In the paper the set with local constraints on the gradient is considered. Such sets are important for applications in plasticity [9], control problems with the state constraints [7], [8], shape optimization [12].

For example in Hencky plasticity we have the following free boundary problem in the form of the variational inequality, see eg. [9] for details.

The problem describes the elastoplastic cylindrical rod with the Huber–von Mises plastic yield condition which in the case of Hook's law in elastic range correspond to the isotropic case and can be formulated as follows.

In a domain $\Omega \subset R^2$ find a solution to the variational inequality

$$v \in K_1 \quad : \quad \int_\Omega \nabla v \cdot (\nabla \bar{v} - \nabla v) dx \geq \int_\Omega \tilde{f}(\bar{v} - v) dx \quad \forall\, \bar{v} \in K_1 \ ,$$

where

$$K_1 = \{ v \in H^1_0(\Omega) \ | \ |\nabla v| \leq 1 \quad \text{a.e. in} \quad \Omega \} \ .$$

The domain Ω corresponds to a cross section of the cylindrical rod. The sought function v is called a stress function. Components of the stress tensor can be represented by the first derivatives of v.

We refer the reader to [4], [6] for the first and the second order optimality conditions with the first and the second order tangent sets.

Let $\Omega \subset R^N$ be a bounded domain. We consider the set

$$B = \{ u \in W^{1,\infty}(\Omega) \,|\, |\nabla u|^2 \leq 1 \} \, .$$

Our aim is to find the tangent cone $T'_B(u)$ for $u \in B$, and the second tangent set $T''_B(f, v)$ for $v \in T'_B(u)$.

For any subset A of a vector space Y, and any $a \in A$, by $T'_A(a)$ we denote the inner tangent cone of A at a, ie., $v \in T'_A(a)$ if and only if for each $t_n \downarrow 0$ there exists a sequence $v_n \in Y$ satisfying

(i) $\lim_{n\to\infty} v_n = v$,

(ii) for each n we have $a + t_n v_n \in A$.

Moreover, for any $v \in T'_A(a)$, $T''_A(a, v)$ denote the second inner tangent set, ie., $w \in T''_A(a, v)$ if and only if for each $t_n \downarrow 0$ there exists a sequence $w_n \in Y$ such that

(i) $\lim_{n\to\infty} w_n = w$,

(ii) for each n we have $a + t_n v + \frac{t_n^2}{2} w_n \in A$.

We have

$$B = F^{-1}(K) = \{u \in W^{1,\infty}(\Omega) \mid F(u) \in K\},$$

where $F : W^{1,\infty}(\Omega) \to L^\infty(\Omega)$, $F(u) = 1 - |\nabla u|^2$, and

$$K = \{f \in L^\infty(\Omega) \mid f \geq 0 \quad a.e.\ in \quad \Omega\}$$

is the positive cone in the space $L^\infty(\Omega)$. For simplicity we shall write $f \geq 0$ on Ω for $f \geq 0$ a.e. on Ω .

The following characterisation of $T'_K(f)$ was given by Cominetti and Penot [6].

Proposition 1.1 (Cominetti,Penot [6]) *The tangent cone $T'_K(f)$ to the positive cone K of $L^\infty(\Omega)$ is the set $\tilde{P}(f)$ of all v's in $L^\infty(\Omega)$ such that for any sequence (Q_n) in $\mathcal{B}$ one has*

$$\|1_{Q_n} f\|_\infty \to 0 \Longrightarrow \|1_{Q_n} v_-\|_\infty \to 0,$$

where $\mathcal{B}$ denotes the family of all measurable subsets of Ω .

In the sequel we use the following refinement of the above proposition with the particular form of Q_n .

Proposition 1.2 *The tangent cone $T'_K(f)$ to the positive cone K of $L^\infty(\Omega)$ is the set $P'(f)$ of all v's in $L^\infty(\Omega)$ such that*

$$\|1_{Q_n} v_-\|_\infty \to 0,$$

where $Q_n = \{x \in \Omega \mid f(x) \leq \frac{1}{n}\}$.

Proof. By Proposition 1.1 we know that

$$T'_K(f) = \tilde{P}(f) = \{v \in L^\infty(\Omega) \mid \|1_{\tilde{Q}_n} f\|_\infty \to 0 \Longrightarrow \|1_{\tilde{Q}_n} v_-\|_\infty \to 0 \text{ for any } \tilde{Q}_n \in \mathcal{B}\}.$$

We shall show that

$$\tilde{P}(f) = P'(f) = \{v \in L^\infty(\Omega) \,|\, \|1_{Q_n} v_-\|_\infty \to 0\} ,$$

where $Q_n = \{x \in \Omega \,|\, f(x) \le \frac{1}{n}\}$. Clearly, $\tilde{P}(f) \subset P'(f)$.

To prove the inclusion $P'(f) \subset \tilde{P}(f)$ let us take any $\tilde{Q}_n \in \mathcal{B}$ such that $\eta_n = \|1_{\tilde{Q}_n} f\|_\infty \to 0$. Without loss of generality we can assume that $\eta_n < 1$. For each $k > 1$ there exists N_k such that $\eta_n \le \frac{1}{k}$ for $n \ge N_k$. We define the sequence $\{a_n\}$ as follows:

$$a_n = \frac{1}{k-1} \quad \text{for all n such that} \quad N_{k-1} \le n < N_k .$$

For each n we have $\eta_n \le a_n$, since for each n there exists k such that $N_{k-1} \le n < N_k$, and $\eta_n \le \frac{1}{k-1} = a_n$.

Hence, all the sets $A_n = \tilde{Q}_n \setminus Q_{a_n}$ are of measure zero and

$$\|1_{\tilde{Q}_n} v_-\|_\infty \le \|1_{Q_{a_n}} v_-\|_\infty + \|1_{A_n} v_-\|_\infty .$$

Since $v \in P'(f)$ we have $\|1_{Q_{a_n}} v_-\|_\infty \to 0$ and we obtain the conclusion.

□

Remark 1.1 *Since* $intK = \{f \in L^\infty(\Omega) \,|\, ess\ inf\{f(x) \mid x \in \Omega\} > 0\}$, *by Proposition 1.2, for* $f \in intK$, $T'_K(f) = L^\infty(\Omega)$.

Remark 1.2 *The condition* $v(x) \ge 0$ *on* $Z(f) = \{x \in \Omega | f(x) = 0\}$ *is clearly necessary but not sufficient for* v *to be in* $T'_K(f)$. *This can be shown by some elementary examples [6].*

2 The first tangent cone to B.

The function $F : W^{1,\infty}(\Omega) \to L^\infty(\Omega)$, $F(u) = 1 - |\nabla u|^2$, is Fréchet differentiable. We have

$$F(u+v) = 1 - |\nabla(u+v)|^2 = 1 - |\nabla u|^2 - 2\nabla u \cdot \nabla v - |\nabla v|^2 .$$

Moreover,

$$\frac{\|\nabla v\|_\infty^2}{\|v\|_{W^{1,\infty}}} \to 0$$

when $\|v\|_{W^{1,\infty}} \to 0$, since $\|v\|_{W^{1,\infty}} \geq \|\nabla v\|_\infty$.

The mapping $F'(u)h = -2\nabla u \cdot \nabla h$ is clearly linear and continuous. The continuity of F' follows from the fact that there exists a constant $c > 0$ such that

$$\|\nabla u \cdot \nabla h\|_\infty \leq c\|\nabla h\|_\infty \leq c\|h\|_{W^{1,\infty}}.$$

We start by showing the following inclusion.

Theorem 2.1 *For any $u \in B$*

$$T_B'(u) \subset \{v \in W^{1,\infty}(\Omega) \,|\, \|1_{Q_n}(\nabla u \cdot \nabla v)_+\|_\infty \to 0\},$$

where $Q_n = \{x \in \Omega \,|\, |\nabla u|^2 \geq 1 - \frac{1}{n}\}$.

Proof. This inclusion follows from the fact (see eg. Cominetti [4]) that in normed spaces X, Y for any subsets $L \subset X$, $M \subset Y$, and a Fréchet differentiable map $f : X \to Y$ we have

$$T'_{L\cap f^{-1}(M)}(x) \subset T'_L(x) \cap f'(x)^{-1}T'_M(f(x)). \tag{1}$$

To make the presentation self-contained we give direct calculations conforming to our problem.

If $v \in T_B'(u)$, for any $t_n \downarrow 0$, there exists a sequence v_n, $v_n \to v$ in $W^{1,\infty}(\Omega)$, such that $u + t_n v_n \in B$ for all n. Hence

$$[1 - |\nabla u|^2] + t_n[-2\nabla u \cdot \nabla v_n + t_n|\nabla v_n|^2] \geq 0,$$

ie.,

$$F(u) + t_n z_n \in K,$$

where $z_n = -2\nabla u \cdot \nabla v_n - t_n|\nabla v_n|^2$. Since $\|\nabla(v_n - v)\|_\infty \leq \|v_n - v\|_{W^{1,\infty}(\Omega)}$, we have $z_n \to z = -2\nabla u \cdot \nabla v = F'(u)v$ in $L^\infty(\Omega)$, which means that

$$z = F'(u)v \in T_K(F(u)).$$

By Proposition 1.2,

$$\|1_{Q_n}(\nabla u \cdot \nabla v)_+\|_\infty \to 0\,,$$

where $Q_n = \{x \in \Omega \,|\, F(u) \le \frac{1}{n}\}$. This proves the required inclusion. □

It is worth noticing that the inclusion opposite to that of (1) does not hold in general. For the conditions ensuring this inclusion see eg. Cominetti [4], Frankowska, Aubin [1].

The result presented below gives a direct proof of the inclusion converse to (1) for our problem.

Theorem 2.2 *For any $u \in B$ we have*

$$\{v \in W^{1,\infty}(\Omega) \,|\, \|1_{Q_n}(\nabla u \cdot \nabla v)_+\|_\infty \to 0\} \subset T'_B(u)\,,$$

where $Q_n = \{x \in \Omega \,|\, 1 - |\nabla u|^2 \le \frac{1}{n}\}$.

Proof. We choose v_n in the form

$$v_n = v - \frac{1}{n}u$$

and we show that $v_n \in T'_B(u)$. Since $T'_B(u)$ is closed, this entails $v \in T'_B(u)$.

To this aim we prove that for each n and all t sufficiently small we have

$$|\nabla(u + t(v - \frac{1}{n}u)|^2 \le 1\,, \tag{2}$$

ie.,

$$t^2(\frac{1}{n^2}|\nabla u|^2 - \frac{2}{n}\nabla u \cdot \nabla v + |\nabla v|^2) + t(-\frac{2}{n}|\nabla u|^2 + 2\nabla u \cdot \nabla v) + |\nabla u|^2 - 1 \le 0\,.$$

We start by checking inequality 2 on the set $Q_m = \{x \in \Omega \,|\, 1 - |\nabla u|^2 \le \frac{1}{m}\}$. Observe that we may assume that $\nabla v \ne 0$. Denote

$$p = |\nabla u|^2\,, \quad q = \nabla u \cdot \nabla v\,, \quad r = |\nabla v|^2\,.$$

With this notation inequality 2 takes the form

$$t^2(\frac{1}{n^2}p - \frac{2}{n}q + r) + t(2q - \frac{2}{n}p) + (p-1) \le 0\,. \tag{3}$$

Note that
$$\begin{aligned}\frac{1}{n^2}p - \frac{2}{n}q + r &= \tfrac{1}{n^2}|\nabla u|^2 - \tfrac{2}{n}\nabla u \nabla v + |\nabla v|^2 \\ &= [\nabla v - \tfrac{1}{n}\nabla u]^2 \\ &= |\nabla v_n|^2\end{aligned}$$
and hence
$$\begin{aligned}\Delta(x) &= 4(q - \tfrac{1}{n}p)^2 - 4(p-1)|\nabla v_n|^2 \\ &= 4(q - \tfrac{1}{n}p)^2 + 4(1-p)|\nabla v_n|^2 \geq 0\,.\end{aligned}$$

Consequently, the positive square root of the left-hand side quadratic form of 3 is equal to
$$t(x) = \frac{\frac{1}{n}p - q + \sqrt{\frac{\Delta(x)}{4}}}{|\nabla v_n|^2}\,.$$

Since $\|1_{Q_m} q_+\| \to 0$ we have $q_+ \leq \frac{1}{2\cdot n^2} < \frac{1}{n^2}$ for $x \in Q_m$ and all m sufficiently large, say, $m \geq m(n)$, and hence, for $x \in Q_m$, $m \geq m(n)$, we have
$$\frac{1}{n}p - q = \frac{1}{n}p - q_+ + q_- \geq \frac{1}{n}(1 - \frac{1}{m}) - \frac{1}{n^2} = \frac{1}{n}(1 - \frac{1}{m} - \frac{1}{n}) \geq \frac{1}{4n}$$
for $m \geq 4\,,\ n \geq 2$.

Moreover,
$$\begin{aligned}\Delta(x) &= 4(q - \tfrac{1}{n}p)^2 - 4(p-1)(\tfrac{1}{n^2}p - \tfrac{2}{n}q + r) \\ &= 4(q^2 - \tfrac{2}{n}pq + \tfrac{1}{n^2}p^2 - \tfrac{1}{n^2}p^2 + \tfrac{1}{n}pq - pr + \tfrac{1}{n^2}p - \tfrac{2}{n}q + r) \\ &= 4(q^2 - pr + \tfrac{1}{n^2}p - \tfrac{2}{n}q + r) \\ &= 4(q^2 + r(1-p) + [\tfrac{1}{n^2}p - \tfrac{2}{n}q_+] + \tfrac{2}{n}q_-)\,.\end{aligned}$$

Since $q_+ \leq \frac{1}{n^2}$ for $x \in Q_m$, $m \geq m(n)$, $m > 4$, and $n > 2$ we have
$$\frac{1}{n^2}p - \frac{2}{n}q_+ \geq \frac{1}{n^2}(1 - \frac{1}{m} - \frac{1}{n}) \geq \frac{1}{4\cdot n^2}\,.$$

Finally, for $\nabla v_n \neq 0$, we have
$$\begin{aligned}t(x) &= \tfrac{\frac{1}{n}p - q + \sqrt{\Delta/4}}{|\nabla v_n|^2} \\ &\geq \tfrac{\frac{1}{4n} + \frac{1}{2n}}{\|\nabla v_n\|^2} \\ &\geq \tfrac{1}{2n\|\nabla v_n\|^2}\,.\end{aligned}$$

This means that for $x \in Q_m$, $m \geq m(n)$, and any positive $t \leq \frac{1}{2n\|\nabla v_n\|_\infty^2}$ we have $u + tv_n \in B$. Observe that we can suppose that $\|\nabla v_n\|_\infty \neq 0$ since we need to analyse only $\|\nabla v\|_\infty \neq 0$.

If $x \notin Q_m$, then $|\nabla u|^2 < 1 - \frac{1}{m}$, ie., $(1 - |\nabla u|^2) > \frac{1}{m}$. Since

$$\begin{aligned}(1-p) - 2t\nabla u\nabla v_n - t^2|\nabla v_n|^2 &\geq \tfrac{1}{m} - 2t|\nabla v_n| \cdot |\nabla u| - t^2\|\nabla v_n\|^2 \\ &\geq \tfrac{1}{m} - 2t\|\nabla v_n\|_\infty \cdot \|\nabla u\|_\infty - t^2\|\nabla v_n\|_\infty^2 ,\end{aligned}$$

by taking $t \leq \frac{1}{m^2\|\nabla v_n\|_\infty}$, we obtain

$$\begin{aligned}\frac{1}{m} - 2t\|\nabla v_n\|_\infty \cdot \|\nabla u\|_\infty - t^2\|\nabla v_n\|_\infty^2 &\geq \tfrac{1}{m} - \tfrac{2\|\nabla u\|_\infty}{m^2} - \tfrac{1}{m^4} \\ &\geq 0 ,\end{aligned}$$

because for all m sufficiently large, $m \geq m_1$, $m \geq (\|\nabla u\|_\infty + \frac{1}{m^2})$.

Clearly, $v_n \to v$ in $W^{1,\infty}(\Omega)$.

□

Remark 2.1 *Since the choice of a sequence v_n is crucial for the inclusion of Theorem 2.2 it could be interesting to know whether other choices of v_n are possible.*

Below we present another possible choice of v_n in the one-dimensional case, where $\Omega = I$ is an interval of reals.

We have

$$B = \{u \in W^{1,\infty}(I) \,|\, F(u) = 1 - u_x^2 \geq 0\} ,$$

or

$$1 - u_x^2 \in K = \{f \in L^\infty(I) \,|\, f \geq 0\} .$$

Hence, for any v such that $\|1_{Q_n}(u_x v_x)_+\|_\infty \to 0$ and any $t_n \downarrow 0$ we need to find a sequence $v_n \in W^{1,\infty}(\Omega)$, $v_n \to v$ in $W^{1,\infty}(\Omega)$, such that $u + t_n v_n \in B$, ie.,

$$1 - u_x^2 + t_n[-2u_x v_{nx} - t_n v_{nx}^2] = 1 - u_x^2 + t_n z_n \geq 0 .$$

For any fixed $0 < \delta < 1$ consider the set $Z = \{x \in I \,|\, u_x^2 > \delta\}$. On Z we seek the sequence $\{z_n\}$ in the form

$$z_n = -2u_x v_x + 1_{Q_n}(2 \cdot u_x v_x)_+ .$$

We have $z_n \to z = -2u_x v_x$ in $L^\infty(\Omega)$. Note that by the construction of Penot and Cominetti [6]

$$1 - u_x^2 + tz_n \geq 0 . \qquad (*)$$

for $t \in]0, \frac{1}{n\|v\|_{L^\infty}}]$. Hence, we want to find $\{v_{nx}\}$ such that

$$-2u_x v_{nx} - tv_{nx}^2 = -2u_x v_x + 2 \cdot 1_{Q_n}(u_x v_x)_+ ,$$

ie.,

$$-tv_{nx}^2 - 2u_x v_{nx} + 2u_x v_x - 2 \cdot 1_{Q_n}(u_x v_x)_+ = 0 ,$$

ie.,

$$tv_{nx}^2 + 2u_x v_{nx} - 2u_x v_x + 2 \cdot 1_{Q_n}(u_x v_x)_+ = 0 ,$$

ie.,

$$tv_{nx}^2 + 2u_x v_{nx} + z_n = 0 .$$

For this quadratic equation

$$\Delta = 4u_x^2 - 4t[-2u_x v_x + 2 \cdot 1_{Q_n}(u_x v_x)_+] = 4u_x^2 - 4tz_n .$$

On Z we have

$$\begin{aligned} u_x^2 - tz_n &= u_x^2 - t(z_n - z) - tz \\ &\geq u_x^2 - t\|z_n - z\|_{L^\infty} - t\|z\|_{L^\infty} \\ &\geq \delta - \varepsilon > 0 , \end{aligned}$$

for $\varepsilon < \delta$ and all n sufficiently large. Hence, the sequence

$$v_{nx} = \frac{-2u_x \pm 2\sqrt{u_x^2 - tz_n}}{2t_n} = \frac{1}{t}[-u_x \pm \sqrt{u_x^2 - tz_n}] .$$

is well-defined on Z.

On $I \setminus Z$ we have $1 - u_x^2 > 1 - \delta$, and hence

$$\begin{aligned} 1 - u_x^2 + t[-2u_x v_x - t_n v_x] &\geq (1-\delta) - 2 \cdot t\|u_x v_x\|_{W^{1,\infty}} - t^2\|v_x^2\|_{W^{1,\infty}} \\ &\geq 0 , \qquad (**) \end{aligned}$$

for all n sufficiently large. Finally, we define

$$v_{nx} = \begin{cases} \frac{1}{t}[-u_x \pm \sqrt{u_x^2 - tz_n}] & x \in Z \\ v_x & x \in I \setminus Z \end{cases}$$

According to $(*)$, $(**)$ we have

$$u + tv_n \in B\,,$$

for $t \in]0, \frac{1}{n\|v\|_\infty}]$. To complete the proof we need to show that $v_n \to v$ in $W^{1,\infty}(I)$.

To this aim we develop the function $\sqrt{\cdot}$ in a neighbourhood of u_x^2. Since

$$\sqrt{x+t} = \sqrt{x} + \frac{1}{2\sqrt{x}}t + o(t)$$

we have

$$v_{nx} = \frac{1}{t_n}[-u_x \pm (|u_x| - \frac{t_n z_n}{2|u_x|} + o(t_n z_n))]\,,$$

where $\frac{o(t_n z_n)}{t_n z_n} \to 0$. By chosing " + " if $u_x > 0$ and " − " if $u_x < 0$ we obtain

$$v_{nx} = \pm\frac{z_n}{2|u_x|} + \frac{o(t_n z_n)}{t_n}\,,$$

and consequently,

$$v_{nx} = \pm\frac{z_n}{2|u_x|} + \frac{o(t_n z_n)}{t_n} \to \pm\frac{z}{2|u_x|} = w\,,$$

in $L^\infty(I)$, and according to the above choice of sign we now have $w = v_x$. We obtain the convergence $v_n \to v$ in $W^{1,\infty}(I)$ by integration.

3 Second order tangent set to the positive cone in L^∞

Following [1],[6] we use also the outer second tangent set $T_A^2(a,v)$ to a subset A of X at $a \in A$ in the direction $v \in X$ defined by the formula

$$T_A^2(a,v) = \limsup_{t \downarrow 0} \frac{A - a - tv}{t^2/2}\,.$$

It is obvious that $T_A''(a,v) \subset T_A^2(a,v)$.

We start with the following characterisation of elements of the second order tangent sets $T_K''(f,v)$, $T_K^2(f,v)$.

Theorem 3.1 *Let* $f \in K$ *and* $v \in T'_K(f)$, $w \in L^\infty(\Omega)$. *The following are equivalent*

(i) $w \in T''_K(f,v)$ ($w \in T^2_K(f,v)$),

(ii) $\lim_{t\downarrow 0} \|\frac{1}{t^2}[f+tv+\frac{t^2}{2}w]_-\|_\infty = 0$ *(there exists a sequence* $t_n \downarrow 0$ *such that* $\lim_n \|\frac{1}{t_n^2}[f+t_nv+\frac{t_n^2}{2}w]_-\|_\infty = 0$).

Proof.

$(i) \to (ii)$. We shall give here the proof for unbracketed part. Let $w \in T''_K(f,v)$, and $t_n \downarrow 0$. There exists $w_n \to w$ in $L^\infty(\Omega)$ such that

$$f + t_n v + \frac{t_n^2}{2} w_n \geq 0 , \quad \text{ie., } \; w_n - w \geq \frac{2}{t_n^2}[-f - t_n v] - w \quad \text{a.e. on } \Omega ,$$

where

$$\frac{2}{t_n^2}[-f - t_n v] - w = \frac{2}{t_n^2}[f + t_n v + \frac{t_n^2}{2}w]_- - \frac{2}{t_n^2}[f + t_n v + \frac{t_n^2}{2}w]_+ .$$

Let

$$\Omega_{t_n} = \{-f - t_n v \geq \frac{t_n^2}{2} w\} .$$

Since $[f + t_n v + \frac{t_n^2}{2}w]_+ = 0$ a.e. on Ω_{t_n}, we obtain

$$w_n - w \geq \frac{2}{t_n^2}[f + t_n v + \frac{t_n^2}{2}w]_- \geq 0 \text{ a.e. on } \Omega_{t_n} \quad (*)$$

We clearly have $[f + t_n v + \frac{t_n^2}{2}w]_+ > 0$ a.e. on $\Omega \setminus \Omega_{t_n}$ and consequently

$$[f + t_n v + \frac{t_n^2}{2}w]_- = 1_{\Omega_{t_n}}[f + t_n v + \frac{t_n^2}{2}w]_- \text{ a.e. on } \Omega .$$

Since $w_n - w \to 0$ in $L^\infty(\Omega)$, by the latter formula and by $(*)$,

$$\|\frac{1}{t_n^2}[f + t_n v + \frac{t_n^2}{2}w]_-\|_\infty = \|1_{\Omega_{t_n}}\frac{1}{t_n^2}[f + t_n v + \frac{t_n^2}{2}w]_-\|_\infty \to 0 ,$$

and the conclusion follows.

$(ii) \to (i)$. Put

$$w_n = w + \frac{2}{t_n^2}[f + t_n v + \frac{t_n^2}{2} w]_- .$$

By assumption, $w_n - w \to 0$ in $L^\infty(\Omega)$. We have

$$y_n = f + t_n v + \frac{t_n^2}{2} w + [f + t_n v + \frac{t_n^2}{2} w]_- \geq 0 \quad \text{a.e. on } \Omega_{t_n} .$$

Moreover, $[f + t_n v + \frac{t_n^2}{2} w]_- = 0$ a.e on $\Omega \setminus \Omega_{t_n}$, and $y_n = [f + t_n v + \frac{t_n^2}{2} w]_+ \geq 0$ a.e. on $\Omega \setminus \Omega_{t_n}$ as well.

□

The following result follows as a corollary.

Theorem 3.2 *Let $f \in K$ and $v \in T'_K(f)$. The following are equivalent*
(i) $0 \in T''_K(f, v)$,
(ii) $\lim_{t \downarrow 0} \| \frac{1}{t^2}[f + tv]_- \|_\infty = 0$.

Proof.
$(i) \to (ii)$. Let $0 \in T''_K(f, v)$. For any $t_n \downarrow 0$ one can find a sequence $w_n \to 0$ such that

$$f + t_n v + \frac{t_n^2}{2} w_n \geq 0 , \quad \text{ie.,} \;\; w_n \geq \frac{2}{t_n^2}[-f - t_n v] \quad \text{a.e. on } \Omega .$$

Since $\frac{2}{t_n^2}[-f - t_n v] = \frac{2}{t_n^2}[f + t_n v]_- - \frac{2}{t_n^2}[f + t_n v]_+$, by putting $\Omega_{t_n} = \{v_+ = 0 ,\; t_n v_- - f \geq 0\}$ we obtain

$$w_n \geq \frac{2}{t_n^2}[f + t_n v]_- \geq 0 \text{ a.e. on } \Omega_{t_n} ,$$

which entails $\| 1_{\Omega_{t_n}} \frac{1}{t_n^2}[f + t_n v]_- \|_\infty \to 0$.

Almost everywhere on $\Omega \setminus \Omega_{t_n}$, either (a) $v_+ > 0$, or (b) $t_n v_- - f < 0$, or both. If (a) holds, then clearly $[f + t_n v]_- = 0$. In view of the inequality

$$f - t_n v_- \leq [f + t_n v]_+ ,$$

if (b) holds, $[f+t_n v]_+ > 0$, and thus $[f+t_n v]_- = 0$. Therefore,

$$[f+t_n v]_- = 1_{\Omega_{t_n}}[f+t_n v]_-\,,$$

and the conclusion follows.

$(ii) \to (i)$. By taking

$$w_n = \frac{2}{t_n^2}[f+t_n v]_-\,,$$

we have $w_n \to 0$ in $L^\infty(\Omega)$ and

$$y_n = f + t_n v + \frac{t_n^2}{2} w_n = [f+t_n v]_+ \geq 0 \text{ a.e. on } \Omega\,.$$

□

Let us note that in general $T''_K(f,v) \neq \emptyset$ does not imply that $0 \in T''_K(f,v)$.

Let f, $v \in L^\infty(\Omega)$. Define

$$\bar{Q}_n = \{x \in \Omega \,|\, f(x) \leq \frac{1}{n}\}\,, \qquad \bar{R}_n = \{x \in \Omega \,|\, v_+(x) \leq \frac{1}{n}\}\,.$$

Proposition 3.1 *Let* f, v, $w \in L^\infty(\Omega)$. *If* $\|1_{\bar{R}_n \cap \bar{Q}_n} w\|_\infty \to 0$, *then*

$$\|1_{R_n \cap Q_n} w\|_\infty \to 0\,,$$

where $Q_n = \{x \in \Omega \,|\, f(x) \leq q_n\}$, $R_n = \{x \in \Omega \,|\, v(x) \leq r_n\}$, *and* q_n, $r_n > 0$, $q_n, r_n \to 0$.

Proof. Immediate. □

Theorem 3.3 *Let* $f \in K$ *and* $v \in T'_K(f)$. *If* $0 \in T''_K(f,v)$, *then*

$$\{w \in L^\infty(\Omega) \,|\, \|1_{\bar{Q}_n \cap \bar{R}_n} w_-\|_\infty \to 0\} \subset T''_K(f,v)\,.$$

Proof. Let $w \in L^\infty(\Omega)$ and $\|1_{\bar{Q}_n \cap \bar{R}_n} w_-\|_\infty \to 0$. According to Theorem 3.1 we need to show that

$$\lim_{t \downarrow 0} \|\frac{1}{t^2}[f + tv + \frac{t^2}{2} w]_-\|_\infty = 0\,.$$

We may suppose that $v\,,\,w \neq 0$. Since $v \in T'_K(f)$, for any sequence $t_n \downarrow 0$ we have

$$\begin{aligned}
\Omega_{t_n} &= \{-f - t_n v \geq \tfrac{t_n^2}{2} w\} \\
&= \{v_+ = 0\,,\, f \leq t_n v_- - \tfrac{t_n^2}{2} w\} \cup \{v_- = 0\,,\, f \leq -t_n v_+ - \tfrac{t_n^2}{2} w\} \\
&\subset \{v_+ = 0\,,\, f \leq t_n v_- - \tfrac{t_n^2}{2} w\} \cup \{f \leq t_n \|v\|_\infty + \tfrac{t_n^2}{2}\|w\|_\infty\,,\, v_+ \leq -t_n/2w\} \\
&\subset \{v_+ \leq t_n \|w\|_\infty\} \cap \{f \leq t_n\|v\|_\infty + \tfrac{t_n^2}{2}\|w\|_\infty\}\,.
\end{aligned}$$

Denoting

$$Q_n = \{f \leq t_n\|v\|_\infty + t_n^2\|w\|_\infty\}\,, \quad R_n = \{v_+ \leq t_n\|w\|_\infty\}\,,$$

we obtain

$$\frac{1}{t_n^2}[f + t_n v + \frac{t_n^2}{2} w]_- \leq \tfrac{1}{t_n^2} 1_{\Omega_{t_n}}[f + t_n v]_- + 1_{Q_n \cap R_n} w_-\,.$$

In view of the assumptions and Proposition 3.1 the conclusion follows.

□

Alternative proof. Let $w \in L^\infty(\Omega)$ and $\|1_{\bar{Q}_n \cap \bar{R}_n} w_-\|_\infty \to 0$. If $v = 0$, by Proposition 1.2, $w \in T''_K(f, 0) = T'_K(f)$. Now we assume that $v\,,\,w \neq 0$.

Let $t_n \downarrow 0$ be a sequence of positive reals. We denote

$$Q_n = \{x \in \Omega \,|\, f(x) \leq t_n\|v\|_\infty + t_n^2\|w\|_\infty\}\,, \quad R_n = \{x \in \Omega \,|\, v_+(x) \leq t_n\|w\|_\infty\}\,.$$

Define the sequence

$$w_n = w + 1_{R_n \cap Q_n} w_- + 1_{Q_n^0 \cap R_n} \frac{2}{t_n^2}[f + t_n v]_-\,,$$

where $Q_n^0 = \{x \in \Omega \,|\, 0 < f(x) \leq t_n\|v\|_\infty + t_n^2\|w\|_\infty\}$, $Q_n = Q_n^0 \cup Z(f)$, where $Z(f) \subset \Omega$ is a subset such that $f = 0$ a.e. on $Z(f)$. We have $w_n \to w$ in $L^\infty(\Omega)$. Put $y_n = f + t_n v + \frac{t_n^2}{2} w_n$.

1^o. Suppose first that $x \notin Q_n^0$. If $x \notin Z(f)$, then

$$\begin{aligned}
y_n &\geq t_n\|v\|_\infty + t_n^2\|w\|_\infty + t_n v + \tfrac{t_n^2}{2} w \\
&\geq 2t_n v_+ + t_n^2\|w\|_\infty - \tfrac{t_n^2}{2} \cdot \|w\|_\infty \\
&\geq 0
\end{aligned}$$

If $x \in Z(f)$, then two situations may occur. If $x \notin R_n$, then $y_n \geq t_n v - \frac{t_n^2}{2}\|w\|_\infty \geq 0$. If $x \in R_n$, then $y_n \geq t_n v + \frac{t_n^2}{2} w_+ \geq 0$.

2^o. Secondly, suppose that $x \in Q_n^0$. If $x \notin R_n$, then $y_n \geq t_n v - \frac{t_n^2}{2}\|w\|_\infty \geq 0$. Finally, if $x \in Q_n^0 \cap R_n$, then

$$\begin{aligned} y_n &\geq [f + t_n v]_+ - [f + t_n v]_- + \tfrac{t_n^2}{2} w_n \\ &= [f + t_n v]_+ + \tfrac{t_n^2}{2} w_+ \geq 0 . \end{aligned}$$

□

To show the necessity of the condition $\|1_{\bar{R}_n \cap \bar{Q}_n} w_-\|_\infty \to 0$ for w to be an element of $T''_K(f, v)$ we start by showing the following auxiliary result.

Proposition 3.2 *For $v \in T'_K(f)$ we have*

$$\|1_{\bar{R}_n \cap \bar{Q}_n} v\|_\infty \to 0 .$$

Proof. Let $v \in T'_K(f)$. We have $\|1_{Q_n} v_-\|_\infty \to 0$, and hence, for $x \in \bar{R}_n \cap \bar{Q}_n$, we have $v_-(x) \leq \eta_n$, for a.a. $x \in \bar{R}_n \cap \bar{Q}_n$, where η_n is a sequence of nonnegative reals, $\eta_n \to 0$. Hence,

$$|v(x)| = v_+(x) + v_-(x) \leq \eta_n + \frac{1}{n} = \kappa_n \quad \text{a.e on} \quad \bar{R}_n \cap \bar{Q}_n ,$$

and $\kappa_n \to 0$.

□

There exist examples showing that in the spaces $C(\Omega)$ and $L^p(\Omega)$ the second tangent sets $T''_K(f, v)$ may be empty for some functions f, and v.

Theorem 3.4 *If $w \in T''_K(f, v)$, then*

$$\|1_{\bar{R}_n \cap \bar{Q}_n} w_-\| \to 0 .$$

Proof

If $w \in T''_K(f, v)$, then for any sequence $t_n \downarrow 0$ one can find $w_n \to w$ such that

$$f + t_n v + \frac{t_n^2}{2} w_n \geq 0 .$$

Hence,

$$f + t_n v + \frac{t_n^2}{2}(w_n - w) \geq -\frac{t_n^2}{2} w\,.$$

Consequently,

$$w \geq (w - w_n) - \frac{2}{t_n^2}[-f - t_n v]\,,$$

and

$$w_- \leq (w_n - w) + \frac{2}{t_n^2} f + \frac{2}{t_n} v\,. \quad (*)$$

On $\bar{D}_n^z = Z(f) \cap \bar{R}_n$ inequality $(*)$ takes the form

$$w_- \leq (w_n - w) + \frac{2}{t_n} v\,.$$

By Proposition 3.2, $\|1_{\bar{D}_n^z} v\|_\infty \to 0$, hence, by taking $t_n = \frac{\|1_{\bar{D}_n^z} v\|_\infty}{\varepsilon}$, where $\varepsilon > 0$ we get

$$w_- \leq (w_n - w) + \varepsilon$$

and since ε is arbitrary, $\|1_{\bar{D}_n^z} w_-\|_\infty \to 0$.

On $\bar{D}_n^0 = \bar{R}_n \cap \bar{Q}_n^0$, where $\bar{Q}_n^0 = \{x \in \Omega \,|\, 0 < f(x) \leq \frac{1}{n}\}$,

$$w_- \leq (w_n - w) + \frac{2}{t_n^2}\frac{1}{n} + \frac{2}{t_n}\frac{1}{n}\,,$$

and by chosing $t_n = \frac{2}{\log n}$ we get

$$\|1_{\bar{D}_n^0} w_-\|_\infty \leq \|w_n - w\|_\infty + \frac{\log^2 n}{n} + \frac{\log n}{n}\,.$$

Observe that one can also choose $t_n = \frac{2\max\{\|1_{\bar{D}_n^0} f\|_\infty^{1/2}, \|1_{\bar{D}_n^0} v\|_\infty^{1/2}\}}{\varepsilon}$. Indeed, if $\|1_{\bar{D}_n^0} f\|_\infty \geq \|1_{\bar{D}_n^0} v\|_\infty$, then

$$\begin{aligned} w_- &\leq (w_n - w) + \tfrac{2}{t_n^2} f + \tfrac{2}{t_n} v \\ &= (w_n - w) + \tfrac{\varepsilon^2}{\|1_{\bar{D}_n^0} f\|_\infty} f + \tfrac{\varepsilon}{\|1_{\bar{D}_n^0} f\|_\infty^{1/2}} v \\ &\leq \|w_n - w\|_\infty + \varepsilon^2 + \tfrac{\varepsilon}{\|1_{\bar{D}_n^0} v\|_\infty^{1/2}} v \\ &\leq \|w_n - w\|_\infty + \varepsilon^2 + \varepsilon \|1_{\bar{D}_n^0} v\|_\infty^{1/2}\,. \end{aligned}$$

Since $\|1_{\bar{D}_n^0} v\|_\infty \to 0$ and ε is arbitrary we get the required convergence.

If $\|1_{\bar{D}_n^0} v\|_\infty \geq \|1_{\bar{D}_n^0} f\|_\infty$, then

$$\begin{aligned} w_- &\leq (w_n - w) + \frac{\varepsilon^2}{\|1_{\bar{D}_n^0} v\|_\infty} f + \frac{\varepsilon}{\|1_{\bar{D}_n^0} v\|_\infty^{1/2}} v \\ &\leq \|w_n - w\|_\infty + \frac{\varepsilon^2}{\|1_{\bar{D}_n^0} f\|_\infty} f + \frac{\varepsilon}{\|1_{\bar{D}_n^0} v\|_\infty^{1/2}} v \\ &\leq \|w_n - w\|_\infty + \varepsilon^2 + \varepsilon \|1_{\bar{D}_n^0} v\|_\infty^{1/2}, \end{aligned}$$

which completes the proof.

□

Summarizing the above results we obtain

Theorem 3.5 *Let $f \in K$, $v \in T'_K(f)$ and $\lim_{t\downarrow 0} \|\frac{1}{t^2}[f + tv]_-\|_\infty = 0$. Then*

$$T''_K(f, v) = \{w \in L^\infty(\Omega) \,|\, \|1_{\bar{R}_n \cap \bar{Q}_n} w_-\|_\infty \to 0\},$$

where $\bar{Q}_n = \{x \in \Omega \,|\, f \leq \frac{1}{n}\}$, and $\bar{R}_n = \{x \in \Omega \,|\, v_+ \leq \frac{1}{n}\}$.

In view of Proposition 3.1 of [4], if $0 \in T''_K(f, v)$, then $T''_K(f, v) = T'_{T'_K(f)}(v)$, and hence, $T''_K(f, v)$ is a closed convex cone for any $f,\ v$, satisfying $\lim_{t\downarrow 0} \|\frac{1}{t^2}[f + tv]_-\|_\infty = 0$.

Remark 3.1 *If $w \in T''_K(f, v)$, then, by Theorem 3.5, $w \geq 0$ a.e. on $Z(f) \cap Z(v)$, where, as previously, $f = 0$ a.e. on $Z(f)$. This latter condition is, however, not sufficient for w to be in $T''_K(f, v)$.*

4 Second tangent set to the set B.

Basing ourselves on the results of the previous section we give formulae for the second tangent sets to B .

Theorem 4.1 *For any $u \in B$ and $v \in T'_B(u)$ we have*

$$T''_B(u, v) \subset \{w \in W^{1,\infty}(\Omega) \,|\, \|1_{D_n}[\nabla u \cdot \nabla w + |\nabla v|^2]_+\|_\infty \to 0\},$$

where $D_n = \tilde{Q}_n \cap \tilde{R}_n$, and

$$\tilde{Q}_n = \{x \in \Omega \,|\, 1 - |\nabla u|^2 \leq \frac{1}{n}\}, \qquad \tilde{R}_n = \{x \in \Omega \,|\, 2[\nabla u \cdot \nabla v]_- \leq \frac{1}{n}\}.$$

Proof. In general, for a twice Fréchet differentiable function F and any $w \in T''_{F^{-1}(K)}(u,v)$ we have

$$F'(u)w + F''(u)(v,v) \in T''_K(F(u), F'(u)v)$$

(see eg. [1], [4]). Indeed, let $w \in T''_B(u,v)$, and $t_n \downarrow 0$. There exists $w_n \to w$ in $L^\infty(\Omega)$ such that $u + t_n v + \frac{2}{t^2} w_n \in B$, ie.,

$$[1-|\nabla u|^2] - 2t_n \nabla u \cdot \nabla v - \frac{t_n^2}{2}[2\nabla u \cdot \nabla w_n + 2|\nabla(v + \frac{t_n}{2} w_n)|^2] \geq 0 .$$

In other words

$$\tilde{u} + t_n \tilde{v} + \frac{t_n^2}{2} \tilde{w_n} \in K ,$$

where $\tilde{u} = 1 - |\nabla u|^2$, $\tilde{v} = -2\nabla u \cdot \nabla v$, and $\tilde{w_n} = -2\nabla u \cdot \nabla w_n - 2|\nabla(v + \frac{t_n}{2} w_n)|^2$.

Moreover, $\tilde{w_n} \to \tilde{w} = -2\nabla u \cdot \nabla w - 2|\nabla v|^2$ in $L^\infty(\Omega)$, and $\tilde{w} \in T''_K(\tilde{u}, \tilde{v})$. By Theorem 3.4,

$$\|1_{D_n}[|\nabla v|^2 + \nabla u \cdot \nabla w]_+\|_\infty \to 0 .$$

$\square$

To complete our considerations we shall prove the following result.

Theorem 4.2 *Let* $u \in B$, $v \in T'_B(u)$, *and* $\lim_{t\downarrow 0} \|\frac{1}{t^2}[1 - |\nabla u|^2 - 2t\nabla u \cdot \nabla v]_-\|_\infty = 0$. *The following inclusion holds*

$$\{w \in W^{1,\infty}(\Omega) \,|\, \|1_{D_n}[|\nabla v|^2 + \nabla u \cdot \nabla w]_+\|_\infty \to 0\} \subset T''_B(u,v) ,$$

where, as previously, $D_n = \tilde{R}_n \cap \tilde{Q}_n$, *and*

$$\tilde{Q}_n = \{x \in \Omega \,|\, 1 - |\nabla u|^2 \leq \frac{1}{n}\} , \qquad \tilde{R}_n = \{x \in \Omega \,|\, 2[\nabla u \cdot \nabla v]_- \leq \frac{1}{n}\} .$$

Proof. Let $u \in B$, $v \in T'_B(u)$, and $\lim_{t\downarrow 0} \|\frac{1}{t^2}[1 - |\nabla u|^2 - 2t\nabla u \cdot \nabla v]_-\|_\infty = 0$. If $v = 0$, then $T''_B(u,0) = T'_B(u)$, and, by Theorem 2.2, the result holds true. Hence, we may suppose that $v \neq 0$.

We shall show that for any w such that $\|1_{D_n}[|\nabla v|^2 + \nabla u \cdot \nabla w]_+\|_\infty \to 0$, there exists a sequence $w_n \to w$ in $W^{1,\infty}(\Omega)$ such that for all t sufficiently small we have

$$y(n,t) = 1 - |\nabla(u + tv + \frac{t^2}{2} w_n)|^2 \geq 0 ,$$

where

$$\begin{aligned} y(n,t) &= 1 - |\nabla u|^2 - 2t\nabla u \cdot \nabla v - t^2[|\nabla v|^2 + \nabla u \cdot \nabla w_n] - \\ & t^2[t\nabla v \cdot \nabla w_n + \tfrac{t^2}{4}|\nabla w_n|^2]\,. \qquad (*) \end{aligned}$$

We may suppose that $\|\nabla v\|_\infty \neq 0$. Set

$$q = \|\nabla u\|_\infty \cdot \|\nabla v\|_\infty\,, \quad p = \|\nabla v\|_\infty^2 + \|\nabla u\|_\infty \cdot \|\nabla w\|_\infty\,.$$

Observe that p, $q \neq 0$. Put

$$Q_n = \{x \in \Omega \,|\, 1 - |\nabla u|^2 \leq \frac{2}{n} \cdot q\}\,, \qquad R_n = \{x \in \Omega \,|\, 2[\nabla u \nabla v]_- \leq \frac{1}{n} \cdot p\}\,.$$

By assumption,

$$\|1_{Q_n \cap R_n}[|\nabla v|^2 + \nabla u \cdot \nabla w]_+\|_\infty = k_n \to 0\,,$$

and, for each n, there exists $\delta(n)$ such that

$$\frac{1}{t^2}[\tilde{u} + t\tilde{v}]_- \leq k_n \quad a.e.\ on \quad \Omega$$

for all $t \leq \delta(n)$, where, as previously, $\tilde{u} = 1 - |\nabla u|^2$, $\tilde{v} = -2\nabla u \cdot \nabla v$. We choose w_n in the form

$$w_n = w - 4k_n u\,.$$

Consider first the case $\|\nabla w\|_\infty = 0$. Then

$$\begin{aligned} y(n,t) = \ & 1 - |\nabla u|^2 - 2t\nabla u \cdot \nabla v - t^2|\nabla v|^2 + \\ & +4k_n t^2 |\nabla u|^2 (1 - k_n t^2) + 4k_n t^3 \nabla u \cdot \nabla v\,. \end{aligned}$$

If $x \notin Q_n \cup R_n$, then $[\nabla u \nabla v]_+ = 0$, and

$$\begin{aligned} y(n,t) &\geq 2/n \cdot q + 2t(1 - 2k_n t^2)[\nabla u \cdot \nabla v]_- - t^2\|\nabla v\|_\infty^2 \\ &\geq 2/n \cdot q + \tfrac{1}{n}t(1 - 2k_n t^2) \cdot \|\nabla v\|_\infty^2 - t^2\|\nabla v\|_\infty^2 \\ &\geq 2/n \cdot q + t\|\nabla v\|_\infty^2 (\tfrac{1}{n} - t - \tfrac{2}{n}k_n t^2) \\ &\geq 0 \text{ for t sufficiently small}\,. \end{aligned}$$

If $x \in Q_n \setminus R_n$, then

$$\begin{aligned} y(n,t) \geq \; & t\|\nabla v\|_\infty^2(\tfrac{1}{n} - t - \tfrac{2}{n}k_n t^2) + 4k_n t^2(1-k_n t^2)(1-\tfrac{2}{n}q) \\ & \geq 0 \text{ for t sufficiently small.} \end{aligned}$$

If $x \in R_n \setminus Q_n$, then

$$\begin{aligned} y(n,t) & \geq \tfrac{2}{n}q - 2t(\nabla u \cdot \nabla v)_+ + 2t(\nabla u \cdot \nabla v)_- \\ & -t^2\|\nabla v\|_\infty^2 + 4k_n t^3[\nabla u \cdot \nabla v]_+ - 4k_n t^3[\nabla u \cdot \nabla v]_- \\ & \geq \tfrac{2}{n}q - 2t[\nabla u \cdot \nabla v]_+ + 2t[\nabla u \cdot \nabla v]_-(1-2k_n t^2) - t^2\|\nabla v\|_\infty^2 \\ & \geq \tfrac{2}{n}q - \tfrac{1}{n}t\|\nabla v\|_\infty^2 - t^2\|\nabla v\|_\infty^2 \\ & \geq \|\nabla v\|_\infty(\tfrac{2}{n}\|\nabla u\|_\infty - \tfrac{1}{n}t\|\nabla v\|_\infty - t^2\|\nabla v\|_\infty) \\ & \geq 0 \text{ for t sufficiently small.} \end{aligned}$$

If $x \in Q_n \cap R_n$, then

$$\begin{aligned} y(n,t) & = [\tilde{u} + t\tilde{v}] - t^2|\nabla v|^2 + 4k_n t^2|\nabla u|^2(1-t^2k_n) + 4t^2k_n t\nabla u \cdot \nabla v \\ & \geq t^2(-k_n - k_n + 4k_n(1-\tfrac{1}{n}q)(1-t^2k_n) - 2\tfrac{k_n}{n}tp) \\ & = 2k_n t^2(1 - \tfrac{2}{n}q - 2tk_n - \tfrac{1}{n}tp) \\ & \geq 0 \text{ for t sufficiently small.} \end{aligned}$$

Consider now the case $\|\nabla w\|_\infty \neq 0$. Applying the formula for w_n in the second term of the right-hand side expression of $(*)$ we get

$$\begin{aligned} y(n,t) & = [1 - |\nabla u|^2] - 2t\nabla u \cdot \nabla v - t^2[|\nabla v|^2 + \nabla u \cdot \nabla w] + \\ & +4k_n t^2|\nabla u|^2 - t^2 z(n,t), \end{aligned}$$

where $z(n,t) = t\nabla v \cdot \nabla w_n + \frac{t^2}{4}|\nabla w_n|^2$.

(i). If $x \notin Q_n \cup R_n$, then

$$\begin{aligned} y(n,t) & \geq \tfrac{2}{n} \cdot q + \tfrac{t}{n} \cdot p - t^2[|\nabla v|^2 + \nabla u \cdot \nabla w] - t^2 z(n,t) \\ & \geq \tfrac{1}{n} \cdot q \cdot t^2 - t^2 z(n,t) \\ & \text{for} \quad t \leq \tfrac{1}{n}. \end{aligned}$$

(ii). If $x \in Q_n \setminus R_n$, then

$$\begin{aligned} y(n,t) & \geq \tfrac{t}{n} \cdot p - t^2[|\nabla v|^2 + \nabla u \cdot \nabla w] + \\ & 4k_n t^2|\nabla u|^2 - t^2 z(n,t) \\ & \geq k_n t^2(1 - \tfrac{4}{n} \cdot q) - t^2 z(n,t) \\ & \text{for } t \leq 1/n. \end{aligned}$$

(iii). If $x \in R_n \setminus Q_n$, then

$$\begin{aligned} y(n,t) &\geq \tfrac{2}{n}\cdot q - 2t[\nabla u \cdot \nabla v]_+ + 2t[\nabla u \cdot \nabla v]_- - t^2 p - t^2 z(n,t) \\ &\geq \tfrac{2}{n}\cdot q - \tfrac{t}{n}\cdot p - t^2 p - t^2 z(n,t) \\ &\geq \tfrac{1}{n}(q - tp) + (\tfrac{1}{n}q - tp) - t^2 z(n,t) \\ &\geq \tfrac{1}{n}\cdot q \cdot t^2 - t^2 z(n,t) \\ &\text{for t sufficiently small}. \end{aligned}$$

(iv). If $x \in Q_n \cap R_n$, then

$$\begin{aligned} y(n,t)/t^2 &\geq \tfrac{1}{t^2}[\tilde{u} + t\tilde{v}] - [|\nabla v|^2 + \nabla u \cdot \nabla w]_+ + [|\nabla v|^2 + \nabla u \cdot \nabla w]_- \\ &+4k_n|\nabla u|^2 - z(n,t) \\ &\geq \tfrac{1}{t^2}[\tilde{u} + t\tilde{v}]_+ - \tfrac{1}{t^2}[\tilde{u} + t\tilde{v}]_- - [|\nabla v|^2 + \nabla u \cdot \nabla w]_+ + [|\nabla v|^2 + \nabla u \cdot \nabla w]_- \\ &+4k_n(1 - \tfrac{2}{n}\cdot q) - z(n,t) \\ &\geq -2k_n + 4k_n - \tfrac{8}{n}\cdot k_n \cdot q - z(n,t) \\ &\geq k_n(1 - \tfrac{4}{n}\cdot q) - z(n,t) \end{aligned}$$

In the cases $(i) - (iv)$ we get $y(n,t) \geq 0$ if

$$\frac{t^2}{4}\|\nabla w_n\|_\infty^2 + t\|\nabla v\|_\infty \cdot \|\nabla w_n\|_\infty - c_i \leq 0\,, \quad (**)$$

where for $i = 1,..,4$, $c_i > 0$ and

$$c_i = \begin{cases} \frac{1}{n}q & \text{for } (i)\,,\,(iii) \\ k_n(1 - \frac{4}{n}q) & \text{for } (ii)\,,\,(iv) \end{cases}$$

The only positive solution to the equation

$$\frac{t^2}{4}\|\nabla w_n\|_\infty^2 + t\|\nabla v\|_\infty \cdot \|\nabla w_n\|_\infty - c_i = 0\,,$$

is

$$t_n^0 = \frac{2\sqrt{\|\nabla v\|_\infty^2 + 4c_i} - 2\|\nabla v\|_\infty}{\|\nabla w_n\|_\infty}\,,$$

which is well-defined since $\|\nabla w_n\|_\infty \neq 0$. Hence, in the cases $(i) - (iv)$ we get the required inequality for all t small enough.

□

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