Gibbs Measures on Attractors in Biological Neural Networks

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Abstract: We consider a class of nonreversible processes with a local interaction as a model of biological neural networks. We study the structure of equilibrium measures on attractors. There exist first and second order phase transitions with respect to the grey level parameter.

Key-words: neural networks, attractor, phase transitions, grey level, cerebellar cortex.

(Résumé : tsvp)

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Mesures de Gibbs sur les Attracteurs dans les Réseaux de Neurones Biologiques.

Résumé : On considère des processus stochastiques non reversibles à interaction locale comme un modèle de reseaux de neurones biologiques.

Mots-clé : réseaux de neurones, transition de phase, attracteur.

1 Definitions and Results

1.1 Model

We consider a discrete time homogeneous Markov chain $\mathcal{L}_\Lambda$, with state space R_+^Λ, where $\Lambda = \Lambda_{N,\nu} = \{-N, ..., N\}^\nu$ is a finite cube in Z_+^ν. We shall consider only the cases $\nu = 1, 2$. The state vector at time t is denoted by

$$s^{(\Lambda)}(t) = (s_i(t), i \in \Lambda) \in R_+^\Lambda. \tag{1}$$

Let $D(i) = \{j \in \mathbf{Z}^\nu : |i-j| = 1\}$ for any $i \in \mathbf{Z}^\nu$, where

$$|i-j| := \sum_{k=1}^{\nu} |i_k - j_k|.$$

If $s_i^{(\Lambda)}(t) > 0$ for all $i \in \Lambda$ then all $s_j^{(\Lambda)}(t)$ decrease deterministically and linearly in time with constant speed 1, i.e.

$$\frac{ds_j^{(\Lambda)}(t)}{dt} = -1, \ j \in \Lambda$$

until some of them (say $s_i^{(\Lambda)}(t+t_0)$, $i \in I \subset \Lambda$) become 0. At this moment $t+t_0$ all $s_j^{(\Lambda)}(t+t_0+0)$ with $j \in I \cup_{i\in I} (D(i) \cap \Lambda)$ receive instantaneous positive increments, more exactly

$$s_j^{(\Lambda)}(t+t_0+0) = s_j^{(\Lambda)}(t+t_0) + \sum_{l \in D(j)\cap I} \eta_{lj}(t+t_0), \tag{2}$$

where $\eta_{ij}(t+t_0)$, $i, j \in \Lambda$, $t+t_0 > 0$, are independent random variables, whose distribution depends only on $\mid i-j \mid$.

Assumption 1 *We assume that all the distributions of r.v. $\eta_{ij}(t)$ are absolutely continuous with respect to Lebesgue measure and bounded.*

Remark 1 *This assumption lets us ignore a.s. all trajectories where more than one spins are simultaneously equal to zero at some moment. Also it allows us to apply the theory of Harris recurrent Markov chains, which is quite similar to the theory of countable Markov chains. Finally, we note that Assumption 1 can be avoided but this demands some additional work unrelated to the essence of the problem.*

According to our assumptions $\eta_{ij}(t+t_0)$, $t+t_0>0$, are independent copies of some variables $\eta_{|i-j|}$ for any $i,j\in\Lambda$. We shall study only the particular case when

$$a_{|i-j|} := E\eta_{|i-j|} = \begin{cases} 0 < a < \infty, & \text{if } i = j, \\ 1, & \text{if } |i-j| = 1. \end{cases}$$

Note scaling invariance $a_{ij} \to ra_{ij}$, see [3].

1.2 Historical Comments

The model we treat here was introduced by Axelrad, Bernard, Cottrell and Giraud [1] as a simple model for the cerebellar cortex. In line with this model whenever $s_i(t)=0$, we call t a moment of firing of the ith neuron, and therefore the state $s_i(t)$ can be interpreted as the time-interval between moment t and the next firing of the ith neuron in the absence of the afferent impulses.

This model was studied in order to investigate the role of the inhibitory synapses in the formation of different regimes of the neuronal activity. The rigorous analysis of this model with a finite number of neurons and constant impulses (which are the variables η_{ij} for $i\neq j$ in our setting) was presented in paper by Cottrell [2] (see also the references therein for motivation of the model). The simulation results of Cottrell [2] show clearly the ability of this model to produce different stable patterns of active neurons by means of the inhibitory connections only.

Karpelevich, Malyshev and Rybko [3] provided a detailed analysis of the generalized version of the model [2] for the case when the matrix of the expected values of the connections (a_{ij}) defines a self-adjoint operator. We essentially use the results and methods of their paper.

¿From the study in [6] it appeared that the mathematical description of the sequences of the spikes (the moments of firing) of the neurons in the above model (which is also called an hourglass model) is equivalent to the one derived from another stochastic neuronal model introduced by Kryukov [4]. The last model takes into account the basic physiological facts about neuronal activity, and it allows one to simulate and somehow explain the nature of such important (and well-known from the experimental studies of brain) regimes as metastability and phase transitions (see [5] and the bibliography therein for the biological justification). Hence, an hourglass model becomes now an adequate mathematical model for studying numerous phenomena in biological neural networks as soon as the distributions η_{ij} are chosen to fit the data.

We notice also, that the results of our paper are valid for the modified model, where it is assumed, that there is no interaction between the simultaneously firing neurons. The later means that (2) defines the states of the jth neurons with $j \notin I$ only, while for any $i \in I$

$$s_i^{(\Lambda)}(t + t_0 + 0) = \eta_{ii}(t + t_0).$$

The later modification takes into account the refractorial period, i.e. the period of time following the firing of a neuron, within which this neuron does not respond to any incoming impulse.

When one looks at a macroscopic part of the brain one can see "darker" regions where a lot of neurons fire and also "whiter" regions where only small number of neurons fire. We would like to formalize this by defining macroimages corresponding to the state of a neural network. On a microscopic scale we see a random image but on a larger scale we see a macro image which is a deterministic one resulting from the Law of Large Numbers.

1.3 Attractors in Finite Networks

Let $(\Omega, \Sigma, P) = (\Omega_\Lambda, \Sigma_\Lambda, P_\Lambda)$ be the underlying probability space of our process. Let $p = p^\Lambda$ be the initial distribution on the state space R_+^Λ. For any $A \subset \Lambda$ let $\Omega(A) \subset \Omega$ be the set of all trajectories ω such that

$$s_i^{(\Lambda)}(t, \omega) \to \infty \text{ for } i \in A \text{ as } t \to \infty, \tag{3}$$

and, for all $\omega \in A$, the set $A = A(\omega)$ is the maximal subset satisfying property (3). It is clear that $\Omega = \bigcup_A \Omega(A)$ where A runs over all subsets of Λ.

Definition 1 *We call the value of the parameter a regular if the following condition holds a.s. For all A such that $\Omega(A) > 0$ the conditional distribution of the vector $(s_i^\Lambda(t), i \in \Lambda \setminus A)$, under the condition that $\omega \in \Omega(A)$, has a weak limit, which is a distribution on $R_+^{\Lambda \setminus A}$.*

Further on we shall consider only regular values of the parameter.

Definition 2 *We call a trap any non-empty set A such that $P(\Omega(A)) > 0$.*

We shall denote $\mathcal{T}_N$ the set of all traps for $\Lambda = \Lambda_{N,\nu}$:

$$\mathcal{T}_N = \{A \subset \Lambda : P(\Omega(A)) > 0\}. \tag{4}$$

On the set of all subsets of Λ we define the measure $\mu_p^\Lambda(A) = P(\Omega(A))$ and call it the exit measure (or scattering measure).

Note that if for fixed A measure $P(\Omega(A))$ is positive for some initial state, then it is also positive for any other initial state. Thus the definition of a trap does not depend on the initial condition.

It is clear from the general theory, that the situation $P(\Omega(\emptyset)) > 0$ cannot occur unless $P(\Omega(\emptyset)) = 1$. Any value of the parameter a which gives ergodicity of $s^{(\Lambda)}(t)$, is regular. In this case $P(\Omega(\emptyset)) = 1$, i.e. there are no traps.

Theorem 1

- *(a) Let $\nu = 1$. Then all the values of the parameter a are regular, except possibly $a_{cr} = 2$, which is called critical value, and*

$$P(\Omega(\emptyset)) = \begin{cases} 1, & \text{if } a > 2, \\ 0, & \text{if } a < 2. \end{cases}$$

- *(b) Let $\nu = 2$. Then the values $a < 3$ and $a > 4$ are regular and*

$$P(\Omega(\emptyset)) = \begin{cases} 1, & \text{if } a > 4, \\ 0, & \text{if } a < 3. \end{cases}$$

- *(c) (periodic boundary) Let $\nu = 2$, and $\Lambda = \Lambda_{N,2}^{per}$ be a set $\Lambda_{N,2}$, where the nodes $(-N, j)$ and $(j, -N)$ are identified with the nodes (N, j) and (j, N), correspondingly, for all $1 \le j \le N$. Then all the values of the parameter a are regular, except possibly $a_{cr}^{per} = 4$, which is called critical, and*

$$P(\Omega(\emptyset)) = \begin{cases} 1, & \text{if } a > 4, \\ 0, & \text{if } a < 4. \end{cases}$$

Theorem 2 *(Stability) Choose a trap $A \in \mathcal{T}_N$ arbitrarily. Let at time zero $s_i^{(\Lambda)}(0) = 0$ if $i \notin A$, and $s_i^{(\Lambda)}(0) = L$ if $i \in A$ for some large L. Then the probability that*

$$s_i^{(\Lambda)}(t) \to \infty \text{ as } t \to \infty \quad \text{if } i \in A,$$
$$\text{and} \quad (s_i^{(\Lambda)}(t),\ i \in \Lambda \setminus A) \quad \text{tends to a stationary distribution as } t \to \infty,$$

equals $1 - \epsilon(L)$, where $\epsilon(L) \to 0$ as $L \to \infty$.

Let $\mathcal{C}^\Lambda := (R_+ \cup \{+\infty\})^\Lambda$ define the compactification of our state space. Then it would be natural (in this compactified space) to call attractors the sets

$$\Gamma_A = \{x : x_i = \infty, i \in A, x_i < \infty, i \in \Lambda \setminus A\},$$

whenever A is a trap.

In one-dimensional case there is a complete description of traps, [3]. There is no complete description of the traps in 2-dimensional case (so far) but we are able, however, to study the low-temperature case.

Proposition 1 *Let $a < 1$ and $\Lambda = \Lambda_{N,2}$. Then the set $B \in \mathcal{T}_N$ if and only if*

a) any node in $\Lambda \backslash B$ is isolated, i.e. for any $i \in \Lambda \backslash B$

$$D(i) \cap (\Lambda \backslash B) = \emptyset;$$

b) any node in B has at least one link with the set $\Lambda \backslash B$, i.e. for any $j \in B$

$$D(j) \cap (\Lambda \backslash B) \neq \emptyset.$$

1.4 Gibbs Measures

For regular a, on the compactification $\mathcal{C}^\Lambda$ of the state space, the limiting measure exists and is uniquely determined by its initial distribution. We denote this measure μ^Λ. We shall show that in one dimension this measure is a Gibbs measure in the thermodynamic limit. Besides space $\mathcal{C}^\Lambda$ we introduce the space of reduced configurations

$$\sigma(s^\Lambda) = \begin{cases} +1, & \text{if } s_i^\Lambda = \infty, \\ 0, & \text{otherwise.} \end{cases}$$

Clearly, there is a one-to-one correspondence between the subsets $A \subseteq \Lambda$ and the configurations $\sigma \in \{0,1\}^\Lambda$, such that for any A

$$\sigma_i(A) = \begin{cases} 1, & \text{if } i \in A, \\ 0, & \text{otherwise.} \end{cases}$$

Inversely, for any $\sigma \in \{0,1\}^\Lambda$ we define

$$A(\sigma) = \{i : \sigma_i = 1\}.$$

We shall call a configuration $\sigma \in \{0,1\}^\Lambda$ also a trap whenever $A(\sigma)$ is a trap, and identify the set of traps $\mathcal{T}_N$ with

$$\{\sigma \in \{0,1\}^\Lambda : \ A(\sigma) \in \mathcal{T}_N\}.$$

Scattering measure induces a measure on the configuration space $\{0,1\}^\Lambda$, so that for any $\sigma \in \{0,1\}^\Lambda$

$$\mu_\Lambda^{(p)}(\sigma) = \begin{cases} \mu_\Lambda^{(p)}(A(\sigma)), & \text{if } A(\sigma) \in \mathcal{T}_N, \\ 0, & \text{if } A(\sigma) \notin \mathcal{T}_N. \end{cases} \tag{5}$$

In particular one can always find the initial distribution (due to stability property and convexity), say $u = u^\Lambda$, such that $\mu_\Lambda^{(u)}$ is the uniform measure on traps. In this case (5) becomes

$$\mu_\Lambda^{(u)}(\sigma) = \begin{cases} \frac{1}{|\mathcal{T}_N|}, & \text{if } \sigma \in \mathcal{T}_N, \\ 0, & \text{otherwise.} \end{cases} \tag{6}$$

Distribution $\mu_\Lambda^{(p)}(\sigma)$ defines a random field ($\xi_i = \xi_i^p,\ i \in \Lambda$) on Λ. For any $B \subset \Lambda$ one can define its correlation functions

$$\mu_{B,\Lambda}^{(p)}(\sigma_B) := P\{\xi_B = \sigma_B\}, \tag{7}$$

where ξ_B is the restriction of the random configuration ξ onto the set B, and σ_B is some fixed configuration on B. In particular case (6) the correlation functions are

$$\mu_{B,\Lambda}^{(u)}(\sigma_B) = \frac{\text{number of traps } \sigma' \in \mathcal{T}_N \text{ with } \sigma'_B = \sigma_B}{|\mathcal{T}_N|}. \tag{8}$$

Lemma 1 *Let $\nu = 1$ and $0 < a < 2$, or $\nu = 2$ and $0 < a < 1$. Then under thermodynamic limit $N \to \infty$, the uniform measures $\mu_\Lambda^{(u)}$ tend weakly to a translation invariant Gibbs distribution with a finite potential equal to 0 or ∞. Furthermore, if $\nu = 1$ then the corresponding thermodynamic limit is a stationary d-Markov chain $\{\xi_n\}_{n \in \mathbf{Z}^1}$ on a finite state space $\{0,1\}$ with the following transition probabilities:*

- *in the case $a < 1$*

$d = 2$ and

$$\begin{aligned} &P\{\xi_{n+1} = 1 \mid \xi_n = \bar{\xi}_n,\ \xi_{n-1} = \bar{\xi}_{n-1}\} \\ &= 1 - P\{\xi_{n+1} = 0 \mid \xi_n = \bar{\xi}_n,\ \xi_{n-1} = \bar{\xi}_{n-1}\} \\ &= \begin{cases} 1, & \textit{if } (\bar{\xi}_n, \bar{\xi}_{n-1}) = (0,1), \\ 1/2, & \textit{if } (\bar{\xi}_n, \bar{\xi}_{n-1}) = (1,0), \\ 0, & \textit{if } (\bar{\xi}_n, \bar{\xi}_{n-1}) = (1,1); \end{cases} \end{aligned} \tag{9}$$

- *in the case $1 \le a < 2$*

$$d = d(a) = 2\beta(a) + 2$$

with $\beta(a) := [\frac{1}{2}(\frac{\pi}{\arccos(a/2)} - 1)]$, *and*

$$P\{\xi_{n+1} = 0 \mid \xi_n = \bar{\xi}_n, \dots, \xi_{n-d+1} = \bar{\xi}_{n-d+1}\} \tag{10}$$

$$= 1 - P\{\xi_{n+1} = 1 \mid \xi_n = \bar{\xi}_n, \dots, \xi_{n-d+1} = \bar{\xi}_{n-d+1}\}$$

$$= \begin{cases} 1, & \textit{if} \quad (\bar{\xi}_n, \dots, \bar{\xi}_{n-d+1}) = (\underbrace{0, \dots, 0}_{n}, 1, \underbrace{0, 1, 0, 1, \dots 0, 1}_{2k}, 0, \dots, 0) \\ & \quad \textit{for } 2 \le n \le 2\beta - 1 \textit{ or } n = 0, \textit{ and } 2 \le 2k \le 2\beta + 1 - n, \\ 1/2, & \textit{if} \quad (\bar{\xi}_n, \dots, \bar{\xi}_{n-d+1}) = (\underbrace{0, 1, 0, 1, \dots 0, 1}_{2k}, 0, \dots, 0) \textit{ for } k \ge 2, \\ 0, & \textit{if} \quad (\bar{\xi}_n, \dots, \bar{\xi}_{n-d+1}) = (0, 1, 0, 0, \dots 0). \end{cases}$$

One-point correlation functions of the corresponding Gibbs field on $\mathbf{Z}^1$ *are*

$$\pi_1(a) := \lim_{N \to \infty} \mu_{i,\Lambda}^{(u)}(1) = \begin{cases} \frac{3}{5}, & \textit{if } a < 1, \\ \frac{3}{2\beta(a)+5}, & \textit{if } 1 \le a < 2. \end{cases} \tag{11}$$

Further we will establish a similar result for the non-uniform measure but in one limiting case only (Lemma 2 below). The one-point correlation functions come into play in our next theorem on phase transitions.

1.5 Macro Images

Rescale $\Lambda_{N,\nu} = \{-N, ..., N\}^\nu$ to the cube $[-1, 1]^\nu \subset R^\nu$:

$$\Lambda_{N,\nu} \to \{\frac{x}{N}, x \in \Lambda_{N,\nu}\} \subset [-1, 1]^\nu. \tag{12}$$

Under this scaling a random trap, which we denote now $M^{(\Lambda)}$, becomes a random subset of the unit cube, i.e.,

$$M^{(\Lambda)} \to \bar{M}^{(\Lambda)} := \{\frac{x}{N}, \ x \in M^{(\Lambda)}\}. \tag{13}$$

Let us define now the scaled random measure $\mu_\Lambda^{(p)\,sc}$ on the unit cube as follows. For any open set $\mathcal{B} \subset [-1, 1]^\nu$ we put

$$\mu_\Lambda^{(p)\,sc}(\mathcal{B}) = \frac{1}{(2N+1)^\nu} \#\{\mathcal{B} \bigcap \bar{M}^{(\Lambda)}.\} \tag{14}$$

1.5.1 First order phase transitions for uniform measures

The following theorem describes first order phase transitions in one- dimensional case for the uniform measures.

Theorem 3 *Let $\nu = 1$ and $a < 2$. Then the sequence of random measures $\mu_\Lambda^{(u)\,sc}$ tends weakly as $N \to \infty$ to nonrandom measure $\mu_0(\cdot) = r(a)\lambda(\cdot)$ on the interval $[-1, 1]$, where $\lambda(\cdot)$ is the Lebesgue measure, and constant $r(a) = \pi_1(a)$ is called the "grey level".*

Let us consider the graph of the function $r(a) = \pi_1(a)$ (see Figure 1).

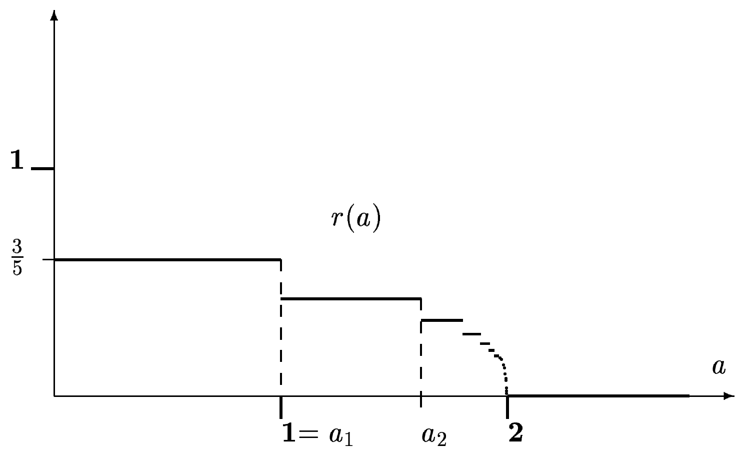

Figure 1. The graph of the function $r(a)$. The trajectory is right-continuous.

From its definition (11) we get the following properties.

1. Function $r(a)$ is nonincreasing and piecewise constant (right- continuous) with discontinuities at some points $1 = a_1 < \ldots < a_k < \ldots < 2$, where for any $k \geq 1$

$$a_k := 2\,cos\,\frac{\pi}{2k+1}\,,$$

and

$$r(a) = \frac{3}{2k+5}, \qquad \text{for all } a \in [a_k, a_{k+1}).$$

2. Notice that for $a > 2$ our system $s^{(\Lambda)}(t)$ with $\Lambda = \Lambda_{N,1}$ is ergodic for any N. Since there are no traps if $a > 2$, and $r(a) \to 0$ as $a \uparrow 2$, it is natural to put $r(a) = 0$ for all $a \geq 2$. Point $a = 2$ is the critical point, and the discontinuities condense to this point from the left.

1.5.2 Scattering measures

Theorem 4 *Let $\nu = 1$ and $\eta_0 < \eta_1$ with probability one. Assume initial distribution p be such that $s_i(0)$, $i \in \mathbf{Z}^\nu$, are identically distributed independent positive variables. Then the sequence of the scaled random measures $\mu_\Lambda^{(p)\,sc}$ tends weakly as $N \to \infty$ to nonrandom measure $\mu(\cdot) = r_0 \lambda(\cdot)$, where $\lambda(\cdot)$ is the Lebesgue measure, and the constant of the grey level is*

$$r_0 = \frac{1 + e^{-1}}{2 + e^{-1}}.$$

In order to prove this theorem we will need the following result analogous to (11).

Lemma 2 *Under the conditions of Theorem 4 as $N \to \infty$, measure $\mu_\Lambda^{(p)}$ tends weakly to a translation invariant Gibbs distribution. More exactly, its thermodynamic limit is a stationary 2-Markov chain $\{\xi_n\}_{n \in \mathbf{Z}^1}$ defined by its conditional distributions:*

$$\begin{aligned} & P\{\xi_{n+1} = 1 \mid \xi_n = \bar{\xi}_n,\ \xi_{n-1} = \bar{\xi}_{n-1}\} \\ &= 1 - P\{\xi_{n+1} = 0 \mid \xi_n = \bar{\xi}_n,\ \xi_{n-1} = \bar{\xi}_{n-1}\} \\ &= \begin{cases} 1, & \text{if } (\bar{\xi}_n, \bar{\xi}_{n-1}) = (0,1), \\ e^{-1}, & \text{if } (\bar{\xi}_n, \bar{\xi}_{n-1}) = (1,0), \\ 0, & \text{if } (\bar{\xi}_n, \bar{\xi}_{n-1}) = (1,1). \end{cases} \end{aligned} \tag{15}$$

One-point correlation functions of the corresponding Gibbs field on $\mathbf{Z}^1$ are

$$\pi_1^0 := \lim_{N \to \infty} P\{\xi_i = 1\} = \frac{1 + e^{-1}}{2 + e^{-1}}. \tag{16}$$

For simulation purposes, it is worth noticing that $r_0 / r(0) \approx 0.9628$. This shows that when the interaction is strong enough, the grey level is close to the one when all the patterns are equally distributed.

1.5.3 Some conjectures

Above results give us that grey level can change with parameters. To show grey level variation in a part of the brain we should do additional rescaling of the initial state.

If the initial condition $s^{(\Lambda)}(0)$ is scaled appropriately with some continuous positive function $g(x), x \in [-1,1]^\nu$, i.e.

$$s_i^{(\Lambda)}(0) = g(\frac{i_1}{N}, \dots, \frac{i_\nu}{N}), \quad i \in \Lambda = \Lambda_{N,\nu}$$

(for example, g can be constant), then the sequence of random measures μ_N tends weakly to some (nonrandom) measure μ on the cube $[-1,1]^\nu$ as $N \to \infty$. Moreover we conjecture that this limiting measure is absolutely continuous with respect to Lebesgue measure.

2 Proofs

2.1 More notations and definitions.

Let $s^W(t) = (s_i^W(t), i \in W)$ be defined analogously to (1), where now W is any subset of Λ ($W \neq \Lambda$). We shall call $s^W(t)$ a restriction of the network $s^{(\Lambda)}(t)$ on the set W.

Whenever the restriction $s^W(t)$, is ergodic we define

$$\pi_i^W = \lim_{T\to\infty} \frac{1}{T} \#\{0 < t < T : s_i^W(t) = 0\} \tag{17}$$

to be the average number of the firings of the ith neuron per time unit in the stationary regime. Further we define the second vector field $v^W = (v_j^W, j \in \Lambda \backslash W)$ (see [3]) along this face W:

$$v_j^W := -1 + \sum_{i\in W} a_{|i-j|}\pi_i^W = -1 + \sum_{i\in D(j)\cap W} \pi_i^W, \quad j \in \Lambda\backslash W. \tag{18}$$

Traps can also be defined in terms of the second vector field. Suppose the process $s^{(\Lambda)}(t)$ is transient, i.e. *a.s.* the sum of its coordinates runs off towards infinity as $t \to \infty$. In particular, this implies that $P(\Omega(\emptyset)) = 0$. The following proposition follows readily the results Karpelevich *et. al.* [3].

Proposition 2 *Let the value of the parameter a be regular such that $s^{(\Lambda)}(t)$ is transient. Then $P(\Omega(\emptyset)) = 0$. Furthermore, a set $A \subset \Lambda$ is a trap if and only if*

- *the restriction $s^{\Lambda \setminus A}(t)$ is ergodic, while*
- *any coordinate of the second vector field along face $\Lambda \setminus A$ is positive:*

$$v_j^{\Lambda \setminus A} > 0, \;\; j \in A.$$

In fact the last part of this proposition was regarded in [3] as a definition of trap.

2.2 Proof of Theorem 1

(a) Let $\nu = 1$. Notice that $|\Lambda_{N,1}| = 2N + 1$ is odd. Then it follows from Theorem 3.3 [3] that Markov chain $s^{(\Lambda)}(t)$ is transient if $a < 2$ and ergodic when $a > 2$. Furthermore, in the transient case $a < 2$ all the possible ways for the components of $s^{(\Lambda)}(t)$ to run off towards infinity are described in [3]. Hence, any positive value $a \neq 2$ is regular, and the assertion (a) of our theorem follows by Proposition 2 and Definition 1.

(b) Let $\nu = 2$. Using the inductive transience condition from Theorem 2.1 [3], we will show that Markov chain $s^{(\Lambda)}(t)$ is transient if $a < 3$. It is easy to see that the restriction $s^{\hat{\Lambda}}$ on the set

$$\hat{\Lambda} := \{(x, y) \in \Lambda_{N,1} : \;\; (x, y) = (-N, -N) + (2k, 2n),$$
$$\text{or } (x, y) = (-N, -N) + (2k + 1, 2n + 1), \text{ for some } k, n \in Z\}$$

is ergodic, since it consists of independent ergodic components. Then for any $i \in \hat{\Lambda}$ we have

$$\pi_i^{\hat{\Lambda}} = 1/a_0 = 1. \tag{19}$$

It is trivial to check now that any component of the second vector field along the face $\hat{\Lambda}$ is positive, since any point in $\Lambda \setminus \hat{\Lambda}$ has at least 3 connected neighbours in $\hat{\Lambda}$. Hence there is at least one trap when $a < 3$, which implies transience of $s^{(\Lambda)}(t)$ due to Theorem 2.1 [3]. The fact that any positive $a < 3$ or $a > 4$ is regular, follows from the study in [3].

The ergodicy of the network when $a > 4$ follows from Theorem 3.2 of [3], since any node has not more than 4 connected neighbours. Thus the assertion (b) follows by Proposition 2 and Definition 1.

(c) Since any node in $\Lambda_{N,2}^{per}$ has exactly 4 connected neighbours, with the same argument as above one can easily check that the set $\hat{\Lambda}$ is a trap for the system if $a < 4$, which is obviously, regular. The ergodicy of the network when $a > 4$ follows again from Theorem 3.2 of [3]. This completes the proof of Theorem 1.

2.3 Proof of Theorem 2

Let us fix $M \in \mathcal{T}_N$ arbitrarily. In particular, it implies according to Proposition 2 that $s^{\Lambda\setminus M}(t)$ is ergodic. Consider for $L > 0$ and arbitrarily fixed $C > 0$

$$\mathbf{P}\{s_j^{(\Lambda)}(L+t) > C,\ j \in M,\ \text{for all } t \geq 0\} \tag{20}$$

$$\geq \mathbf{P}\{s_j^{(\Lambda)}(L+t) > C,\ j \in M,\ \text{for all } t \geq 0\ \mid$$

$$s_j^{(\Lambda)}(L) > \frac{L}{2}\sum_{i\in D(j)} \pi_i^{\Lambda\setminus M},\ j \in M\}$$

$$\times \mathbf{P}\{s_j^{(\Lambda)}(L) > \frac{L}{2}\sum_{i\in D(j)} \pi_i^{\Lambda\setminus M},\ j \in M\}.$$

Recall, that according to the definition (2)

$$s_j^{(\Lambda)}(t) = L - t + \sum_{\tau_n \leq t : s_j^{(\Lambda)}(\tau_n)=0} \eta_0^{(j,n)} + \sum_{\tau_n \leq t : s_i^{(\Lambda)}(\tau_n)=0,\ i\in D(j)} \eta_1^{(j,n)},$$

for all $j \in M$, where $\eta_0^{(j,n)}$ and $\eta_1^{(j,n)}$, $j \in \Lambda, n \geq 1$, are independent copies of the variables η_1 and η_0, respectively, with $\mathbf{E}\eta_1 = 1$ and $\mathbf{E}\eta_0 = a$. Hence it is easy to see that for

$$B_j := \sum_{i\in D(j)\cap\Lambda} \pi_i^{\Lambda\setminus M},$$

we have

$$\mathbf{P}\{s_j^{(\Lambda)}(L) > \frac{LB_j}{2},\ j \in M\} \tag{21}$$

$$= \mathbf{P}\{\sum_{\tau_n \leq L : s_i^{(\Lambda\setminus M)}(\tau_n)=0,\ i\in D(j)} \eta_1^{(j,n)} > \frac{LB_j}{2},\ j \in M\}$$

$$\geq 1 - \epsilon_1(L),$$

where the last inequality holds for some positive $\epsilon_1(L) \to 0$ as $L \to \infty$ due to ergodicity of $s^{(\Lambda\setminus M)}(t)$ and the Strong Law of Large Numbers. Let $L > \frac{4C}{B}$, where $B := \min_{j\in\Lambda\setminus M} B_j$. Then making use of (21) we derive from (20):

$$\mathbf{P}\{s_j^{(\Lambda)}(L+t) > C,\ j \in M,\ \text{for all } t \geq 0\} \tag{22}$$

$$\geq (1-\epsilon_1(L))\mathbf{P}\{s_j^{(\Lambda)}(L+t) > C,\ j \in M,\ \text{for all } t \geq LB/4\}$$

$$\geq (1-\epsilon_1(L))\mathbf{P}\{\frac{LB_j}{2} - t$$

$$+ \sum_{L\leq\tau_n\leq L+t: s_i^{(\Lambda\setminus M)}(\tau_n)=0,\ i\in D(j)} \eta_1^{(j,n)} > C,\ j\in M,\ \text{ for all } t\geq LB/4\}$$

$$\geq (1-\epsilon_1(L)) \times \mathbf{P}\{\frac{1}{t}$$

$$\sum_{L\leq\tau_n\leq L+t: s_i^{(\Lambda\setminus M)}(\tau_n)=0,\ i\in D(j)} \eta_1^{(j,n)} > 1 + \frac{4C}{LB},\ j\in M,\ \text{ for all } t\geq LB/4\}$$

$$\geq (1-\epsilon_1(L))(1-\epsilon_2(L))$$

for some $\epsilon_2(L)\to 0$ as $L\to\infty$, where the last inequality follows from the ergodicity of $s_i^{(\Lambda\setminus M)}(t)$, and the Strong Law of Large Numbers. Taking into account that the constants A_1, A_2 and C are fixed arbitrarily, we derive the stability property from (22). Theorem 2 is proved.

2.4 Proof of Proposition 1

First we will show that condition $a)$ is necessary and sufficient for the set $\Lambda\setminus B$ to be ergodic. Let us fix B arbitrarily. Consider

$$\Lambda\setminus B = \{(x,y)\in\Lambda : z(x,y)=0\} = \cup_j W_j, \tag{23}$$

where the W_j are the connected components of graph $\Lambda\setminus B$. This means that for any j and for any $w, w'\in W_j$ there exists a path between the nodes w and w' along the links, whereas for any $i\neq j$ and any $w_i\in W_i$ and $w_j\in W_j$ there does not exist a path between w_i and w_j.

Suppose that condition $a)$ is not satisfied, i.e., there exists at least one connected subgraph W_j with $|W_j|>1$ in the above decomposition (23), and suppose also that the corresponding Markov chain $s^{\Lambda\setminus B}(t)$ is ergodic.

Let

$$\hat{\mathbf{Z}}^2 := \{(x,y) : (x,y)=(2n,2k) \text{ or } (x,y)=(2n-1,2k-1),\ (k,n)\in\mathbf{Z}^2\}.$$

Define

$$W' := \{\{\Lambda\setminus B\}\cap\hat{\mathbf{Z}}^2\}\cup\{\cup_{j:|W_j|=1}W_j\}.$$

Clearly, $M_{W'}$ is ergodic, since it consists of the independent ergodic components. Notice, that $\{\Lambda \backslash B\} \backslash W' = \emptyset$ if and only if the condition $a)$ is fulfilled. Consider now second vector field $v^{W'}$

$$(v^{W'})_{(x,y)} = -1 + \sum_{(x',y') \in D(x,y)} \pi^{W'}_{(x',y')} \tag{24}$$

for any $(x, y) \in \{\Lambda \backslash B\} \backslash W'$. Obviously, $\pi^{W'}_{w'} = 1/a$ for any $w' \in W'$. Note also, that for any $(x, y) \in \{\Lambda \backslash B\} \backslash W'$ we have

$$D(x, y) \cap W' \neq \emptyset$$

due to our definition of the set W'. Hence (24) yields

$$(v^{W'})_{(x,y)} \geq -1 + 1/a > 0$$

for any $(x, y) \in \{\Lambda \backslash B\} \backslash W'$. This together with ergodicity of $s^{W'}(t)$ implies according to Theorem 2.1 [3], that the Markov chain $s^{\Lambda \backslash B}(t)$ is transient, which contradicts our assumption. Thus condition $a)$ is necessary for the ergodicity of the corresponding Markov chain $s^{\Lambda \backslash B}(t)$.

The sufficiency of condition $a)$ for the ergodicity of $s^{\Lambda \backslash B}(t)$ is obvious, since $s^{\Lambda \backslash B}(t)$ consists of the independent ergodic components.

Suppose now that the graph $\Lambda \backslash B$ satisfies condition $a)$ and consider the second vector field $(v^{\Lambda \backslash B})_{(x,y)}$ for an arbitrary $(x, y) \in B$:

$$(v^{\Lambda \backslash B})_{(x,y)} = -1 + \sum_{(x',y') \in D(x,y)} \pi^{\Lambda \backslash B}_{(x',y')},$$

where $\pi^{\Lambda \backslash B}_{(x',y')} = 1/a$ due to condition a). It is easy to see that the components of the second vector field are positive, i.e., $(v^{\Lambda} \backslash B)_{(x,y)} > 0$ for any $(x, y) \in B$ if and only if $D(x, y) \cap \{\mathit{\Lambda} \backslash B\} \neq \emptyset$, which is equivalent to our condition $b)$. This ends the proof.

2.5 Proof of Lemma 1

We shall use the fact that the restrictions required on a configuration $\xi \in \{0, 1\}^{\Lambda}$ in order to be a trap (i.e., to belong to $\mathcal{T}_N$), are of a local nature. This will allow us to represent the uniform measure on traps as a Gibbs field with two states $\{0, 1\}$ on Λ. The Hamiltonian of this Gibbs field can be written as

$$H_{\Lambda} = \sum_{B} \Phi_B$$

where $\Phi_B,\ B \subseteq \Lambda$, are local functions on R_+^Λ. In the case of the uniform distribution on traps the following properties hold:

- the diameters of B are uniformly bounded;
- the functions Φ_B can take only the values 0 and ∞;
- except for a neighbourhood of the boundary, the system of these functions is translation invariant;
- there exists a unique limiting Gibbs field. This is easier to describe in terms of d-Markov chains, which we will show below.

Once we have proved that the measure on the reduced configuration is Gibbs, it is evident that it is also Gibbs on the set of all the attractors. This is due to the property, that the distribution between two neighbouring infinities depends only on the number of points between these infinities.

(I) Let $\nu = 1$ and, correspondingly $\Lambda = \Lambda_{N,1} = \{-N, \dots, N\}$. Recall that in this case according to Proposition 3.1 [3] (and due to Proposition 2) a configuration $\xi \in \{0,1\}^\Lambda$ is a trap for $s^\Lambda(t)$ with parameter $a < 1$ if and only if the following conditions are fulfilled:

a_1) any of $\xi_{-N} = \xi_{-N+1} = 1$ and $\xi_{N-1} = \xi_N = 1$ is impossible;
b_1) if $\xi_i = 0$ then $\xi_{i-1} = \xi_{i+1} = 1$;
c_1) if $\xi_i = 1$ then either $\xi_{i-1} = 0$ or $\xi_{i+1} = 0$;

while in the case $1 \le a < 2$ the corresponding conditions are:
a) $\xi_{-N} = \xi_N = 0$;
b) if $\xi_i = 0$ then $\xi_{i-1} = \xi_{i+1} = 1$;
c) the number of the subsequent components of ξ with spin 0 is either 1 or $2\beta(a)$;
d) if $\xi_i = \xi_{i+1} = \ldots = \xi_{i+2\beta(a)-1} = 0$ then $\xi_{i-3} = \xi_{i+2\beta(a)+2} = 1$ if $i-3 > -N$ and $i + 2\beta(a) + 2 < N$.

Next we define for any set $B \in \Lambda$ its d'-neighbourhood

$$B_{d'} := \{x - d', x - d' + 1, \dots, x + d',\ x \in B\}, \tag{25}$$

and d'-boundary

$$\partial_{d'} B = B_{d'} \backslash B, \tag{26}$$

with

$$d' = \begin{cases} 1, & \text{if } a < 1, \\ d'(a) = \beta(a) + 2, & \text{if } 1 \le a < 2, \end{cases}$$

which is determined by conditions $a_1) - c_1)$ and $a) - d)$. Let us define now on the configuration space $\{0,1\}^\Lambda$ the following conditional probabilities: for any subset $B \in \Lambda$ and any given configuration $\sigma_{\Lambda\backslash B} = \bar{\sigma}_{\Lambda\backslash B}$ we set

$$p_B^\Lambda(\sigma_B \mid \bar{\sigma}_{\Lambda\backslash B}) \tag{27}$$

$$= \begin{cases} (\#\{\xi \in \mathcal{T}_N : \xi_j = \bar{\sigma}_j,\ j \in \Lambda\backslash B\})^{-1}, & \text{if } (\sigma_B, \bar{\sigma}_{\Lambda\backslash B}) \in \mathcal{T}_N, \\ 0, & \text{otherwise.} \end{cases}$$

Notice, that these probabilities are uniform in a sense that $p_B^\Lambda(\sigma_B \mid \bar{\sigma}_{\Lambda\backslash B})$ does not depend on the configuration σ_B itself as long as the entire configuration $(\sigma_B, \bar{\sigma}_{\Lambda\backslash B})$ is a trap. Making use of conditions $a_1) - c_1)$ or $a) - d)$, depending on the parameter a, it is easy to find (potential) functions $\Phi_{\{x\}_{d'}}(\sigma) = \Phi(\sigma_y,\ y \in \{x\}_{d'})$ for $x \in \Lambda$, $\sigma \in \{0,1\}^\Lambda$, such that

$$H_B(\sigma) := \sum_{x \in B} \Phi_{\{x\}_{d'}}(\sigma) = \begin{cases} 0, & \text{if } \sigma \in \mathcal{T}_N, \\ \infty, & \text{otherwise,} \end{cases} \tag{28}$$

for any $\sigma \in \{0,1\}^\Lambda$. This allows us to show that the conditional probabilities (27) define Gibbs distribution in B with boundary configuration $\bar{\sigma}_{\partial B_{d'}}$, since

$$p_B^\Lambda(\sigma_B \mid \bar{\sigma}_{\Lambda\backslash B}) = \frac{e^{-H_B(\sigma_B, \bar{\sigma}_{\Lambda\backslash B})}}{Z_B(\bar{\sigma}_{\Lambda\backslash B})} \tag{29}$$

$$= \frac{e^{-H_B(\sigma_B, \bar{\sigma}_{\partial_{d'} B})}}{Z_B(\bar{\sigma}_{\partial_{d'} B})} = p_B^\Lambda(\sigma_B \mid \bar{\sigma}_{\partial_{d'} B}),$$

which implies the representation of the measure $\mu_\Lambda^{(u)}$ as a conditional Gibbs distribution. (For example, in the case when $a < 1$, and, correspondingly, $d' = 1$,

$$\mu_\Lambda^{(u)}\{\sigma\} = p_\Lambda^{\Lambda_1}(\sigma_\Lambda \mid \sigma_{-N-1} = \sigma_{N+1} = 1) \tag{30}$$

for any $\sigma \in \{0,1\}^\Lambda$, where potential functions $\Phi_{\{x\}_1}(\sigma)$ are chosen to be

$$\Phi_{\{x\}_1}(\sigma) = \frac{h_{\{x\}_1}}{(1 - h_{\{x\}_1})(2 - h_{\{x\}_1})},$$

with

$$h_{\{x\}_1} := (1 - \sigma_x)(2 - \sigma_{x-1} - \sigma_{x+1}) + \prod_{y \in \{x\}_1} \sigma_y.)$$

This proves the first statement of our Lemma in the case $\nu = 1$.

(II) Let $\nu = 2$, $\Lambda = \Lambda_{N,2}$, and $a < 1$. Taking into account Proposition 2 we can reformulate the statement of Proposition 1 as follows: a configuration $(\xi_x,\ x \in \Lambda) \in \{0,1\}^\Lambda$ is a trap if and only if for any $x \in \Lambda$ the following conditions are fulfilled:
a_2) if $\xi_x = 0$ then $\xi_y = 1$ for any $y \in D(x) \cap \Lambda$;
b_2) if $\xi_x = 1$ then $\xi_y = 0$ at least for one $y \in D(x) \cap \Lambda$.

Next we define for any set $B \in \Lambda$ its d'-neighbourhood with $d' = 1$ (analogue of (25)):

$$B_1 := \{x \cup D(x),\ x \in B\}.$$

The d'-boundary is defined as in (26). Further, statements (27)-(29) hold as well for the case $\nu = 2$, where the potential functions $\Phi_{\{x\}_1}(\sigma)$ are chosen in a correspondence with $a_2) - b_2)$, for example,

$$\Phi_{\{x\}_1}(\sigma) = \frac{h_{\{x\}_1}}{\prod_{m=1}^{4}(m - h_{\{x\}_1})},$$

with

$$h_{\{x\}_1} := (1 - \sigma_x)(4 - \sum_{y \in D(x)} \sigma_y) + \prod_{y \in \{x\}_1} \sigma_y.$$

This completes the proof of the first statement of our Lemma.

For the rest of the proof let $\nu = 1$. We shall derive now (9) and (10).

Consider first the stationary Markov chain $\{\hat{\xi}\}_{n\geq 0}$ naturally associated with the introduced Gibbs distribution on $\mathbf{Z}^1$ due to its d'-Markov property. More precisely, associate the parameter of time n for the Markov chain $\{\hat{\xi}\}_{n\geq 0}$ with index $i \in \Lambda$ running from the left to the right. Then

Case 1: $a < 1$. The state space of $\{\hat{\xi}\}_{n\geq 0}$ is $\mathcal{S}(a) := \{0, E_1, E_2\}$ where 0 corresponds to the state 0 (for the components of ξ), while E_1 and E_2 stand for the first and for the second 1 in the row (from the left to the right), correspondingly. The transition probabilities are

$$p(0, E_1) = p(E_2, 0) = 1, \quad p(E_1, 0) = p(E_1, E_2) = 1/2, \tag{31}$$
$$p(u, v) = 0 \text{ otherwise.}$$

Case 2: $1 \leq a < 2$ The state space of $\{\hat{\xi}\}_{n\geq 0}$ is $\mathcal{S}(a) := \{e_1, O_1, O_2,\ \ldots, O_{2\beta+2}\}$, where the states $O_k, k = 1, \ldots, 2\beta$ stand for the k-th consecutive state 0 in the row, state $O_{2\beta+1}$ corresponds to the first state 1 after the row of 2β consecutive states 0, $O_{2\beta+2}$ corresponds to the first 0 after the row of 2β consecutive states 0 ended by 1, and state e_1 corresponds to state 1 following consecutive states $1, 0$. The transition probabilities in this case are

$$p(e_1, O_1) = p(O_{2\beta+2}, 1) = p(O_k, O_{k+1}) = 1, \quad k = 2, \ldots, 2\beta+1, \tag{32}$$

$$p(O_1, e_1) = p(O_1, O_2) = 1/2, \quad p(u, v) = 0 \text{ otherwise.}$$

Notice that value $1/2$ appears in the above transition probabilities due to the assumption of the uniform distribution on the traps.

Obviously, (9) and (10) follow directly from (31) and (32), correspondingly.

Finally, we will prove (11). Let $p(u), u \in \mathcal{S}(a)$, denote the stationary measure of the introduced Markov chain $\{\hat{\xi}_n\}$. Then we obtain the values of the one-point correlation functions of the Gibbs distribution simply as the corresponding stationary measure, i.e.

$$\pi_1(a) := \begin{cases} p(E_1) + p(E_2), & \text{if } a < 1, \\ p(e_1) + p(O_{2\beta+1}), & \text{if } 1 \le a < 2. \end{cases} \tag{33}$$

Now it is straightforward to compute the stationary distributions, which gives result (11). This completes the proof of the lemma.

2.6 Proof of Theorem 3

In the sequel $\Lambda = \Lambda_{N,1}$. Let ξ be a random variable with distribution $\mu_\Lambda^{(u)}$ on $\{0, +1\}^\Lambda$. Then for any open interval $\mathcal{B} = (B_1, B_2) \in [-1, 1]$ we derive from the definition (14):

$$\mu_\Lambda^{(p)\,cs}(\mathcal{B}) =_d \frac{\#\{[B_1(2N+1)]^+ \le i \le [B_2(2N+1)]^- : \xi_i = 1\}}{(B_2 - B_1)(2N+1)} \lambda(\mathcal{B}), \tag{34}$$

where $[x]^+ := \min\{z \in \mathbf{N} : z > x\}$ and $[x]^- := \min\{z \in \mathbf{N} : z < x\}$ for any $x \in \mathbf{R}$. Further we derive from (34):

$$\mu_\Lambda^{(p)\,cs}(\mathcal{B}) = \frac{\sum_{n=[B_1(2N+1)]^+}^{[B_2(2N+1)]^-} \xi_n}{(B_2 - B_1)(2N+1)} \lambda(\mathcal{B}). \tag{35}$$

Recall that in the previous proof we introduced the stationary Markov chain $\{\hat{\xi}\}_{n \ge 0}$ associated with the Gibbs distribution on $\mathbf{Z}^1$ defined in Lemma 1. Notice then that Lemma 1 implies the following distributional equality:

$$\lim_{N\to\infty} \frac{\sum_{n=[B_1(2N+1)]^+}^{[B_2(2N+1)]^-} \xi_n}{(B_2-B_1)(2N+1)} = \lim_{N\to\infty} \frac{\sum_{n=0}^{(B_2-B_1)(2N+1)]} \hat{\xi}_n}{(B_2-B_1)(2N+1)}, \tag{36}$$

which by the ergodic theorem and results (33) and (11) implies

$$\lim_{N\to\infty} \frac{\sum_{n=[B_1(2N+1)]^+}^{[B_2(2N+1)]^-} \xi_n}{(B_2-B_1)(2N+1)} = \pi_1(a). \tag{37}$$

Combining now (35) and (37) we readily derive the statement of the theorem, which finishes the proof.

2.7 Proof of Theorem 4

It is obvious that the proof of Theorem 4 follows line by line as in the previous proof as soon as we have the result of Lemma 2.

2.7.1 Proof of Lemma 2

We will show that the limiting correlation functions can be represented as joint distributions of the components of some Markov chain. We shall state this more precisely.

Note that since the set of traps $\bar{\mathcal{T}}_N$ (for any fixed N) is again defined by conditions $a_1) - c_1)$ one can argue as in the proof of Lemma 1, that there exists unique thermodynamical limit

$$p_B(\sigma_B) := \lim_{N\to\infty} P\{M_i^{(\Lambda)} = \sigma_i,\ i \in B\},$$

which is translationary invariant Gibbs distribution. Again, to show that this distribution is a 2-Markov chain as it is stated in our Lemma, we shall introduce a stationary Markov chain, call it $\{\xi'_z\}_{z\in\mathbf{Z}^1}$, analogous to the chain $\{\hat{\xi}_z\}_{z\in\mathbf{Z}^1}$ in the proof of Lemma 1, such that

$$\mu_{B,\Lambda}^{(p)}(\sigma_B) = P\{\xi'_i = \sigma_i,\ i \in B\}.$$

Obviously, $\{\xi'_z\}_{z\in\mathbf{Z}^1}$ has the same state space $\mathcal{S} := \{0, E_1, E_2\}$ as in **Case 1** $(a < 1)$ in Lemma 1.

Let $p'(u,v)$, $u,v \in \mathcal{S}(a)$, denote the transition probabilities of the chain $\{\xi'\}_{n\geq 0}$. Recall that we put in the previous proof of Lemma 1

$$p(E_1,0) = p(E_1,E_2) = 1/2$$

assuming uniform distribution of traps, i.e., equal frequency of isolated 1 and "coupled" 1 in any trap. Here we will derive the transition probabilities $p'(u,v)$ taking into account the dynamics of our system. We will prove, that

$$p'(0,E_1) = p'(E_2,0) = 1, \ \ p'(E_1,0) = 1 - e^{-1}, \ p'(E_1,E_2) = e^{-1}, \tag{38}$$

and

$$p'(u,v) = 0 \ \text{ otherwise.}$$

First we will show that $p'(E_1,0) = 1 - e^{-1}$. Then the rest of the statement (38) obviously follows from this and conditions $a_1) - c_1)$.

Consider the dynamics of our system $s^{\Lambda}(t)$ with $\Lambda = \Lambda_{N,1}$ for N fixed arbitrarily. First we will show how the system "falls" into trap. The trajectories $s^{\Lambda}(t)$ decrease at every coordinate until one of them, say the J_1th, hits zero, i.e., until moment $t_1 := \min_i s_i^{\Lambda}(0)$. Clearly, J_1 is random and distributed uniformly over Λ so that

$$P\{J_1 = i\} = \frac{1}{|\Lambda|}, \text{ for any } i \in \Lambda.$$

At time t_1 the neighbours of J_1, i.e. the jth coordinates with $j \in \{J_1-1, J_1+1\} \cap \Lambda$, receive increments $\eta^{(1)}_{jJ_1}$, and according to our assumption

$$s_j^{\Lambda}(t_1) = s_j^{\Lambda}(t_1-) + \eta^{(1)}_{jJ_1} > \eta^{(1)}_{J_1J_1} = s^{\Lambda}_{J_1}(t_1).$$

The later implies that the J_1th coordinate hits zero infinitely often, while its neighbour coordinates $s_j(t)$ never hits zero after moment t_1 because of the condition $\eta_1 > \eta_0$. Thus the nodes $j \in \{J_1-1, J_1+1\} \cap \Lambda$ belong to a trap. This gives us the following information about the (random) trap $M^{(\Lambda)} = \xi$:

$$\xi_{J_1} = 0, \ \ \xi_j = 1, \ j \in \{J_1-1, J_1+1\} \cap \Lambda.$$

Notice that the J_1th node divide Λ into (at most) two parts: $\Lambda^{(1,1)} := (-N, \ldots, J_1 - 2)$ and $\Lambda^{(1,2)} := (J_1+2, \ldots, N)$, which behave independently after moment t_1. Clearly, one of the last two sets can be empty. In the case when $J_1 - 2 = -N$ (or $J_1 + 2 = N$), naturally we set $\xi_{-N} = 0$ (or $\xi_N = -1$). Since moment t_1 the dynamics of the system on any set $\Lambda^{(1,k)}$ with $|\Lambda^{(1,k)}| > 1$, $k = 1, 2$, is repeated. More precisely, any coordinate decrease until the moment when one of them, say J_2^k, hits zero. (Note, that again J_2^k is distributed uniformly over the sites of $\Lambda^{(1,k)}$ so that $P\{J_2^k = j\} = \frac{1}{|\Lambda^{(1,k)}|}$ for any $j \in \Lambda^{(1,k)}$.) Then we have

$$\xi_{J_2^k} = 0, \text{ and } \xi_j = 1 \text{ for } j \in \{J_2^k - 1, J_2^k + 1\} \cap \Lambda^{(1,k)}.$$

Again, the J_2^k node divide set $\Lambda^{(1,k)}$ into two parts, and the dynamics on any of these parts are repeated independently as just described. If some part contains only one node, say i, we define the corresponding ith coordinate of the trap ξ to be $\xi_i = 0$. Thus, continuing this procedure we find the trap of the system.

Next using the above description of the local structure of traps, we obtain:

$$p'(E_1, 0) = \lim_{N\to\infty} P\{\xi_{J_1+2} = 0\}, \tag{39}$$

(where J_1 is random value on $\Lambda = \Lambda_{N,1}$ as defined above). To compute this probability we note that it can be rewritten as follows:

$$p'(E_1, 0) = \lim_{K\to\infty} P\{\xi_1 = 0 \mid \xi_0 = 1, \xi_K = 0\}, \tag{40}$$

where we use the fact that the thermodynamical limit is traslationary invariant. Consider for $k \geq 1$

$$p_k := P\{\xi_1 = 0 \mid \xi_0 = 1, \xi_k = 0\}.$$

It is easy to derive from our construction that

$$p_1 = 1, p_2 = 0, p_3 = 1. \tag{41}$$

Further for $k \geq 4$ we obtain the following recurrent formula:

$$p_k = \sum_{n=1}^{k-2} P\{\xi_1 = 0 \mid \xi_0 = 1, \xi_n = 0\} P\{\xi_n = 0\}, \tag{42}$$

where we associate the event $\{\xi_n = 0\}$ with the event that the minimum of the i.i.d.r.v's. numerated by $\{1, \ldots, k-2\}$ occurs at the site n. Then we obtain from (42)

$$p_k = \frac{1}{k-2}(p_1 + \ldots + p_{k-2}) =: \frac{1}{k-2} S_{k-2}. \tag{43}$$

The last equality gives us the following recurrent relation:

$$p_{n+1} = \frac{1}{n-1}(S_{n-2} + p_{n-1}) = \frac{1}{n-1}((n-2)p_n + p_{n-1}), \tag{44}$$

from where we derive for $\delta_{n+1} := p_{n+1} - p_n$:

$$\delta_{n+1} = -\frac{1}{n-1}\delta_n. \tag{45}$$

Taking into account (41) we obtain from (45)

$$\delta_{n+1} = \frac{(-1)^n}{(n-1)!},$$

which readily gives us

$$\lim_{n\to\infty} p_n = p_4 + \sum_{n=5}^{\infty} \delta_n = 1 - e^{-1}. \tag{46}$$

This together with (40) finishes the proof of (38).

Furthermore, it is easy to see that (15) follows from (38).

Computing the invariant measure for the Markov chain $\{\xi'\}_{n\geq 0}$ (just as we did in the proof of Lemma 1), we derive (16). This completes the proof of Lemma 2, which in turn proves Theorem 4.

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