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Tangent cones in Besov spaces

Murali Rao* and J. Sokołowski †

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Abstract: Tangent cones for obstacle problem arise when studying differentiability of metric projection. Characterising the tangent cones is the first step in these considerations. This has been done in some of the Sobolev spaces using Hilbert space methods. In this article we describe these tangent cones precisely, using non–linear potential theoretic ideas, for all Besov spaces $B_{\alpha}^{p,q}$, $1 < p < \infty$, $1 < q < \infty$, $\alpha > 0$.

Key-words: tangent cone, Radon measure, capacity, capacitary potential, balayage, kernel, Besov space.

(Résumé : tsvp)

* Department of Mathematics, 358 Little Hall, University of Florida, Gainesville, FL 32611, USA

† Institut Elie Cartan, Laboratoire de Mathématiques, Université Henri Poincaré Nancy I, B.P. 239, 54506 Vandoeuvre lès Nancy Cedex, France and Systems Research Institute of the Polish Academy of Sciences, ul. Newelska 6, 01-447 Warszawa, Poland; e-mail: sokolows@iecn.u-nancy.fr

Unité de recherche INRIA Lorraine
Technopôle de Nancy-Brabois, Campus scientifique,
615 rue de Jardin Botanique, BP 101, 54600 VILLERS LÈS NANCY (France)
Téléphone : (33) 83 59 30 30 – Télécopie : (33) 83 27 83 19
Antenne de Metz, technopôle de Metz 2000, 4 rue Marconi, 55070 METZ
Téléphone : (33) 87 20 35 00 – Télécopie : (33) 87 76 39 77

Les cones tangents dans les espaces de Besov

Résumé : Dans cet article nous caractérisons les cones tangents dans les espaces des Besov pour des convexes déterminés par "obstacles'. Pour cela, nous utilisons la théorie non linéaire du potentiel.

Mots-clé : cone tangent, mesure de Radon, capacité, potentiel capacitaire, balayage, noyau, espace de Besov

1 Introduction

The main result of the paper is the characterisation of tangent cones in Besov spaces for convex subsets determined by *obstacles*.

The main ideas stem from non–linear potential theory.

In Section 2 we introduce *kernels* on $\mathbb{R}^N$ with values in $l^{q'}(L^{p'}(\mathbb{M}))$ spaces. The conditions on these kernels are general enough to include the Besov $B_\alpha^{p,q}$ spaces, $1 < p < \infty$, $1 < q < \infty$, $\alpha > 0$. The action of any $l^q(L^p(\mathbb{M}))$ function on this kernel then determines a *potential* on $\mathbb{R}^N$ and the action of any measure on $\mathbb{R}^N$ determines a *potential* on $l^{q'}(L^{p'}(\mathbb{M}))$. The main result of this section, Theorem 1, characterizes the elements of $l^{q'}(L^{p'}(\mathbb{M}))$ which are non–negative on elements of $l^q(L^p(\mathbb{M}))$ giving rise to non–negative potentials on $\mathbb{R}^N$, as potentials of non–negative Radon measures on $\mathbb{R}^N$. Theorem 1, whose proof is quite simple will be seen, in Sections 3 and 4 to be quite useful.

In Section 3, it is shown how Theorem 1 can be used to introduce *balayage* and capacitary potential in this setup. These happen to be elements of smallest norm in suitable closed convex sets. It is shown that there are given as non–linear potentials. Even though capacitary potential appears in [1] in a more general setting, balayage has not, to our knowledge, been discussed in the non–linear setting. And the form of the capacitary potential alone does not seem sufficient for the results of section 4 – we need the form of the balayage. We propose to develop this non–linear balayage further in a later article.

Finally in Section 4, the results of Sections 2 and 3 and the characterization of Besov $B_\alpha^{p,q}$ spaces given in ([1], Chapter 4) allows us to precisely describe the form of the tangent cone $\mathfrak{T}_{\mathfrak{K}}(z)$ for any z in the convex set

$$\mathfrak{K} = \{f \in B_\alpha^{p,q}(\mathbb{R}^N) | f(x) \geq \psi(x) \quad \text{q.e.}\} \quad 1 < p, q < \infty, \alpha > 0\ .$$

This result of the paper generalizes the previous results [2],[3],[4], obtained in the framework of the Hilbert space theory of Sobolev spaces combined with the linear potential theory to the general setting of non–linear potential theory in the Besov spaces.

2 A general result

We derive a result for $l^q(L^p)$ spaces which will be useful for applications. The same result can be proved for the L^p spaces using the same arguments.

Let $\mathbb{M}$ be a measure space with a σ–finite measure ν. Let

$$X = \mathbb{R}^N \times \mathbb{M} \times \mathbb{N} ,$$

where $\mathbb{N}$ is the set of non–negative integers.

Fix $1 < p, q < \infty$. The spaces $l^q(L^p(\mathbb{M}))$ are defined by: the set of $f = f(y,n), y \in \mathbb{M}, n \in \mathbb{N}$ such that the following norm is finite

$$\|f\|_{p,q} = \left(\Sigma_{n\in\mathbb{N}} \|f(\cdot, n\|_p^q\right)^{\frac{1}{q}} < \infty$$

ie., the sequence $\|f(\cdot,n)\|_p \in l^q$. The space $l^q(L^p(\mathbb{M}))$ is a reflexiv Banach space with the dual space $(l^q(L^p(\mathbb{M})))' = l^{q'}(L^{p'}(\mathbb{M}))$.

For any $f = f(y,n)$ we write

$$\Sigma_{n\in\mathbb{N}} \int f(y,n)\nu(dy) = \int f(y,n)\nu(dydn) \ .$$

Definition 1 *By a kernel T on X we understand a non–negative function $T(x,y,n), x \in \mathbb{R}^N, y \in \mathbb{M}, n \in \mathbb{N}$, such that*

1. *$0 \le T$; for each fixed (y,n), $T(\cdot,y,n)$ is continuous on $\mathbb{R}^N$ with compact support. For each $x \in \mathbb{R}^N, T(x,y,n)$ is measurable in (y,n).*
2. *For each compact set $K \subset \mathbb{R}^N$*

$$\int_K T(x,y,n)dx \in l^{q'}(L^{p'}(\mathbb{M})) \ .$$

3. *There exists a non–negative $A \in l^q(L^p(\mathbb{M}))$ such that*

$$\Phi(x) = \Sigma_{n\in\mathbb{N}} \int T(x,y,n)A(y,n)\nu(dy) = \int T(x,y,n)A(y,n)\nu(dydn)$$

is stricly positive on $\mathbb{R}^N$.

Remark 1 *For all non–negative $f \in l^q(L^p(\mathbb{M}))$ the integral*

$$Tf = \int T(x,y,n) f(y,n) \nu(dydn)$$

is well–defined on $\mathbb{R}^N$. Tf is a lower semicontinuous function on $\mathbb{R}^N$, from condition 1 above.

For each non–negative measure μ on $\mathbb{R}^N$ the integral

$$\check{T}\mu = \int T(x,y,n) \mu(dx)$$

is a well–defined non–negative function on $\mathbb{M} \times \mathbb{N}$.

Remark 2 *Condition 2 above implies that for each $f \in l^q(L^p(\mathbb{M}))$,*

$$Tf \in L^1_{\text{loc}}(\mathbb{R}^N), \tag{1}$$

Moreover, the Hölder inequality implies that T maps $l^q(L^p(\mathbb{M}))$ continuously into $L^1(\mu)$ for each $\mu \in \mathfrak{M}$ where

$$\mathfrak{M} = \{\mu | \mu \text{ non–negative Radon measure }, \check{T}\mu \in l^{q'}(L^{p'}(\mathbb{M}))\} .$$

2.1 Quasi–null sets

Now, we need the notion of a quasi–null set or a set of capacity zero.

Definition 2 *A set $\mathcal{E}$ is called a quasi–null set if $\mathcal{E} \subset \mathcal{B}$ for a Borel set $\mathcal{B}$ such that*

$$\mu(\mathcal{B}) = 0 \quad \text{for all} \quad \mu \in \mathfrak{M} .$$

A countable union of quasi null sets is quasi–null.

A property holds quasi everywhere written q.e. if it holds except perhaps for a quasi–null set.

If $f \in l^q(L^p(\mathbb{M}))$ the set

$$\{x \mid T|f|(x) = \infty\}$$

is easily seen, using the Hölder inequality, to be quasi–null. In particular, for each $f \in l^q(L^p(\mathbb{M}))$ Tf is q.e. well defined.

Condition 2 above implies, of course, that quasi–null sets are of Lebesque measure zero.

We record another simple consequence:

Consequence. If a sequence $\{f_n\} \subset l^q(L^p(\mathbb{M}))$ tends to $f \in l^q(L^p(\mathbb{M}))$ then for a subsequence Tf_n tends to Tf pointwise quasi–everywhere.

Indeed, choose a subsequence $\{f_{n_i}\}$ so that

$$\Sigma_i \|f_{n_i} - f_{n_{i-1}}\|_{p,q} < \infty \ .$$

Then, by Fubini's theorem and the Hölder inequality, for any $\mu \in \mathfrak{M}$,

$$\Sigma_i \int |Tf_{n_i} - Tf_{n_{i-1}}| d\mu \leq \Sigma_i \int T|f_{n_i} - f_{n_{i-1}}| d\mu \leq \Sigma_i \|f_{n_i} - f_{n_{i-1}}\|_{p,q} \|\check{T}\mu\|_{p',q'} < \infty$$

Thus the μ–measure of the set

$$\{x \mid \Sigma_i |Tf_{n_i} - Tf_{n_{i-1}}|(x) = \infty\}$$

is zero, for every $\mu \in \mathfrak{M}$.

2.2 Main result

With this setup we have the following theorem.

Theorem 1 *For $g \in l^{q'}(L^{p'}(\mathbb{M}))$ the following are equivalent:*

1. $\int g(y,n) f(y,n) \nu(dydn) \geq 0$ *for all* $f \in l^q(L^p(\mathbb{M}))$ *such that* $Tf \geq 0$ *q.e. on* $\mathbb{R}^N$.
2. $g = \check{T}\mu$ *for some non–negative Radon measure* μ *on* $\mathbb{R}^N$.

Proof. We give the proof in simple steps.

Step 1 Let

$$\mathfrak{C} = \{h \in l^{q'}(L^{p'}(\mathbb{M})) | h = \check{T}\mu \text{ for some non–negative Radon measure } \mu \text{ on } \mathbb{R}^N\}$$

We claim $\mathfrak{C}$ is a closed convex cone in $l^{q'}(L^{p'}(\mathbb{M}))$.

The only item not at once clear is the closedness of $\mathfrak{C}$. Let $\{h_k\} \subset \mathfrak{C}$ converge to $h \in l^{q'}(L^{p'}(\mathbb{M}))$. By choosing a subsequence if necessary we may assume $h_k(y,n)$ converges to $h(y,n)$ μ–a.e.y for all n. We have if $h_k = \check{T}\mu_k$:

$$\int \Phi(x)\mu_k(dx) = \int h_k(y,n)A(y,n)\nu(dydn) \le \|h_k\|_{p',q'}\|A\|_{p,q} \ . \tag{2}$$

The right side is bounded in k because $\{h_k\}$ converges in $l^{q'}(L^{p'}(\mathbb{M}))$.

Now, Φ being a strictly positive lower semicontinuous function, is bounded below on compacts. From (2) one concludes that $\{\mu_k(K)\}$ is bounded for each compact K. By choosing a subsequence if necessary we may assume that there is a Radon measure μ such that

$$\lim_k \int \varphi(x)\mu_k(dx) = \int \varphi(x)\mu(dx)$$

for each continuous function φ with compact support in $\mathbb{R}^N$. Since by assumption, for each $(y,n) \in \mathbb{M} \times \mathbb{N}$, $T(x,y,n)$ is continuous in x and has compact support, and since $h_k(y,n)$ converges for ν–a.e.y and all n to $h(y,n)$, we have for ν–a.e.y and all n

$$h(y,n) = \lim_k h_k(y,n) = \lim_k \check{T}\mu_k(y,n) = \check{T}\mu(y,n) \ .$$

This shows that $h \in \mathfrak{C}$ and thus $\mathfrak{C}$ is closed.

Step 2

$$\begin{aligned} & \{f \in l^q(L^p(\mathbb{M})) | \int F(y,n)f(y,n)\nu(dydn) \ge 0 \text{ for all } F \in \mathfrak{C}\} \\ = \ & \{g \in l^q(L^p(\mathbb{M})) | Tg \ge 0 \text{ q.e. on } \mathbb{R}^N\} \end{aligned} \tag{3}$$

The set on the left in (3) clearly contains that on the right. On the other hand let $f \in l^q(L^p(\mathbb{M}))$ be such that

$$\int F(y,n)f(y,n)\nu(dydn) \ge 0 \quad \text{ for all } F \in \mathfrak{C} \ .$$

Since, $|f| \in l^q(L^p(\mathbb{M}))$ and $F \in l^{q'}(L^{p'}(\mathbb{M}))$

$$\infty > \int |f(y,n)|F(y,n)\nu(dydn) = \int |f(y,n)|T(x,y,n)\mu(dx)\nu(dydn)$$

Fubini's theorem permits interchange of the order of integration. Therefore, for all non–negative measures μ on $\mathbb{R}^N$ such that $\check{T}\mu \in l^{q'}(L^{p'}(\mathbb{M}))$

$$0 \leq \int f(y,n)\check{T}\mu(y,n)\nu(dydn = \int (Tf)(x)\mu(dx)\ . \tag{4}$$

If $\mu \in \mathfrak{M}$, for any Borel set $\mathcal{B}$, $\mu_{\mathcal{B}}$ also belongs to $\mathfrak{M}$, where $\mu_{\mathcal{B}}$ is the restriction of μ to $\mathcal{B}$. From (4) we see that the set $\{Tf < 0\}$ has μ–measure zero for every $\mu \in \mathfrak{M}$ ie., $Tf \geq 0$ q.e. Thus the sets are identical.

Step 3 This is the last step in the proof. To this end let $g \in l^{q'}(L^{p'}(\mathbb{M}))$ and suppose

$$\int g(y,n)f(y,n)\nu(dydn) \geq 0 \text{ for each } f \text{ such that } Tf \geq 0 \text{ q.e.} \tag{5}$$

We want to show that $g \in \mathfrak{C}$. If not, by the Hahn–Banach theorem, there is a function $\varphi \in l^q(L^p(\mathbb{M}))$ and $\alpha \in \mathbb{R}$ such that

$$\int F(y,n)\varphi(y,n)\nu(dydn) > \alpha \text{ for all } F \in \mathfrak{C} \tag{6}$$

$$\text{but } \int g(y,n)\varphi(y,n)\nu(dydn) < \alpha\ .$$

Now $\mathfrak{C}$ is a cone so $\lambda F \in \mathfrak{C}$ for all $\lambda > 0$. From the first inequality in (6) with F replaced by λF, we get

$$\lambda \int F(y,n)\varphi(y,n)\nu(dydn) > \alpha,\ \text{for all } \lambda > 0\ . \tag{7}$$

Dividing by λ and letting $\lambda \to \infty$ we infer that

$$\int F(y,n)\varphi(y,n)\nu(dydn) \geq 0 \quad \text{for all } F \in \mathfrak{C}$$

which implies from step 2

$$T\varphi(x) \geq 0 \quad \text{q.e.on } \mathbb{R}^N .$$

This last inequality in turn implies by (5) that

$$\int g(y,n)\varphi(y,n)\nu(dydn) \geq 0 .$$

But then, by the second inequality in (6) we must have $\alpha > 0$. This, however, cannot be valid by (7) since λ can be chosen arbitrarily small.

This contradiction completes the proof. □

3 Applications

Keeping the above notation, let $\mathbb{B}$ denote the space of functions on $\mathbb{R}^N$:

$$\mathbb{B} = \{Tf | f \in l^q(L^p(\mathbb{M}))\} . \tag{8}$$

If $u \in \mathbb{B}, u = Tf$, we define

$$\|u\| = \|u\|_{\mathbb{B}} \equiv \|f\|_{p,q} .$$

$\mathbb{B}$ is then a Banach space. By suitable choices of the kernel T and the space $\mathbb{M}$ we get the Besov space $B_\alpha^{p,q}$, $\alpha > 0$. We return to this later.

Remark 3 *Condition 2 on the kernel T guarantees that*

$$\mathbb{B} \subset L^1_{\mathrm{loc}}(\mathbb{R}^N) .$$

We have seen that if a sequence $\{u_k\}$ in $\mathbb{B}$ converges to $u \in \mathbb{B}$ then a subsequence of $u_k(x)$ converges to $u(x)$ for quasi every x.

Remark 4 *Using the strict convexity of L^p–spaces it is not difficult to see that, every closed convex set in $\mathbb{B}$ has a unique element of smallest norm.*

Theorem 1 permits the introduction of *balayage* into $\mathbb{B}$:

Let h be any measurable function on $\mathbb{R}^N$ and let

$$\mathfrak{C}_h = \{u \in \mathbb{B} | u \geq h \text{ q.e. } \} . \tag{9}$$

Assume $\mathfrak{C}_h$ is not empty.

From the above remarks we infer that $\mathfrak{C}_h$ is a closed convex subset of $\mathbb{B}$ and there is a unique element of smallest norm in $\mathfrak{C}_h$.

This element of smallest norm we call balayage of h and denote it by $\mathcal{R}h$.

Let us look at this a bit more closely. Let

$$\mathcal{R}h = T\varphi, \quad \varphi \in l^q(L^p(\mathbb{M})) .$$

Then $T\varphi \geq h$ q.e. and for any $t > 0$ and any $f \in l^q(L^p(\mathbb{M}))$ such that $Tf \geq 0$ q.e.

$$T(\varphi + tf) \geq h \quad \text{q.e.}$$

In the other words $\varphi + tf \in \mathfrak{C}_h$. By the definition of φ

$$\|\varphi\|_{p,q} \leq \|\varphi + tf\|_{p,q},\ t > 0,\ Tf \geq 0 \text{ q.e.} \tag{10}$$

Written in full (10) is the same as

$$\Sigma_n \|\varphi(\cdot, n)\|_p^q \leq \Sigma_n \|\varphi(\cdot, n) + tf(\cdot, n)\|_p^q \tag{11}$$

for all $t > 0$ and all f such that $Tf \geq 0$ q.e. Derivative relative to t of the n–th term on the right side is

$$q\|\varphi(\cdot, n) + tf(\cdot, n)\|_p^{q-p} \int |\varphi(y, n) + tf(y, n)|^{p-2}(\varphi(y, n) + tf(y, n))f(y, n)\nu(dy) , \tag{12}$$

whose absolute value is by the Hölder inequality dominated by

$$q\|\varphi(\cdot, n) + tf(\cdot, n)\|_p^{q-p}\|f(\cdot, n)\|_p\|\varphi(\cdot, n) + tf(\cdot, n)\|_p^{\frac{p}{p'}}$$

$$= q\|\varphi(\cdot, n) + tf(\cdot, n)\|_p^{q-1}\|f(\cdot, n)\|_p .$$

Using this estimate and the Hölder inequality we see that the series of derivatives of the terms on right side of (11) is uniformly convergent on compacts

Therefore, term by term differentiation of the right side in (11) is permissible and (11) says the derivative at $t=0$ is non–negative. From (12)

$$q\Sigma_n \|\varphi(\cdot,n)\|_p^{q-p} \int |\varphi(y,n)|^{p-2}\varphi(y,n)f(y,n)\nu(dy) \geq 0 \quad \text{if } Tf \geq 0 \ q.e. \tag{13}$$

The function

$$a = a(y,n) = \|\varphi(\cdot,n)\|_p^{q-p}|\varphi(y,n)|^{p-2}\varphi(y,n) \tag{14}$$

belongs to $l^{q'}(L^{p'}(\mathbb{M}))$ as can be verified using the Hölder inequality.

(13) and Theorem 1 imply

$$a = \check{T}\mu \tag{15}$$

for some non–negative Radon measure μ on $\mathbb{R}^N$. Using (14) and (15) we get

$$\varphi(y,n) = \|\check{T}\mu(\cdot,n)\|_{p'}^{q'-p'}(\check{T}\mu(y,n))^{p'-1} \tag{16}$$

This we state as:

Theorem 2 *Let h be any measurable function and suppose the closed convex set $\mathfrak{C}_h$ is not empty. There is a unique element $T\varphi$ of smallest norm in $\mathfrak{C}_h$ where φ is given by (16) for some non–negative Radon measure μ on $\mathbb{R}^N$.*

Remark 5 *We have not used any special properties of $\mathbb{R}^N$ or the Lebesque measure. Therefore, in Theorems 1 and 2, $\mathbb{R}^N$ can be replaced by a locally compact second countable space provided with a σ–finite measure satisfying condition 2 of Definition 1.*

In particular, if K is a compact subset of $\mathbb{R}^N$ which is not quasi null, there is a measure $\eta \in \mathfrak{M}$ such that $\eta(K) > 0$. Replace $\mathbb{R}^N$ and the Lebesque measure by K and the restriction of η to K to get the following, stronger version of Theorem 2:

Theorem 3 *Let K be a non–quasi null compact subset of $\mathbb{R}^N$, and let h be measurable and suppose the set*

$$\mathcal{C}_h = \{u \in \mathbb{B} | u \geq h \quad \text{q.e. on } K\}$$

is not empty. Then, there is a measure μ on K such that the unique element of the smallest norm in $\mathcal{C}_h$ is given by $T\varphi$ where

$$\varphi(y,n) = \|\check{T}\mu(\cdot,n)\|_{p'}^{q'-p'}(\check{T}\mu(y,n))^{p'-1} \ . \tag{17}$$

Specialising to $h = 1_K$ in Theorem 3 we get (by condition 3 of Definition 1 the set $\mathcal{C}_h$ is not empty).

Corollary 4 *To each compact set K corresponds a measure μ on K such that $T\varphi$, where φ is given by (17) satisfies*

$$T\varphi \geq 1, \quad \text{q.e. on } K$$

and $\|\varphi\|_{p,q}$ is minimum.

This unique element of $\mathbb{B}$ is called the *Capacitary Potential* of K and the *Capacity* of K is defined to be

$$C(K) = \|\varphi\|_{p,q}^{q} \ . \tag{18}$$

Using (17) we get

$$C(K) = \|\varphi\|_{p,q}^{q} = \|\check{T}\mu\|_{p',q'}^{q'} = \int T\varphi(x)d\mu(x) \ . \tag{19}$$

For more information on capacities we refer to [1].

4 Tangent cones in Besov spaces

Much of the considerations below are valid in the more general setup of the previous sections.

In this section we will denote by $\mathbb{B}$ the Besov space $B_\alpha^{p,q}(\mathbb{R}^N)$, where $\alpha > 0, 1 < p < \infty, 1 < q < \infty$. Our reference for Besov spaces is ([1], Chapter 4). $\mathbb{B}$ is a Banach space. The following characterisation of $\mathbb{B}$ will be used (see theorem 4.4.1, page 105 of [1]).

Fix a non–negative C^∞ function η on $\mathbb{R}^N$ with support in the unit ball $B(0,1)$. We will assume

$$\int \eta(x)dx = 1 \ .$$

Let

$$\eta_n(x) = 2^{nN}\eta(2^n x)$$

for $n \geq 0$ so that $\eta_0 \equiv \eta$.

Then a function $u \in \mathbb{B}$ iff there is a function sequence

$$f = \{f_n\} \in l^q(L^p(\mathbb{R}^N))$$

such that

$$u = \Sigma_0^\infty 2^{-n\alpha}\eta_n * f_n \tag{20}$$

We may take the $\mathbb{B}$–norm of u to be $\|f\|_{p,q}$. All choices of η give equivalent norms.

Remark 6 *u defined by (20) is in L^p. To see this let $g \in L^{p'}$. We have, denoting the $\left(L^p, L^{p'}\right)$ pairing by $\langle\cdot,\cdot\rangle$,*

$$\begin{aligned}\langle|u|,|g|\rangle \leq \Sigma_0^\infty 2^{-n\alpha}\langle\eta_n * |f_n|, |g|\rangle &\leq \Sigma_0^\infty 2^{-n\alpha}\|f_n\|_p\|\check{\eta}_n * |g|\|_{p'} \\ &\leq \left(\Sigma_0^\infty \|f_n\|_p^q\right)^{\frac{1}{q}}\left(\Sigma_0^\infty 2^{-n\alpha q'}\|g\|_{p'}^{q'}\right)^{\frac{1}{q'}} < \infty\end{aligned}$$

We have used:

$$\|\check{\eta}_n * |g|\|_{p'} \leq \|g\|_{p'}$$

because $\int \check{\eta}_n(x)dx = 1$, here $\check{\eta}_n(x) = \eta_n(-x)$.

Defining for $(x,y,n) \in \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{N}$

$$\begin{aligned}T(x,y,n) &= 2^{-n\alpha}\eta_n(x-y), \quad n \geq 0 \\ \nu(dy) &= dy \quad \text{Lebesque measure .}\end{aligned}$$

We can verify that Conditions 1,2,3 of Definition 1 are satisfied:

Condition 1 is immediately seen to be satisfied.

If $f \in L^p(\mathbb{R}^N)$

$$\left(\left|\int T(x,y,n)f(x)dx\right|\right)^p \leq 2^{-n\alpha p}\int \eta_n(x-y)|f(x)|^p dx$$

so that

$$\int dy \left(\left| \int T(x,y,n)f(x)dx \right|^p \right) \leq 2^{-n\alpha p} \|f\|_p^p$$

because $\int \eta_n(x-y)dy = 1$ for every n. Condition 2 follows.

Condition 3 is similarly easy to verify.

Thus the Besov space $B_\alpha^{p,q}$ fall under our setup. The meaning of the notion such as *q.e.* will be as before. For ease of reference let us emphasize that elements of $\mathbb{B}$ are of the form Tf for f in $l^q(L^p(\mathbb{R}^N))$ with norm $\|f\|_{p,q}$.

4.1 Tangent cone

Let $\psi \in \mathbb{B}$ and $\mathfrak{K}$ denote the closed convex set:

$$\mathfrak{K} = \{f \in \mathbb{B} | f(x) \geq \psi(x) \quad \text{q.e.}\} \tag{21}$$

Given $z \in \mathfrak{K}$, the *tangent cone* $\mathfrak{T}_\mathfrak{K}(z)$ *is the closure of the set*

$$\mathfrak{C}_\mathfrak{K}(z) = \{\varphi \in \mathbb{B} | \exists t > 0 \quad \text{such that} \quad z + t\psi \in \mathfrak{K}\} \ . \tag{22}$$

Both $\mathfrak{C}_\mathfrak{K}(z)$ and $\mathfrak{T}_\mathfrak{K}(z)$ are convex cones and contain all non–negative elements of $\mathbb{B}$. Put

$$\Xi = \{x | z(x) = \psi(x)\} \ . \tag{23}$$

Clearly every $v \in \mathfrak{T}_\mathfrak{K}(z)$ is non–negative q.e. on Ξ.

Since $\mathfrak{T}_\mathfrak{K}(z)$ is a closed convex cone, for each $V \in \mathbb{B}$, $V - \mathfrak{T}_\mathfrak{K}(z)$ is a closed convex set and contains a unique element of smallest norm. This element u_0 is the "projection" of V on the tangent cone:

$$\|V - u_0\| \leq \|V - u\|, \quad u \in \mathfrak{T}_\mathfrak{K}(z) \ . \tag{24}$$

As observed before, each non–negative element of $\mathbb{B}$ belongs to $\mathfrak{T}_\mathfrak{K}(z)$. Suppose then that $Tf \geq 0$ q.e. Since $\mathfrak{T}_\mathfrak{K}(z)$ is a cone, $u_0 + tTf \in \mathfrak{T}_\mathfrak{K}(z)$ for all $t > 0$:

From (24)

$$\|V - u_0\| \leq \|V - u_0 - tTf\| \ , \quad t > 0 \ , \quad Tf \geq 0 \quad \text{q.e.}$$

Arguing as in the last section we get

Theorem 5 *There is a Radon measure μ_0 such that*

$$V - u_0 = -T\varphi_0 \ , \tag{25}$$

where

$$\varphi_0 = \varphi_0(y,n) = \|\check{T}\mu_0(\cdot,n)\|_{p'}^{q'-p'} \left(\check{T}\mu_0(y,n)\right)^{p'-1} \ . \tag{26}$$

More can be said about the measure μ_0:

Theorem 6 *Let μ_0 be as in Theorem 5. Then*

1.

$$\int u d\mu_0 \geq 0 \ , \quad \forall u \in \mathfrak{T}_{\mathfrak{K}}(z) \ . \tag{27}$$

2. *μ_0 is concentrated on Ξ.*

3.

$$\int u_0 d\mu_0 = 0 \quad \text{ie., } \mu_0 \quad \text{is concentrated on} \quad \{u_0 = 0\} \ . \tag{28}$$

4. *If $\mu_0 \neq 0$,*

$$\int V d\mu_0 = -\int T\varphi_0(x)\mu_0(dx) = -\|\check{T}\mu\|_{p',q'}^{q'} < 0 \tag{29}$$

Proof.

1. Note that $\mathfrak{T}_{\mathfrak{K}}(z)$ being a cone, $u_0 + tu \in \mathfrak{T}_{\mathfrak{K}}(z)$ for each $t > 0$ and $u \in \mathfrak{T}_{\mathfrak{K}}(z)$. Hence

$$\|V - u_0\| \leq \|V - u_0 - tu\| \ .$$

Write $V - u_0 = -T\varphi$ and follow the proof leading to inequality (13) of the last section to get (27).

2. It is known that $\mathcal{D} = \mathcal{D}(\mathbb{R}^N)$ is a multiplier for $\mathbb{B}$: $u \in \mathbb{B}, \varphi \in \mathcal{D}$ implies $\varphi u \in \mathbb{B}$. See ([5], page 140).

Let $\varphi \in \mathcal{D}$, then $\varphi(z-\psi) \in \mathbb{B}$. Hence if $t = \|\varphi\|_\infty^{-1}$,

$$z - \psi + t\varphi(z-\psi) = (1+t\varphi)(z-\psi) \geq 0 \ .$$

It follows that

$$\varphi(z-\psi) \in \mathfrak{T}_{\mathfrak{K}}(z), \quad \varphi \in \mathcal{D} \ .$$

From the already established property 1

$$\int \varphi(z-\psi) d\mu_0 \geq 0, \quad \varphi \in \mathcal{D} \ .$$

This can only happen if $\int \varphi(z-\psi)d\mu_0 = 0$ ie., μ_0 is concentrated on Ξ.

3. $u_0 \in \mathfrak{T}_{\mathfrak{K}}(z)$, hence $tu_0 \in \mathfrak{T}_{\mathfrak{K}}(z)$ for all $t > 0$.
Therefore,

$$\|V - u_0\| \leq \|V - u_0 + tu_0\| \quad \text{if } t \leq 1 \ .$$

Writing $V - u_0 = -T\varphi_0$ and following proof leading to the inequality (13) of the last section (as in 1. above) we get

$$\int u_0 d\mu_0 \leq 0 \ .$$

But u_0 being in $\mathfrak{T}_{\mathfrak{K}}(z)$, (27) and the last inequality give (28). Since $u_0 \geq 0$ q.e. on Ξ (all elements of $\mathfrak{T}_{\mathfrak{K}}(z)$ satisfy this) we see that μ_0 is concentrated on the set $\{u_0 = 0\}$.

4. Integrate both sides of (25) relative to μ_0 and use (28) to get (29). □

The following corollary characterizes the tangent cone $\mathfrak{T}_{\mathfrak{K}}(z)$:

Corollary 7 *$V \in \mathfrak{T}_{\mathfrak{K}}(z)$ if and only if*

$$V \geq 0 \quad \text{q.e. on} \quad \Xi$$

Proof.

Immediate from (29). □

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Unit´e de recherche INRIA Lorraine, Technop ˆole de Nancy-Brabois, Campus scientifique,
615 rue du Jardin Botanique, BP 101, 54600 VILLERS LÈS NANCY
Unit´e de recherche INRIA Rennes, Irisa, Campus universitaire de Beaulieu, 35042 RENNES Cedex
Unit´e de recherche INRIA Rh ˆone-Alpes, 655, avenue de l'Europe, 38330 MONTBONNOT ST MARTIN
Unit´e de recherche INRIA Rocquencourt, Domaine de Voluceau, Rocquencourt, BP 105, 78153 LE CHESNAY Cedex
Unit´e de recherche INRIA Sophia-Antipolis, 2004 route des Lucioles, BP 93, 06902 SOPHIA-ANTIPOLIS Cedex

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