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Abstract: Recently in the paper [6] there was established the following Gamma-type result. Given the number N of a homogeneous Poisson process' points defining a random figure, its volume is $\Gamma(N, \lambda)$ distributed, where λ is the intensity of the process. The goal of this paper is to give an alternative description of the class of the random sets for which the Gamma-type results hold. We show that it corresponds to the class of *stopping sets* with respect to the natural filtration of the point process with certain scaling properties. The proof is very short and uses the martingale technique for directed processes, in particular, the analog of the Doob's optional sampling theorem proved in [4]. Along with an elegance, this approach provides a new inside into the nature of geometrical objects constructed with respect to a point process. We show, in particular, that in the Poisson case the probability of a point to be covered by a stopping set does not depend on whether it is point of the Poisson process or not.

Key-words: stopping set, martingales, directed processes, Palm distributions, point process, Poisson process

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Ensembles d'arrêt: résultats de type Gamma et propriétés de recouvrement

Résumé : Récemment les résultats de type Gamma suivants ont été démontrés dans l'article [6]. Étant donné le nombre N des points d'un processus de Poisson homogène qui définissent une figure aléatoire, le volume de cette dernière est distribué selon une loi $\Gamma(N, \lambda)$, où λ est l'intensité du processus. Le but de cet article est de donner une autre description de la classe des ensembles aléatoires pour lesquels des résultats de ce type sont valides. Nous montrons que cette classe est peut être définie comme celle des *ensembles d'arrêt* par rapport à la filtration naturelle du processus ponctuel avec des certains propriétés de similarité. Le preuve est très courte et utilise une technique de martingale pour les processus directionnels, en particulier, l'analogue du théorème d'arrêt de Doob, montré dans [4]. Cette approche donne une nouveau point de vue sur la nature des objets géométriques construits à partir des points d'un processus. Nous montrons par exemple que dans le cas Poissonnien, la probabilité qu'un point soit contenu dans un ensemble d'arrêt ne dépend pas du fait que ce point est un point du processus ou non.

Mots-clé : ensemble d'arrêt, martingales, processus directionnel, distributions de Palm, processus ponctuel, processus de Poisson

1 Preliminaries

Let E be a locally compact separable topological space (LCS-space), $\mathbb{F}$, $\mathbb{K}$, $\mathbb{G}$ being the system of its closed, compact and open subsets, respectively. In a LCS-space there exists a countable base $\mathcal{D} = \{D_m\}$ of the topology, i. e. a countable family of compact sets such that for any $K \in \mathbb{K}$ there exists a monotone decreasing sequence $\{D_{m_j}\}$, $D_{m_j} \in \mathcal{D}$, $D_{m_j} \supseteq D_{m_{j+1}}$ such that $K = \cap_j D_{m_j}$ (that is denoted by $D_{m_j} \downarrow K$ in the sequel). Moreover, we also have $K_{(n)} \downarrow K$, where

$$K_{(n)} = \bigcap_{m \le n} \{D_m \in \mathcal{D} : \ D_m \supseteq K\} . \tag{1}$$

Indeed, since for any j we have $K \subseteq K_{(m_j)} \subseteq D_{m_j}$ then $\cap_j K_{(m_j)} = \cap_j D_{m_j} = K$ and since $K_{(n)}$ is monotone decreasing we also have (1). Property (1) reflects the fact that the set $\mathbb{K}$ is a *separable from above lattice* (with respect to the *inclusion* relation).

Let $(\Omega, \{\mathcal{F}_K\}, \mathbf{P})$ be an abstract filtered probability space. The *filtration* here is an ensemble of $\mathbf{P}$-completed σ-algebras $\mathcal{F}_K$ indexed by compact sets $K \in \mathbb{K}$ that possesses the following two properties:

- monotonicity: $\mathcal{F}_{K_1} \subseteq \mathcal{F}_{K_2}$ for any two compact $K_1 \subseteq K_2$;
- continuity from above: $\mathcal{F}_K = \cap_{n=1}^{\infty} \mathcal{F}_{K_n}$ if $K_n \downarrow K$.

A *random closed set* Ξ is a measurable mapping $\Xi : \ (\Omega, \{\mathcal{F}_K\}, \mathbf{P}) \mapsto [\mathbb{F}, \sigma_f]$, where σ_f is the σ-algebra generated by the system $\{F \in \mathbb{F} : \ F \cap K \neq \emptyset\}$, $K \in \mathbb{K}$. We say that Ξ is $\{\mathcal{F}_K\}$-adapted, if the random set $\Xi \cap K$ is $\mathcal{F}_K$-measurable for all $K \in \mathbb{K}$.

Definition 1. A random compact set Δ is called a *stopping set* (more precisely, $\{\mathcal{F}_K\}$-stopping set) if the event $\{\Delta \subseteq K\}$ is $\mathcal{F}_K$ measurable for all $K \in \mathbb{K}$.

Let $\mathcal{F} = \vee_{K \in \mathbb{K}} \mathcal{F}_K$. The *stopping σ-algebra* is the following collection:

$$\mathcal{F}_\Delta = \{A \in \mathcal{F} : \ A \cap \{\Delta \subseteq K\} \in \mathcal{F}_K \ \forall K \in \mathbb{K}\} .$$

It is easy to verify that $\mathcal{F}_\Delta$ is really a σ-algebra and that a compact set Δ is a stopping set if and only if it is $\mathcal{F}_\Delta$-measurable.

To avoid unnecessary trivialities we always assume that $\Delta \neq \emptyset$ almost surely.

Definition 2. A set indexed random process X_K, $K \in \mathbb{K}$ is called a *martingale* (more precisely, a $\big(\mathbf{P}, \{\mathcal{F}_K\}\big)$-*martingale*) if for all K_1, $K_2 \in \mathbb{K}$ such that $K_1 \subseteq K_2$ one has

$$\mathbf{E}[X_{K_2} \bigm| \mathcal{F}_{K_1}] = X_{K_1} \quad \mathbf{P}-a.\ s.$$

Since $\mathbb{K}$ is separable from above and the filtration is continuous from above then by Lemma 2.13 of [4] there exists an $\{\mathcal{F}_K\}$-adapted measurable modification $\widetilde{X}$ of a martingale X such that $\lim_{n\to\infty} \widetilde{X}_{K_{(n)}}(\omega) = \widetilde{X}_K(\omega)$ for every $\omega \in \Omega$ for which the limit above exists. In the sequel by *martingale* we mean, in fact, this adapted modification of the process.

The following statement, that is the analog of the Doob's optional sampling theorem, is an adaptation of the Kurtz' theorem [4, Theorem 2.15] to our situation:

Theorem 1. *Let $\Delta_1,\ \Delta_2$ be two a. s. compact stopping sets such that $\Delta_1 \subseteq \Delta_2$ almost surely. Let X_K be a martingale for which there exists a sequence $D_m \in \mathcal{D}$ such that*

$$\lim_{m\to\infty} \mathbf{E}\left[|X_{D_m}|\ \mathbb{1}\{\Delta_2 \not\subseteq D_m\}\right] = 0\,. \tag{2}$$

Assume, moreover, that

$$\lim_{n\to\infty} X_{K_{(n)}}(\omega) = X_K(\omega) \tag{3}$$

for any (K,ω) for which the limit above exists. Then

$$\mathbf{E}\left[X_{\Delta_2}\ \middle|\ \mathcal{F}_{\Delta_1}\right] = X_{\Delta_1}\ a.\ s. \tag{4}$$

provided $\mathbf{E}\,|X_{\Delta_2}| < \infty$.

Note that although $\mathbb{K}$ is a Hausdorff space (cf. [5, Th. 1.2.1]) but it is *not* a *topological* lattice. Indeed, the "operation of maximum" $\cup$ is continuous in the Borel topology on $\mathbb{K}$, (i. e. the topology generated by "intervals" $\mathbb{K}_L \stackrel{\text{def}}{=} \{K \in \mathbb{K} :\ K \subseteq L\},\ L \in \mathbb{K}$) but the "operation of minimum" $\cap$ is only upper semi-continuous (cf. [5, Corrolary 1 of Th. 1.2.4]). That does not, however, affect the proof of part (a) of the Kurtz' theorem that we use. Mention here that he recent state of the martingale theory for directed processes is summarized in book [2].

An important example of a martingale satisfying (2) is provided by a *likelihood ratio.* Namely, let $\mathbf{Q}$ and $\mathbf{P}$ be two probability measures on $\mathcal{F}$ such that $\mathbf{Q} \underset{loc}{\ll} \mathbf{P}$, i. e. for any $K \in \mathbb{K}$ the restriction $\mathbf{Q}^K$ of $\mathbf{Q}$ onto $\mathcal{F}_K$ is absolutely continuous with respect to the restriction $\mathbf{P}^K$ of $\mathbf{P}$ on the same σ-algebra . Denote the likelihood ratio by

$$L_K = \frac{d\mathbf{Q}^K}{d\mathbf{P}^K}\,, \tag{5}$$

The following proposition allows us to interpret L_Δ as the Radon-Nikodym derivative of $\mathbf{Q}^\Delta$ with respect to $\mathbf{P}^\Delta$ – the restrictions of $\mathbf{Q}$ and $\mathbf{P}$ onto $\mathcal{F}_\Delta$.

Proposition 2. *Assume the likelihood ratio* (5) *satisfies Condition* (3)*. Then for any $\mathbf{P}$-a. s. compact stopping set Δ one has*

$$L_\Delta = \frac{d\mathbf{Q}^\Delta}{d\mathbf{P}^\Delta}\,.$$

Proof. It is easy to verify that if Δ is a stopping set then Δ_K defined by

$$\Delta_K = \begin{cases} \Delta, & \text{if } \Delta \subseteq K \\ K, & \text{otherwise}\,. \end{cases}$$

is also a stopping set for any $K \in \mathbb{K}$. Next, if $A \in \mathcal{F}_\Delta$ then for any $K \in \mathbb{K}$ we have

$$A \cap \{\Delta \subseteq K\} \in \mathcal{F}_{\Delta_K}\,. \tag{6}$$

Indeed,

$$\big(A \cap \{\Delta \subseteq K\}\big) \cap \{\Delta_K \subseteq L\} = A \cap \{\Delta \subseteq K\} \cap \{\Delta \subseteq L\} \\ = A \cap \{\Delta \subseteq K \cap L\} \in \mathcal{F}_{K\cap L} \subseteq \mathcal{F}_L$$

for all $L \in \mathbb{K}$, that was needed. Therefore preserving the sign $\mathbf{E}$ for the expectation with respect to $\mathbf{P}$, we have the following chain of identities:

$$\mathbf{Q}\{A \cap \{\Delta \subseteq K\}\} = \mathbf{Q}^K\{A \cap \{\Delta \subseteq K\}\} \\ = \mathbf{E}^K L_K \mathbb{1}\{A \cap \{\Delta \subseteq K\}\} = \mathbf{E}\,\mathbf{E}\left[\mathbb{1}\{A \cap \{\Delta \subseteq K\}\}L_K \;\middle|\; \mathcal{F}_{\Delta_K}\right] \\ = \mathbf{E}\,\mathbb{1}\{A \cap \{\Delta \subseteq K\}\}\,\mathbf{E}\left[L_K \;\middle|\; \mathcal{F}_{\Delta_K}\right] \quad (7)$$

by (6). It is easy to see that L_K is a uniformly integrable $\mathbf{P}$-martingale so by Theorem 1 we have $\mathbf{E}\left[L_K \;\middle|\; \mathcal{F}_{\Delta_K}\right] = L_{\Delta_K}$. Thus the last expression of (7) equals

$$\mathbf{E}\,\mathbb{1}\{A \cap \{\Delta \subseteq K\}\}L_{\Delta_K} = \mathbf{E}\,\mathbb{1}\{A \cap \{\Delta \subseteq K\}\}L_{\Delta}\,.$$

Finally, substituting a sequence $K_n \uparrow E$ (that exists by local compactness of the space E) in place of K, we obtain for a $\mathbf{P}$-a. s. compact Δ and any $A \in \mathcal{F}_\Delta$ that

$$\mathbf{Q}\{A\} = \mathbf{E}\,\mathbb{1}\{A\}L_\Delta\,,$$

so the statement is proved. □

2 Gamma-type result and hitting properties

Let $\Lambda(\bullet)$ be a Radon measure on the Borel σ-algebra of E and let $\mathbf{P}_\rho$ be a family of the *Poisson process* distributions with the intensity measure $\rho\Lambda(\bullet)$, where ρ is a positive parameter. In this section $\{\mathcal{F}_K\}$ is the point process *canonical natural filtration*, i. e. Ω is the space of all locally finite counting measures on the Borel σ-algebra of subsets of E, Π is the identical mapping from Ω to Ω and $\mathcal{F}_K = \sigma\{\Pi(L),\ L \subseteq K,\ L \in \mathbb{K}\}$. Note that $\mathbf{P}_\rho \underset{loc}{\ll} \mathbf{P}_\nu$ for any $\rho,\ \nu > 0$ and the likelihood ratio is given by

$$L_K = \frac{d\mathbf{P}_\rho^K}{d\mathbf{P}_\nu^K}(\Pi) = \left(\frac{\rho}{\nu}\right)^{\Pi(K)} e^{-(\rho-\nu)\Lambda(K)},\ \forall K \in \mathbb{K}. \quad (8)$$

Since Λ is locally finite then with probability 1 there is no accumulation point of the process' points in any compact set. Therefore condition (3) holds for martingale L_K and by Proposition 2 for a compact stopping set Δ and a $\mathcal{F}_\Delta$-measurable F we have

$$\mathbf{E}_\rho F = \mathbf{E}_\nu \left(\frac{\rho}{\nu}\right)^{\Pi(\Delta)} e^{-(\rho-\nu)\Lambda(\Delta)} F \quad (9)$$

provided at least one part of this identity exists (compare with Theorem 1 in the paper [6]).

Now we are in position to prove the Gamma-type result, following the lines of Theorem 2 of the paper [6].

Theorem 3 (The Gamma-type result). *Let Δ be an a. s. compact stopping set and assume that*

$$\mathbf{P}_\rho\{\Pi(\Delta)=n\}>0 \text{ and does not depend on } \rho. \tag{10}$$

Then $\Lambda(\Delta)$ given $\Pi(\Delta)=n$ has the Gamma $\Gamma(n,\rho)$ distribution.

Proof. By (9) for any z we can write

$$\mathbf{E}_\rho\left[e^{z\Lambda(\Delta)}\bigm|\Pi(\Delta)=n\right]=\frac{\mathbf{E}_\rho\left[e^{z\Lambda(\Delta)}\,\mathbb{I}\{\Pi(\Delta)=n\}\right]}{\mathbf{P}_\rho\{\Pi(\Delta)=n\}}$$
$$=\frac{\mathbf{E}_\nu\left[e^{z\Lambda(\Delta)}\,\mathbb{I}\{\Pi(\Delta)=n\}\rho^n\nu^{-n}e^{-(\rho-\nu)\Lambda(\Delta)}\right]}{\mathbf{P}_\nu\{\Pi(\Delta)=n\}}\,.$$

Choosing now $\nu=\rho-z$ we see that the last expression simplifies to $(1-z/\rho)^{-n}$ that is the Laplace transform of the $\Gamma(n,\rho)$ distribution, *Q. E. D.* □

A generic example when (10) holds is the following. Assume there exists a family of automorphisms $\phi_\rho:\ E\mapsto E$ such that $\Lambda(\phi_\rho B)=\rho\Lambda(B)$ for any ρ and all Borel B. Let Π_1 be the Poisson process with the intensity measure Λ. It is easy to see that if $\mathbf{P}_1$ is its distribution, then the process $\Pi_\rho \stackrel{\text{def}}{=} \phi_\rho\Pi$ has the intensity measure $\rho\Lambda$ and hence, the distribution $\mathbf{P}_\rho$. Let Δ_ρ be a family of compact stopping sets and assume that $\mathbf{P}_\rho$-distribution of Δ_ρ coincides with the $\mathbf{P}_1$-distribution of $\phi_\rho\Delta_1$ for all $\rho>0$. Then, defining Δ_ρ as $\phi_\rho\Delta_1$, we obviously have

$$\Pi_\rho(\Delta_\rho)=\phi_\rho\Pi_1(\phi_\rho\Delta_1)=\Pi_1(\Delta_1)\ \mathbf{P}_1\text{–}a.\ s.$$

for any ρ, and hence (10) holds. This is the case, for example, when $E=\mathbb{R}^d\times Z$, and Λ is the direct product of the Lebesgue measure on $\mathbb{R}^d$ and a "mark distribution" Q on Z, ϕ_ρ is the dilation with coefficient $\rho^{-1/d}$ and the stopping set dilates together with dilations of the process' points. Possible examples include the minimal closed disc centered in the origin containing exactly n points of a homogeneous Poisson process, the circumball over the Delaunay simplex containing the origin, a typical "Voronoi flower" (also known as the *fundamental region* of a Voronoi cell) or their unions, and many others (see Examples 1, 2 and Sections 4, 5 in [6]).

The likelihood ratio (8) is a particular case of martingales of the type

$$X_K=\frac{\prod_{x_i\in\Pi\cap K}\exp f(x_i)}{\mathbf{E}\prod_{x_i\in\Pi\cap K}\exp f(x_i)}\,, \tag{11}$$

where f is any locally finite function. It corresponds to the Radon-Nikodym derivative (with respect to $\mathbf{P}$) of the distribution of the Poisson process with the intensity measure μ such that $\frac{d\mu}{d\Lambda}=\exp f(x)$. Since $\mathbf{E}\,X_K=1$ for all $K\in\mathbb{K}$ then under condition (2) we also have

$\mathbf{E}\, X_\Delta = 1$ for a compact stopping set Δ. Using the explicit expression for the *generating functional* of the Poisson process (cf. [7, p.117]) we obtain that for any function f

$$\mathbf{E}\, X_\Delta = \mathbf{E} \exp\left[\int_\Delta f(x)\, \Pi(dx) - \int_\Delta (e^{f(x)} - 1)\, \Lambda(dx)\right] = 1\,. \tag{12}$$

Put, for example, $f(x) = \sum_i a_i\, \mathbb{1}_{A_i}(x)$, where $a_i \in \mathbb{R}$ and $\{A_i\},\ i = 1, \dots, I$ is a finite family of disjoint compact sets. Then (12) becomes

$$\mathbf{E}\left[\prod_{i=1}^{I} (1 - z_i)^{\Pi(A_i \cap \Delta)}\, e^{z_i \Lambda(A_i \cap \Delta)}\right] = 1\,, \tag{13}$$

where $z_i = 1 - e^{a_i}$.

The development of the function in the LHS of (13) in the point $z_1 = \cdots = z_I = 0$ is given by

$$\mathbf{E} \prod_{i=1}^{I} \sum_{m_i=0}^{\infty} \frac{z_i^{m_i}}{m_i!} \sum_{k_i=0}^{m_i} \binom{m_i}{k_i} (-1)^{k_i} N_i^{(k_i)} V_i^{m_i - k_i}\,,$$

where

$$N^{(k)} = \begin{cases} 1 & \text{if } k = 0 \\ N(N-1)\dots(N-k+1) & \text{if } k > 0\,; \end{cases}$$

and $N_i = \Pi(A_i \cap \Delta),\ V_i = \Lambda(A_i \cap \Delta)$, for short. Identity (13) implies that the coefficient of every product $\prod_{i=1}^{I} z_i^{m_i}$ equals 0:

$$\mathbf{E} \prod_{i=1}^{I} \sum_{k_i=0}^{m_i} \binom{m_i}{k_i} (-1)^{k_i} N_i^{(k_i)} V_i^{m_i - k_i} = 0 \tag{14}$$

provided $\sum_{i=1}^{I} m_i > 0$.

Take the simplest case $I = 1$. The first two coefficients yield

$$\mathbf{E}\, \Pi(A \cap \Delta) = \mathbf{E}\, \Lambda(A \cap \Delta) \quad \text{and} \tag{15}$$

$$\mathbf{E}(\Pi(A \cap \Delta) - \Lambda(A \cap \Delta))^2 = \mathbf{E}\, \Pi(A \cap \Delta) = \mathbf{var}\, \Pi(A \cap \Delta)\,. \tag{16}$$

Considering the coefficient of $z_1 \dots z_I$ we get

$$\mathbf{E} \prod_{i=1}^{I} (\Lambda(A_i \cap \Delta) - \Pi(A_i \cap \Delta)) = 0\,,$$

in particular,

$$\mathbf{cov}\{\Lambda(A_i \cap \Delta) - \Pi(A_i \cap \Delta), \Lambda(A_j \cap \Delta) - \Pi(A_j \cap \Delta)\} = 0 \quad \text{for all } i \neq j.$$

Note, that by the refined Campbell formula (cf. [1, Proposition 12.1.IV]) we can rewrite the LHS of (15) as

$$\mathbf{E}\int \mathbb{1}_A(x)\,\mathbb{1}_{\Delta(\omega)}(x)\,\Pi(dx) = \int_A \mathbf{P}^x\{x\in\Delta\}\,\Lambda(dx)\,,$$

where $\mathbf{P}^x$ is the *local Palm distribution* of the process. Since it equals the RHS of (15) that is

$$\int_A \mathbf{P}\{x\in\Delta\}\,\Lambda(dx)$$

for any A, then we obtain the following curious result: for a compact stopping set Δ one has

$$\mathbf{P}^x\{x\in\Delta\} = \mathbf{P}\{x\in\Delta\} \tag{17}$$

for Λ-almost all x. Loosely speaking, the probability of a point x to be covered by a stopping set does not depend on whether x *is* a point of the Poisson process or not. The generalization of this fact is shown in Theorem 4 below but now we give a couple examples.

Example 1. Let Π be a Poisson process in $\mathbb{R}^d$ such that its intensity measure is infinite and charges the whole $\mathbb{R}^d$. The *Delaunay triangulation* consists of the 1-dimensional edges of the simplexes with the vertices in the points of Π such that the corresponding circumballs contain no point of Π inside. Let Δ be the circumball over the simplex $\mathcal{S}_0$ containing the origin. Then Δ is a compact stopping set and the probability that Δ contains a point x equals by (17) the probability that a process' point x is a vertex of $\mathcal{S}_0$.

Example 2. Let Π be as in the previous example. Consider the *Voronoi flower* $\mathcal{F}$ centered in the origin, i.e. the figure formed by the closed balls that have the origin and exactly d points of the process on their boundaries and no process' point inside. The centers of these balls are the vertices of the Voronoi cell constructed with respect to the process $\Pi\cup\{0\}$ with the nucleus in the origin $\{0\}$. Expectation of geometrical characteristics of this cell correspond to the Palm distribution of these characteristics. Under our assumptions on Π the flower $\mathcal{F}$ is an almost surely compact stopping set. By (17) and the fact that the Delaunay triangulation is dual to the Voronoi tessellation, we may conclude that the probability of a point x to be covered by $\mathcal{F}$ equals the probability that a Poisson process' point x is neighboring to the origin in the Delaunay triangulation.

Theorem 4. *Let ∇_x be an operator $\nabla_x\mathbf{Q} \stackrel{\text{def}}{=} \mathbf{Q}-\mathbf{Q}^x$, where $\mathbf{Q}^x$ is the local Palm distribution corresponding to the distribution $\mathbf{Q}$. Denote $\nabla_{x_1\ldots x_{k-1}x_k} = \nabla_{x_k}(\nabla_{x_1\ldots x_{k-1}})$. Then for a compact stopping set Δ, all $m\geq 1$ and for Λ^m-almost all distinct $x_1,\ldots,x_m$ the following identity holds:*

$$(\nabla_{x_1\ldots x_m}\mathbf{P})\,\{x_1,\ldots,x_m\in\Delta\} = 0\,.$$

In particular, one has (17) *and*

$$\mathbf{P}^{x_1x_2}\{x_1,\ x_2\in\Delta\}-\mathbf{P}^{x_1}\{x_1,\ x_2\in\Delta\}\\ -\mathbf{P}^{x_2}\{x_1,\ x_2\in\Delta\}+\mathbf{P}\{x_1x_2\in\Delta\}=0\quad\Lambda\times\Lambda-a.e..$$

Note that we do not assume here neither homogeneity of the Poisson process nor any scaling properties of the stopping set.

Proof. Denote the k-th order *factorial measure* of the counting measure Π by

$$\Pi^{(k)}(dx_1\dots dx_k)=\Pi(dx_1)\,(\Pi-\delta_{x_1})(dx_2)\dots(\Pi-\textstyle\sum_{i=1}^{k-1}\delta_{x_i})(dx_k)\,,$$

where $\delta_x(\cdot)=\mathbb{1}_{\{x\}}(\cdot)$ is the unit measure concentrated on $\{x\}$. Its expectation is known as the k-th order *factorial moment measure* and equals $\Lambda^k(dx_1\dots dx_k)\overset{\text{def}}{=}\prod_{i=1}^k\Lambda(dx_i)$ in the Poisson case (cf. [7, p.39]). Then $N_i^{(k_i)}=\Pi^{(k_i)}(A_i\cap\Delta)$ and the LHS of (14) can be rewritten (with convention $\Pi^{(0)}=\Lambda^0\equiv 1$) as

$$\begin{aligned}&\mathbf{E}\prod_{i=1}^{I}\sum_{k_i=0}^{m_i}\binom{m_i}{k_i}(-1)^{k_i}\\&\times\int_{A_i^{m_i}}\!\!\cdots\int\mathbb{1}\{x_1^i,\dots,x_{m_i}^i\in\Delta\}\,\Pi^{(k_i)}(dx_1^i\dots dx_{k_i}^i)\Lambda^{m_i-k_i}(dx_{k_i+1}^i\dots dx_{m_i}^i)\\&=\mathbf{E}\prod_{i=1}^{I}\int_{A_i^{m_i}}\!\!\cdots\int\sum_{s_i}(-1)^{|s_i|}\,\mathbb{1}\{x_1^i,\dots,x_{m_i}^i\in\Delta\}\,\Pi^{(|s_i|)}(d\boldsymbol{x}_{s_i})\Lambda^{m_i-|s_i|}(d\boldsymbol{x}_{s_i^c})\,,\end{aligned}$$

where the last sum is taken over all subsets $s_i\subseteq\{1,\dots,m_i\}$ with cardinality $|s_i|$ and $\boldsymbol{x}_{s_i}=\{x_j^i:\ j\in s_i\}$, $\boldsymbol{x}_{s_i^c}=\{x_j^i:\ j\notin s_i\}$. Since all A_i are disjoint the last expression is seen to be equal to

$$\begin{aligned}&\mathbf{E}\int_{A_1^{m_1}\times\cdots\times A_I^{m_I}}\!\!\cdots\int\sum_{s_1,\dots,s_I}(-1)^{|s_1|+\cdots+|s_I|}\,\mathbb{1}\{x_{l_i}^i\in\Delta\ \forall i=1,\dots,I,\ l_i=1,\dots,m_i\}\\&\times\prod_{i=1}^{I}\Pi^{(|s_i|)}(d\boldsymbol{x}_{s_i})\Lambda^{m_i-|s_i|}(d\boldsymbol{x}_{s_i^c})\\&=\mathbf{E}\int_{A_1^{m_1}\times\cdots\times A_I^{m_I}}\!\!\cdots\int\sum_{s\subseteq\{1,\dots,m\}}(-1)^{|s|}\,\mathbb{1}\{x_1,\dots,x_m\in\Delta\}\,\Pi^{(|s|)}(d\boldsymbol{x}_s)\Lambda^{m-|s|}(d\boldsymbol{x}_{s^c})\,.\end{aligned}\tag{18}$$

Here $m=\sum_{i=1}^I m_i$, $\{x_1,\dots,x_m\}=\{x_1^1,\dots,x_{m_1}^1,\dots,x_1^I,\dots,x_{m_I}^I\}$ and the sum is over all subsets s of $\{1,\dots,m\}$. In the Poisson case for an integrable function $g(x_1,\dots,x_k,\omega)$:

$E^k \times \Omega \mapsto \mathbb{R}$ one has

$$\mathbf{E} \int g(x_1, \dots, x_k, \omega)\, \Pi^{(k)}(dx_1 \dots dx_k) = \int \mathbf{E}^{x_1 \dots x_k}\, g(x_1, \dots, x_k, \omega)\, \Lambda^k(dx_1 \dots dx_k)\,,$$

where $\mathbf{E}^{x_1 \dots x_k}$ is the expectation with respect to the *k-fold Palm distribution* $\mathbf{P}^{x_1 \dots x_k}$ (cf. [3, p.110]). Applying this formula to (18) we finally obtain

$$\begin{aligned} &\int\limits_{A_1^{m_1} \times \dots \times A_I^{m_I}} \cdots \int \sum_{s \subseteq \{1,\dots,m\}} (-1)^{|s|} \mathbf{P}^{\boldsymbol{x}_s} \{x_1, \dots, x_m \in \Delta\}\, \Lambda^m(dx_1 \dots dx_m) \\ &= \int\limits_{A_1^{m_1} \times \dots \times A_I^{m_I}} \cdots \int (\nabla_{x_1 \dots x_m} \mathbf{P})\, \{x_1, \dots, x_m \in \Delta\}\, \Lambda^m(dx_1 \dots dx_m)\,. \end{aligned}$$

Since by (14) this is zero for all $A_1, \dots, A_I$ then the expression under the integral is zero Λ^m-almost everywhere. The proof is complete. □

A Appendix

In the appendix we give some auxiliary results concerning mainly measurability properties of the concerned objects. Throughout this section we use the notation $\sigma\{A,\ A \in \mathbb{A}\}$ for the σ-algebra generated by a system of sets $\{A\}_{A \in \mathbb{A}}$ and $\sigma\{\Xi\}$ for $\sigma\{\{\Xi \cap K \neq \emptyset\},\ K \in \mathbb{K}\}$.

The necessity of the next Proposition stems from the fact that the process $\Pi(K, \omega)$ considered as a process on $\mathbb{K} \times \Omega$ may *not* be progressively measurable, i. e. its restriction on $\mathbb{K}_L \times \Omega$ may not be $\mathcal{T}_L \otimes \mathcal{F}_L$-measurable. So the composition $\Pi(\Delta)$ may not be measurable neither. However, considering Π as a random set, Proposition 5 assures that $\Pi(\Delta)$ *is* measurable.

Proposition 5. *Let Ξ be a $\{\mathcal{F}_K\}$-adapted random closed set and Δ is an a.s. compact $\{\mathcal{F}_K\}$-stopping set. Then*

$$\sigma\{\Xi \cap \Delta\} \subseteq \mathcal{F}_\Delta\,.$$

The proof of this theorem is broken into several lemmas. The first lemma shows that the structure of the σ-algebra $\sigma\{\Xi\}$ is, in some sense, compatible with the structure of $\mathcal{F}_\Delta$.

Lemma 1. *If Ξ is a.s. compact then*

$$\sigma\{\Xi\} = \sigma\{\{\Xi \subseteq K\},\ K \in \mathbb{K}\}\,.$$

Proof. It is well known that in the case of a LCS space $\sigma\{\Xi\}$ is also generated by the system $\{\Xi \cap G = \emptyset\},\ G \in \mathbb{G}\}$ (cf. [5, p.27]). We have, therefore,

$$\sigma\{\Xi\} = \{\Xi \cap G = \emptyset\},\ G \in \mathbb{G}\} = \sigma\{\{\Xi \subseteq F\},\ F \in \mathbb{F}\}\,.$$

Since X is locally compact then there exists a sequence of compact sets W_n such that $X = \cup_n W_n$. But Ξ is a.s. compact, thus

$$\{\Xi \subseteq F\} = \bigcup_n \{\Xi \subseteq F \cap W_n\} \in \sigma\{\Xi \subseteq K\}, K \in \mathbb{K}\},$$

that ends the proof of the lemma. □

We need also the following lemma (in fact, only the necessity part of it):

Lemma 2. *Define*

$$\Delta_K = \begin{cases} \Delta, & \text{if } \Delta \subseteq K \\ K, & \text{otherwise}. \end{cases} \tag{19}$$

Then Δ is a compact stopping set if and only if Δ_K is $\mathcal{F}_K$-measurable for all $K \in \mathbb{K}$.

Proof. Necessity. We are to show that for any $K \in \mathbb{K}$ we have $\sigma\{\Delta_K\} \subseteq \mathcal{F}_K$. By Lemma 1 it is equivalent to $\{\Delta_K \subseteq L\} \in \mathcal{F}_K \ \forall K,\ L \in \mathbb{K}$. Indeed,

$$\begin{aligned}\{\Delta_K \subseteq L\} = \{\Delta_K \subseteq L,\ \Delta \subseteq K\} \cup \{\Delta_K \subseteq L,\ \Delta \nsubseteq K\}& \\ = \{\Delta \subseteq L \cap K\} \cup \{K \subseteq L\} \cap \{\Delta \nsubseteq K\}&.\end{aligned}$$

The first set belongs to $\mathcal{F}_{K\cap L}$ and thus to $\mathcal{F}_K$ by the definition of a stopping set. By the same reason the third set also belongs to $\mathcal{F}_K$, while the second is either $\emptyset$ or the whole Ω. The inclusion is proved.

Sufficiency. By Lemma 1 the $\mathcal{F}_K$-measurability of a compact random set Δ_K is equivalent to the fact that for all $L \in \mathbb{K}$ one has

$$\{\Delta_K \subseteq L\} = \{\Delta \subseteq L \cap K\} \in \mathcal{F}_K.$$

Thus taking $K_n \downarrow L$ we have

$$\{\Delta \subseteq L\} = \bigcap_n \{\Delta \subseteq K_n\} \in \bigcap_n \mathcal{F}_{K_n} = \mathcal{F}_L$$

by continuity of the filtration, so Δ is a stopping set. The lemma is proved. □

Proof of Theorem 5. It is sufficient to show that a generic element of $\sigma\{\Xi \cap \Delta\}$ that is $\{\Xi \cap \Delta \cap L \neq \emptyset\}$ belongs to $\mathcal{F}_\Delta$. Indeed,

$$\{\Xi \cap \Delta \cap L \neq \emptyset\} \cap \{\Delta \subseteq K\} = \{(\Xi \cap K) \cap (\Delta_K \cap L) \neq \emptyset\} \cap \{\Delta \subseteq K\}.$$

The set $\Xi \cap K$ is $\mathcal{F}_K$-measurable since Ξ is $\{\mathcal{F}_K\}$-adapted. $\Delta_K \cap L$ is $\mathcal{F}_K$-measurable by Lemma 2. The intersection of measurable sets is also measurable. Finally, $\{\Delta \subseteq K\} \in \mathcal{F}_K$ since Δ is a stopping set, *Q. E. D.* □

Although we do not use it in our considerations, the following proposition is interesting on its own. It allows one in the case of the natural filtration generated by a random closed set Ξ to interpret a stopping set Δ as such a function $\Delta(\Xi)$ that $\Delta(\Xi|_\Delta) = \Delta(\Xi)$, where $\Xi|_\Delta$ is the restriction of Ξ onto a random set Δ (note that $\Xi|_\Delta$ is *not* just $\Xi\cap\Delta$ since we also know Δ outside of $\Xi \cap \Delta$). This corresponds to the intuition that to decide whether a compact set is stopping one needs to know only configuration of Ξ inside it. This observation make a link to the object Δ_ρ of our considerations in [6] and shows that it is in fact a stopping set.

Proposition 6. *Let $\{\mathcal{F}_K\}$ be the natural filtration of the random closed set Ξ, i. e. $\mathcal{F}_K = \sigma\{\Xi \cap K\}$ for all $K \in \mathbb{K}$. Then for a compact $\{\mathcal{F}_K\}$-stopping set Δ one has*

$$\mathcal{F}_\Delta = \sigma\{\Xi \cap K, K \subseteq \Delta,\ K \in \mathbb{K}\}\,.$$

The difficult part of the proof is to show the inclusion

$$\mathcal{F}_\Delta \subseteq \sigma\{\Xi \cap K, K \subseteq \Delta,\ K \in \mathbb{K}\}\,. \tag{20}$$

Again we begin with a few lemmas from which the statement of the theorem will follow easily. Given a stopping set Δ_1 define the following σ-algebra :

$$\mathcal{F}_{\Delta_1^-} = \sigma\{A_L \cap \{L \subseteq \overset{\circ}{\Delta}_1\},\ A_L \in \mathcal{F}_L,\ L \in \mathbb{K}\}\,.$$

Lemma 3. *Let $\Delta,\ \Delta_1$ be two stopping sets such that $\Delta \subset \overset{\circ}{\Delta}_1$ almost surely. Then $\mathcal{F}_\Delta \subseteq \mathcal{F}_{\Delta_1^-}$.*

Proof. We are to check that an event $A \in \mathcal{F}$, such that $A \cap \{\Delta \subseteq K\} \in \mathcal{F}_K$ for any $K \in \mathbb{K}$ belongs to $\mathcal{F}_{\Delta_1^-}$.

Since the space X is LCS then there exists a countable system of compacts $\mathcal{Q}$ such that for any compact K and open $G \supset K$ there exists a compact set $L \in \mathcal{Q}$ such that $K \subseteq L \subset G$. Since up to negligible sets

$$\Omega = \bigcup_{L\in\mathcal{Q}} \{\Delta \subseteq L \subset \overset{\circ}{\Delta}_1\}$$

then almost surely

$$A = \bigcup_{L\in\mathcal{Q}} A \cap \{\Delta \subseteq L \subset \overset{\circ}{\Delta}_1\} = \bigcup_{L\in\mathcal{Q}} (A \cap \{\Delta \subseteq L\}) \cap \{L \subset \overset{\circ}{\Delta}_1\}\,. \tag{21}$$

Here $A \cap \{\Delta \subseteq L\} \in \mathcal{F}_L$ since $A \in \mathcal{F}_\Delta$ and therefore $(A \cap \{\Delta \subseteq L\}) \cap \{L \subset \overset{\circ}{\Delta}_1\}$ belongs to $\mathcal{F}_{\Delta_1^-}$ by its definition. Therefore A also belongs to $\mathcal{F}_{\Delta_1^-}$ as a countable union (21) of $\mathcal{F}_{\Delta_1^-}$-measurable sets. □

Lemma 4. *For the natural filtration of Ξ and a stopping set Δ_1 one has*

$$\mathcal{F}_{\Delta_1^-} = \sigma\{\Xi \cap K,\ K \subset \overset{\circ}{\Delta}_1,\ K \in \mathbb{K}\}\,.$$

Proof. Trivial, because $\sigma\{A_K \cap \{K \subset \overset{\circ}{\Delta}_1\},\ A_K \in \mathcal{F}_K\}$ is also generated by $\{\{\Xi \cap K\} \cap \{K \subset \overset{\circ}{\Delta}_1\},\ K \in \mathbb{K}\}$ since $\mathcal{F}_K = \sigma\{\Xi \cap K\}$. □

Proof of Proposition 6. As it was shown in [4, Lemma 2.14], for the separable from above lattice and a stopping set Δ such that there exists $\overline{O}_m \supset \Delta$ for some $m \le n$, the set $\Delta_{(n)}$ is a stopping set and $\mathcal{F}_\Delta = \cap_n \mathcal{F}_{\Delta_{(n)}}$. Now by Lemma 3

$$\mathcal{F}_\Delta \subseteq \bigcap_n \mathcal{F}_{\Delta_{(n)}^-}\,.$$

But by Lemma 4 the last intersection is just

$$\bigcap_n \sigma\{\Xi \cap K,\ K \subset \overset{\circ}{\Delta}_{(n)},\ K \in \mathbb{K}\} = \sigma\{\Xi \cap K,\ K \subseteq \Delta,\ K \in \mathbb{K}\}\,,$$

that proves (20).

To show the opposite inclusion to (20) it is sufficient to show that $\{\Xi \cap K\} \cap \{K \subseteq \Delta\} \in \mathcal{F}_\Delta$, i.e. for any $L \in \mathbb{K}$

$$\{\Xi \cap K\} \cap \{K \subseteq \Delta \subseteq L\} \in \mathcal{F}_L\,. \tag{22}$$

If $K \not\subseteq L$ then the set above is $\emptyset$ that is in $\mathcal{F}_L$. If $K \subseteq L$ then $\Xi \cap K \subseteq \Xi \cap L$, so $\{\Xi \cap K\} \in \mathcal{F}_L = \sigma\{\Xi \cap L\}$, and (22) will be proved if we show that $\{K \subseteq \Delta \subseteq L\} \in \mathcal{F}_L$. But this event is equivalent to say that Δ is not contained in any subset of L that does not contain K:

$$\{K \subseteq \Delta \subseteq L\} = \bigcap_{\substack{D_n \in \mathcal{D} \\ K \setminus D_n \neq \emptyset}} \{\Delta \not\subseteq D_n \cap L\} \cap \{\Delta \subseteq L\}$$

that is $\mathcal{F}_L$-measurable. The proposition is completely proved. □

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References

[1] D. J. Daley and D. Vere-Jones. *An Introduction to the Theory of Point Processes.* Springer, New York, 1988.

[2] G.A. Edgar and L. Sucheston. *Stopping times and directed processes.* Cambridge Univ. Press, Cambridge, 1992.

[3] O. Kallenberg. *Random measures.* Akademie-Verlag and Academic Press, 1983.

[4] T. Kurtz. The optional sampling theorem for martingales indexed by directed sets. *Ann. Probab.*, 8:675–681, 1980.

[5] G. Matheron. *Random Sets and Integral Geometry.* Wiley, New York, 1975.

[6] J. Møller and S. Zuyev. Gamma-type results and other related properties of Poisson processes. *Adv. in Appl. Probab.*, 28:662–673, 1996.

[7] D. Stoyan, W.S. Kendall, and J. Mecke. *Stochastic Geometry and its Applications.* Wiley series in probability and mathematical statistics. Wiley, Chichester, second edition edition, 1995.

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